DAMPED OSCILLATOR WITH DELTA-KICKED FREQUENCY

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Abstract

Exact solutions of the Schrödinger equation for quantum damped oscillator subject to frequency δ–kick describing squeezed states are obtained. The cases of strong, intermediate, and weak damping are investigated.

1 Introduction

The aim of the paper is to consider parametric excitation of damped quantum oscillator. The parametric excitation is choosen in the form of very short pulse simulated by δ–kick of frequency. The damping is considered in the frame of Caldirola–Kanai model [1], [2]. This model is a partial case of the multidimensional system described by nonstationary Hamiltonian which is a general quadratic form in coordinates and momenta operators considered in [3], [4]. The problem of quantum oscillator with a time-dependent frequency was solved in [3]–[16]. In [3], [4] it was shown that the solutions for systems with quadratic Hamiltonian are expressed in terms of classical trajectory of the system. The case under consideration is interesting due to possibility of finding the classical trajectory in explicit form. The goal of this work is to extend the analysis of [13] to more simple one-oscillator case but taking into account the dissipation and to study the influence of the damping on the squeezing phenomenon for the kicked oscillator. Here the quantum dispersion of coordinate of damped oscillator is obtained in explicit form and the influence on squeezing phenomenon of strong, intermediate, and weak damping is studied.

2 Integrals of Motion

Let us consider the quantum damped parametric oscillator in the frame of Caldirola-Kanai model [1], [2] using the method of integrals of motion [3], [4], [7]. The Hamiltonian of the system is

$$\widehat{H} = \frac{1}{2}me^{2\gamma t}\omega^2(t)\widehat{x}^2 + \frac{1}{2m}e^{-2\gamma t}\widehat{p}^2, \tag{1}$$

where m is the mass of the oscillator, γ is the damping coefficient, $\widehat{x}$ and $\widehat{p}$ are the coordinate and momentum operators, and $\omega(t)$ is time-dependent frequency of the oscillator. The equation of motion for the classical coordinate x and momentum p are of the form

$$\dot{x} = pe^{-2\gamma t}, \quad \dot{p} = -\omega^2(t)e^{2\gamma t}x, \quad \ddot{x} + 2\gamma\dot{x} + \omega^2(t)x = 0. \tag{2}$$

The Heisenberg equation of motion for the position and momentum operators have the same form. Let us look in the Schrödinger representation for the integral of motion $\widehat{A}(t)$ which is linear in coordinate and momentum operators and satisfies the equation $[i\hbar\partial/\partial t - \widehat{H}, \widehat{A}] = 0$. At the initial moment of time, this integral of motion is equal to usual boson annihilation operator. Then for the operator $\widehat{A}$, one obtains the expression

$$\widehat{A}(t) = \frac{i}{\sqrt{2}} \left(\frac{\varepsilon(t)\widehat{p}l}{\hbar} - \frac{\dot{\varepsilon}e^{2\gamma t}\widehat{x}}{l\Omega(0)} \right), \tag{3}$$

where $\varepsilon(t)$ is the solution to the equation of motion

$$\ddot{\varepsilon}(t) + 2\gamma\dot{\varepsilon}(t) + \omega^2(t)\varepsilon(t) = 0, \quad \Omega^2(0) = \omega^2(0) - \gamma^2, \tag{4}$$

with initial conditions $\varepsilon(0) = 1$, $\dot{\varepsilon}(0) = i\Omega(0)$. In order the operator (3) and its hermitian conjugate satisfy at any time t the boson commutation relation, $\varepsilon(t)$ must satisfy the additional condition

$$e^{2\gamma t} (\dot{\varepsilon}\varepsilon^* - \dot{\varepsilon}^*) = 2i\Omega(0). \tag{5}$$

The eigenstates of operator (3) are the complete set of the squeezed correlated states of damped oscillator. Solving the equation $\widehat{A}(t)\Psi_\alpha(x,t) = \alpha\Psi_\alpha(x,t)$, where α is complex number, one can obtain these eigenstates in the explicit form

$$\Psi_\alpha(x,t) = (\pi\varepsilon^2 l^2)^{-1/4} \exp\left(\frac{i\dot{\varepsilon}e^{2\gamma t}x^2}{2\varepsilon l^2\Omega(0)} + \frac{\sqrt{2}\alpha x}{\varepsilon l} - \frac{\varepsilon^*\alpha^2}{2\varepsilon} - \frac{|\alpha|^2}{2} \right), \tag{6}$$

where $l^2 = \hbar/m\Omega(0)$. The wave functions in coordinate representation are gaussian packets with time- dependent coefficients in quadratic form under the exponential function. The density propability has consequently the gaussian form, too and the quantum dispersion of coordinate in the state (6) can be immediately obtained. It is of the form

$$\sigma_{x^2} = \langle\Psi_\alpha \mid \widehat{x}^2 \mid \Psi_\alpha\rangle - \langle\Psi_\alpha \mid \widehat{x} \mid \Psi_\alpha\rangle^2 = \frac{l^2 \mid \varepsilon \mid^2}{2}. \tag{7}$$

One can obtain for the quantum dispersion of momentum and for the squeezing coefficient,

$$\sigma_{p^2} = \langle\Psi_\alpha \mid \widehat{p}^2 \mid \Psi_\alpha\rangle - \langle\Psi_\alpha \mid \widehat{p} \mid \Psi_\alpha\rangle^2 = \frac{\hbar^2 e^{4\gamma t} \mid \dot{\varepsilon} \mid^2}{2l^2\Omega^2(0)}, \quad k = \frac{\sigma_{x^2}(t)}{\sigma_{x^2}(0)} = \mid \varepsilon \mid^2. \tag{8}$$

If $\mid \varepsilon \mid ^2 < 1$, which means that the disperssion of coordinate at the same moment of time t is less than at the initial one, the squeezing phenomenon appears. Due to this the states (6) are called squeezed correlated states as well as in the case without damping. Then all physical characteristics of the system are expressed through the solution of classical equation of motion $\varepsilon(t)$. The only remaining problem is to find explicit expression for $\varepsilon(t)$. In the following sections the explicit expressions for classical trajectories will be found for different regimes of damping.

3 The Case of Weak Damping

We consider a quantum damped oscillator with time-dependent frequency which varies in the specific manner of δ–kick

$$\omega^2(t) = \omega_0^2 - 2\kappa\delta(t),$$

where ω_0 is constant part of frequency, δ is Dirac delta-function. For $\varepsilon(t)$, we have the equation

$$\ddot{\varepsilon}(t) + 2\gamma\dot{\varepsilon}(t) + \omega_0^2\varepsilon(t) - 2\kappa\delta(t) = 0. \tag{9}$$

In this section we consider the case of weak damping, when $\omega_0 > \gamma$. Before and after δ–kick of frequency the solution to Eq. (9) is given by

$$\varepsilon_k(t) = A_k e^{-\gamma t + i\Omega t} + B_k e^{-\gamma t - i\Omega t}, \quad k = 0, 1, \tag{10}$$

where in the case of weak damping $\Omega = (\omega_0^2 - \gamma^2)^{1/2}$. Due to continuity conditions,

$$\varepsilon_0(0) = \varepsilon_1(0), \quad \dot{\varepsilon}_1(0) - \dot{\varepsilon}_0(0) = 2\kappa\varepsilon_0(0). \tag{11}$$

The coefficients A_k and B_k must satisfy the relations which can be expressed in matrix form

$$\begin{pmatrix} A_1 \\ B_1 \end{pmatrix} = \begin{pmatrix} 1 - i\kappa/\Omega & -i\kappa/\Omega \\ i\kappa/\Omega & 1 + i\kappa/\Omega \end{pmatrix} \begin{pmatrix} A_0 \\ B_0 \end{pmatrix}. \tag{12}$$

If $\varepsilon(-0) = 1$, $\dot{\varepsilon}(-0) = i\Omega$ at the initial instant, then $A_0 = 1 - i\gamma/2\Omega$, $B_0 = i\gamma/2\Omega$, one has for the classical trajectory after δ–kick,

$$\varepsilon_1(t) = \left[1 - \frac{i(\kappa + \gamma/2)}{\Omega}\right] \exp(-\gamma t + i\Omega t) - \frac{i(\kappa + \gamma/2)}{\Omega} \exp(-\gamma t - i\Omega t). \tag{13}$$

If before the first δ–kick the oscillator was in the state (6) with $\varepsilon(t) = e^{-\gamma t}(e^{i\Omega t} + \frac{\gamma}{\Omega}\sin\Omega t)$, the parametric excitation will transform it into a squeezed correlated state determined by (6) with $\varepsilon(t)$ given by (13). One can calculate the quantum dispersion of coordinate in excited correlated squeezed state, it is

$$\sigma_{x^2}(t) = \frac{\hbar e^{-2\gamma t}}{2m\Omega}\left[1 + \frac{\sin^2\Omega t}{\Omega^2}(2\kappa + \gamma)^2 + (2\kappa + \gamma)\frac{\sin 2\Omega t}{\Omega}\right]. \tag{14}$$

From the above expressions, we see that the maximum and minimum of $\sigma_{x^2}(t)$ and of squeezing coefficient $k^2(t) = \sigma_{x^2}(t)/\sigma_{x^2}(0)$ depend on ratio of the force of δ–kick and damping constant to the frequency of oscillations, while lower limit of squeezing coefficient is

$$\begin{aligned} k^2 &= \left[1 + 2\frac{(\kappa + \gamma/2)^2}{\Omega^2} - 2\frac{(\kappa + \gamma/2)}{\Omega^2}\sqrt{(\kappa + \gamma/2)^2 + \Omega^2}\right] \\ &\otimes \exp\left\{\frac{\gamma}{\Omega}\cos^{-1}\left[\frac{\Omega}{\sqrt{(\kappa + \gamma/2)^2 + \Omega^2}}\right] - \frac{\pi\gamma}{\Omega}(2n - 1)\right\}, \end{aligned} \tag{15}$$

$n = 0, 1, \ldots$. From the above formulae, one can see that the squeezing phenomenon can be achieved for all values of damping coefficient. So choosing kicks of frequency (increasing the force of δ–kick) we can squeeze quantum noise in coordinate even in the case of large (but smaller then ω_0) damping coefficient γ.

In the case of zero damping, formula (15) coincides with the result of [5] and [13] (for two-mode system). In the case of zero damping ($\gamma = 0$) for the limit of free particle ($\omega_0 = 0$), one δ–kick of frequency does not produce squeezing [17].

4 The Case of Strong Damping

Let us consider quantum damped oscillator in the regime of strong damping, when $\gamma > \omega_0$. In this case the solution to Eq. (9) before and after δ–kick of frequency is $\varepsilon_k = A_k e^{(\Omega-\gamma)t} + B_k e^{-(\gamma+\Omega)t}$ with frequency $\Omega = (\gamma^2 - \omega_0^2)^{1/2}$. Making the same procedure as in Section 2 one can obtain that after δ–kick, coefficients A_1 and B_1 are connected with the initial ones through the matrix equation

$$\begin{pmatrix} A_1 \\ B_1 \end{pmatrix} = \begin{pmatrix} 1+\kappa/\Omega & \kappa/\Omega \\ -\kappa/\Omega & 1-\kappa/\Omega \end{pmatrix} \begin{pmatrix} A_0 \\ B_0 \end{pmatrix}. \tag{16}$$

Taking the initial conditions in the form $\varepsilon(0) = 1, \quad \dot{\varepsilon}(0) = i\Omega$ one has $A_0 = \frac{1}{2}(1 + i + \gamma/\Omega)$, $B_0 + \frac{1}{2}(1 - i - \gamma/\Omega)$. The classical trajectory $\varepsilon(t)$ after δ–kick of frequency is

$$\varepsilon(t) = e^{-\gamma t}\left[\cosh \Omega t + \sinh \Omega t \left(i + \frac{\gamma}{\Omega} + \frac{2\kappa}{\Omega}\right)\right]. \tag{17}$$

The dispersion of coordinate after δ–kick of frequency takes the form

$$\sigma_{x^2}(t) = \frac{\hbar e^{-2\gamma t}}{2m\Omega}\left[\cosh 2\Omega t + \left(\frac{2\kappa+\gamma}{\Omega}\right)^2 \frac{\cosh 2\Omega t - 1}{2} + \left(\frac{2\kappa+\gamma}{2}\right)\sqrt{\cosh^2 2\Omega t - 1}\right]. \tag{18}$$

Since $\cosh \alpha \geq 1$, the dispersion cannot be less than $\hbar e^{-2\gamma t}/2m\Omega$, squeezing (by δ– kick of frequency) cannot exist in the system under study in the regime of strong damping.

5 Parametric Excitation of Free Particle Motion

In the last section we consider the case when the constant part of frequency is equal to zero but parametric excitation acts on the free particle motion. The gaussian wave packets for such systems without parametric excitation were considered in [18]– [20]. The equation for classical trajectory in case $\omega_0 = 0$ is

$$\ddot{\varepsilon}(t) + 2\gamma\dot{\varepsilon}(t) - 2\kappa\delta(t) = 0. \tag{19}$$

Before and after δ–kick the solution to this equation is given by expression: $\varepsilon_k = A_k + B_k e^{-2\gamma t}$. Applying the procedure used in Section 2 and continuity conditions one can obtain the relation

$$\begin{pmatrix} A_1 \\ B_1 \end{pmatrix} = \begin{pmatrix} 1+\kappa/\gamma & \kappa/\gamma \\ -\kappa/\gamma & 1-\kappa/\gamma \end{pmatrix} \begin{pmatrix} A_0 \\ B_0 \end{pmatrix}. \qquad (20)$$

Taking into account coeficients $A_0 = 1 + i/2$ and $B_0 = -i/2$, which coincide with the initial conditions considered above, the expession for classical trajectory after δ–kick can be obtained

$$\varepsilon(t) = 1 + \frac{\kappa}{\gamma}(1 - e^{-2\gamma t}) + \frac{i}{2}(1 - e^{-2\gamma t}). \qquad (21)$$

The excited states are determined by formula (6) with dispersion of coordinate (7), where $\varepsilon(t)$ is given by (21). The squeezing coefficient is

$$k^2 = 1 + \frac{\kappa^2}{\gamma^2}(1 - e^{-2\gamma t})^2 + \frac{2\kappa}{\gamma}(1 - e^{-2\gamma t}). \qquad (22)$$

From this expression, one can see that squeezing coefficient $k^2 > 1$ $(e^{-2\gamma t} < 1)$, for $t > 0$ and $\gamma > 0$. The squeezing can not be obtained for free damped particle by one δ–kick of frequency.

6 Conclusion

We have considered in the frame of Caldirola–Kanai model the parametric excitation of damped oscillator and discussed the influence of different regims of damping on the possibility of appearing the squeezing phenomenon in this system. It is worthy to note that different aspects of the damped oscillator problem was considered in [7], [18]–[27]. Here the parametric excitation is choosen in the special form (δ–kick of frequency), which permits to obtain explicit expressions for squeezing coefficient and quantum coordinate dispersion for different regimes of damping. It is shown that in the region of small damping the squeezing can be obtained for all $\gamma < \omega_0$ by choosing different force of δ–kick. In the region of strong damping and for damped free particle motion, it is impossible to have squeezing phenomenon by δ– kick of frequency.

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