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A Non-Axisymmetric Linearized Supersonic Wave Drag Analysis

Mathematical Theory

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Abstract

A mathematical theory is developed to perform the calculations necessary to determine the wave drag for slender bodies of non-circular cross section. The derivations presented in this report are based on extensions to supersonic linearized small perturbation theory. A numerical scheme is presented utilizing Fourier decomposition to compute the pressure coefficient on and about a slender body of arbitrary cross section.

Introduction

Supersonic linearized small perturbation theory has been used to calculate the wave drag for simple aircraft configurations of slender fuselages and thin wings since the 1950s. Complex aircraft configurations have been analyzed by linearized small perturbation theory since the 1970s. The interference effects between aircraft fuselage, wing, and engine nacelle found in complex configurations can be simulated by the superposition of pressure fields from and on interfering bodies. Only axisymmetric slender body theory has been used to model the interference of aircraft fuselages and engine nacelles in the past. However, current designs for supersonic transport aircraft incorporate engine nacelle configurations which are not axisymmetric. In order to analyze the engine/airframe interference effects for these current nacelle designs other methods have been used. In particular, Euler Computational Fluid Dynamic techniques have been used to compute the interference wave drags for complex aircraft configurations. Using CFD techniques greatly complicates the analysis set up and run times, thus limiting the number of possible configurations which can be analyzed. The motivation for this work is to find an extended solution technique whereby linearized small perturbation theory can be used to calculate the pressure field about slender, non-axisymmetric bodies. This would extend the modeling capablities of existing aerodynamic configuration analysis programs based on linearized small perturbation methods.

Governing Differential Equation and General Solution

The following are the basic relations and theory developed in the non-axisymmetric wave drag analysis. The problem is formulated in cylindrical coordinates, as is typical for slender body theories. Figure 1 shows a slender body whose cross sections is not axisymmetric, and the coordinate system used in the following analyses. The x axis is aligned with the free stream flow direction. The governing partial differential equation for the perturbation velocity potential is:

$$\left(1 - M^2\right) \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} = 0 \tag{1}$$

Implicit in the derivation of Equation 1 are the assumptions that the gas flow is; supersonic ($M > 1$), ideal, irrotational, and inviscid. Ward (Reference 1) gives a general solution of the above equation:

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$$\phi(x,r,\theta) = \sum_{n=0}^{\infty} \cos n\theta \int_0^{x-\beta r} \frac{1}{2} \left\{ \left[\frac{x-\xi+\sqrt{(x-\xi)^2-\beta^2r^2}}{\beta r} \right]^n + \left[\frac{x-\xi-\sqrt{(x-\xi)^2-\beta^2r^2}}{\beta r} \right]^n \right\} \frac{f_n(\xi)d\xi}{\sqrt{(x-\xi)^2-\beta^2r^2}}$$
$$+\sum_{n=0}^{\infty} \sin n\theta \int_0^{x-\beta r} \frac{1}{2} \left\{ \left[\frac{x-\xi+\sqrt{(x-\xi)^2-\beta^2r^2}}{\beta r} \right]^n + \left[\frac{x-\xi-\sqrt{(x-\xi)^2-\beta^2r^2}}{\beta r} \right]^n \right\} \frac{g_n(\xi)d\xi}{\sqrt{(x-\xi)^2-\beta^2r^2}} \quad (2)$$

Where:

$$\beta^2 = M^2 - 1 \quad (3)$$

The terms $f_n(\xi), g_n(\xi)$ in Equation 2 are singularity distributions along the body axis. The perturbation velocities are found by:

$$u = \frac{\partial \phi}{\partial x} \quad (4)$$

$$v = \frac{\partial \phi}{\partial r} \quad (5)$$

$$w = \frac{1}{r}\frac{\partial \phi}{\partial \theta} \quad (6)$$

For axisymmetric flows (Reference 2) the governing P.D.E and solution are:

$$\left(1-M^2\right)\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} = 0 \quad (7)$$

$$\phi(x,r) = \int_0^{x-\beta r} \frac{f(\xi)d\xi}{\sqrt{(x-\xi)^2-\beta^2r^2}} \quad (8)$$

$$u(x,r) = \int_0^{x-\beta r} \frac{f'(\xi)d\xi}{\sqrt{(x-\xi)^2-\beta^2r^2}} \quad (9)$$

$$v(x,r) = -\frac{1}{r}\int_0^{x-\beta r} \frac{(x-\xi)f'(\xi)d\xi}{\sqrt{(x-\xi)^2-\beta^2r^2}} \quad (10)$$

For $n = 0$ the solution (Equation 2) to the non-axisymmetric problem reduces to:

$$\phi_0(x,r,\theta) = \int_0^{x-\beta r} \frac{f_0(\xi)d\xi}{\sqrt{(x-\xi)^2-\beta^2r^2}} \quad (11)$$

By comparison with Equation 8 it is seen that the zeroth-order solution in Equation 2 recovers the exact solution to the axisymmetric problem. Higher-order terms, $n > 0$, in Equation 2 represent non-axisymmetric corrections to the zeroth-order axisymmetric solution.

Determination of the Perturbation Velocities

Returning to the non-axisymmetric problem, let:

$$\phi(x,r,\theta)=\sum_{n=0}^{\infty}\phi_n(x,r,\theta) \tag{12}$$

Where now:

$$\begin{aligned}\phi_n(x,r,\theta)=\cos n\theta\int_0^{x-\beta r}\frac{1}{2}\Bigg\{&\left[\frac{x-\xi+\sqrt{(x-\xi)^2-\beta^2r^2}}{\beta r}\right]^n\\&+\left[\frac{x-\xi-\sqrt{(x-\xi)^2-\beta^2r^2}}{\beta r}\right]^n\Bigg\}\frac{f_n(\xi)d\xi}{\sqrt{(x-\xi)^2-\beta^2r^2}}\\+\sin n\theta\int_0^{x-\beta r}\frac{1}{2}\Bigg\{&\left[\frac{x-\xi+\sqrt{(x-\xi)^2-\beta^2r^2}}{\beta r}\right]^n\\&+\left[\frac{x-\xi-\sqrt{(x-\xi)^2-\beta^2r^2}}{\beta r}\right]^n\Bigg\}\frac{g_n(\xi)d\xi}{\sqrt{(x-\xi)^2-\beta^2r^2}}\end{aligned} \tag{13}$$

By change of variable:

$$\xi=x-\beta r\cosh\sigma \tag{14}$$

$$\sigma=\cosh^{-1}\left(\frac{x-\xi}{\beta r}\right) \tag{15}$$

$$\frac{d\xi}{d\sigma}=-\beta r\sinh\sigma \tag{16}$$

$$d\xi=-\sqrt{(x-\xi)^2-\beta^2r^2}d\sigma \tag{17}$$

Note that:

$$\cosh n\sigma=\frac{1}{2}\left\{\left[\frac{x-\xi+\sqrt{(x-\xi)^2-\beta^2r^2}}{\beta r}\right]^n+\left[\frac{x-\xi-\sqrt{(x-\xi)^2-\beta^2r^2}}{\beta r}\right]^n\right\} \tag{18}$$

The limits of integration in Equation 13 become:

$$\xi=0 \quad\rightarrow\quad \sigma=\cosh^{-1}\left(\frac{x}{\beta r}\right)$$
$$\xi=x-\beta r \quad\rightarrow\quad \sigma=\cosh^{-1}(1)=0$$

Yields a new form of the perturbation potential:

$$\phi_n(x,r,\theta) = \cos n\theta \int_0^{\cosh^{-1}(x/\beta r)} f_n\left(x - \beta r\cosh\sigma\right)\cosh n\sigma d\sigma + \sin n\theta \int_0^{\cosh^{-1}(x/\beta r)} g_n\left(x - \beta r\cosh\sigma\right)\cosh n\sigma d\sigma \quad (19)$$

The axial perturbation velocity may now be found by:

$$u(x,r,\theta) = \sum_{n=0}^{\infty} u_n(x,r,\theta) \quad (20)$$

Where:

$$u_n(x,r,\theta) = \frac{\partial \phi_n}{\partial x} \quad (21)$$

$$u_n(x,r,\theta) = \cos n\theta \frac{\partial}{\partial x} \int_0^{\cosh^{-1}(x/\beta r)} f_n\left(x - \beta r\cosh\sigma\right)\cosh n\sigma d\sigma + \sin n\theta \frac{\partial}{\partial x} \int_0^{\cosh^{-1}(x/\beta r)} g_n\left(x - \beta r\cosh\sigma\right)\cosh n\sigma d\sigma \quad (22)$$

Since the upper limit in the integrals in Equation 22 is a function of the variable of differentiation, Liebnitz rule* must be used. (The following applies to both of the integrals above. Only the first term will be treated explicitly.)

$$\frac{\partial}{\partial x} \int_0^{\cosh^{-1}(x/\beta r)} f_n\left(x - \beta r\cosh\sigma\right)\cosh n\sigma d\sigma = \quad (23)$$
$$= \int_0^{\cosh^{-1}(x/\beta r)} \frac{\partial}{\partial x}\left[f_n\left(x - \beta r\cosh\sigma\right)\cosh n\sigma\right] d\sigma + f_n\left(x - \beta r\cosh\cosh^{-1}(x/\beta r)\right)\cosh n\cosh^{-1}(x/\beta r)\frac{\partial}{\partial x}\cosh^{-1}(x/\beta r)$$

For closed nosed bodies, $f_n(0) = 0$, and so:

$$u_n(x,r,\theta) = \cos n\theta \int_0^{\cosh^{-1}(x/\beta r)} \frac{\partial}{\partial x} f_n\left(x - \beta r\cosh\sigma\right)\cosh n\sigma d\sigma + \sin n\theta \int_0^{\cosh^{-1}(x/\beta r)} \frac{\partial}{\partial x} g_n\left(x - \beta r\cosh\sigma\right)\cosh n\sigma d\sigma \quad (24)$$

Recall the change of variable in Equation 14:

$$\frac{\partial}{\partial x} f_n\left(x - \beta r\cosh\sigma\right) = \frac{\partial}{\partial \xi} f_n(\xi)\frac{\partial \xi}{\partial x} = f_n'(\xi) \quad (25)$$

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$$\frac{\partial}{\partial \eta}\int_{a(\eta)}^{b(\eta)} f(x,\eta)\,dx = \int_a^b \frac{\partial f}{\partial \eta}dx + f(b,\eta)\frac{\partial b}{\partial \eta} - f(a,\eta)\frac{\partial a}{\partial \eta}$$

$$u_n(x,r,\theta) = \cos n\theta \int_0^{\cosh^{-1}(x/\beta r)} f_n'(x - \beta r \cosh\sigma) \cosh n\sigma d\sigma \tag{26}$$
$$+ \sin n\theta \int_0^{\cosh^{-1}(x/\beta r)} g_n'(x - \beta r \cosh\sigma) \cosh n\sigma d\sigma$$

Reversing the change of variable in Equations 14 through 18 yields:

$$\begin{aligned} u_n(x,r,\theta) = {} & \cos n\theta \int_0^{x-\beta r} \frac{1}{2} \left\{ \left[\frac{x - \xi + \sqrt{(x-\xi)^2 - \beta^2 r^2}}{\beta r} \right]^n \right. \\ & \left. + \left[\frac{x - \xi - \sqrt{(x-\xi)^2 - \beta^2 r^2}}{\beta r} \right]^n \right\} \frac{f_n'(\xi) d\xi}{\sqrt{(x-\xi)^2 - \beta^2 r^2}} \\ & + \sin n\theta \int_0^{x-\beta r} \frac{1}{2} \left\{ \left[\frac{x - \xi + \sqrt{(x-\xi)^2 - \beta^2 r^2}}{\beta r} \right]^n \right. \\ & \left. + \left[\frac{x - \xi - \sqrt{(x-\xi)^2 - \beta^2 r^2}}{\beta r} \right]^n \right\} \frac{g_n'(\xi) d\xi}{\sqrt{(x-\xi)^2 - \beta^2 r^2}} \end{aligned} \tag{27}$$

Similarly, the radial perturbation velocity can be found:

$$v(x,r,\theta) = \sum_{n=0}^{\infty} v_n(x,r,\theta) \tag{28}$$

Where:

$$v_n(x,r,\theta) = \frac{\partial \phi_n}{\partial r} \tag{29}$$

$$v_n(x,r,\theta) = \cos n\theta \frac{\partial}{\partial r} \int_0^{\cosh^{-1}(x/\beta r)} f_n(x - \beta r \cosh\sigma) \cosh n\sigma d\sigma \tag{30}$$
$$+ \sin n\theta \frac{\partial}{\partial r} \int_0^{\cosh^{-1}(x/\beta r)} g_n(x - \beta r \cosh\sigma) \cosh n\sigma d\sigma$$

And by similar reasoning used in Equation 23:

$$v_n(x,r,\theta) = \cos n\theta \int_0^{\cosh^{-1}(x/\beta r)} \frac{\partial}{\partial r} f_n(x - \beta r \cosh\sigma) \cosh n\sigma d\sigma \tag{31}$$
$$+ \sin n\theta \int_0^{\cosh^{-1}(x/\beta r)} \frac{\partial}{\partial r} g_n(x - \beta r \cosh\sigma) \cosh n\sigma d\sigma$$

Recalling the change of variable in Equation 14:

$$\frac{\partial}{\partial r} f_n(x - \beta r \cosh\sigma) = \frac{\partial}{\partial \xi} f_n(\xi) \frac{\partial \xi}{\partial r} = f_n'(\xi)(-\beta \cosh\sigma) \tag{32}$$

$$v_n(x,r,\theta) = -\cos n\theta \int_0^{\cosh^{-1}(x/\beta r)} \beta \cosh\sigma f_n'\left(x - \beta r\cosh\sigma\right)\cosh n\sigma d\sigma - \sin n\theta \int_0^{\cosh^{-1}(x/\beta r)} \beta \cosh\sigma g_n'\left(x - \beta r\cosh\sigma\right)\cosh n\sigma d\sigma \tag{33}$$

Reversing the change of variable in Equations 14 through 18 yields:

$$\begin{aligned} v_n(x,r,\theta) = &-\cos n\theta \int_0^{x-\beta r} \frac{1}{2r}\left\{\left[\frac{x-\xi+\sqrt{(x-\xi)^2-\beta^2r^2}}{\beta r}\right]^n + \left[\frac{x-\xi-\sqrt{(x-\xi)^2-\beta^2r^2}}{\beta r}\right]^n\right\}\frac{(x-\xi)\,f_n'(\xi)d\xi}{\sqrt{(x-\xi)^2-\beta^2r^2}} \\ &-\sin n\theta \int_0^{x-\beta r} \frac{1}{2r}\left\{\left[\frac{x-\xi+\sqrt{(x-\xi)^2-\beta^2r^2}}{\beta r}\right]^n + \left[\frac{x-\xi-\sqrt{(x-\xi)^2-\beta^2r^2}}{\beta r}\right]^n\right\}\frac{(x-\xi)\,g_n'(\xi)d\xi}{\sqrt{(x-\xi)^2-\beta^2r^2}} \end{aligned} \tag{34}$$

Finally, the aximuthal perturbation velocity can be found:

$$w(x,r,\theta) = \sum_{n=0}^{\infty} w_n(x,r,\theta) \tag{35}$$

Where:

$$w_n(x,r,\theta) = \frac{1}{r}\frac{\partial\phi_n}{\partial\theta} \tag{36}$$

And by direct differentiation:

$$\begin{aligned} w_n(x,r,\theta) = &-\sin n\theta \int_0^{x-\beta r} \frac{n}{2r}\left\{\left[\frac{x-\xi+\sqrt{(x-\xi)^2-\beta^2r^2}}{\beta r}\right]^n + \left[\frac{x-\xi-\sqrt{(x-\xi)^2-\beta^2r^2}}{\beta r}\right]^n\right\}\frac{f_n(\xi)d\xi}{\sqrt{(x-\xi)^2-\beta^2r^2}} \\ &+\cos n\theta \int_0^{x-\beta r} \frac{n}{2r}\left\{\left[\frac{x-\xi+\sqrt{(x-\xi)^2-\beta^2r^2}}{\beta r}\right]^n + \left[\frac{x-\xi-\sqrt{(x-\xi)^2-\beta^2r^2}}{\beta r}\right]^n\right\}\frac{g_n(\xi)d\xi}{\sqrt{(x-\xi)^2-\beta^2r^2}} \end{aligned} \tag{37}$$

Examining the perturbation velocities for $n = 0$, the lowest order term in the non-axisymmetric formulation, it is found that:

$$u_0(x,r,\theta) = \int_0^{x-\beta r} \frac{f_0'(\xi)d\xi}{\sqrt{(x-\xi)^2 - \beta^2 r^2}} \tag{38}$$

$$v_0(x,r,\theta) = -\frac{1}{r}\int_0^{x-\beta r} \frac{(x-\xi)f_0'(\xi)d\xi}{\sqrt{(x-\xi)^2 - \beta^2 r^2}} \tag{39}$$

$$w_0(x,r,\theta) = 0 \tag{40}$$

Comparing the above with Equations 9 and 10 shows that the zeroth-order perturbation velocities recover the exact solution to the axisymmetric problem. Higher-order terms, $n > 0$, in Equations 27, 34 and 37 represent non-axisymmetric corrections to the zeroth-order axisymmetric solution.

Body Surface Tangent Flow Boundary Condition

The flow along the surface of a body cannot penetrate the body, and must therefore follow the contour of the body. The requirement that the flow be tangent to the body, on the body surface, can be used as a boundary condition from which the singularity distributions may be determined. The tangent flow boundary condition may be written as:

$$\vec{V} \cdot \vec{\nabla} S = 0 \tag{41}$$

Where $S(x,r,\theta) = 0$ defines the body surface. For cylindrical coordinates in terms of the perturbation velocities:

$$\vec{V} \cdot \vec{\nabla} S = \left(\tilde{U} + \tilde{u}\right)\frac{\partial S}{\partial x} + \tilde{v}\frac{\partial S}{\partial r} + \frac{\tilde{w}}{r}\frac{\partial S}{\partial \theta} = 0 \tag{42}$$

And since $\tilde{u} << \tilde{U}$ the boundary condition becomes:

$$\frac{\partial S}{\partial x} + v\frac{\partial S}{\partial r} + \frac{w}{r}\frac{\partial S}{\partial \theta} = 0 \tag{43}$$

Where $v = \tilde{v}/\tilde{U}$, etc. Defining the body surface as:

$$S(x,r,\theta) = r - R(x,\theta) = 0 \tag{44}$$

Where $R(x,\theta)$ describes the body radius at a given x,θ location. It then follows that:

$$\frac{\partial S}{\partial x} = -\frac{\partial R}{\partial x} \tag{45}$$

$$\frac{\partial S}{\partial r} = 1 \tag{46}$$

$$\frac{\partial S}{\partial \theta} = -\frac{\partial R}{\partial \theta} \tag{47}$$

Substituting the above into Equation 43 yields:

$$-\frac{\partial R}{\partial x} = -v + \frac{w}{r}\frac{\partial R}{\partial \theta} \tag{48}$$

Where v, w are the known velocity perturbation functions previously defined in Equations 28 and 35. The boundary condition is applied on the body surface, $r = R(x,\theta)$, and so:

$$\frac{\partial}{\partial x}R(x,\theta) = v(x,r,\theta) - w(x,r,\theta)\frac{1}{R(x,\theta)}\frac{\partial}{\partial \theta}R(x,\theta) \tag{49}$$

It is convienent to introduce the following change of notation:

$$u(x,r,\theta) = \sum_{n=0}^{\infty} \cos n\theta \mathcal{I}_n^u[f_n'] + \sum_{n=0}^{\infty} \sin n\theta \mathcal{I}_n^u[g_n'] \tag{50}$$

$$v(x,r,\theta) = \sum_{n=0}^{\infty} \cos n\theta \mathcal{I}_n^v[f_n'] + \sum_{n=0}^{\infty} \sin n\theta \mathcal{I}_n^v[g_n'] \tag{51}$$

$$w(x,r,\theta) = \sum_{n=0}^{\infty} \cos n\theta \mathcal{I}_n^w[g_n] - \sum_{n=0}^{\infty} \sin n\theta \mathcal{I}_n^w[f_n] \tag{52}$$

Where the integral operators for each perturbation velocity are defined as:

$$\mathcal{I}_n^u[h] = \int_0^{x-\beta r} \frac{1}{2}\left\{\left[\frac{x-\xi+\sqrt{(x-\xi)^2-\beta^2 r^2}}{\beta r}\right]^n + \left[\frac{x-\xi-\sqrt{(x-\xi)^2-\beta^2 r^2}}{\beta r}\right]^n\right\}\frac{h(\xi)d\xi}{\sqrt{(x-\xi)^2-\beta^2 r^2}} \tag{53}$$

$$\mathcal{I}_n^v[h] = \int_0^{x-\beta r} -\frac{1}{2r}\left\{\left[\frac{x-\xi+\sqrt{(x-\xi)^2-\beta^2 r^2}}{\beta r}\right]^n + \left[\frac{x-\xi-\sqrt{(x-\xi)^2-\beta^2 r^2}}{\beta r}\right]^n\right\}\frac{(x-\xi)\,h(\xi)d\xi}{\sqrt{(x-\xi)^2-\beta^2 r^2}} \tag{54}$$

$$\mathcal{I}_n^w[h] = \int_0^{x-\beta r} \frac{n}{2r}\left\{\left[\frac{x-\xi+\sqrt{(x-\xi)^2-\beta^2 r^2}}{\beta r}\right]^n + \left[\frac{x-\xi-\sqrt{(x-\xi)^2-\beta^2 r^2}}{\beta r}\right]^n\right\}\frac{h(\xi)d\xi}{\sqrt{(x-\xi)^2-\beta^2 r^2}} \tag{55}$$

Or, using binominal notation:

$$u(x,r,\theta) = \sum_{n=0}^{\infty}\begin{pmatrix}\cos\\ \sin\end{pmatrix} n\theta \begin{pmatrix}\mathcal{I}_n^u[f_n']\\ \mathcal{I}_n^u[g_n']\end{pmatrix} \tag{56}$$

$$v(x,r,\theta) = \sum_{n=0}^{\infty}\begin{pmatrix}\cos\\ \sin\end{pmatrix} n\theta \begin{pmatrix}\mathcal{I}_n^v[f_n']\\ \mathcal{I}_n^v[g_n']\end{pmatrix} \tag{57}$$

$$w(x,r,\theta) = \sum_{n=0}^{\infty}\begin{pmatrix}\cos\\ \sin\end{pmatrix} n\theta \begin{pmatrix}\mathcal{I}_n^w[g_n]\\ -\mathcal{I}_n^w[f_n]\end{pmatrix} \tag{58}$$

It is also appropriate to write $R(x,\theta)$ as a Fourier series:

$$\boxed{R(x,\theta) = \sum_{n=0}^{\infty} a_n(x)\cos n\theta + \sum_{n=0}^{\infty} b_n(x)\sin n\theta} \tag{59}$$

$$R(x,\theta) = \sum_{n=0}^{\infty} \begin{pmatrix}\cos\\ \sin\end{pmatrix} n\theta \begin{pmatrix}a_n\\ b_n\end{pmatrix} \tag{60}$$

Then:

$$\frac{\partial}{\partial x}R(x,\theta) = \sum_{n=0}^{\infty} \begin{pmatrix}\cos\\ \sin\end{pmatrix} n\theta \begin{pmatrix}\partial a_n/\partial x\\ \partial b_n/\partial x\end{pmatrix} \tag{61}$$

$$\frac{\partial}{\partial \theta}R(x,\theta) = \sum_{n=0}^{\infty} \begin{pmatrix}\cos\\ \sin\end{pmatrix} n\theta \begin{pmatrix}nb_n\\ -na_n\end{pmatrix} \tag{62}$$

Substituting Equations 57, 58, and 60 through 62 into Equation 49 results in:

$$\sum_{n=0}^{\infty} \begin{pmatrix}\cos\\ \sin\end{pmatrix} n\theta \begin{pmatrix}\partial a_n/\partial x\\ \partial b_n/\partial x\end{pmatrix} = \sum_{n=0}^{\infty} \begin{pmatrix}\cos\\ \sin\end{pmatrix} n\theta \begin{pmatrix}\mathcal{I}_n^v[f_n']\\ \mathcal{I}_n^v[g_n']\end{pmatrix} - \frac{\sum_{n=0}^{\infty} \begin{pmatrix}\cos\\ \sin\end{pmatrix} n\theta \begin{pmatrix}\mathcal{I}_n^w[g_n]\\ -\mathcal{I}_n^w[f_n]\end{pmatrix} \cdot \sum_{n=0}^{\infty} \begin{pmatrix}\cos\\ \sin\end{pmatrix} n\theta \begin{pmatrix}nb_n\\ -na_n\end{pmatrix}}{\sum_{n=0}^{\infty} \begin{pmatrix}\cos\\ \sin\end{pmatrix} n\theta \begin{pmatrix}a_n\\ b_n\end{pmatrix}} \tag{63}$$

The above equation is the exact boundary condition for tangential flow on the surface of a body. The last term consists of products and quotients of infinite series, and presents a problem in attempting to determine the singularity distributions f_n', g_n'. If the body surface is very nearly axisymmetric, then $\partial R/\partial\theta \simeq 0$ and $w << v$. This limitation will be called the quasi-axisymmetric boundary condition. In this case the last term in Equation 63 can be dropped and the following boundary condition results:

$$\sum_{n=0}^{\infty} \begin{pmatrix}\cos\\ \sin\end{pmatrix} n\theta \begin{pmatrix}\partial a_n/\partial x - \mathcal{I}_n^v[f_n']\\ \partial b_n/\partial x - \mathcal{I}_n^v[g_n']\end{pmatrix} = 0 \tag{64}$$

Note that the above boundary condition is exact for axisymmetric bodies, and is an approximation for non-axisymmetric bodies. In Equation 64, since cosine and sine are orthogonal functions, and since the infinite series are equal to zero, it necessarily follows that each term in the series must be equal to zero also. Thus Equation 64 reduces to the following pair of independent sets of linear equations:

$$\frac{\partial a_n}{\partial x} = \mathcal{I}_n^v[f_n'], \qquad n = 0, 1, 2, \ldots\infty \tag{65}$$

$$\frac{\partial b_n}{\partial x} = \mathcal{I}_n^v[g_n'], \qquad n = 1, 2, 3, \ldots\infty \tag{66}$$

The above equations provide unique relations between the singularity distributions and the derivatives of the body Fourier coefficients. For known singularity distributions, a body surface is defined. For a defined body surface, a unique set of singularity distributions must exist. To determine the singularity distribution from a known body, start by expanding Equation 65. A similar analysis exists for Equation 66.

$$\begin{aligned}\frac{\partial a_n(x)}{\partial x} = &-\int_0^{x-\beta r} \frac{1}{2r}\left[\frac{x-\xi+\sqrt{(x-\xi)^2-\beta^2 r^2}}{\beta r}\right]^n \frac{(x-\xi)\, f_n'(\xi)d\xi}{\sqrt{(x-\xi)^2-\beta^2 r^2}} \\ &-\int_0^{x-\beta r} \frac{1}{2r}\left[\frac{x-\xi-\sqrt{(x-\xi)^2-\beta^2 r^2}}{\beta r}\right]^n \frac{(x-\xi)\, f_n'(\xi)d\xi}{\sqrt{(x-\xi)^2-\beta^2 r^2}}\end{aligned} \tag{67}$$

Discretization of the Singularity Distributions

There is a relatively simple technique which can be used to determine $f_n'(\xi)$ if $\partial a_n(x)/\partial x$ is known. If $f_n'(\xi)$ is assumed piecewise constant over a small interval (ξ_{i-1}, ξ_i), then $f_n'(\xi_i)$ can be pulled out of the integrals as follows.

$$\frac{\partial a_n(x)}{\partial x} = -\sum_{i=1}^{k} f_n'(\xi_i) \int_{\xi_{i-1}}^{\xi_i} \frac{1}{2r} \left[\frac{x-\xi+\sqrt{(x-\xi)^2-\beta^2 r^2}}{\beta r}\right]^n \frac{(x-\xi)\,d\xi}{\sqrt{(x-\xi)^2-\beta^2 r^2}}$$
$$-\sum_{i=1}^{k} f_n'(\xi_i) \int_{\xi_{i-1}}^{\xi_i} \frac{1}{2r} \left[\frac{x-\xi-\sqrt{(x-\xi)^2-\beta^2 r^2}}{\beta r}\right]^n \frac{(x-\xi)\,d\xi}{\sqrt{(x-\xi)^2-\beta^2 r^2}} \tag{68}$$

Where:

$$\xi_0 = 0 \tag{69}$$

$$\xi_k = x - \beta r \tag{70}$$

Note that x, r are fixed locations on the body surface. It is convient to introduce another integral notation. Let:

$$I_n^{v+}\Big|_{\xi_{i-1}}^{\xi_i} = \int_{\xi_{i-1}}^{\xi_i} \frac{1}{2r} \left[\frac{x-\xi+\sqrt{(x-\xi)^2-\beta^2 r^2}}{\beta r}\right]^n \frac{(x-\xi)\,d\xi}{\sqrt{(x-\xi)^2-\beta^2 r^2}} \tag{71}$$

$$I_n^{v-}\Big|_{\xi_{i-1}}^{\xi_i} = \int_{\xi_{i-1}}^{\xi_i} \frac{1}{2r} \left[\frac{x-\xi-\sqrt{(x-\xi)^2-\beta^2 r^2}}{\beta r}\right]^n \frac{(x-\xi)\,d\xi}{\sqrt{(x-\xi)^2-\beta^2 r^2}} \tag{72}$$

Then Equation 68 becomes:

$$\frac{\partial a_n(x)}{\partial x} = -\sum_{i=1}^{k} f_n'(\xi_i) \left(I_n^{v+} + I_n^{v-}\right)\Big|_{\xi_{i-1}}^{\xi_i} \tag{73}$$

By breaking up the above summation:

$$-\frac{\partial a_n(x)}{\partial x} = f_n'(\xi_k) \left(I_n^{v+} + I_n^{v-}\right)\Big|_{\xi_{k-1}}^{\xi_k} + \sum_{i=1}^{k-1} f_n'(\xi_i) \left(I_n^{v+} + I_n^{v-}\right)\Big|_{\xi_{i-1}}^{\xi_i} \tag{74}$$

Or finally:

$$\boxed{f_n'(\xi_k) = -\frac{\sum_{i=1}^{k-1} f_n'(\xi_i) \left(I_n^{v+} + I_n^{v-}\right)\Big|_{\xi_{i-1}}^{\xi_i} + \frac{\partial a_n(x)}{\partial x}}{\left(I_n^{v+} + I_n^{v-}\right)\Big|_{\xi_{k-1}}^{\xi_k}}} \tag{75}$$

Thus $f_n'(\xi_k)$ can be found in terms of known integrals and $f_n'(\xi_{k-1}), f_n'(\xi_{k-2}), \ldots$. Therefore, the entire singularity distribution can be determined by marching down along the body axis. The corresponding relation for the g_n' terms is:

$$g_n'(\xi_k) = -\frac{\sum_{i=1}^{k-1} g_n'(\xi_i)\left(I_n^{v^+} + I_n^{v^-}\right)\Big|_{\xi_{i-1}}^{\xi_i} + \frac{\partial b_n(x)}{\partial x}}{\left(I_n^{v^+} + I_n^{v^-}\right)\Big|_{\xi_{k-1}}^{\xi_k}} \tag{76}$$

With the singularity distributions known it is possible to calculate the pressure coefficient field on and about the body. A first order pressure coefficient is defined as:

$$c_p(x,r,\theta) = -2u(x,r,\theta) \tag{77}$$

And from Equation 50:

$$c_p(x,r,\theta) = -2\sum_{n=0}^{\infty} \cos n\theta \mathcal{I}_n^u[f_n'] - 2\sum_{n=0}^{\infty} \sin n\theta \mathcal{I}_n^u[g_n'] \tag{78}$$

Since f_n', g_n' are discrete:

$$\begin{aligned} c_p(x,r,\theta) = & -2\sum_{n=0}^{\infty} \cos n\theta \sum_{i=1}^{k} f_n'(\xi_i) \int_{\xi_{i-1}}^{\xi_i} \frac{1}{2}\left[\frac{x-\xi+\sqrt{(x-\xi)^2-\beta^2 r^2}}{\beta r}\right]^n \frac{d\xi}{\sqrt{(x-\xi)^2-\beta^2 r^2}} \\ & -2\sum_{n=0}^{\infty} \cos n\theta \sum_{i=1}^{k} f_n'(\xi_i) \int_{\xi_{i-1}}^{\xi_i} \frac{1}{2}\left[\frac{x-\xi-\sqrt{(x-\xi)^2-\beta^2 r^2}}{\beta r}\right]^n \frac{d\xi}{\sqrt{(x-\xi)^2-\beta^2 r^2}} \\ & -2\sum_{n=0}^{\infty} \sin n\theta \sum_{i=1}^{k} g_n'(\xi_i) \int_{\xi_{i-1}}^{\xi_i} \frac{1}{2}\left[\frac{x-\xi+\sqrt{(x-\xi)^2-\beta^2 r^2}}{\beta r}\right]^n \frac{d\xi}{\sqrt{(x-\xi)^2-\beta^2 r^2}} \\ & -2\sum_{n=0}^{\infty} \sin n\theta \sum_{i=1}^{k} g_n'(\xi_i) \int_{\xi_{i-1}}^{\xi_i} \frac{1}{2}\left[\frac{x-\xi-\sqrt{(x-\xi)^2-\beta^2 r^2}}{\beta r}\right]^n \frac{d\xi}{\sqrt{(x-\xi)^2-\beta^2 r^2}} \end{aligned} \tag{79}$$

Simplifying the above expression by the following integral definitions:

$$I_n^{u^+}\Big|_{\xi_{i-1}}^{\xi_i} = \int_{\xi_{i-1}}^{\xi_i} \frac{1}{2}\left[\frac{x-\xi+\sqrt{(x-\xi)^2-\beta^2 r^2}}{\beta r}\right]^n \frac{d\xi}{\sqrt{(x-\xi)^2-\beta^2 r^2}} \tag{80}$$

$$I_n^{u^-}\Big|_{\xi_{i-1}}^{\xi_i} = \int_{\xi_{i-1}}^{\xi_i} \frac{1}{2}\left[\frac{x-\xi-\sqrt{(x-\xi)^2-\beta^2 r^2}}{\beta r}\right]^n \frac{d\xi}{\sqrt{(x-\xi)^2-\beta^2 r^2}} \tag{81}$$

Yields the following expression for the pressure coefficient:

$$c_p(x,r,\theta) = -2\sum_{n=0}^{\infty} \cos n\theta \sum_{i=1}^{k} f_n'(\xi_i)\left(I_n^{u^+} + I_n^{u^-}\right)\Big|_{\xi_{i-1}}^{\xi_i} - 2\sum_{n=0}^{\infty} \sin n\theta \sum_{i=1}^{k} g_n'(\xi_i)\left(I_n^{u^+} + I_n^{u^-}\right)\Big|_{\xi_{i-1}}^{\xi_i} \tag{82}$$

Evaluation of the Integrals

All that remains is to find closed form solutions for the integrals $I_n^{v^+}\Big|_{\xi_{i-1}}^{\xi_i}$, $I_n^{v^-}\Big|_{\xi_{i-1}}^{\xi_i}$, $I_n^{u^+}\Big|_{\xi_{i-1}}^{\xi_i}$, $I_n^{u^-}\Big|_{\xi_{i-1}}^{\xi_i}$, given in Equations 71, 72, 80, and 81. Starting with the evaluation of the integral in Equation 71, where now $c = \beta r$:

$$I_n^{v+}\Big|_{\xi_{i-1}}^{\xi_i} = \frac{1}{2rc^n}\int_{\xi_{i-1}}^{\xi_i}\left[x-\xi+\sqrt{(x-\xi)^2-c^2}\right]^n \frac{(x-\xi)\,d\xi}{\sqrt{(x-\xi)^2-c^2}} \tag{83}$$

Introducing the first change of variable:

$$\sigma = x - \xi \tag{84}$$

Yields:

$$I_n^{v+}\Big|_{\xi_{i-1}}^{\xi_i} = -\frac{1}{2rc^n}\int_{\sigma_{i-1}}^{\sigma_i}\left[\sigma+\sqrt{\sigma^2-c^2}\right]^n \frac{\sigma d\sigma}{\sqrt{\sigma^2-c^2}} \tag{85}$$

Introducing the second change of variable:

$$\rho = \sqrt{\sigma^2-c^2} \tag{86}$$

$$d\sigma = \frac{\sqrt{\sigma^2-c^2}}{\sigma}d\rho \tag{87}$$

Yields:

$$I_n^{v+}\Big|_{\xi_{i-1}}^{\xi_i} = -\frac{1}{2rc^n}\int_{\rho_{i-1}}^{\rho_i}\left[\rho+\sqrt{\rho^2+c^2}\right]^n d\rho \tag{88}$$

Introducing the third change of variable, and utilizing one of the indefinite integrals found in the CRC Standard Math Tables* :

$$\lambda = \tan^{-1}\frac{\rho}{c} \tag{89}$$

Yields:

$$I_n^{v+}\Big|_{\xi_{i-1}}^{\xi_i} = -\frac{c}{2rc^n}\int_{\lambda_{i-1}}^{\lambda_i}\left[c\tan\lambda + c\sec\lambda\right]^n \sec^2\lambda d\lambda \tag{90}$$

$$I_n^{v+}\Big|_{\xi_{i-1}}^{\xi_i} = -\frac{c}{2r}\int_{\lambda_{i-1}}^{\lambda_i}\left[\frac{1+\sin\lambda}{\cos\lambda}\right]^n \frac{d\lambda}{\cos^2\lambda} \tag{91}$$

Introducing the fourth, and final, change of variable:

$$\eta = \frac{1+\sin\lambda}{\cos\lambda} \tag{92}$$

$$d\lambda = \cos\lambda\frac{d\eta}{\eta} \tag{93}$$

Yields:

$$I_n^{v+}\Big|_{\xi_{i-1}}^{\xi_i} = -\frac{c}{2r}\int_{\eta_{i-1}}^{\eta_i}\frac{\eta^{n-1}}{\cos\lambda}d\eta \tag{94}$$

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*

$$\int f\left(x,\sqrt{x^2+a^2}\right)dx = a\int f\left(a\tan u, a\sec u\right)\sec^2 u du, \qquad u=\tan^{-1}\frac{x}{a}, \qquad a>0$$

From Equation 92, and utilizing simple trigonometric identities, it can be shown that:

$$\cos\lambda = \frac{2\eta}{\eta^2 + 1} \tag{95}$$

Substituting Equation 95 into Equation 94, and recalling that $c = \beta r$, yields a simple expression which can be directly integrated in closed form:

$$I_n^{v^+}\Big|_{\xi_{i-1}}^{\xi_i} = -\frac{\beta}{4}\int_{\eta_{i-1}}^{\eta_i} \eta^{n-2}\left(\eta^2+1\right) d\eta \tag{96}$$

For $n = 0$:

$$I_n^{v^+}\Big|_{\xi_{i-1}}^{\xi_i} = -\frac{\beta}{4}\left[\eta - \frac{1}{\eta}\right]_{\eta_{i-1}}^{\eta_i} \tag{97}$$

For $n = 1$:

$$I_n^{v^+}\Big|_{\xi_{i-1}}^{\xi_i} = -\frac{\beta}{4}\left[\frac{1}{2}\eta^2 + \ln\eta\right]_{\eta_{i-1}}^{\eta_i} \tag{98}$$

For $n \geq 2$:

$$I_n^{v^+}\Big|_{\xi_{i-1}}^{\xi_i} = -\frac{\beta}{4}\left[\frac{1}{n+1}\eta^{n+1} + \frac{1}{n-1}\eta^{n-1}\right]_{\eta_{i-1}}^{\eta_i} \tag{99}$$

Reversing the change of variables in Equations 92, 89, 86, and 84 yields the limits of integration in Equations 97, 98, and 99:

$$\eta_{i-1} = \frac{x - \xi_{i-1} + \sqrt{\left(x-\xi_{i-1}\right)^2 - \beta^2 r^2}}{\beta r} \tag{100}$$

$$\eta_i = \frac{x - \xi_i + \sqrt{\left(x-\xi_i\right)^2 - \beta^2 r^2}}{\beta r} \tag{101}$$

Continuing on with the evaluation of the integral in Equation 72:

$$I_n^{v^-}\Big|_{\xi_{i-1}}^{\xi_i} = \frac{1}{2rc^n}\int_{\xi_{i-1}}^{\xi_i}\left[x - \xi - \sqrt{\left(x-\xi\right)^2 - c^2}\right]^n \frac{\left(x-\xi\right) d\xi}{\sqrt{\left(x-\xi\right)^2 - c^2}} \tag{102}$$

Using the same change of variable in Equations 84, 86, and 89 yields the following relations:

$$I_n^{v^-}\Big|_{\xi_{i-1}}^{\xi_i} = -\frac{1}{2rc^n}\int_{\sigma_{i-1}}^{\sigma_i}\left[\sigma - \sqrt{\sigma^2 - c^2}\right]^n \frac{\sigma d\sigma}{\sqrt{\sigma^2 - c^2}} \tag{103}$$

$$I_n^{v^-}\Big|_{\xi_{i-1}}^{\xi_i} = -\frac{1}{2rc^n}\int_{\rho_{i-1}}^{\rho_i}\left[\sqrt{\rho^2 + c^2} - \rho\right]^n d\rho \tag{104}$$

$$I_n^{v^-}\Big|_{\xi_{i-1}}^{\xi_i} = -\frac{c}{2rc^n}\int_{\lambda_{i-1}}^{\lambda_i}\left[c\sec\lambda - c\tan\lambda\right]^n \sec^2\lambda d\lambda \tag{105}$$

$$I_n^{v^-}\Big|_{\xi_{i-1}}^{\xi_i} = -\frac{c}{2r}\int_{\lambda_{i-1}}^{\lambda_i}\left[\frac{1-\sin\lambda}{\cos\lambda}\right]^n \frac{d\lambda}{\cos^2\lambda} \tag{106}$$

Introducing a new fourth change of variable:

$$\mu = \frac{1-\sin\lambda}{\cos\lambda} \tag{107}$$

$$d\lambda = -\cos\lambda \frac{d\mu}{\mu} \tag{108}$$

Yields:

$$I_n^{v^-}\Big|_{\xi_{i-1}}^{\xi_i} = \frac{c}{2r}\int_{\mu_{i-1}}^{\mu_i} \frac{\mu^{n-1}}{\cos\lambda} d\mu \tag{109}$$

From Equation 107, it can be shown that:

$$\cos\lambda = \frac{2\mu}{\mu^2+1} \tag{110}$$

Substituting Equation 110 into Equation 109 yields a simple expression which can be directly integrated in closed form:

$$I_n^{v^-}\Big|_{\xi_{i-1}}^{\xi_i} = \frac{\beta}{4}\int_{\mu_{i-1}}^{\mu_i} \mu^{n-2}\left(\mu^2+1\right) d\mu \tag{111}$$

For $n = 0$:

$$I_n^{v^-}\Big|_{\xi_{i-1}}^{\xi_i} = \frac{\beta}{4}\left[\mu - \frac{1}{\mu}\right]_{\mu_{i-1}}^{\mu_i} \tag{112}$$

For $n = 1$:

$$I_n^{v^-}\Big|_{\xi_{i-1}}^{\xi_i} = \frac{\beta}{4}\left[\frac{1}{2}\mu^2 + \ln\mu\right]_{\mu_{i-1}}^{\mu_i} \tag{113}$$

For $n \geq 2$:

$$I_n^{v^-}\Big|_{\xi_{i-1}}^{\xi_i} = \frac{\beta}{4}\left[\frac{1}{n+1}\mu^{n+1} + \frac{1}{n-1}\mu^{n-1}\right]_{\mu_{i-1}}^{\mu_i} \tag{114}$$

Reversing the change of variables in Equations 107, 89, 86, and 84 yields the limits of integration in Equations 112, 113, and 114:

$$\mu_{i-1} = \frac{x - \xi_{i-1} - \sqrt{\left(x-\xi_{i-1}\right)^2 - \beta^2 r^2}}{\beta r} \tag{115}$$

$$\mu_i = \frac{x - \xi_i - \sqrt{\left(x-\xi_i\right)^2 - \beta^2 r^2}}{\beta r} \tag{116}$$

Proceding with the evaluation of the integral in Equation 80:

$$I_n^{u^+}\Big|_{\xi_{i-1}}^{\xi_i} = \frac{1}{2c^n}\int_{\xi_{i-1}}^{\xi_i} \left[x - \xi + \sqrt{\left(x-\xi\right)^2 - c^2}\right]^n \frac{d\xi}{\sqrt{\left(x-\xi\right)^2 - c^2}} \tag{117}$$

Using the same change of variable in Equations 84, 86, 89, and 92 yields the following:

$$I_n^{u^+}\Big|_{\xi_{i-1}}^{\xi_i} = -\frac{1}{2c^n}\int_{\sigma_{i-1}}^{\sigma_i} \left[\sigma + \sqrt{\sigma^2 - c^2}\right]^n \frac{d\sigma}{\sqrt{\sigma^2 - c^2}} \tag{118}$$

$$I_n^{u^+}\Big|_{\xi_{i-1}}^{\xi_i} = -\frac{1}{2c^n}\int_{\rho_{i-1}}^{\rho_i} \left[\rho + \sqrt{\rho^2 + c^2}\right]^n \frac{d\rho}{\sqrt{\rho^2 + c^2}} \tag{119}$$

$$I_n^{u+}\Big|_{\xi_{i-1}}^{\xi_i} = -\frac{1}{2c^n}\int_{\lambda_{i-1}}^{\lambda_i} [c\tan\lambda + c\sec\lambda]^n \sec\lambda d\lambda \tag{120}$$

$$I_n^{u+}\Big|_{\xi_{i-1}}^{\xi_i} = -\frac{1}{2}\int_{\lambda_{i-1}}^{\lambda_i} \left[\frac{1+\sin\lambda}{\cos\lambda}\right]^n \frac{d\lambda}{\cos\lambda} \tag{121}$$

$$I_n^{u+}\Big|_{\xi_{i-1}}^{\xi_i} = -\frac{1}{2}\int_{\eta_{i-1}}^{\eta_i} \eta^{n-1} d\eta \tag{122}$$

For $n = 0$:

$$I_n^{u+}\Big|_{\xi_{i-1}}^{\xi_i} = -\left[\frac{1}{2}\ln\eta\right]_{\eta_{i-1}}^{\eta_i} \tag{123}$$

For $n = 1$:

$$I_n^{u+}\Big|_{\xi_{i-1}}^{\xi_i} = -\left[\frac{1}{2}\eta\right]_{\eta_{i-1}}^{\eta_i} \tag{124}$$

For $n \geq 2$:

$$I_n^{u+}\Big|_{\xi_{i-1}}^{\xi_i} = -\left[\frac{1}{2n}\eta^n\right]_{\eta_{i-1}}^{\eta_i} \tag{125}$$

The limits of integration for Equations 123, 124, and 125 are the same as those given in Equations 100 and 101. Finally, the evaluation of the last integral in Equation 81:

$$I_n^{u-}\Big|_{\xi_{i-1}}^{\xi_i} = \frac{1}{2c^n}\int_{\xi_{i-1}}^{\xi_i} \left[x - \xi - \sqrt{(x-\xi)^2 - c^2}\right]^n \frac{d\xi}{\sqrt{(x-\xi)^2 - c^2}} \tag{126}$$

Using the same change of variable in Equations 84, 86, 89, and 107 yields the following:

$$I_n^{u-}\Big|_{\xi_{i-1}}^{\xi_i} = -\frac{1}{2c^n}\int_{\sigma_{i-1}}^{\sigma_i} \left[\sigma - \sqrt{\sigma^2 - c^2}\right]^n \frac{d\sigma}{\sqrt{\sigma^2 - c^2}} \tag{127}$$

$$I_n^{u-}\Big|_{\xi_{i-1}}^{\xi_i} = -\frac{1}{2c^n}\int_{\rho_{i-1}}^{\rho_i} \left[\sqrt{\rho^2 + c^2} - \rho\right]^n \frac{d\rho}{\sqrt{\rho^2 + c^2}} \tag{128}$$

$$I_n^{u-}\Big|_{\xi_{i-1}}^{\xi_i} = -\frac{1}{2c^n}\int_{\lambda_{i-1}}^{\lambda_i} [c\sec\lambda - c\tan\lambda]^n \sec\lambda d\lambda \tag{129}$$

$$I_n^{u-}\Big|_{\xi_{i-1}}^{\xi_i} = -\frac{1}{2}\int_{\lambda_{i-1}}^{\lambda_i} \left[\frac{1-\sin\lambda}{\cos\lambda}\right]^n \frac{d\lambda}{\cos\lambda} \tag{130}$$

$$I_n^{u-}\Big|_{\xi_{i-1}}^{\xi_i} = \frac{1}{2}\int_{\mu_{i-1}}^{\mu_i} \mu^{n-1} d\mu \tag{131}$$

For $n = 0$:

$$I_n^{u-}\Big|_{\xi_{i-1}}^{\xi_i} = \left[\frac{1}{2}\ln\mu\right]_{\mu_{i-1}}^{\mu_i} \tag{132}$$

For $n = 1$:

$$I_n^{u-}\Big|_{\xi_{i-1}}^{\xi_i} = \left[\frac{1}{2}\mu\right]_{\mu_{i-1}}^{\mu_i} \tag{133}$$

For $n \geq 2$:

$$I_n^{u^-}\Big|_{\xi_{i-1}}^{\xi_i} = \left[\frac{1}{2n}\mu^n\right]_{\mu_{i-1}}^{\mu_i} \tag{134}$$

The limits of integration for Equations 132, 133, and 134 are the same as those given in Equations 115 and 116. This completes the evaluation of all the integrals necessary to formulate the problem solution numerically.

Summary

An analysis technique has been developed employing supersonic linearized small perturbation theory which is not restricted to slender bodies of circular cross section. The method developed here can be employed in the analysis of wave drag for slender, arbitrary cross section bodies. The method permits the calculation of the first-order pressure coefficient on or about the body. The pressure coefficient can be integrated over the surface of the body to compute the wave drag resulting from the body. Also, the pressure coefficient field can be calculated about the body, and this pressure field can then be used in other existing analyses and computer programs to determine the aerodynamic interference drag for complex aircraft configurations.

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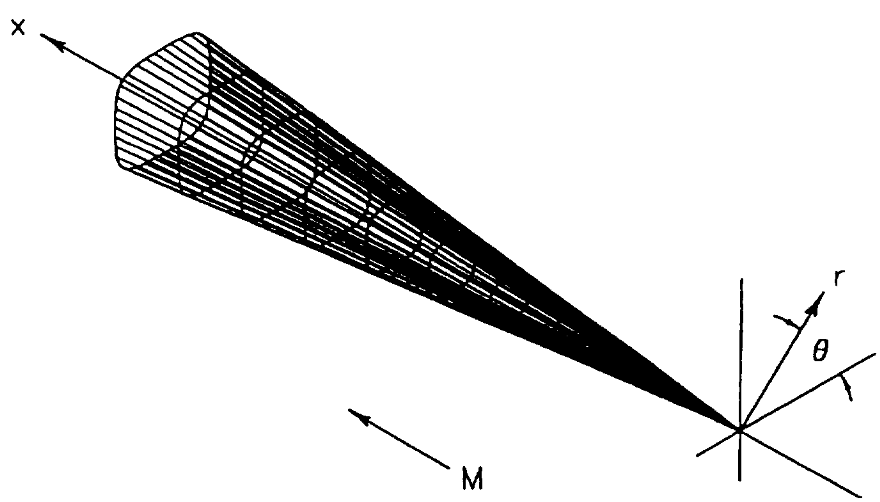

Figure 1: Cylindrical Coordinate System

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A Mathematical theory is developed to perform the calculations necessary to determine the wave drag for slender bodies of non-circular cross section. The derivations presented in this report are based on extensions to supersonic linearized small perturbation theory. A numerical scheme is presented utilizing Fourier decomposition to compute the pressure coefficient on and about a slender body of arbitrary cross section.

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