GENERALIZED ENTROPIC UNCERTAINTY RELATIONS WITH TSALLIS' ENTROPY

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Abstract

A generalization of the entropic formulation of the Uncertainty Principle of Quantum Mechanics is considered with the introduction of the q-entropies recently proposed by Tsallis. The concomitant generalized measure is illustrated for the case of phase and number operators in Quantum Optics. Interesting results are obtained when making use of q-entropies as the basis for constructing generalized entropic uncertainty measures.

1 Introduction

The Uncertainty Principle (UP) can be stated quantitatively in the following fashion

$$\mathcal{U}(\hat{A}, \hat{B}; \psi) \geq \mathcal{B}(\hat{A}, \hat{B}) \tag{1.1}$$

where $\mathcal{U}$ is an estimation of the uncertainty in the result of a simultaneous measurement of two incompatible observables $\hat{A}$ and $\hat{B}$, when the system is in a state $|\psi\rangle$. What the UP asserts is that such an estimation is limited by an irreducible lower bound, the infimum $\mathcal{B}$, which merely depends on both operators. $\mathcal{U}$ must attain a fixed minimum value ($\mathcal{U}_{\min} \equiv 0$) if and only if $|\psi\rangle$ is a common eigenstate of $\hat{A}$ and $\hat{B}$, and $\mathcal{B}$ vanishes when the observables share at least one eigenvector.

The extension of Heisenberg's inequality to describe the UP for *arbitrary* pairs of operators (when their commutator is not a c-number) has been criticized because its r.h.s. is not a fixed lower bound [1]. Much effort has been devoted to present quantitative formulations of the UP (see, for example, refs. [1]–[8]). A central idea underlying these works is that the most natural measure of uncertainty is precisely the *missing information* [9] that remains once a measurement is made.

Deutsch first proposed [1] the use of Shannon's information-theory entropy [9] ($S(\{p_i\}) \equiv -\sum_{i=1}^{N} p_i \ln p_i$, for any probability distribution $\{p_i\}$) to measure uncertainty, in the following way

$$\mathcal{U}_1(\hat{A}, \hat{B}; \psi) = S(\hat{A}; \psi) + S(\hat{B}; \psi) \tag{1.2}$$

with the entropies calculated for the distributions $\{p_{\hat{A},i} = |\langle a_i|\psi\rangle|^2\}$ and $\{p_{\hat{B},j} = |\langle b_j|\psi\rangle|^2\}$, which correspond to the projections of $|\psi\rangle$ onto the bases of eigenvectors of $\hat{A}$ and $\hat{B}$, respectively. With reference to an $\hat{A}$-measurement, a system in a state with a probability distribution $\{\delta_{ii_0}\}$ has a "minimum lack of information" (or "maximum knowledge"), and then $S(\hat{A}; \psi) = S_{\min} = 0$. On the other hand, a uniform distribution $\{1/N\}$ characterizes a situation of "maximum ignorance", with $S(\hat{A};\psi) = S_{\max} = \ln N$.

It has been shown [1] that $\mathcal{U}_1$ satisfies

$$\mathcal{U}_1(\hat{A},\hat{B};\psi) \geq 2 \ln \frac{2}{1+c} \tag{1.3}$$

with $c = \sup_{ij} |\langle a_i|b_j\rangle|$. It was conjectured first by Kraus [3] and demonstrated later by Maassen and Uffink [4] that a better bound can be given,

$$\mathcal{U}_1(\hat{A},\hat{B};\psi) \geq 2 \ln \frac{1}{c} \tag{1.4}$$

Kraus specifically considered having two *complementary* observables: exact knowledge of the measured value of one of them implies maximum uncertainty in the other measurement, and consequently $|\langle a_i|b_j\rangle| = 1/\sqrt{N}$, for all $i,j = 1,\ldots,N$.

It seems natural to look for alternative descriptions of the UP expressed in entropic terms. In Section 2, we analyze the quantitative formulation of uncertainty in the spirit of Information Theory, with the aid of the recently introduced Tsallis' entropy [10], which is regarded as information measure [11]. We illustrate with a simple example, namely the phase–number uncertainty measures within the Pegg–Barnett formalism, and outline some conclusions in Section 3.

2 Tsallis' entropy as measure of uncertainty

A quite interesting generalization of the conventional entropy form has been recently advanced by Tsallis [10]. For any normalized probability distribution $\{p_i\}$, Tsallis' entropy reads

$$S_q(\{p_i\}) = \frac{1 - \sum_{i=1}^{N} p_i{}^q}{q-1} \tag{2.1}$$

where q is any real number, characterizing a particular statistics. (The sum must be carried out over non-zero probabilities.) The $q \to 1$ limit of (2.1) yields the Boltzmann–Shannon's logarithmic expression.

The physics is an extensive one only for $q = 1$ [10, 12]. Tsallis' entropy is related to the more familiar Rényi's entropy by $S_q^R = (\ln[1+(1-q)S_q])/(1-q)$. A crucial difference distinguishes these two *alternative* entropies, however. Tsallis' entropy always possesses a definite concavity, being a concave (convex) function of the probabilities for $q > 0$ ($q < 0$), which is not the case for Rényi's one. It is thus the former the generalized entropy recently employed in several distinct physical contexts. The generalized statistics associated to (2.1) has been shown to satisfy appropriate forms of Ehrenfest theorem [11], Jaynes' information-theory duality relations [11], von Neumann's equation [13], and the fluctuation–dissipation theorem [14, 15], among others. H-theorems and irreversibility have been in this connection also discussed [16, 17], as well as a possible connection with quantum groups [18], for instance. This nonextensive statistics has allowed, within an astrophysical context, to overcome the inability of the conventional, extensive one, to adequately deal (without infinities) with self-gravitating stellar systems, in what constituted the first physical application of the $q \neq 1$-theory [19]. A second application refers to Lévy flights, relevant for a variety of systems [20].

Some properties of the q-entropies are: i) $S_q \geq 0$ for any q and $\{p_i\}$, with $S_q = 0$ for $p_i = \delta_{ii_0}$ (certainty); ii) S_q reaches the extreme value $(1 - N^{1-q})/(q-1)$ for every q and $p_i = 1/N$ (equiprobability); iii) S_q is a non-increasing function of $q > 0$ for each $\{p_i\}$; iv) For two independent distributions $\{p_i\}$ and $\{p'_j\}$ (such that the joint probability is $p_{ij} = p_i p'_j$), it verifies that $S_q(\{p_{ij}\}) = S_q(\{p_i\}) + S_q(\{p'_j\}) + (1-q)\, S_q(\{p_i\})\, S_q(\{p'_j\})$.

We consider the new entropy as measure of uncertainty. Let us recall first that Heisenberg's relation, as well as the entropic relations given above, refer to *independent* measurements of the observables $\hat{A}$ and $\hat{B}$ on different microsystems in the same state $|\psi\rangle$. The UP states that the probability distributions obtained when $|\psi\rangle$ is projected on the corresponding eigenbases cannot be both arbitrarily peaked, given operators $\hat{A}$ and $\hat{B}$ "sufficiently non-commuting" [3]. The uncertainty measure appearing in eq. (1.2) takes into account the total information entropy associated to two independent probability distributions. Shannon's entropy is additive and $\mathcal{U}_1$ is just $S(\hat{A}) + S(\hat{B})$. We introduce now Tsallis' entropy to measure the amount of uncertainty, in the same spirit. The generalized expression reads

$$\mathcal{U}_q(\hat{A}, \hat{B}; \psi) \equiv S_q(\hat{A}; \psi) + S_q(\hat{B}; \psi) + (1-q)\, S_q(\hat{A}; \psi)\, S_q(\hat{B}; \psi) \tag{2.2}$$

where q is a positive parameter and the entropies are given by (2.1) for the probability sets $\{p_{\hat{A},i}\}$ and $\{p_{\hat{B},j}\}$. It is immediately seen that $\mathcal{U}_q \geq 0$, with $\mathcal{U}_q = 0$ if and only if $|\psi\rangle$ is a common eigenstate of $\hat{A}$ and $\hat{B}$. Besides this, $\mathcal{U}_q$ never exceeds $(1 - N^{2(1-q)})/(q-1)$. (We mention that these ideas can be extended to deal with pairs of observables with continuous spectra. However, one must be careful when defining the (generalized) information entropy for non-discrete distributions $\{p(x)\}$ [17, 21].)

A (weak) bound can be imposed on (2.2), namely

$$\mathcal{U}_q(\hat{A}, \hat{B}; \psi) \geq \frac{1}{q-1}\left(1 - \left(\frac{2}{1+c}\right)^{2(1-q)}\right) \tag{2.3}$$

which holds for any $q > 0$. By recourse to Riesz' theorem (as used in ref. [4]), it can be demonstrated that a better bound for $\mathcal{U}_q$ exists, at least in the region $1/2 \leq q \leq 1$:

$$\mathcal{U}_q(\hat{A}, \hat{B}; \psi) \geq \frac{1}{q-1}\left(1 - \left(\frac{1}{c}\right)^{2(1-q)}\right) \tag{2.4}$$

3 Example and conclusions

We shall apply our ideas to the phase and number operators in Quantum Optics. The treatment of optical states can be accomplished by recourse to the Pegg–Barnett (PB) formalism [22]. This implies working in a finite but arbitrarily large $(s+1)$-dimensional Hilbert space $\mathcal{H}^{s+1}$ spanned by the number states $|0\rangle_s, |1\rangle_s, \ldots, |s\rangle_s$, and taking the limit $s \to \infty$ at the end. The *Hermitian* phase operator is defined as

$$\hat{\Phi} = \sum_{m=0}^{s} \theta_m\, |\theta_m\rangle_s\, {}_s\langle\theta_m|, \qquad |\theta_m\rangle_s = \frac{1}{\sqrt{s+1}} \sum_{n=0}^{s} e^{in\theta_m} |n\rangle_s \tag{3.1}$$

The corresponding eigenvalues are $\theta_m = \theta_0 + 2\pi m/(s+1)$. **(Hereafter, the arbitrary reference** phase θ_0 will be set equal to 0.) The phase and number operators, $\hat{\Phi}$ and $\hat{N}$, are mutually complementary, with overlap $c = 1/\sqrt{s+1}$.

It is found that, for a system in a state $|\psi\rangle \in \mathcal{H}^{s+1}$, $\mathcal{U}_1(\hat{\Phi}, \hat{N}; \psi, s) \geq \ln(s+1)$ which diverges when $s \to \infty$. In order to extract some information out of this relation, Abe examined [5] the entropy differences from a certain reference state before going to the infinite-s limit. Number and phase eigenstates (which actually saturate that inequality) were chosen. Within the framework of Tsallis' information entropy, for a given $q > 0$ and a state $|\theta_m\rangle_s$, for instance, the entropies are given by $S_q(\hat{\Phi}; \theta_m, s) = 0$ and $S_q(\hat{N}; \theta_m, s) = (1 - (s+1)^{1-q})/(q-1)$. Consequently,

$$\lim_{s\to\infty} \mathcal{U}_q(\hat{\Phi}, \hat{N}; \theta_m, s) = \begin{cases} \infty, & \text{if } 0 < q \leq 1 \\ \frac{1}{q-1}, & \text{if } q > 1 \end{cases} \tag{3.2}$$

The same obtains for a number eigenstate. We stress that, considering generalized information entropies with $q > 1$, the divergence in the uncertainty for number or phase states is removed.

Let us consider the generalized entropic uncertainty measures for a system prepared in a phase coherent state (PCS). These states, recently found by Kuan and Chen [21], are given by

$$|z\rangle_s = \frac{1}{\sqrt{e_s(|\tilde{z}|^2)}} \sum_{m=0}^{s} \frac{\tilde{z}^m}{\sqrt{m!}} |\theta_m\rangle_s \tag{3.3}$$

where $\tilde{z} \equiv \sqrt{2\pi/(s+1)}\, z$ is a complex number and the normalizing function is given by $e_s(x) = \sum_{n=0}^{s} x^n/n!$. The projections of a PB PCS on phase and number eigenstates are

$$p_m \equiv |_s\langle\theta_m|z\rangle_s|^2 = \frac{1}{e_s(|\tilde{z}|^2)} \frac{|\tilde{z}|^{2m}}{m!} \tag{3.4}$$

and

$$p'_n \equiv |_s\langle n|z\rangle_s|^2 = \frac{1}{(s+1)\, e_s(|\tilde{z}|^2)} \left| \sum_{k=0}^{s} \frac{\tilde{z}^k e^{-in\theta_k}}{\sqrt{k!}} \right|^2 \tag{3.5}$$

respectively, with $m, n = 0, 1, \ldots, s$.

The $\hat{\Phi} - \hat{N}$ Heisenberg's inequality has been discussed for the $s = 1$ case [21]. We have analyzed the shapes of the phase and number q-entropies, for many different values of s. Within a given statistical frame of index q, the entropies $S_q(\hat{\Phi}; z, s)$ and $S_q(\hat{N}; z, s)$, will depend on both $|z|$ and s. The complementariness of $\hat{\Phi}$ and $\hat{N}$ is clearly seen. The phase entropy vanishes both for $|z| = 0$ (as it should for the vacuum phase state $|\theta_0\rangle$) and for $|z|$ sufficiently large. The number entropy has a minimum in the intermediate region (those PCS for which the entropy approach zero can be interpreted as "number-like" states). Those states are also of relatively low uncertainty. It can be seen that $(1 - (s+1)^{1-q})/(q-1)$ is a lower bound for the generalized uncertainty measure (see eq. (2.4)). This is obtained for arbitrary size of the PB space, s, or statistical parameter, q. Fig. 1 displays the q-entropies and the uncertainty $\mathcal{U}_q(\hat{\Phi}, \hat{N}; z, s)$ as a function of $|z|$, assuming particular values for both q and s.

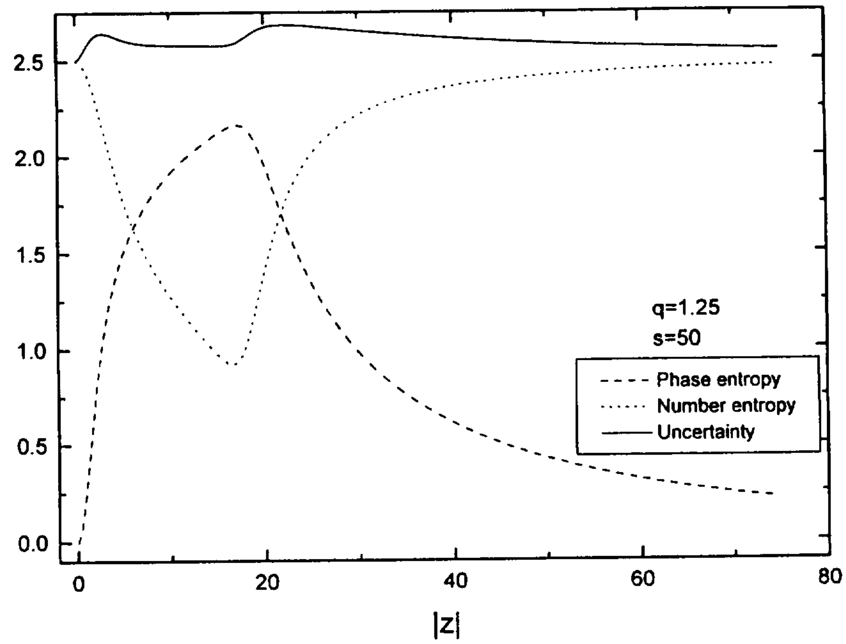

FIG. 1. Phase and number q-entropies and generalized uncertainty measure, for a PB PCS, as a function of coherence.

As a conclusion, generalized entropies recently introduced by Tsallis have been discussed in order to establish general uncertainty relations for the measurement of two quantum incompatible observables. Number and phase operators within the Pegg–Barnett formalism have been investigated in some detail. Interesting results are obtained when making use of q-entropies as the basis for constructing generalized entropic uncertainty measures.

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