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Visualization of 3-D Tensor Fields

Principal Investigator: Lambertus Hesselink*
Stanford University, Stanford, California 94305-4035
Phone: (415)723-4850 Email: bert@kaos.stanford.edu

Introduction

Second-order tensor fields have applications in many different areas of physics, such as general relativity and fluid mechanics. The wealth of multivariate information in tensor fields makes them more complex and abstract than scalar and vector fields. Visualization is a good technique for scientists to gain new insights from them.

Visualizing a 3-D continuous tensor field is equivalent to simultaneously visualizing its three eigenvector fields. In the past, research has been conducted in the area of two-dimensional tensor fields.[1] It was shown that degenerate points, defined as points where eigenvalues are equal to each other, are the basic singularities underlying the topology of tensor fields. Moreover, it was shown that eigenvectors never cross each other except at degenerate points. Since we live in a three-dimensional world, it is important for us to understand the underlying physics of this world. In this report, we describe a new method for locating degenerate points along with the conditions for classifying them in three-dimensional space. Finally, we discuss some topological features of three-dimensional tensor fields, and interpret topological patterns in terms of physical properties.

*Professor, Electrical Engineering and Aeronautics and Astronautics Departments.

Theoretical Background

Definitions

Three-dimensional tensor fields can be represented by 3×3 matrices. In Cartesian coordinates these take the following form:

$$\mathbf{T}(\vec{x}) = \begin{pmatrix} T_{11}(x,y,z) & T_{12}(x,y,z) & T_{13}(x,y,z) \\ T_{12}(x,y,z) & T_{22}(x,y,z) & T_{23}(x,y,z) \\ T_{13}(x,y,z) & T_{23}(x,y,z) & T_{33}(x,y,z) \end{pmatrix} \quad (1)$$

Definition 1 *A degenerate point of a tensor field $\mathbf{T} : E \rightarrow \mathcal{L}(\mathbf{R}^m, \mathbf{R}^m)$, where E is an open subset of $\mathbf{R}^m$, is a point $\vec{x}_0 \in E$ where at least two of the m eigenvalues of $\mathbf{T}$ are equal to each other.*[1]

Definition 2 *(Hyperstreamline) A geometric primitive of finite size sweeps along the longitudinal eigenvector field, $\vec{v}_l$, while stretching in the transverse plane under the combined action of the two transverse eigenvectors, $\vec{v}_{t_1}$ and $\vec{v}_{t_2}$. Hyperstreamlines are surfaces that envelop the stretched primitives along the trajectories. We refer to hyperstreamlines as "major", "medium" or "minor" depending on the corresponding longitudinal eigenvector field that defines their trajectories and color hyperstreamlines by means of a user-defined function of the three eigenvalues, usually the amplitude of the longitudinal eigenvalue.*[1]

Locating Degenerate Points

A three-dimensional symmetric tensor field (Equation (1)) has 6 independent variables, three of which are on its diagonal. As a result, various types of degenerate points may exist. These types correspond to the following conditions:

$$\lambda_1(\vec{x}_0) = \lambda_2(\vec{x}_0) > \lambda_3(\vec{x}_0) \quad (2)$$

$$\lambda_1(\vec{x}_0) > \lambda_2(\vec{x}_0) = \lambda_3(\vec{x}_0) \quad (3)$$

$$\lambda_1(\vec{x}_0) = \lambda_2(\vec{x}_0) = \lambda_3(\vec{x}_0) \quad (4)$$

The characteristic equation of a 3-D symmetric tensor can be expressed in the following form

$$A(\lambda) = -\lambda^3 + a\lambda^2 + b\lambda + c \quad (5)$$

where a, b and c are expressed in terms of the 6 independent tensor components.

By applying the conditions for double and triple degeneracy, Equations (2, 3, and 4), we obtain the corresponding conditions respectively:

$$B_1(x,y,z) = \frac{2a^3 + 9ab + 2d^{3/2}}{27} + c = 0 \quad (6)$$

$$B_2(x,y,z) = \frac{2a^3 + 9ab - 2d^{3/2}}{27} + c = 0 \quad (7)$$

$$B_3(x,y,z) = a^2 + 3b = 0 \quad (8)$$

From the expressions for B_1, B_2 and B_3, we can also get that: $B_1(x,y,z) = 0$ is a maximum for B_1, $B_2(x,y,z) = 0$ is a minimum for B_2 and $B_3(x,y,z) = 0$ is a maximum for B_3.

Now the problem is to find extrema in a 3D continuous field from the discrete data sets. On a 3-D discrete mesh, the search for the various extrema is conducted by processing one grid cell at a time for each spatial function.

This method can successfully find the points of triple degeneracy and is especially useful when extended to locate points of double degeneracy where the local tensor appears in the diagonal form only when transformed into its eigenvector space.

Separating Surfaces

The classification of degenerate points in 2-D tensor fields[1] can be extended to 3-D tensor fields. The building blocks are the fundamental elements as defined for 2-D.[1] However, the separating surfaces in 3-D tensor fields have a general structure as they could appear at various angles. Each of the surfaces is characterized by patterns similar to those of hyperbolic or parabolic sectors and is bounded by hyperstreamlines that are emanating from or terminated at the degenerate point. Consequently, a point of triple degeneracy can be classified by the number and type of separating surfaces surrounding it.

In Figures 1 we show the eigenvector patterns in the vicinity of a point of triple degeneracy with 4 bounding hyperstreamlines. These hyperstreamlines form 6 hyperbolic separating surfaces. Figures 2 shows a point of triple

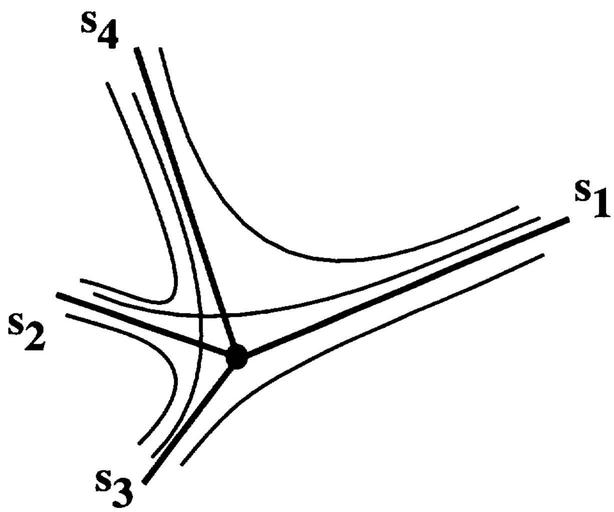

Figure 1: A point of triple degeneracy with 6 hyperbolic separating surfaces.

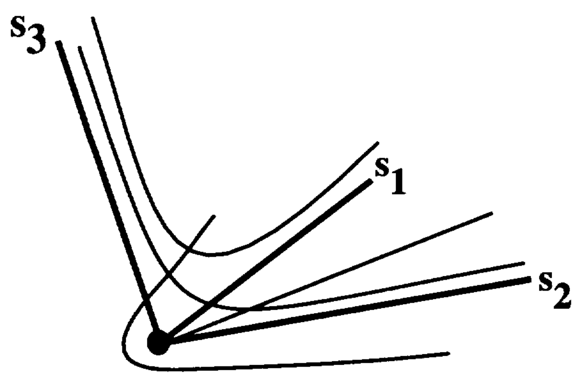

Figure 2: A point of triple degeneracy with 2 hyperbolic separating surfaces and one parabolic surface.

degeneracy with only 3 bounding hyperstreamlines which form 2 hyperbolic separating surfaces and one parabolic surface.

The trajectories on the surfaces are locally 2-D, while off the surfaces they are fully 3-D and are determined by their closest surface.

Topology of 3-D Tensor Fields

We choose the elastic stress tensor induced by two compressive forces on the top of a semi-infinite plane to illustrate the advantages of using topological skeletons in visualizing 3-D tensor fields. In principle, hyperstreamline trajectories of the stress tensor show the transmission of forces inside the material. Figure 3 shows two hyperstreamlines corresponding to the most compressive eigen direction, the minor eigenvector $\overline{v}_3$. The two forces, indicated by the arrows, act on the surface at $\mathbf{P}_1 = (0.5, 0.0, -1.05)$ and $\mathbf{P}_2 = (-0.5, 0.0, -1.05)$ in the $+z$ direction (downward). The domain of interest (described by the bounding frame) extends between $(-1.0, -1.0, -1.114367)$ and $(1.0, 1.0, 0.0)$ so it includes the key features of the stress tensor field, i.e., the degenerate points. It is assumed that the region where $z < -1.05$ is in tension and that no stresses are transferred across the plane $z = -1.05$. The color of the hyperstreamlines encodes the magnitude of the most compressive eigenvalue, λ_3, while their cross section encodes the magnitude and direction of the transverse eigenvectors. The hyperstreamlines converge toward regions of high stresses where the forces are applied. Note the sharp change in color and cross-section size of the hyperstreamlines as they approach the acting points of the forces.

Analysis reveals that the tensor field contains two points of triple degeneracy and that these points reside on the surface of the semi-infinite plane. Moreover, the eigenvalues at these points (the location of which is given by: $\mathbf{D}_1 = (0.0, 0.5, -1.05)$, $\mathbf{D}_2 = (0.0, -0.5, -1.05)$) are equal to zero. This means that these points are stress free, a fact that can be verified by an examination of the stress equations. We have therefore acquired physical insight into the stress tensor field just by an examination of a basic topological feature, a point of triple degeneracy.

Figure 4 shows hyperstreamlines that are obtained by tracing the major eigenvector field. The location and direction of the forces are indicated by the arrows and the location of the points of triple degeneracy are marked by

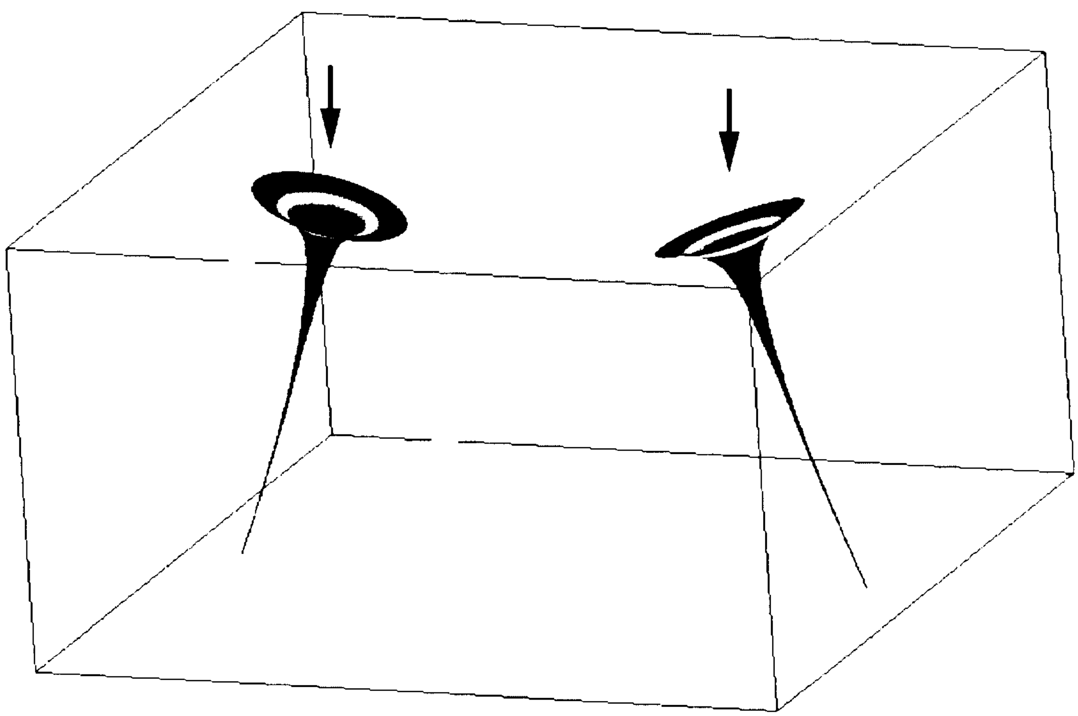

Figure 3: Stress tensor induced by two compressive forces; minor hyper-streamlines

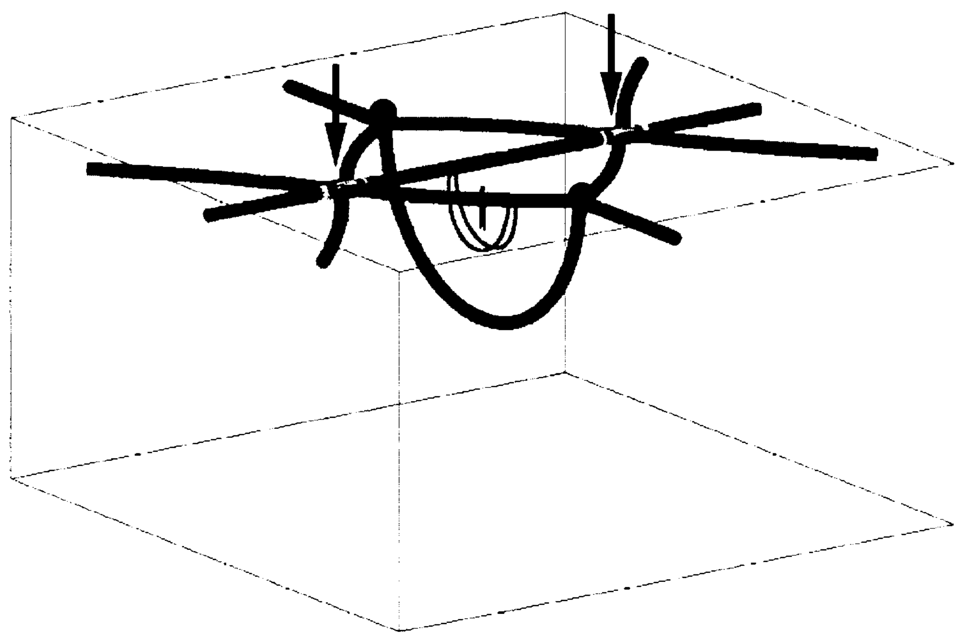

Figure 4: Stress tensor induced by two compressive forces; major hyper-streamlines

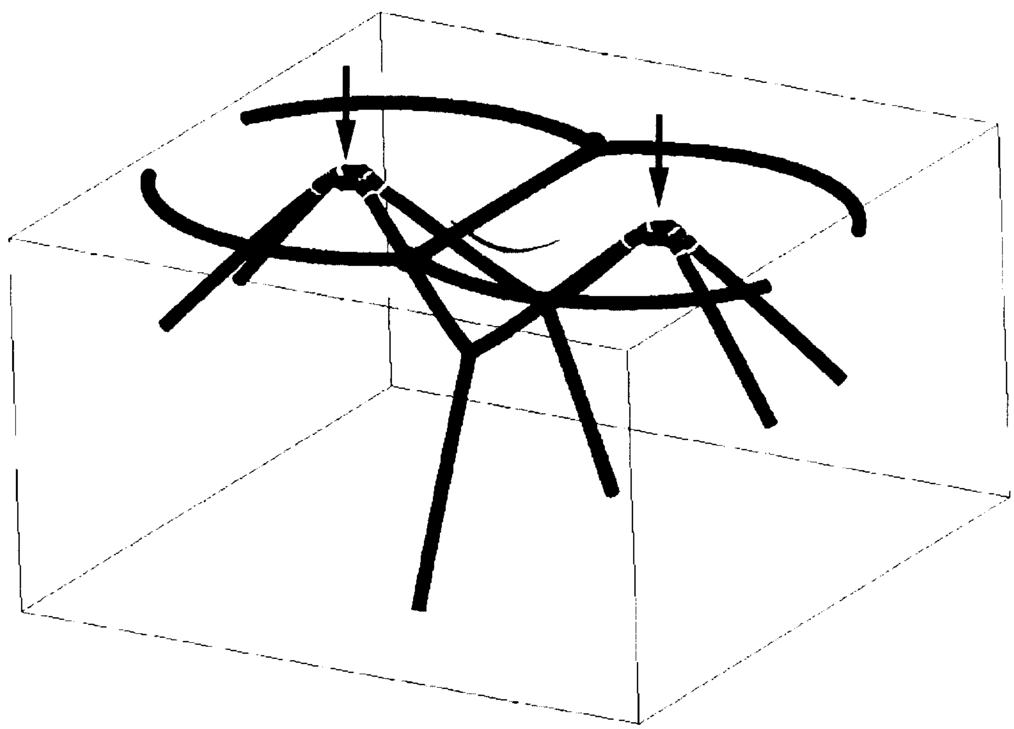

Figure 5: Stress tensor induced by two compressive forces; minor hyperstreamlines

spheres. The hyperstreamlines are presented with a constant cross section to avoid visual clutter resulting from the high eigenvalues in the vicinity of the points of the acting forces. They are, however, still color encoded by the major eigenvalue. Each of the 2 degenerate points has 4 bounding hyperstreamlines(separatrices), three of which lie on the surface $z = -1.05$ in a trisector pattern and the forth, which is pointing in the $+z$ direction, connects the points of triple degeneracy, and delineates one of the two symmetry planes (the other goes through the points of action of the forces).

To further clarify the tensor topology, the skeletons of the minor and medium hyperstreamlines are presented in Figures 5 and 6 respectively. We

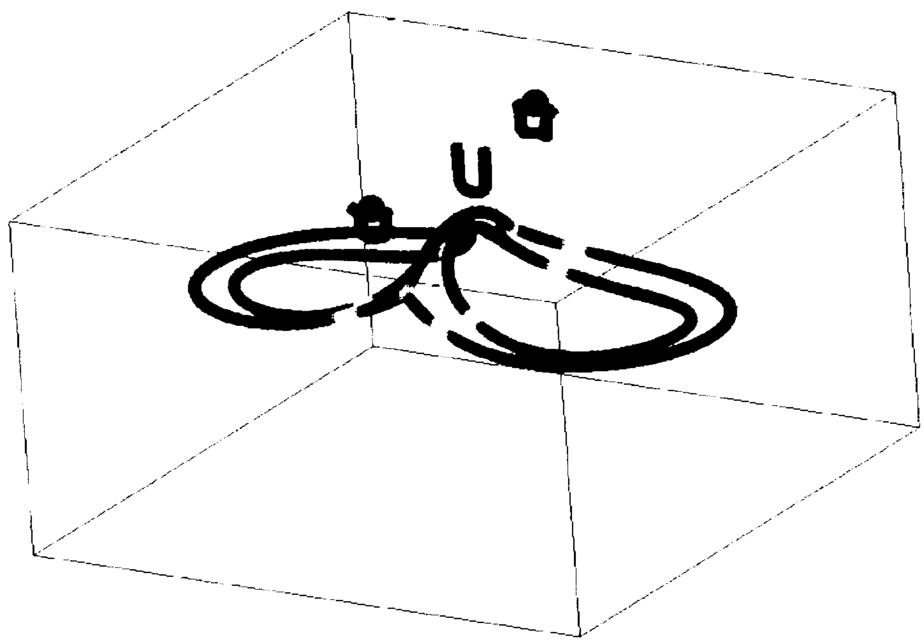

Figure 6: Stress tensor induced by two compressive forces; medium hyper-streamlines

can see from Figure 5 that the minor hyperstreamlines form a trisector-point like pattern in the vicinity of the points of triple degeneracy. They also indicate that a locus of points of double degeneracy ($\lambda_2 = \lambda_3$) connects the points of triple degeneracy. This is evident from the two trisector points that lie in the symmetry planes just below the points of triple degeneracy. The existence of the line of double degeneracy is further verified by noting the two points of double degeneracy in the medium hyperstreamlines skeleton (Figure 6).

Summary

In this report we describe the novel methods we developed and apply them to determine the topology of tensor data sets. We made use of advanced representations to determine the significance of degenerate points and topological skeletons in terms of physical features.

By extracting the geometric structure of tensor data, we produce simple and austere depictions that allow observers to infer the behavior of any hyperstreamlines in the field. It enables important elements of 3D stress distribution to be envisaged without visual clutter.

Degenerate points represent the singularities of the tensor field. In the 3-D elastic stress tensor case we were able to identify points of zero stresses and to illustrate transmission of forces inside the material.

Note: a paper based on this report has been accepted for presentation in IEEE Visualization '96.

On Going Work

The results presented above indicate the existence of continuous lines of double degeneracy. The method we are using to locate degenerate points is suitable only in the case of isolated points. We are currently developing techniques to locate lines of double degeneracy to assist as in studying continuous topological features, i.e. lines and surfaces of double degeneracy.

Future Work

The classification of the various types of points of triple degeneracy requires a continuous presentation of the separating surfaces. Currently, these surfaces are described only by their bounding hyperstreamlines. A more complete representation is one that is based on the use of textures.[1,2] The idea is to define the general separating surfaces and then use texture to illustrate the topology of the eigenvector field in the vicinity of points of triple degeneracy.

References

[1] T. Delmarcelle, *The Visualization of Second-Order Tensor Fields.* PhD thesis, Stanford University, 1994.

[2] B. Cabral and L. C. Leedom, "Imaging vector fields using line integral convolution," *Computer Graphics (SIGGRAPH'93 Proc.)*, pp. 263–272, 1993.