On the stochastic quantization method: characteristics and applications to singular systems

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Abstract

Introducing the generalized Langevin equation, we extend the stochastic quantization method so as to deal with singular dynamical systems beyond the ordinary territory of quantum mechanics. We also show how the uncertainty relation is built up to the quantum-mechanical limit with respect to fictitious time, irrespective of its initial value, within the framework of the usual stochastic quantization method.

1 Basic ideas of stochastic quantization method (SQM)

The Parisi-Wu stochastic quantization method (SQM) [1, 2] was so designed as to give quantum mechanics as the thermal equilibrium limit of a hypothetical stochastic process with respect to a new (fictitious) time other than the ordinary time. The Background idea is that a d-dimensional quantum system is equivalent to a $(d+1)$-dimensional classical system with random noise. We can consider the SQM to be a third method of quantization remarkably different from the conventional theories, i.e., the canonical and path-integral ones. The SQM has the following advantages:

1. We can quantize any dynamical system only on the basis of *equation of motion*, while the canonical method is based on *Hamiltonian* and the path-integral method on *Lagrangian*.

2. We can quantize the gauge field without resorting to the conventional gauge fixing procedure [3].

We deal with the dynamical system described by Euclidean action $S_E[q]$, where $q(x) = \{q_i; i = 1, 2, \cdots\}$ are dynamical variables and x is the ordinary time for particles or 4-dimensional coordinates for fields. As the first step, we show that SQM gives the same result as given by the conventional path-integral method:

$$< G > \; = \; C \int \mathcal{D}q G(q) \exp(-S_E[q]/\hbar) \; , \tag{1}$$

$$\Delta(x, x') \; = \; C \int \mathcal{D}q q(x) q(x') \exp(-S_E[q]/\hbar) \; , \tag{2}$$

where $< G >$ is the quantum-mechanical expectation value of an observable $G(q)$, $\Delta(x, x')$ is the propagator and C the normalization constant. In this paper we also observe how the uncertainty relation is built up to the quantum-mechanical limit within the framework of the hypothetical stochastic process of SQM.

According to the prescription of SQM, we set up the basic Langevin equation in the following way:

$$\begin{aligned} \frac{\partial q_i(x,t)}{\partial t} &= -\left.\frac{\delta S_E[q]}{\delta q_i(x)}\right|_{q=q(x,t)} + \eta_i(x,t) \ , & (3) \\ <\eta_i(x,t)>_\eta &= 0 \ , <\eta_i(x,t)\eta_j(x',t')>_\eta = 2\alpha\delta_{ij}\delta(x-x')\delta(t-t') \ , & (4) \end{aligned}$$

where t stands for the fictitious time, η_i for Gaussian white noises and α for the diffusion constant. Using its solution in the thermal equilibrium limit, we get the same expectation value as given by the conventional path-integral method. To show this situation more clearly, we need to use the Fokker-Planck equation corresponding to the Langevin equation.

Defining the probability distribution functional $\Phi[\phi, t]$ by

$$\int \mathcal{D}qG(q)\Phi[q,t] = < G(q_i^\eta(x,t) >_\eta \ , \tag{5}$$

we can derive the Fokker-Planck equation as

$$\frac{\partial}{\partial t}\Phi[q,t] = \hat{F}\Phi[q,t] \ , \quad \hat{F} = \alpha \int dx \sum_i \frac{\delta}{\delta q_i(x)}\{\frac{\delta}{\delta q_i(x)} + \frac{1}{\alpha}\frac{\delta S_E[q]}{\delta q_i(x)}\} \ , \tag{6}$$

where $\hat{F}$ is the Fokker-Planck operator. If the drift force $K_i(q,t) = -(\delta S_E[q]/\delta q_i)_{q=q(x,t)}$ has a damping effect, i.e. $(\delta S_E[q]/\delta q_i)_{q=q(x,t)} > 0$, we get the thermal equilibrium limit $(t \to \infty)$ as follows:

$$\Phi_{eq}[q] = C\exp(-\frac{1}{\alpha}S_E[q]) \ . \tag{7}$$

Putting $\alpha = \hbar$, therefore, we obtain the prescription of SQM:

$$\begin{aligned} \lim_{t\to\infty} < G(q_i^\eta(x,t) >_\eta &= \lim_{t\to\infty} \int \mathcal{D}qG(q)\Phi[q,t] \\ = \int \mathcal{D}qG(q)\Phi_{eq}[q] &= C\int \mathcal{D}qG(q)\exp(-\frac{1}{\hbar}S_E[q]) = < G > \ . & (8) \end{aligned}$$

2 Building-up of the uncertainty relation in the hypothetical stochastic process

Quantizing one-dimensional harmonic oscillator by means of SQM, let us see the dependence of the uncertainty relation on the fictitious time. The Euclidean action of the one-dimensional harmonic oscillator is given by

$$S[q] = \int dx_0 \left[\frac{M}{2}(\frac{dq}{dx_0})^2 + \frac{1}{2}M^2\omega^2 q^2\right] \ . \tag{9}$$

According to the prescription of SQM, we set up the Langevin equation of this harmonic oscillator as follows:

$$\begin{aligned} \frac{\partial}{\partial t}q(x_0,t) &= M^2\left[\frac{\partial}{\partial x_0^2} - \omega^2\right]q(x_0,t) + \eta(x_0,t) \ , & (10) \\ <\eta(x_0,t)>_\eta &= 0, \quad <\eta(x_0,t)\eta(x_0',t')>_\eta = 2\hbar\delta(x_0-x_0')\delta(t-t') \ . & (11) \end{aligned}$$

Solving this, we easily obtain the following dependence of the uncertainty relation on t:

$$(\Delta q(t))^2(\Delta p(t))^2 = \left[\frac{1}{2\pi}\int dk\rho(k)e^{-2M(k^2+\omega^2)t} + \frac{\hbar}{2M\omega}\frac{2}{\sqrt{\pi}}\int_0^{\sqrt{2M\omega^2 t}} e^{-z^2}dz\right]$$
$$\times\left[\frac{1}{2\pi}\int dk k^2\rho(k)e^{-2M(k^2+\omega^2)t} - \frac{\hbar M\omega}{2}\frac{2}{\sqrt{\pi}}\int_0^{\sqrt{2M\omega^2 t}} e^{-z^2}dz\right] , \tag{12}$$

where $\rho(k) = (\Delta q(k,0))^2$ is the initial value at $t = 0$.

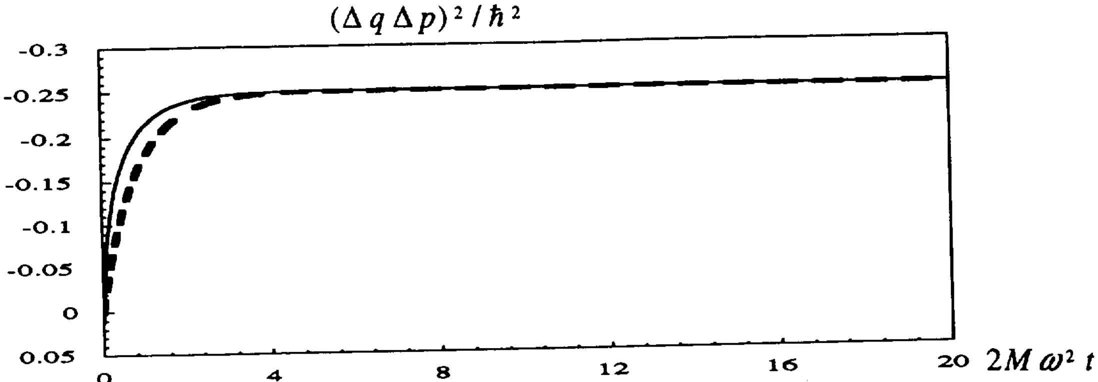

FIG. 1. We show how the uncertainty relation were built up with respect to the fictitious time. The solid line corresponds to the uncertainty relation for $\Delta q(k,0) = 0$, the dotted line to that for $\Delta q(k,0) \neq 0$. Note that the negative sign of $(\Delta q\Delta p)^2$ is due to the Wick rotation ($x_0 \to -ix_0$).

In FIG. 1, we can clearly see that the uncertainty relation in the hypothetical stochastic process approaches to the quantum-mechanical limit, irrespective of its initial values.

3 Generalized stochastic quantization method

We have many basic Langevin equations to give the same quantum mechanics [2]. By making use of this kind of freedom, we can go beyond the ordinary territory of quantum mechanics.

A generalized Langevin equation to give the same quantum mechanics is given by

$$\frac{\partial}{\partial t}\phi(x,t) = -\int d^dx'\ K(x,x';\phi)\frac{\delta S}{\delta\phi(x',t)} + \int d^dx'\frac{\delta K(x,x';\phi)}{\delta\phi(x',t)} + \int d^dx'\ G(x,x';\phi)\ \eta(x',t)\ , \tag{13}$$

$$< \eta(x,t) >= 0\ , \qquad < \eta(x,t)\eta(x',t') >= 2\delta^d(x-x')\delta(t-t')\ . \tag{14}$$

Note that we put $\hbar = 1$ hereafter. As an example, let us discuss the bottomless system described by scalar field $\phi(x)$ with the following action

$$S_E[\phi] = S_{\text{free}}[\phi] + S_{\text{int}}[\phi]\ , \tag{15}$$

x being d-dimensional Euclidean space-time point. $S_{\text{free}}[\phi]$ is the free part of the action and $S_{\text{int}}[\phi]$ the bottomless interaction part. We know that we can hardly quantize the bottomless system by means of the conventional quantization method. For

$$K(x,x';\phi) = \delta^d(x-x')K[\phi]\ , \qquad G(x,x';\phi) = \delta^d(x-x')K^{1/2}[\phi]\ , \tag{16}$$

we simplify the above generalized Langevin equation as

$$\frac{\partial}{\partial t}\phi(x,t) = -K[\phi]\frac{\delta S_{\mathrm{E}}[\phi]}{\delta\phi(x,t)} + \frac{\delta K[\phi]}{\delta\phi(x,t)} + K^{1/2}[\phi]\ \eta(x,t), \tag{17}$$

i.e.

$$\frac{\partial}{\partial t}\phi(x,t) = -K[\phi]\frac{\delta S_{\mathrm{K}}[\phi]}{\delta\phi(x,t)} + K^{1/2}[\phi]\ \eta(x,t), \tag{18}$$

where we have put $S_{\mathrm{K}} = S_{\mathrm{E}} - \ln K$. Provided that the drift force has a damping effect, that is to say, $S_{\mathrm{K}} = S_{\mathrm{E}} - \ln K > 0$, this Langevin equation has the thermal equilibrium limit. To satisfy this condition in the bottomless system, we may choose the Kernel as $K[\phi] = \exp\{S_{\mathrm{int}}\}$. In this case the generalized Langevin equation becomes

$$\frac{\partial}{\partial t}\phi(x,t) = -K[\phi]\frac{\delta S_{\mathrm{free}}[\phi]}{\delta\phi(x,t)} + K^{1/2}[\phi]\ \eta(x,t)\ . \tag{19}$$

Based on this equation, we can perform the numerical simulations of bottomless scalar field models and the bottomless hermitian matrix model.

4 Application to bottomless systems

A simple bottomless example [4] is given by

$$S[\phi] = S_2[\phi] - S_4[\phi]\ , \quad S_2[\phi] = \frac{1}{2}m^2\phi^2\ , \quad S_4[\phi] = \frac{\lambda}{4}\phi^4\ , \quad \lambda > 0\ , \tag{20}$$

where ϕ is a zero-dimensional field. If we put $K[\phi] = \exp(\lambda_K\phi^4)$, the well-posed condition mentioned in the preceding section becomes

$$S_K = S_E - \ln K = \frac{1}{2}m^2\phi^2 + \frac{1}{4}(\lambda_K - \lambda)\phi^4 > 0\ , \quad \text{i.e.} \quad \lambda_K \geq \lambda\ . \tag{21}$$

If we choose λ_K equal to λ, the Langevin equation reduces to

$$\frac{\partial}{\partial t}\phi(t) = -m^2\exp[-S_4]\phi(t) + \exp[-S_4/2]\eta(t)\ . \tag{22}$$

Based on this Langevin equation, we have numerically simulated the stochastic process of ϕ.

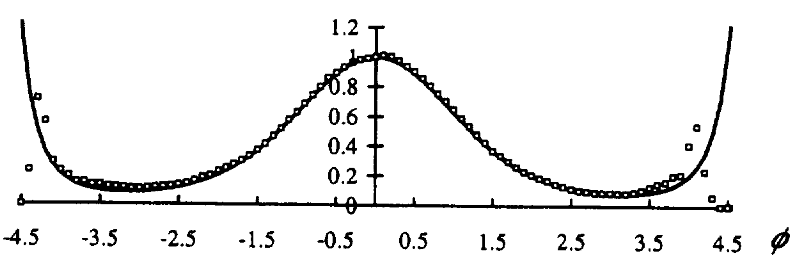

FIG. 2. Distribution of numerical solution of the Langevin equation (22) (open circles) for $\lambda = 0.1, m = 1$. For comparison, we plot the path-integral measure $\exp\{-S\}$ of the bottomless action (20) (solid line) for the same parameters.

FIG. 2 shows that the form of the distribution of numerical solution is consistent with the form of the path-integral measure $\exp\{-S\}$ of the bottomless action (20) in the central region of ϕ. From the prescription of SQM that at the thermal equilibrium limit we get the same expectation value as given by the path-integral method, we conclude that, in the central region of ϕ, the probability distribution of solutions of the generalized Langevin equation (22) is consistent with the path-integral measure $\exp\{-S\}$ even of the bottomless action.

As the next example, let us consider the bottomless hermitian matrix model [5], which is regarded as an important model of two-dimensional quantum gravity [6]. The partition function of $N \times N$ hermitian matrix model is given by

$$Z = \int d\phi \exp\{-S[\phi]\} \ , \quad d\phi \equiv \prod_{i=1}^{N} d\phi_{ii} \prod_{1 \le i < j \le N} d(\mathrm{Re}\phi_{ij}) d(\mathrm{Im}\phi_{ij}) \ . \tag{23}$$

Independent variables of the hermitian matrix model are $\mathrm{Re}\phi_{ij}$, $\mathrm{Im}\phi_{ij}$ $(i < j)$ and ϕ_{ii} with $i, j = 1, 2, \cdots N$. The action of the bottomless hermitian matrix model is given by

$$S[\phi] = S_{\mathrm{free}}[\phi] + S_{\mathrm{int}}[\phi] \ , \quad S_{\mathrm{free}}[\phi] = N\mathrm{tr}[\frac{1}{2}\phi^2] \ , \quad S_{\mathrm{int}}[\phi] = -N\mathrm{tr}[g\phi^4] \equiv -S_4[\phi] \ , \quad g > 0 \ . \tag{24}$$

For kernel $K[\phi] = \exp\{-S_4[\phi]\}$, the generalized Langevin equation becomes

$$\frac{\partial}{\partial t}\phi_{ii}(t) = -Ne^{-S_4}\phi_{ii}(t) + e^{-\frac{1}{2}S_4}\eta_{ii}(t) \ , \tag{25}$$

$$\frac{\partial}{\partial t}\phi_{ij}^{R,I}(t) = -2Ne^{-S_4}\phi_{ij}^{R,I}(t) + e^{-\frac{1}{2}S_4}\eta_{ij}^{R,I}(t) \ , \quad (i < j) \ . \tag{26}$$

The statistical properties of the Gaussian white noises must be subjected to

$$< \eta_{ii}(t) >_\eta = 0 \ , \quad < \eta_{ii}(t)\eta_{jj}(t') >_\eta = 2\delta_{ij}\delta(t - t') \ , \tag{27}$$

$$< \eta_{ij}^{A}(t) >_\eta = 0 \ , \quad < \eta_{ij}^{A}(t)\eta_{kl}^{B}(t') >_\eta = 2\delta^{AB}\delta_{ik}\delta_{jl}\delta(t - t') \ , \quad (i < j, k < l) \ , \tag{28}$$

$$< \eta_{ij}^{A}(t)\eta_{mm}(t') >_\eta = 0 \ , \quad (i < j) \ , \quad (A, B = R, I) \ . \tag{29}$$

One of the most remarkable results is observed in the deviation of $< \mathrm{tr}\phi^2 > /N$ from the planar calculation [6], as shown in FIG. 3.

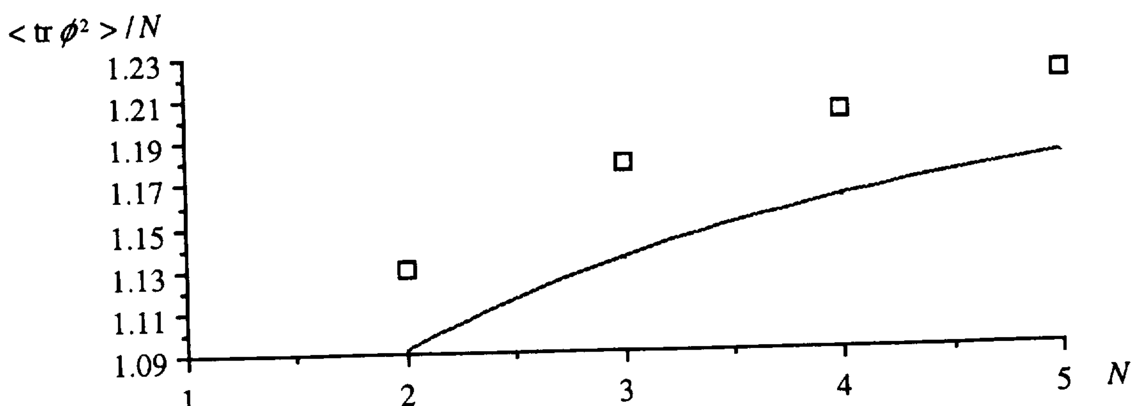

FIG. 3. Expectation values $< \mathrm{tr}\phi^2 > /N$ for various values of N(open squares). The solid line shows the planar result [6]

This deviation has so far been anticipated only from theoretical conjecture.

5 Conclusion

We have observed, within the framework of SQM, that the uncertainty relation will be built up to the quantum-mechanical limit, irrespective of its initial value, in a hypothetical stochastic process with respect to the fictitious time.

Introducing generalized (kerneled) Langevin equations, we have extended SQM so as to deal with singular dynamical systems beyond the ordinary territory of quantum mechanics. We also have attempted to quantize a few singular systems, such as bottomless systems, by means of SQM which is based on the generalized Langevin equations.

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