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Is the non-dipole magnetic field random?

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SUMMARY

Statistical modelling of the Earth's magnetic field **B** has a long history (see e.g. McDonald 1957; Gubbins 1982; McLeod 1986; Constable & Parker 1988). In particular, the spherical harmonic coefficients of scalar fields derived from **B** can be treated as Gaussian random variables (Constable & Parker 1988). In this paper, we give examples of highly organized fields whose spherical harmonic coefficients pass tests for independent Gaussian random variables. The fact that coefficients at some depth may be usefully summarized as independent samples from a normal distribution need not imply that there really is some physical, random process at that depth. In fact, the field can be extremely structured and still be regarded for some purposes as random. In this paper we examined the radial magnetic field B_r produced by the core, but the results apply to any scalar field on the core–mantle boundary (CMB) which determines **B** outside the CMB.

Key words: Earth's core, Earth's magnetic field, geomagnetic field, geomagnetism.

1 INTRODUCTION

In the absence of perfect and complete magnetic data, researchers have sought to develop statistical models of **B** in order to make predictions or test assumptions about **B**. The radial magnetic field on a sphere $S(r)$ of radius r can be expressed as an infinite sum of spherical harmonic functions:

$$B_r(r) = \sum_{l=1}^{l=\infty} \sum_{m=-l}^{m=l} f_l^m(r) X_l^m(\hat{\mathbf{r}}), \tag{1}$$

where

$$X_l^m(\hat{\mathbf{r}}) = \left[2(2l+1)\frac{(l-m)!}{(l+m)!}\right]^{1/2} P_l^m(\cos\theta)\cos m\lambda, \quad m>0, \tag{2}$$

$$X_l^{-m}(\hat{\mathbf{r}}) = \left[2(2l+1)\frac{(l-m)!}{(l+m)!}\right]^{1/2} P_l^m(\cos\theta)\sin m\lambda, \quad m>0, \tag{3}$$

$$X_l^0(\hat{\mathbf{r}}) = (2l+1)^{1/2} P_l^m(\cos\theta), \tag{4}$$

$$P_l^m(\mu) = \frac{1}{2^l l!}(1-\mu^2)^{m/2}\partial_\mu^{l+m}(\mu^2-1)^l. \tag{5}$$

Here $\hat{\mathbf{r}}$ is the unit vector with colatitude θ and longitude λ.

If b is the radius of the Earth, surface and satellite measurements give $f_l^m(b)$ for $1 \le l \le L$, where L may be as high as 40 or 50 (Schmitz, Meyer & Cain 1989; Cain *et al.* 1989a). However, the data strongly suggest (Langel & Estes 1982; Cain *et al.* 1989b) that, for the **B** observed on $S(b)$, the $f_l^m(b)$ come mostly from the crust if $l \ge 15$ and mostly from the core if $1 \le l \le 12$. In the latter range of l, the pre-Maxwell equations imply that

$$f_l^m(a) = (b/a)^{l+2} f_l^m(b), \tag{6}$$

where $S(a)$ is the CMB. The axial dipole is well-known to be anomalously large, but for $2 \le l \le 12$ and $-l \le m \le l$, and for various p and q, subsets of the products $(l+1)^{p/2}(2l+1)^{q/2} f_l^m(a)$ with restricted ranges of l have been shown to pass the Kolmogorov–Smirnov (K–S) test (Kendall & Stuart 1979) for independent identically distributed (i.i.d.) Gaussian random variables with mean 0. The classical K–S test must be slightly modified because the common variance of the $f_l^m(a)$ is inferred from the sample (Mason & Bell 1986). Constable & Parker (1988) gave the first such result, with $p=-1$, $q=2$, and $2 \le l \le 8$. Walker & Backus (1993) obtained the same result for $p=1$, $1 \le q \le 6$ and $2 \le l \le 10$. The present authors have verified that the coefficients $f_l^m(a)$ derived from Langel & Estes (1985) pass the K–S test if $2 \le l \le 12$. For these coefficients, the maximum separation between the empirical cumulative distribution function and the Gaussian distribution function with mean 0 and sample variance is 0.0785. The probability that a separation this large will occur by chance is 0.2125.

This paper gives examples to show that the passing of such tests for randomness need not mean that B_r on $S(a)$ looks random in any intuitive sense. Thus one cannot infer from such results alone that the physical process producing B_r is physically random. On the positive side, this result suggests that much of the statistical apparatus for treating random fields can be applied to B_r, even if it does not look particularly random.

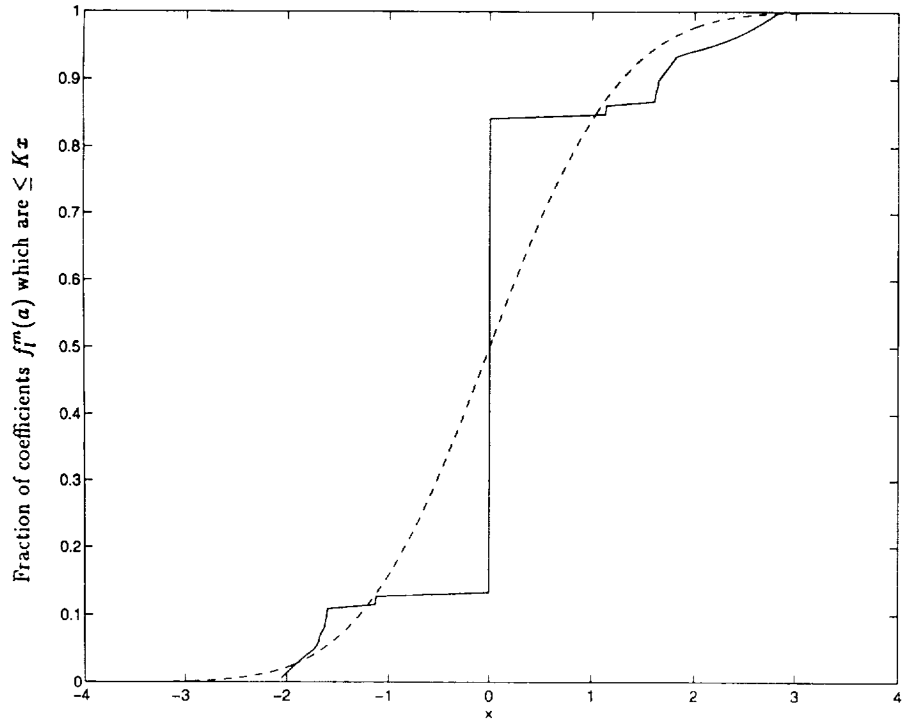

Figure 1. The empirical cumulative distribution function of the coefficients $f_l^m(a)$ with $2 \leq l \leq 12$ produced by a core spot with strength K at 90° colatitude and 0° longitude (solid line). Also shown is the Gaussian cumulative distribution function with the same variance and mean 0 (dashed line). The maximum deviation between these two curves is 0.3727, and the probability that a deviation at least this large will occur by chance if the core spot coefficients really are drawn at random from an i.i.d. mean 0 Gaussian population is less than 10^{-5}.

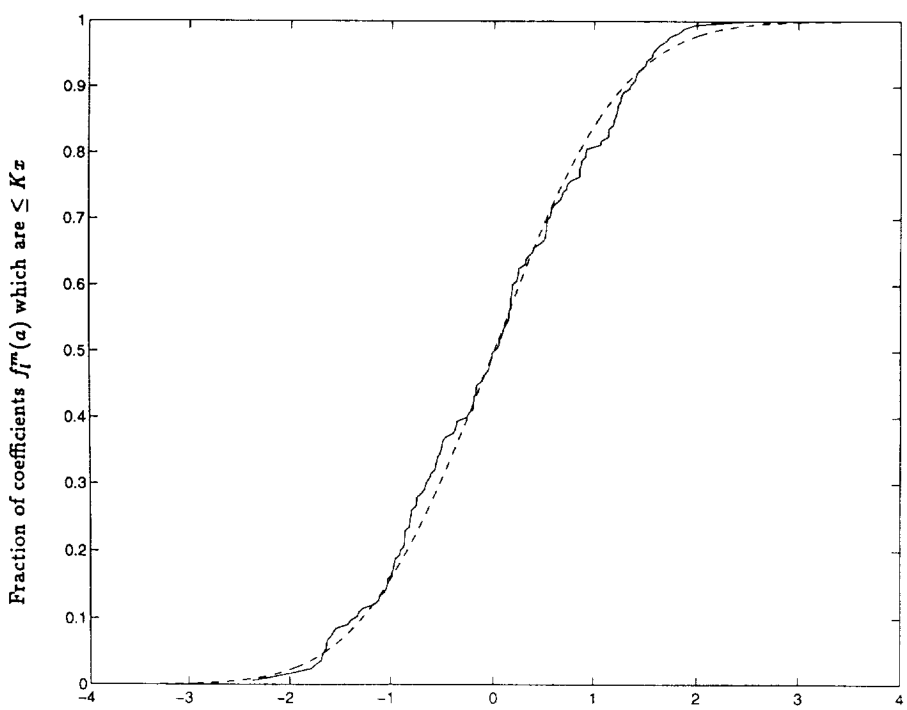

Figure 2. The empirical cumulative distribution function of the coefficients $f_l^m(a)$ with $2 \leq l \leq 12$ produced by a core spot with strength K at 72° colatitude and 229° longitude (solid line). Also shown is the Gaussian cumulative distribution function (dashed line). The maximum deviation between these two curves is 0.0544 and the probability that a deviation at least this large will occur by chance if the core spot coefficients really are drawn at random from an i.i.d. mean 0 Gaussian population is 0.6274.

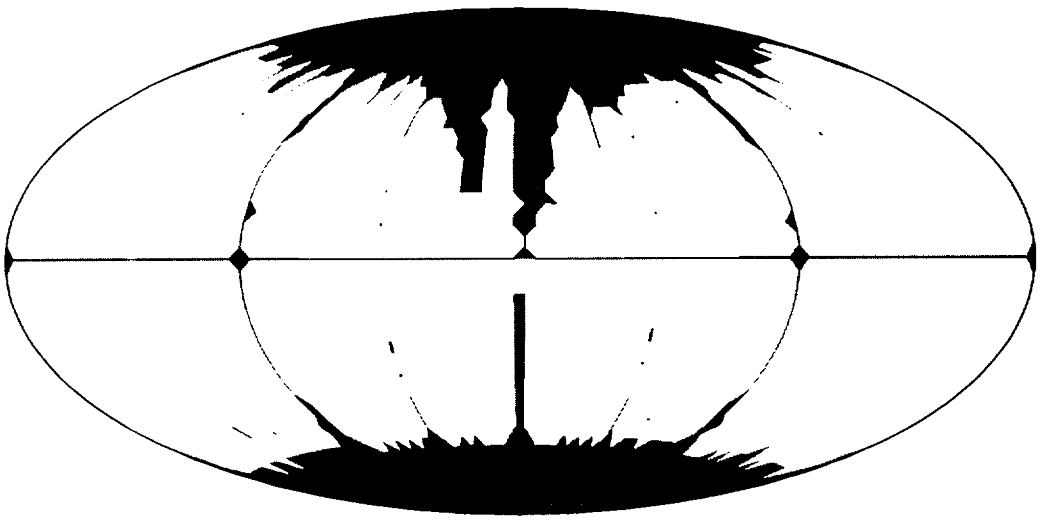

Figure 3. Hammer–Aitoff equal-area plot of the CMB centred on the Greenwich meridian. Core spots in the grey area permit rejection at the 5 per cent significance level of the hypothesis that their coefficients $f_l^m(a)$ are i.i.d. Gaussian with mean 0. The test is Kolmogorov–Smirnov, and is confined to the 165 spot coefficients with $2 \leq l \leq 12$ and $-l \leq m \leq l$.

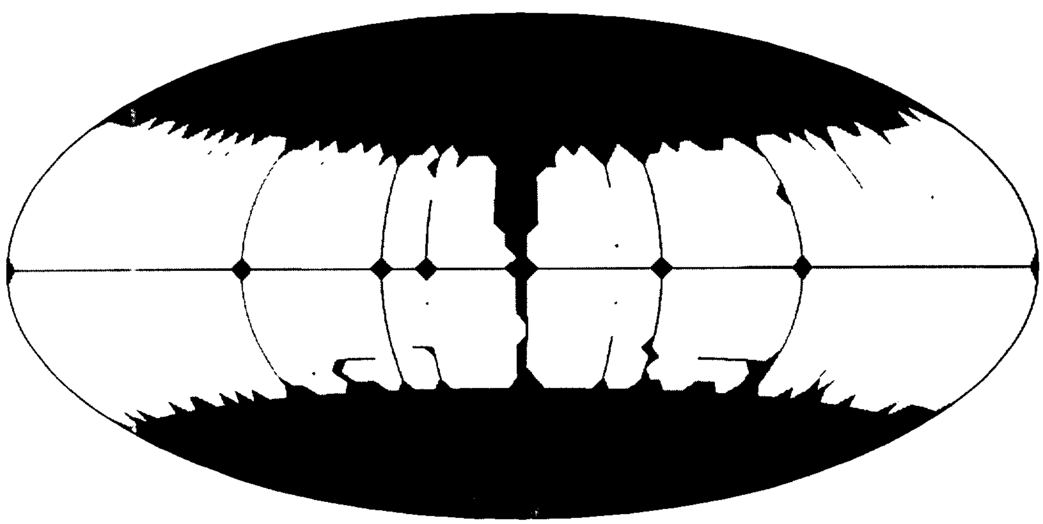

Figure 4. Hammer–Aitoff equal-area plot of the CMB centred on the Greenwich meridian. Core spots in the grey area permit rejection at the 5 per cent significance level of the hypothesis that their coefficients $f_l^m(a)$ are i.i.d. Gaussian with mean 0. The test is Kolmogorov–Smirnov, and is confined to the 957 spot coefficients with $2 \leq l \leq 30$ and $-l \leq m \leq l$.

2 CORE SPOT

For our example of a non-random field which passes tests for randomness, we make use of the spherical Dirac delta function $\delta(\hat{\mathbf{r}}, \hat{\mathbf{s}})$, defined by the requirement that

$$\delta(\hat{\mathbf{r}}, \hat{\mathbf{s}}) = 0 \qquad \text{if } \hat{\mathbf{r}} \neq \hat{\mathbf{s}}, \tag{7}$$

and

$$\int_{S(1)} dA(\hat{\mathbf{r}})\delta(\hat{\mathbf{r}}, \hat{\mathbf{s}}) = 4\pi. \tag{8}$$

We define a core spot with strength K at $a\hat{\mathbf{s}}$ on $S(a)$ by requiring that for all $a\hat{\mathbf{r}}$ on $S(a)$

$$B_r(a\hat{\mathbf{r}}) = K[\delta(\hat{\mathbf{r}}, \hat{\mathbf{s}}) - 1], \tag{9}$$

where K is a constant with units of magnetic field strength. With this definition, the net flux of B_r through $S(a)$ is 0. Multiplying both sides of eq. (9) by $X_l^m(\hat{\mathbf{r}})$ and integrating over the unit sphere gives the spherical harmonic coefficients of the core spot at $a\hat{\mathbf{s}}$:

$$f_l^m(a) = KX_l^m(\hat{\mathbf{s}}). \tag{10}$$

Many choices of core-spot location yield harmonic coefficients for which it is easy to reject the hypothesis that they are i.i.d. Gaussian random variables. Fig. 1 shows the empirical cumulative distribution function of the coefficients $f_l^m(a)$ $(2 \leq l \leq 12)$ generated by a core spot at 90° colatitude and 0° longitude. Also shown is the theoretical cumulative distribution with mean 0 and variance calculated from the coefficients. The maximum separation between the curves is 0.3727, and the K–S test gives a probability of less than 10^{-5} that separation this large would occur by chance. However, as

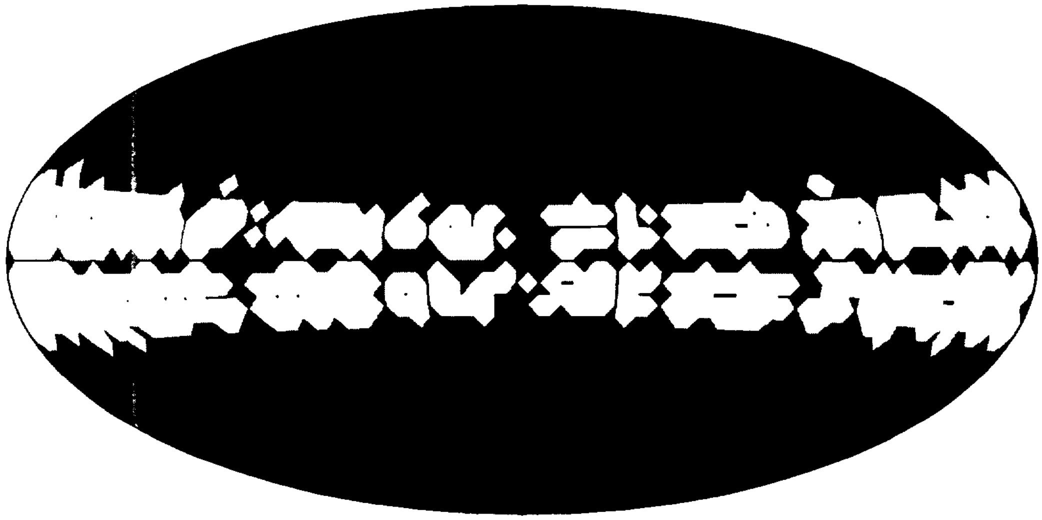

Figure 5. Hammer–Aitoff equal-area plot of the CMB centred on the Greenwich meridian. Core spots in the grey area permit rejection at the 5 per cent significance level of the hypothesis that their coefficients $f_l^m(a)$ are i.i.d. Gaussian with mean 0. The test is Kolmogorov–Smirnov, and is confined to the 3717 spot coefficients with $2 \leq l \leq 60$ and $-l \leq m \leq l$.

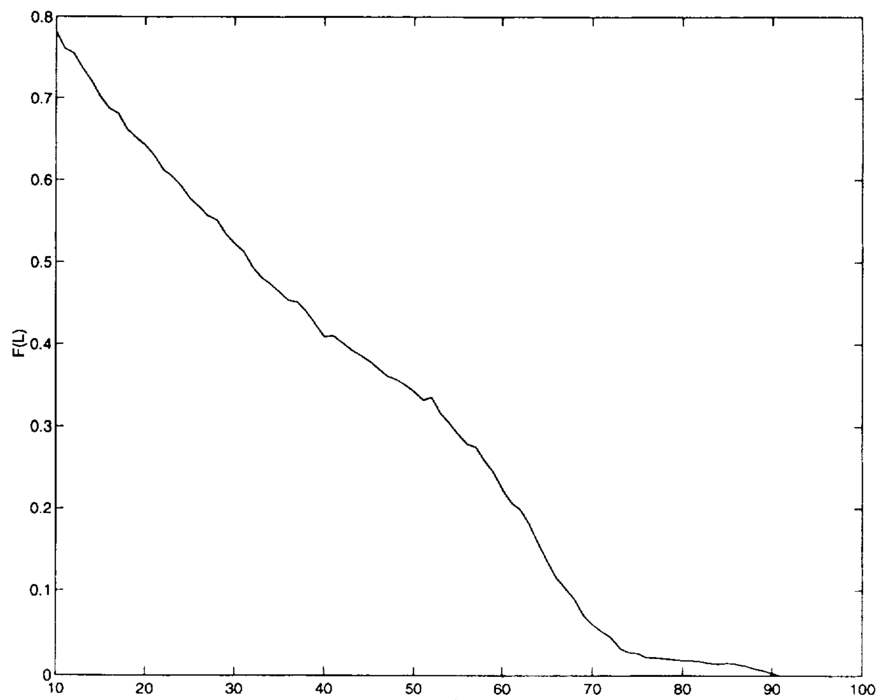

Figure 6. Fraction $F(L)$ of 1000 randomly chosen core spots whose coefficients $f_l^m(a)$ look Gaussian when we test only the L^2+2L-3 coefficients for which $2 \leq l \leq L$ and $-l \leq m \leq l$. The spot coefficients are deemed to 'look Gaussian' if at the 5 per cent significance level the modified Kolomogorov–Smirnov test fails to reject the hypothesis that those coefficients are drawn independently from a Gaussian population with mean 0 and sample second moment. Probabilities for this plot were calculated using asymptotic formulas from Mason & Bell (1986).

shown by Fig. 2, a different choice of core-spot location yields coefficients $f_l^m(a)$ that pass the K–S test for i.i.d. Gaussian random variables. Fig. 3 is a Hammer–Aitoff equal-area projection of the CMB. Areas shaded in grey correspond to core-spot locations where, at the 5 per cent significance level, one can reject the hypothesis that the coefficients $f_l^m(a)$ produced by the core spot are i.i.d. Gaussian random variables. Most locations produce coefficients that cannot be rejected as i.i.d. Gaussian random variables in this way.

For mathematical completeness, we have extended our calculations beyond $l=12$, into the range of harmonic degrees l where the core is not 'visible' at the surface of the Earth. Although the fraction of core-spots where the hypothesis of i.i.d. Gaussian variables can be rejected increases with harmonic degree, there are still many core spots for which the hypothesis cannot be rejected (see Figs 4 and 5). Finally, Fig. 6 shows the percentage of core spots for which the hypothesis of i.i.d. Gaussian random variables can be rejected at the

5 per cent significance level as a function of maximum harmonic degree. As might be expected, the percentage tends to zero as the maximum harmonic degree increases. A large percentage of core spots produce coefficients $2 \leq l \leq 30$ that pass the K–S test (Figs 4 and 6).

3 CONCLUSIONS

Very structured fields can have spherical harmonic coefficients that pass the K–S test for harmonic degrees $2 \leq l \leq 60$. This fact extends the usefulness of statistical models to some highly structured fields, but it means that tests of normality over a finite range of harmonic degrees cannot assure the absence of coherent structure.

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