QUANTUM PROBABILITY CANCELLATION DUE TO A SINGLE-PHOTON STATE

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Abstract

When an N-photon state enters a lossless symmetric beamsplitter from one input port, the photon distribution for the two output ports has the form of Bernouli Binormial, with highest probability at equal partition ($N/2$ at one outport and $N/2$ at the other). However, injection of a single photon state at the other input port can dramatically change the photon distribution at the outputs, resulting in zero probability at equal partition. Such a strong deviation from classical particle theory stems from quantum probability amplitude cancellation. The effect persists even if the N-photon state is replaced by an arbitrary state of light. A special case is the coherent state which corresponds to homodyne detection of a single photon state and can lead to the measurement of the wave function of a single photon state.

1 Introduction

Interference effect of light has played an important role in the conceptual development of quantum theory. Richard Feynmann once wrote[1] that the Young's double slit experiment "has in it the heart of quantum mechanics". But the phenomena of interference do not simply stop at Young's double slit experiment and its variations. Much richer phenomena occur in higher-order interference[2–6] when there are more than one particle involved in the process. For example, Greenberger et al.[7] recently proposed new demonstration of locality violation by quantum theory with superposition state of three or more particles.

In the meantime, along a quite different line, Ou and Mandel[8] have investigated a startling quantum interference effect where a strong field interferes with a considerably weak field. It was shown[3] and demonstrated[8] that for certain nonclassical fields, the interference fringe visibility does not change even though the ratio of the intensities of the two interfering fields is much greater than 1, in conflict with the intuitive picture from classical wave theory for interference. In this case, the seemingly insignaficant weak field plays an essential role for the interference effect even though its intensity is negligibly small. Therefore, the presence of the weak field can dramatically change the outcome of the result.

In this paper, we will present another example of how existence of a weak field can make a significant difference. It deals with $N+1$ photons with N being a positive integer. We will consider a situation when an N-photon state interferes with a single photon state with the help

of a symmetric lossless beamsplitter (see Fig.1). A special case of $N = 1$ has been experimentally investigated as an example of fourth-order interference.[9] However, quite different from the two-photon coincidence measurement technique used in fourth-order interference, we will exam photon probability distribution at two output ports of the beamsplitter. Although no interference pattern exists, the phenomenon discussed here attributes to quantum interference of multi-particle ($N + 1$ particles). We will also extend the discussion to an arbitrary state input in replacement of the N-photon state.

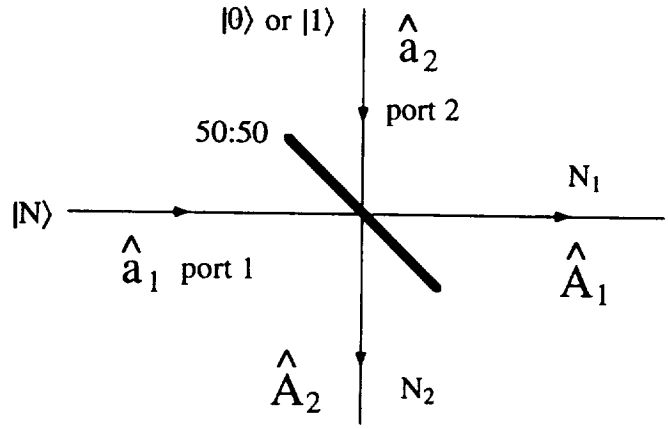

FIG. 1. Layout for the interference between N-photon state and a single photon state via a beamsplitter.

2 Photon Probability Distribution for a Symmetric Lossless Beamsplitter

It is well-known that when a number of particles, say N, enter a 50:50 lossless beamsplitter from one input port, the particles are randomly sent to the two output ports with equal probability, resulting in the simple Bernoulli binormial distribution as

$$P_0(N_1, N_2) = \frac{N!}{2^N N_1! N_2!}\, \delta_{N_1+N_2,N}. \tag{1}$$

N_1 is the number of particles exiting from output port 1 while N_2 is for port 2. In the case of photon, the above result suggests that each photon acts independently as a classical particle. The wave behavior of light does not show up here because of the absence of superposition. This distribution has its maximum when $N_1 = N_2 = N/2$ (equal partition). So it is most likely to find equal number of photons on each side of the beamsplitter. For large N and $|N_1 - N_2| << N$, Eq.(1) becomes

$$P_0(N_1, N_2) = \frac{2}{\sqrt{2N\pi}}\, e^{-(N_1-N_2)^2/2N}\, \delta_{N_1+N_2,N} \tag{2}$$

which is a Gaussian. The extra factor of 2 is because $P_0(N_1, N_2) = 0$ for every other value of $N_1 - N_2$.

Next, we let a single photon state enter the input port 2 of the beamsplitter. We will look for the probability $P_1(N_1, N_2)$ that N_1 photons exit at output port 1 while the other N_2 photons

at port 2 with $N_1 + N_2 = N + 1$. Let us for a brief moment consider the outcome from classical particle theory. As a classical particle, the input single photon will have 50% of probability going out at either ports. Because the single photon is independent of the other N photons, we simply add the probabilities to obtain the final result:

$$P_1^{cl}(N_1, N_2) = \frac{1}{2}\frac{N!}{2^N(N_1-1)!N_2!} + \frac{1}{2}\frac{N!}{2^N(N_2-1)!N_1!} = \frac{(N+1)!}{2^{N+1}N_1!N_2!}, \tag{3}$$

which is in the exactly same form as that in Eq.(1). Therefore the existence of the single photon at the other port does not influence the photon probability distribution at all. The single photon from port 1 acts as if it were part of the N photons from the port 1. This is because classical particles are independent of each other and it doesn't matter which port it enters.

On the other hand, the outcome is totally different if we treat the photons as quantum particles. We cannot simply add the probabilities. The principle of quantum mechanics requires that the probability amplitudes be added. For simplicity, let us first consider the case when N is an odd integer and $N_1 = N_2 = (N+1)/2$. The probability amplitude has two contributions as shown in Fig.2: (a) the single photon input at port 2 goes directly to output port 2 while $N_1 - 1 = (N-1)/2$ of the N photons input at port 1 are reflected and go to output port 2 and $N_2 = (N+1)/2$ photons to port 1, or (b) the single photon is reflected and goes to output port 1 while $N_2 - 1 = (N-1)/2$ photons go to output port 1 and $N_2 = (N+1)/2$ photons are reflected to port 2. From Eq.(1), we find that these two possibilities have equal probability thus their probability amplitudes have equal absolute value. For their phases, however, because there is a $\pi/2$ phase shift for the reflected field and no phase shift for the transmitted one at a symmetric beam splitter, the total phase shift for the $N+1$ photons at the output ports will be different for the two possibilities. Referring to Fig.1, we find that the total phase shift for the first possibility mentioned above is $\varphi_a = (N_2 - 1)\pi/2 = (N-1)\pi/4$ while for the second possibility, $\varphi_b = \pi/2 + N_2\pi/2 = (N+3)\pi/4$. The phase difference between the two possibilities is thus $\varphi_b - \varphi_a = \pi$. Therefore, the two probability amplitudes will cancel each other, resulting zero probability for $N_1 = N_2 = (N+1)/2$. This result is completely different from that of a classical particle theory in Eq.(3). As seen above, the probability cancellation at $N_1 = N_2$ results from the quantum interference of $N+1$ particles.

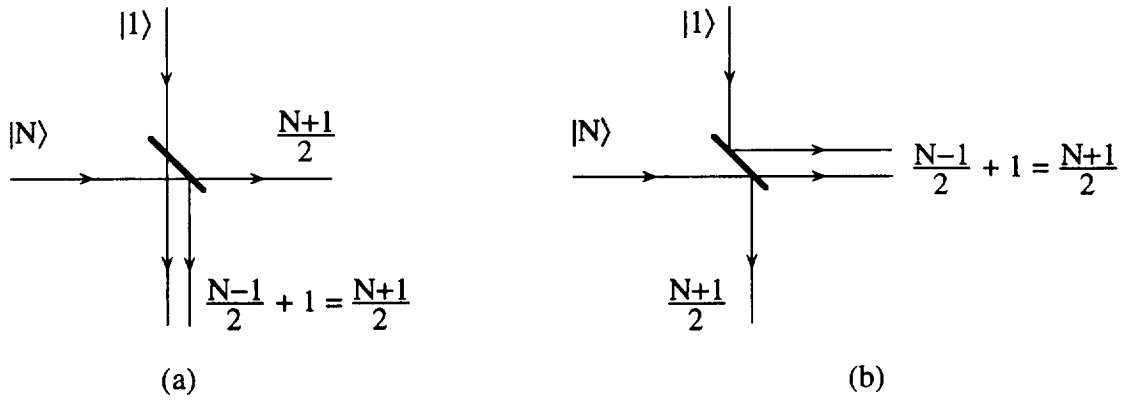

FIG. 2. Two contributions to the output photon distribution.

For the other cases when $N_1 \neq N_2$, we cannot use the simple argument as above. But we may derive the output state along the line of Ref.10 and find the probability distribution $P_1(N_1, N_2)$. Or we can use the formula

$$P_1(N_1, N_2) = \langle : \frac{(\hat{A}_1^\dagger \hat{A}_1)^{N_1}}{N_1!} e^{-\hat{A}_1^\dagger \hat{A}_1} \frac{(\hat{A}_2^\dagger \hat{A}_2)^{N_2}}{N_2!} e^{-\hat{A}_2^\dagger \hat{A}_2} : \rangle, \tag{4}$$

where

$$\hat{A}_1 = (\hat{a}_1 + i\,\hat{a}_2)/\sqrt{2}, \quad \hat{A}_2 = (\hat{a}_2 + i\,\hat{a}_1)/\sqrt{2}$$

are the annihilation operators for the output modes for a symmetric lossless beamsplitter. The input modes represented by $\hat{a}_1, \hat{a}_2$ are in the state of $|\Phi\rangle = |N\rangle_1 |1\rangle_2$. After some lengthy calculation, we have

$$P_1(N_1, N_2) = \frac{N!}{2^{N+1} N_1! N_2!} (N_1 - N_2)^2 \delta_{N_1+N_2, N+1}. \tag{5}$$

The above expression can also be derived from the general formula given by Campos, Saleh and Teich in Ref.11 for arbitrary numbers $\{n_1, n_2\}$ of input photons at the two input ports, with the setting of $\tau = 1/2, n_1 = N, n_2 = 1$. When $N, N_1, N_2 >> 1$, Eq.(5) can be approximated by

$$P_1(N_1, N_2) \approx \frac{(N_1 - N_2)^2}{N} \frac{2}{\sqrt{2N\pi}} \; e^{-(N_1-N_2)^2/2N} \delta_{N_1+N_2, N+1}. \tag{6}$$

Notice that when N is an odd integer, $P_1(N_1, N_2) = 0$ for $N_1 = N_2 = (N+1)/2$, exactly as predicted from the simple argument of probability superposition given in the previous paragraph. When N is an even integer, $P_1(N/2+1, N/2) = N!/2^{N+1}(N/2+1)!(N/2)! \neq 0$, but because $P_1(N/2+1, N/2)/P_0(N/2+1, N/2) = 1/(N+1) << 1$ for $N >> 1$, or the probability with a single photon input is much smaller than that with vacuum state input, the probabilities for $N_1 \approx N_2$ are quite different in the two cases with or without the single photon state at port 2. Actually, the whole probability distribution in Eq.(5) is different from the probability distribution in Eq.(1), as seen in Fig.3. The maximum probability for $P_1(N_1, N_2)$ occurs at $|N_1 - N_2| \approx \sqrt{N}$ or $N_1 \approx (N \pm \sqrt{N})/2$ while for $P_0(N_1, N_2)$ it occurs at $N_1 = N_2 \approx N/2$. The existence of a single photon dramatically changes the pattern of the output photon distribution.

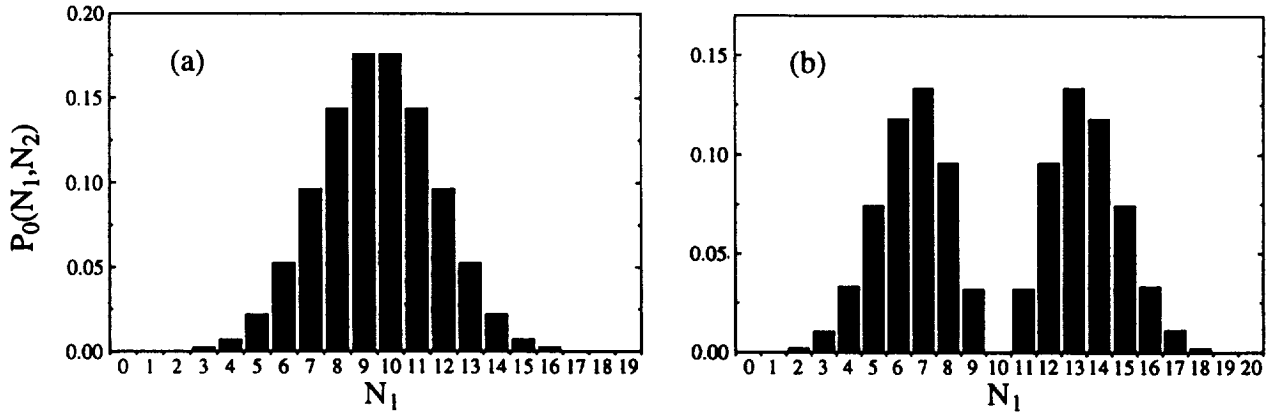

FIG. 3. Output photon distribution for N-photon state input at port 1 with (a) vacuum state or (b) single photon state at port 2 (N=19).

3 Interference of a Single-Photon State with Arbitrary State

The above quantum probability cancellation effect due to a single photon state is not stricted to N-photon state as input state. Let us consider an arbitrary state of light input at port 1. Its state is generally described by the Glauber P-distribution $P_{in}(\alpha)$. But before going into lengthy calculation, we may take a guess about the photon distribution of the output fields by the following argument: since the vacuum state and the single photon state are completely incoherent in the sense that they have a totally random phase distribution, the output fields due to interference of one of these states with any other state will not have any coherence information of the input state. Therefore, the output photon distribution of the beamsplitter will lose all the coherence information of the input state and will depend simply on the photon statistics P_n^{in} of the input state at port 1. So combining this fact with Eqs.(1,5), we come up with the output photon distributions in the form of

$$P_0(N_1, N_2) = \frac{(N_1 + N_2)!}{2^{N_1+N_2} N_1! N_2!} P^{in}_{N_1+N_2} \tag{7a}$$

for vacuum input at port 2 and

$$P_1(N_1, N_2) = \frac{(N_1 + N_2 - 1)!}{2^{N_1+N_2} N_1! N_2!} (N_1 - N_2)^2 P^{in}_{N_1+N_2-1} \tag{7b}$$

for single photon state input at port 2. Of course, we may rigorously derive the output photon distribution by following the procedure described in Ref.10 to first find the state of the output fields of the beamsplitter in terms of the P-distribution. The photon distribution for the output fields can then be calculated through Eq.(4). It can be shown that Eq.(7) is indeed the correct form for the output photon distribution.

By comparing Eqs.(7a) and (7b), we easily find that $P_1(N_1 = N_2) = 0$ for single photon state input at port 2 while

$$P_0(N_1 = N_2) = \sum_{N_1=0}^{\infty} \frac{(2N_1)!}{2^{2N_1}(N_1!)^2} P^{in}_{2N_1} \neq 0$$

for vacuum input. Therefore, the existence of the single photon state at port 2 does make a difference in the output photon distribution even for arbitrary input state at port 1, and the probability for $N_1 = N_2$ is exactly equal to zero. The cancellation of the probability for $N_1 = N_2$ is because of the destructive interference between the N photons and the single photon as we discussed above.

In an actual experiment, however, it is difficult to measure the complete distribution $P(N_1, N_2)$, but the distribution $P(N_1 - N_2 = M)$ can be measured by balanced homodyne detection.[12,13] From Eqs.(7a,b) we find that

$$\begin{aligned} P_0(N_1 - N_2 = M) &= \sum_{N_1=M}^{\infty} \frac{(2N_1 - M)!}{2^{2N_1-M} N_1! (N_1 - M)!} P^{in}_{2N_1-M} \\ P_1(N_1 - N_2 = M) &= M^2 \sum_{N_1=M}^{\infty} \frac{(2N_1 - M - 1)!}{2^{2N_1-M} N_1! (N_1 - M)!} P^{in}_{2N_1-M-1} \end{aligned} \tag{8}$$

for $M \geq 0$. For $M < 0$, the symmetry between N_1, N_2 in Eq.(7) leads to $P(M) = P(-M)$.

Next, we will evaluate $P_0(M), P_1(M)$ for some special states. For N-photon state input with $N >> 1$, we have $P_n^{in} = \delta_{n,N}$, and Eq.(8) gives results similar to Eqs(1,5):

$$\begin{aligned} P_0(M) &= \frac{N!}{2^N(N/2+M/2)!(N/2-M/2)!} \\ &\approx \frac{2}{\sqrt{2N\pi}}\ e^{-M^2/2N} \qquad \text{for } N >> 1 \\ P_1(M) &= \frac{M^2 N!}{2^{N+1}(N/2+M/2+1/2)!(N/2-M/2+1/2)!} \\ &\approx \frac{2}{\sqrt{2N\pi}}\frac{M^2}{N}\ e^{-M^2/2N} \qquad \text{for } N >> 1. \end{aligned} \tag{9}$$

For coherent state input, $P_{in}^n = \bar{n}^n e^{-\bar{n}}/n!$ with $\bar{n}$ being the average photon number. Therefore, we have

$$\begin{aligned} P_0(M) &= \sum_{N_2=0}^{\infty} \frac{(2N_1+M)!}{2^{2N_1+M}N_2!(N_1+M)!}\frac{\bar{n}^{2N_2+M}e^{-\bar{n}}}{(2N_2+M)!} = e^{-\bar{n}}I_M(\bar{n}) \\ P_1(M) &= \sum_{N_2=0}^{\infty} \frac{M^2(2N_1+M-1)!}{2^{2N_1+M}N_2!(N_1+M)!}\frac{\bar{n}^{2N_2+M-1}e^{-\bar{n}}}{(2N_2+M-1)!} = \frac{M^2}{\bar{n}}e^{-\bar{n}}I_M(\bar{n}), \end{aligned} \tag{10}$$

where $I_M(\bar{n})$ is the Bessel function with purely imaginary argument and has the form of

$$I_M(\bar{n}) \equiv \int_{-\pi}^{\pi} d\varphi e^{-iM\varphi}e^{\bar{n}\cos\varphi} \approx \frac{1}{\sqrt{2\bar{n}\pi}}\ e^{\bar{n}-M^2/2\bar{n}} \quad \text{when } \bar{n} >> 1. \tag{11}$$

Therefore, for large $\bar{n}$,

$$\begin{aligned} P_0(M) &\approx \frac{1}{\sqrt{2\bar{n}\pi}}\ e^{-M^2/2\bar{n}} \\ P_1(M) &\approx \frac{1}{\sqrt{2\bar{n}\pi}}\frac{M^2}{\bar{n}}\ e^{-M^2/2\bar{n}}. \end{aligned} \tag{12}$$

Eq.(12) has the same form as Eq.(9) for large N besides the factor of 2 which is explained earlier right after Eq.(2). This is not surprising if we consider the fact that when the photon number is large, the interference scheme discussed above becomes homodyne detection scheme. Since both vacuum state and single photon state have random phase distribution, homodyne detections with N-photon state ($N >> 1$) and coherent state as local oscillators are equivalent. As a matter of fact, the output photon distributions will always have the form of Eq.(12) for any state as local oscillator, provided that the average photon number is large and photon number fluctuation is much less than average photon number ($\sqrt{\langle \Delta n^2 \rangle} << \bar{n}$). We can see this point from Eq.(8): when $\sqrt{\langle \Delta n^2 \rangle} << \bar{n}$, P_{in}^n has a narrow peak around $\bar{n}$ and is a fast changing function as compared with

other terms in the summation, therefore the contribution to the summation is only from the few terms around $\bar{n}$, so that we can pull all other terms out of the sum, that is,

$$P_0(M) \approx \frac{\bar{n}!}{2^{\bar{n}}(\bar{n}/2-M)!(\bar{n}/2+M)!}\sum_n P_{in}^n/2 \approx \frac{1}{\sqrt{2\bar{n}\pi}}\ e^{-M^2/2\bar{n}} \quad \text{when } \bar{n} >> 1, \tag{13a}$$

and similarly

$$P_1(M) \approx \frac{1}{\sqrt{2\bar{n}\pi}}\frac{M^2}{\bar{n}}\ e^{-M^2/2\bar{n}} \quad \text{when } \bar{n} >> 1. \tag{13b}$$

We can also understand this result from the fact that any fluctuation in local oscillator is cancelled in balanced homodyne detection scheme.[12]

Furthermore, if we set $\bar{n} \to \infty$, we can replace the discrete variable M with a continuous one defined by $x = M/\sqrt{\bar{n}}$ and the probability distributions in Eqs.(13a,b) lead to probability densities of continuous variable x as

$$P_0(x) = \frac{1}{\sqrt{2\pi}}\ e^{-x^2/2}, \quad P_1(x) = \frac{x^2}{\sqrt{2\pi}}\ e^{-x^2/2} \tag{14}$$

which correspond to the square of the absolute value of the wavefunction for the ground state and single photon state, respectively. Thus by measuring $P(M)$ in homodyne detection, we can deduce the wavefunction of the input state at port 2. This is exactly the technique of optical tomography used by Smithey et al.[13] But here we applied it to a single photon state (input at port 2) and proved that the outcome does not depend on the state of the local oscillator (input field at port 1) as long as the average photon number is large and the fluctuation is not very large for the local oscillator (i.e., the condition for the approximation in Eqs.(13a,b)).

However, there is an exception to the above. It is well-known that for thermal light, we have

$$\langle \Delta n^2 \rangle = \bar{n}(\bar{n}+1)$$

so that $\sqrt{\langle \Delta n^2 \rangle} \approx \bar{n}$ and we cannot use the approximation in Eqs.(13a,b). For thermal light, $P_{in}^n = \bar{n}^n/(\bar{n}+1)^{n+1}$, so from Eq.(8), we have

$$\begin{aligned} P_0(M) &= \sum_{N_2=0}^{\infty} \frac{(2N_2+M)!}{2^{2N_2+M}(N_2+M)!N_2!}\frac{\bar{n}^{2N_2+M}}{(\bar{n}+1)^{2N_2+M+1}} \\ &= \frac{x^M}{\bar{n}+1}\mathcal{F}\left(\frac{M+1}{2}, \frac{M}{2}+1, M+1; 4x^2\right) \\ P_1(M) &= M^2\sum_{N_2=0}^{\infty} \frac{(2N_2+M-1)!}{2^{2N_2+M}(N_2+M)!N_2!}\frac{\bar{n}^{2N_2+M-1}}{(\bar{n}+1)^{2N_2+M}} \\ &= \frac{Mx^M}{\bar{n}}\mathcal{F}\left(\frac{M+1}{2}, \frac{M}{2}, M+1; 4x^2\right) \end{aligned} \tag{15}$$

where $x = \bar{n}/2(\bar{n}+1)$ and $\mathcal{F}(\alpha, \beta, \gamma; z)$ is the hypergeometric function. With some re-arrangement, we can prove that Eq.(15) have a simpler form as

$$P_0(M) = \frac{1}{\sqrt{2\bar{n}+1}}\ q^M$$

$$(M \geq 0) \qquad (16)$$

$$P_1(M) = \frac{M}{\bar{n}}\ q^M$$

with $q = 1 + 1/\bar{n} - \sqrt{2\bar{n}+1}/\bar{n}$. For large $\bar{n}$, q^M becomes $e^{-M/\sqrt{\bar{n}/2}}$ so that Eq.(15) is changed to

$$P_0(M) = \frac{1}{\sqrt{2\bar{n}+1}}\ e^{-M/\sqrt{\bar{n}/2}}$$

$$(M \geq 0) \qquad (17)$$

$$P_1(M) = \frac{M}{\bar{n}}\ e^{-M/\sqrt{\bar{n}/2}}.$$

Therefore, The output photon distribution for thermal light input is different from that of coherent state input. But the general trend in the change of the shape from $P_0(M)$ to $P_1(M)$ is similar in both states (Fig.4). The quantum interference effect due to single photon is the same.

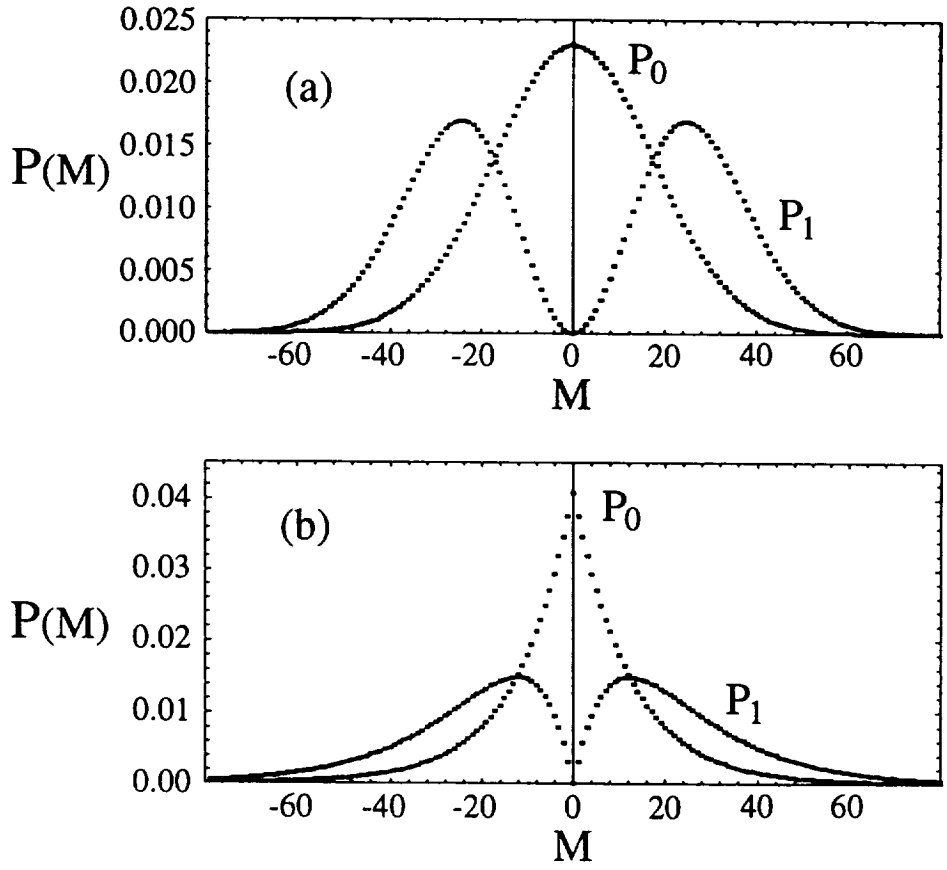

FIG. 4. Probability distribution $P_{0,1}(M)$ for the balanced homodyne detection of vacuum state and single photon state with (a) coherent state or (b) thermal state as local oscillator. $\bar{n} = 300$.

It is interesting to note that the weak nonclassical state (single photon state) plays an important role in the interference with a *strong classical* field (coherent state or thermal state) in contrast to the case discussed in Ref.3 where the nonclassical interference occurs between a *strong nonclassical* field and a weak classical field. Even though the nonclassical field is weak here, the result is very nonclassical in the sense that the probability of detecting equal intensities in the two outputs is zero ($P_1(M = 0) = 0$). It can be proved that in the similar situation (one field is weak and the other is strong), classical wave theory predicts that the probability is largest for equal intensity output at the two ports.

So far we have only discussed the single mode situations. In practice, we always have wide spectrum. Since two different sources of light are involved in the interference, the observation of the probability cancellation effect requires the overlap of both spatial and temporal mode structure of the two fields as well as near unit quantum efficiency of the detectors.

Acknowledgments

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