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Capture Conditions for Merging Trajectory Segments to Model Realistic Aircraft Descents

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A typical commercial aircraft trajectory consists of a series of flight segments. An aircraft switches from one segment to another when certain specified variables reach their desired values. Trajectory synthesis for air traffic control automation must be consistent with practical pilot procedures. We examine capture conditions for merging trajectory segments to model commercial aircraft descent in trajectory synthesis. These conditions translate into bounds on measurements of atmospheric wind, pressure, and temperature. They also define ranges of thrust and drag feasible for a descent trajectory. Capture conditions are derived for the Center-TRACON Automation System developed at NASA Ames Research Center for automated air traffic control. Various uses of capture conditions are discussed. A Boeing 727-200 aircraft is used to provide numerical examples of capture conditions.

Nomenclature

A, B	= defined constants
$B_1(h)$	= upper bound on wind gradient
$B_2(h)$	= lower bound on wind gradient
D	= aircraft drag
g	= acceleration because of gravity
h	= altitude
L	= aircraft lift
M	= Mach number
m	= aircraft mass
p	= air pressure
$\bar{p}$	= air pressure normalized by sea-level pressure
R	= gas constant
r	= wind magnitude bound
s	= path distance
T	= engine thrust
V_{cas}	= calibrated airspeed
V_g	= ground speed
V_t	= true airspeed
V_w	= wind speed
x	= state vector in a segment
y	= generic capture variable
δ	= lower capture bound
ϵ	= capture accuracy
γ_a	= air-relative flight-path angle
γ_i	= inertial flight-path angle
κ	= thrust setting
ϕ	= aircraft bank angle
Ψ	= inertial velocity heading
Θ	= atmospheric temperature
$\bar{\Theta}$	= temperature normalized by sea-level value
θ_{rw}	= relative wind angle
$(\,)_0$	= constant values
$(\,)_s$	= sea-level values
$(\,)^*$	= desired values

Introduction

GROWTH in nationwide air traffic has put severe burdens on the current air traffic control systems and created heavy workloads for air traffic controllers. Further increase in the throughput of the existing air traffic control infrastructure will be greatly enhanced by computer automation.

Researchers at NASA Ames Research Center have developed a Center-TRACON Automation System (CTAS) for automated air traffic control.[1] CTAS generates landing times and descent advisories for arrival aircraft over extended terminal areas consisting of a Center airspace and a TRACON airspace. Center stands for Air Route Traffic Control Center and covers the en route airspace area about 200 n miles around an airport. TRACON stands for Terminal Radar Approach Control and covers the terminal airspace area about 40 n miles around an airport. Computed advisories allow aircraft to descend efficiently and to meet specified arrival times while avoiding conflict. CTAS has been shown to be able to decrease controllers' workload and to increase aircraft landing rates.

CTAS comprises several generic modular tools. The Traffic Management Advisor schedules landing sequences and times for arrival aircraft. The Descent Advisor provides advisories for descending aircraft in the Center airspace. The Final Approach Spacing Tool provides advisories in the TRACON airspace for merging traffic onto the final approach. All of these tools rely on the aircraft trajectories calculated in the Trajectory Synthesizer module.

An air traffic control automation system has two essential components: a trajectory synthesizer and a scheduler. The trajectory synthesizer calculates aircraft trajectories that are efficient, conflict-free, and consistent with air traffic control and pilot procedures. It also can vary descent speeds to meet scheduled arrival times. The scheduler generates a sequence of landing orders and times based on aircraft trajectories predicted by the trajectory synthesizer. Therefore, the success of an air traffic control automation system hinges on the reliability of trajectory synthesis algorithms.

Trajectory synthesis for air traffic control automation should be consistent with practical pilot procedures. In a realistic commercial aircraft descent, a complete trajectory is divided into a series of flight segments. Mathematically, each flight segment can be described by two constant control variables selected from among engine thrust setting, Mach number or calibrated airspeed, and altitude rate or inertial flight-path angle.[2,3] These segments can be specified by controllers via voice communication and can be flown by pilots with standard instrumentation. Furthermore, airline specifications often combine a number of segments in a specified order to form descent

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profiles. Three types of descent profiles are currently modeled in CTAS: fast, nominal, and slow. Each type of profile covers a range of arrival times.

For trajectory synthesis in air traffic control automation, point-mass aircraft equations are adequate. Inputs to trajectory synthesis include a series of waypoint locations that define the horizontal path, initial aircraft states from radar measurements, aircraft thrust and drag models, measurements of atmospheric conditions, and constraints because of aircraft performance and air traffic control. With given descent and/or cruise speeds, aircraft equations of motion are numerically integrated segment by segment to form complete trajectories. Then, descent and/or cruise speeds are iterated to achieve specified final times. Therefore, outputs of the trajectory synthesis are aircraft trajectories that meet specified final times.

For two flight segments to connect properly, certain variables, called capture variables, need to reach their desired values. Specifically, a capture variable is monitored during the integration of a segment. Once the capture variable reaches the desired value within a certain tolerance, the capture is declared successful and the integration of a new segment is started. Each segment has a specified capture variable, depending on its location in a descent profile.

Clearly, the convergence of trajectory synthesis algorithms depends on three related elements: 1) numerical integration of every flight segment, 2) proper connections of flight segments, and 3) iteration over speeds to meet a specified time of arrival. Stability of numerical integration depends on the aircraft equations, the integration scheme, and the integration step size. In CTAS, a second-order Runge–Kutta integration scheme is used and integration step sizes are varied according to the nature of each flight segment. These choices achieve a proper balance between accuracy and computational speed. The search over speeds to achieve a desired time of arrival uses function values only and is based on bracketing the solution with an ever-decreasing interval. It is limited to four iterations. If a desired speed cannot be found within four iterations, a different horizontal path can be used and the trajectory synthesizer module can be called upon again. In contrast, the trajectory synthesis algorithm cannot obtain a complete aircraft trajectory if any two segments fail to connect in capture. Therefore, successful segment connections represent a crucial requirement to convergence of trajectory synthesis.

The connection of two segments in a numerical integration is guaranteed if the capture variable in the first segment changes monotonically. In this paper, conditions required for capture variables to behave monotonically are presented for CTAS. Capture requirements are unique to the Descent Advisor trajectory synthesis. These conditions translate into bounds on predicted weather conditions, including wind components, pressure, and temperature, as well as feasible ranges of thrust minus drag.

The remainder of the paper states the capture requirement mathematically, presents details of capture conditions for different segments and profiles, and uses a Boeing 727 aircraft example to illustrate these capture conditions. A summary of key results is given at the end.

Problem Statement

Trajectory synthesis for the Descent Advisor calculates aircraft trajectories from the current aircraft position to a specified boundary point (Metering Fix) between the Center airspace and the TRACON airspace, for one aircraft at a time. A computed trajectory reaches the crossing point at specified altitude, speed, and time of arrival.[4] Over the Center airspace, hundreds of these four-dimensional aircraft trajectories need to be generated every 12 s, which is the cycle of radar sweeps. As a result, trajectory-synthesis algorithms need to be computationally efficient. In CTAS, the Center Trajectory Synthesizer integrates a set of point-mass aircraft equations of motion, simplified to speed up computations[5]:

$$\dot{V}_t = \frac{T(h, M; \kappa) - D(h, V_t, L)}{m} - g\gamma_a + \frac{\mathrm{d}}{\mathrm{d}t}(V_w \cos\theta_{rw}) \quad (1)$$

$$\dot{h} = \gamma_a V_t = \gamma_i V_g \quad (2)$$

Table 1 Flight segments in a descent

No.	Segment type	Integration variables	Constant controls	Capture variables
1	Cruise	s	$\dot{h} = 0$, (M or V_{cas})	$s^{\uparrow}$
2	Descent accel	s, V_t, h	κ, ($\dot{h}$ or γ_i)	$M^{\uparrow}$, $V_{cas}^{\uparrow}$, or $h^{\downarrow}$
3	Constant Mach	s, h	M, (κ, $\dot{h}$, or γ_i)	$V_{cas}^{\uparrow}$ or $h^{\downarrow}$
4	Constant V_{cas}	s, h	V_{cas}, (κ, $\dot{h}$, or γ_i)	$M^{\downarrow}$ or $h^{\downarrow}$
5	Level deceleration	s, V_t	κ, $\dot{h} = 0$	$M^{\downarrow}$, $V_{cas}^{\downarrow}$, or $s^{\uparrow}$

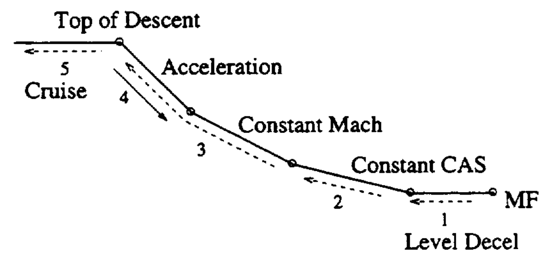

Fig. 1 Fast-type descent trajectory for high initial altitudes.

$$\dot{s} = \sqrt{V_t^2 - (V_w \sin\theta_{rw})^2} - V_w \cos\theta_{rw} = V_g \quad (3)$$

$$\dot{\Psi} = \frac{L \sin\phi}{m V_g} \quad (4)$$

and

$$L = mg/\cos\phi \quad (5)$$

In the National Oceanic and Atmospheric Administration weather models currently used in CTAS, wind components (V_w, θ_{rw}), pressure p, temperature Θ, and actual altitude are measured at a grid of 60×60 km horizontal stations. At each station, they are given at a series of pressure altitudes.

In the above equations, (V_t, h) describes the vertical path and (s, Ψ) describes the horizontal path. The ground speed V_g couples the horizontal and vertical paths. The horizontal route is synthesized first on the basis of an approximate vertical profile. In particular, the heading angle Ψ is approximated by a combination of straight lines and constant-radius turns at waypoints.[6]

The vertical trajectory is divided into a series of flight segments[2,3] to be consistent with current pilot procedure. Each segment is defined by setting two control variables constant. These variables come from the following three groups: engine thrust setting, speed (Mach or calibrated airspeed), and vertical rate (altitude rate or inertial flight-path angle). Table 1 lists five flight segments in a realistic descent, and for each segment, variables that need to be integrated, capture variables that are discussed below, and control variables that are set constant. Arrows next to the capture variables indicate whether they increase or decrease in that segment with respect to real time.

Furthermore, a series of segments are combined in a predetermined order to form descent profiles. There are three types of profiles used in CTAS: fast, nominal, and slow. Figure 1 summarizes a fast-descent profile for high initial altitudes. In comparison, a fast-descent profile for low initial altitudes does not contain a constant Mach descent segment. A nominal descent profile does not contain a descending-acceleration segment. A slow descent profile can be obtained from a nominal descent profile by including a level-deceleration segment after the cruise segment and removing the Mach descent segment. Each profile defines a range of possible descent-calibrated airspeed variations. The descent-calibrated airspeed is varied within these ranges to achieve a specified arrival time.

A trajectory of fast, nominal, or slow type is numerically integrated segment by segment. The integration is either forward or backward in time, and the order in which the segments are integrated depends on what information is known. In the fast profile shown in Fig. 1, both the initial aircraft states and the desired final states at the crossing fix are known. For a certain descent speed, the location of the top of descent needs to be determined so that the trajectory meets initial and final conditions. As a result, the fast trajectory is integrated backward from the level-deceleration segment, as indicated by the arrows in Fig. 1. Then, the descent-calibrated airspeed and/or descent Mach are varied to meet a specified final time. Currently in CTAS, the search over descent speed is based on bracketing the descent speed with an ever-decreasing interval, and is limited to four iterations.

Vertical trajectory synthesis involves the calculation of time histories for three variables: true airspeed, altitude, and path distance. The specifications of a flight segment often remove the need to integrate one of the three variables directly. For example, true airspeed can be solved algebraically in a constant Mach descent or calibrated airspeed descent segment. Mach number and calibrated airspeed, V_{cas}, are related to true airspeed as follows[7]:

$$V_t = M\sqrt{1.4R\Theta(h)} \tag{6}$$

$$V_{cas}^2 = 7R\Theta_s\left(\{\bar{p}(h)[(0.2M^2+1)^{3.5}-1]+1\}^{\frac{2}{7}}-1\right) \tag{7}$$

$$M^2 = 5\left(\left\{\frac{1}{\bar{p}(h)}\left[\left(\frac{V_{cas}^2}{7R\Theta_s}+1\right)^{3.5}-1\right]+1\right\}^{\frac{2}{7}}-1\right) \tag{8}$$

In integrating a segment, a certain variable is monitored to determine the junction of the current segment with an adjacent segment. In a fast trajectory, for example, the calibrated airspeed V_{cas} is monitored during the backward integration of the level-deceleration segment. When V_{cas} reaches a certain descent V_{cas} value, integration of the level-deceleration segment ends and that of the constant V_{cas} descent segment begins. All other segments are connected similarly. Therefore, a trajectory calculation is successful only if the monitored variables can reach or capture their desired values.

In a given descent profile, each segment normally has one variable that needs to be monitored. These variables are called capture variables. When the same segment is put into a different profile, it may require a different capture variable. Table 1 lists all possible capture variables for each segment. A capture variable is chosen from Mach, calibrated airspeed, altitude, or path distance.

Segment connections will be successful if the capture variable in a segment changes monotonically. Let x represent the state vector of a segment consisting of some or all of $\{V_t, h, s\}$. The capture variable is y and is thus one of $\{M, V_{cas}, h, s\}$. Then, equations of a segment can be described in the following compact form:

$$\dot{x} = f(x, t) \qquad t_1 \le t \le t^* \qquad x(t_1) \text{ known} \tag{9}$$

$$y = v(x) \tag{10}$$

The time can go either forward or backward. The final time is defined through

$$y(t^*) = y^* \tag{11}$$

Numerically, a capture is declared if

$$|y(t^*) - y^*| < \epsilon \tag{12}$$

For the ease of discussion, we assume $y(t_1) < y^*$. Then, a sufficient capture condition requires

$$\dot{y} \ge \delta > 0 \tag{13}$$

Monotonicities of capture variables are sufficient but not necessary for segment connections. A capture variable could vary in an arbitrary manner and still reach a specified value before an allowed time. However, it is highly desirable to have predictable capture variables both for trajectory-synthesis algorithms and in practical flights.

Derivation of Capture Conditions

In this section, we derive all possible capture conditions for every segment. Physical meanings of these capture conditions are discussed, and alternative flight strategies are suggested when the conditions are not met.

Path distance capture is discussed first. Path distance s is calculated from the numerical integration of Eq. (3). The square root in Eq. (3) must be well defined, and the ground speed must be positive to capture a specified path distance. These requirements impose different limits on headwind and tailwind. It turns out that if there is a positive constant $r < 1$ such that

$$V_w < rV_t \tag{14}$$

the square root will be meaningful and

$$0 < (1-r)V_t \le V_g \le (1+r)V_t \tag{15}$$

A wind magnitude smaller than the true airspeed guarantees path distance capture. Path distance capture is needed for every segment. Therefore, Eq. (14) must be satisfied in every segment. In most flight cases, this condition is satisfied. When predicted wind speed becomes comparable to the true airspeed, the airplane should not attempt to fly through the region, for safety reasons.

A. Cruise Segment

For the cruise segment, Eq. (3) is integrated either forward or backward to capture a specified path distance value. Altitude and speed (Mach or V_{cas}) are held constant. Therefore, the capture condition in a cruise segment is the same as that stated above for capturing the path distance.

B. Descending-Acceleration Segment

In a descending-acceleration segment, true airspeed, altitude, and path distance are all integrated numerically. Engine thrust is set at minimum (idle) and wind gradient is ignored. True airspeed and path distance are integrated from Eqs. (1) and (3), respectively, and altitude history is integrated by either specifying a desired descent rate $\dot{h}^*$ or from Eq. (2) with a desired inertial flight-path angle γ_i^*. Mathematically,

$$\dot{h} = \dot{h}^* \qquad \text{or} \qquad \gamma_i^* V_g \tag{16}$$

Correspondingly,

$$\gamma_a = \frac{\dot{h}^*}{V_t} \qquad \text{or} \qquad \frac{\gamma_i^* V_g}{V_t} \tag{17}$$

Possible capture variables are Mach number, calibrated airspeed, and altitude (Table 1).

Capturing a Mach number requires $\dot{M} \ge \delta_m$. Differentiating Eq. (6) gives

$$\dot{M} = \frac{\dot{V}_t}{\sqrt{1.4R\Theta(h)}} - \frac{V_t}{2\sqrt{1.4R\Theta(h)}}\left(\frac{\mathrm{d}\Theta/\mathrm{d}h}{\Theta}\right)\dot{h} \tag{18}$$

By combining Eqs. (1), (2), and (18), we have

$$T(h, M; \kappa_0) \ge D(h, V_t; L) + m\dot{h}\left[\frac{g}{V_t} + \frac{V_t}{2}\left(\frac{\mathrm{d}\Theta/\mathrm{d}h}{\Theta}\right)\right] + m\delta_m\sqrt{1.4R\Theta(h)} \tag{19}$$

Capturing a calibrated airspeed requires $\dot{V}_{cas} \ge \delta_c$. From Eqs. (6) and (7), we have

$$\dot{V}_{cas} = \frac{1}{f_{DA}(h, V_t)}\left(\dot{V}_t - \left\{\frac{V_t}{2}\left(\frac{\mathrm{d}\Theta/\mathrm{d}h}{\Theta}\right) + \frac{z}{(1+z)^{\frac{5}{7}}} \times \left[\frac{R\Theta(h)}{V_t}\right]\left[-\frac{\mathrm{d}p/\mathrm{d}h}{p(h)}\right]\right\}\dot{h}\right) \tag{20}$$

where

$$z(h, V_t) \triangleq \left[\frac{V_t^2}{7R\Theta(h)} + 1\right]^{3.5} - 1 > 0 \tag{21}$$

$$f_{DA}(h, V_t) \triangleq (\bar{\Theta}/\bar{p})(V_{cas}/V_t)\left(V_t^2/7R\Theta + 1\right)^{-2.5}$$
$$(\bar{p}z + 1)^{\frac{5}{7}} > 0 \tag{22}$$

Therefore

$$T(h, M; \kappa_0) \geq D(h, V_t; L) + m\delta_c f_{DA}(h, V_t)$$
$$+ m\dot{h}\left[\frac{g}{V_t} + \frac{V_t}{2}\left(\frac{\mathrm{d}\Theta/\mathrm{d}h}{\Theta}\right) + E(V_t, h)\right] \tag{23}$$

In a descent flight, $\dot{h} < 0$. For subsonic airspeed and standard atmosphere,

$$\frac{g}{V_t} + \frac{V_t}{2}\left(\frac{\mathrm{d}\Theta/\mathrm{d}h}{\Theta}\right) > 0 \tag{24}$$

and

$$E(V_t, h) \triangleq \frac{z}{(1+z)^{\frac{5}{7}}}\left[\frac{R\Theta(h)}{V_t}\right]\left[-\frac{\mathrm{d}p/\mathrm{d}h}{p(h)}\right] > 0 \tag{25}$$

Therefore, conditions in Eqs. (19) and (23) require that the minimum thrust in a descending-acceleration flight be larger than a quantity that is smaller than the aerodynamic drag. Physically, there must be sufficient excess energy for an aircraft to increase Mach number or calibrated airspeed. When the minimum thrust is used during the descending flight, the aircraft extracts potential energy to gain Mach number or calibrated speed.

If Eq. (19) is not satisfied during the trajectory synthesis, minimum thrust with specified descent rate $\dot{h}$ does not provide sufficient energy for the aircraft to increase Mach number. One can specify a larger rate of descent or a higher thrust setting. It is also possible to specify a desired calibrated airspeed for capture. The minimum thrust required in Eq. (19) is larger than that in Eq. (23). As a result, it is more difficult to capture a Mach number than a calibrated airspeed during a descent acceleration. If Eq. (23) is not satisfied, one should either specify a larger rate of descent or use a higher thrust setting. Under the same conditions, a heavier aircraft is more likely to satisfy Eq. (19) or (23) than a lighter aircraft.

Capturing a specified altitude is automatic if a desired descent rate is selected so that $\dot{h}^* \leq -\delta_h$. If a desired inertial flight-path angle is selected, then $\gamma_i^* V_g \leq -\delta_h$, which results in

$$V_g \geq \frac{\delta_h}{|\gamma_i^*|} \tag{26}$$

This is the same condition as in path distance capture in Eq. (15), if δ_h is properly selected. It requires that wind magnitude be smaller than the true airspeed, as in Eq. (14). In capturing a specified altitude, Eq. (19) or (23) still needs to be satisfied for Mach number or V_{cas} to increase. The desired value of descent rate or inertial flight-path angle affects the specific requirement on thrust minus drag in Eq. (19) or (23).

C. Constant Mach Descent Segment

In a constant Mach descent segment, altitude and path distance are numerically integrated and true airspeed is solved as a function of altitude from Eq. (6). We have

$$\frac{\mathrm{d}V_t}{\mathrm{d}h} = \frac{1}{2}M_0\sqrt{1.4R}\left(\frac{\mathrm{d}\Theta/\mathrm{d}h}{\sqrt{\Theta}}\right) = \frac{V_t}{2}\left(\frac{\mathrm{d}\Theta/\mathrm{d}h}{\Theta}\right) \tag{27}$$

Altitude is integrated with one of the following three choices: 1) setting the engine thrust at minimum, 2) following a desired descent rate, and 3) following a desired inertial flight-path angle. Mathematically,

$$\dot{h} = \gamma_a^* V_t, \qquad \dot{h}^*, \qquad \text{or} \qquad \gamma_i^* V_g \tag{28}$$

and

$$\gamma_a = \gamma_a^*, \qquad \frac{\dot{h}^*}{V_t}, \qquad \text{or} \qquad \frac{\gamma_i^* V_g}{V_t} \tag{29}$$

where $\gamma_a^*(h; M_0)$ corresponds to setting engine thrust at minimum and is given by

$$\gamma_a^*(h; M_0) = \left(\frac{1}{m}\right)\left\{\frac{T(h, M_0; \kappa_0) - D(h, V_t; L)}{V_t[(\mathrm{d}V_t/\mathrm{d}h) - (\mathrm{d}/\mathrm{d}h)(V_w\cos\theta_{rw})] + g}\right\} \tag{30}$$

Possible capture variables are calibrated airspeed and altitude.

Differentiating Eq. (7) while keeping Mach number constant gives

$$\dot{V}_{cas} = \frac{\dot{h}}{f_{MD}(h, M_0)}\left(\frac{\mathrm{d}\bar{p}}{\mathrm{d}h}\right) \tag{31}$$

where

$$f_{MD}(h, M_0) \triangleq \frac{\sqrt{7}}{B\sqrt{R\Theta_s}}[\bar{p}(h)B + 1]^{\frac{5}{7}}$$
$$\times\sqrt{[\bar{p}(h)B + 1]^{\frac{2}{7}} - 1} > 0 \tag{32}$$

$$B \triangleq \left(1 + 0.2M_0^2\right)^{3.5} - 1 > 0 \tag{33}$$

Capturing a desired calibrated airspeed requires $\dot{V}_{cas} \geq \delta_c$, which gives

$$\frac{\mathrm{d}\bar{p}}{\mathrm{d}h} \leq \delta_c\frac{f_{MD}(h, M_0)}{\dot{h}} < 0 \tag{34}$$

where the descent rate $\dot{h}$ is given by Eq. (28). In other words, measured pressures must decrease as altitude increases.

Equation (34) defines a bound for the altitude gradient of measured pressure at each altitude level. The specific bound depends on which one of κ, $\dot{h}$, or γ_i is set to constant. Because atmospheric pressure should decrease with altitude, this equation is normally satisfied. Actually, it can be used to smooth measurement data of atmospheric pressure for trajectory integrations.

Altitude capture requires $\dot{h} \leq -\delta_h$. The three choices for integration of altitude are discussed next.

1) If the engine thrust setting is held constant, we have

$$\dot{h} = \gamma_a^* V_t \leq -\delta_h \tag{35}$$

which requires that $\gamma_a^*(h; M_0)$ be negative. In other words, the numerator and the denominator in Eq. (30) should have different signs. In most flight conditions, the minimum thrust is less than the drag:

$$T(h, M; \kappa_0) - D(h, V_t; L) < 0$$

Therefore it is required that

$$V_t\left[\frac{\mathrm{d}V_t}{\mathrm{d}h} - \frac{\mathrm{d}}{\mathrm{d}h}(V_w\cos\theta_{rw})\right] + g > 0 \tag{36}$$

From Eqs. (30), (35), and (36), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}h}(V_w\cos\theta_{rw}) < \frac{g}{V_t} + \frac{\mathrm{d}V_t}{\mathrm{d}h} \triangleq B_1(h) \tag{37}$$

$$\frac{\mathrm{d}}{\mathrm{d}h}(V_w\cos\theta_{rw}) \geq \frac{g}{V_t} + \frac{\mathrm{d}V_t}{\mathrm{d}h}$$
$$+ \frac{T(h, M_0; \kappa_0) - D(h, V_t; L)}{m\delta_h} \triangleq B_2(h) \tag{38}$$

In addition, a mathematical singularity occurs in the calculation of altitude rate if

$$\frac{\mathrm{d}}{\mathrm{d}h}(V_w\cos\theta_{rw}) \to B_1(h) \tag{39}$$

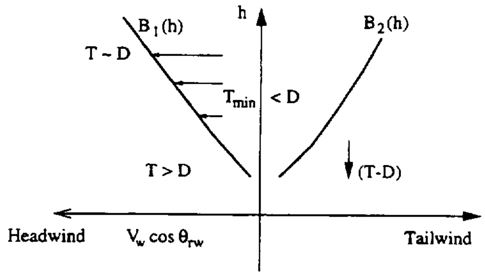

Fig. 2 Wind gradient effect on thrust minus drag.

Airspeed and its gradient are functions of altitude for a given Mach number; thus the right-hand sides of Eqs. (37) and (38) are functions of altitude and atmospheric temperature. The last term on the right-hand side of Eq. (38) is negative. These equations establish a feasible wind gradient range as a function of altitude for known temperature profiles. These limits are satisfied by small wind magnitudes. The case of a minimum thrust larger than the drag can be similarly studied.

If these limits are violated, altitude capture in the trajectory synthesis may fail. Physically, the aircraft has to climb or descend slowly to maintain a constant Mach number by taking advantage of the wind gradient. If the singularity in Eq. (39) occurs, the calculated altitude rate will be very large. There are two alternative methods to proceed with trajectory calculations, depending on the nature of wind-gradient-limit violations.

If measured wind gradients violate Eq. (37) or (38) only for a short period of time, wind data can be smoothed to satisfy these equations. If the violations occur over a long period of time, the aircraft may lose its physical ability to follow a constant Mach number while descending with minimum thrust. In either case, a desired descent rate or inertial flight-path angle can be followed. Correspondingly, the required thrust is higher than the minimum if Eq. (37) is violated, and a larger drag is needed if Eq. (38) is violated. Details are discussed below.

2) When a descent rate $\dot{h}^*$ is to be followed, altitude capture is guaranteed if

$$\dot{h}^* \leq -\delta_h \tag{40}$$

3) If a desired inertial flight-path angle is to be followed, altitude capture requires

$$\dot{h} = \gamma_i^* V_g \leq -\delta_h \Rightarrow V_g \geq \left(\delta_h / -\gamma_i^*\right) \tag{41}$$

which is the same condition as path distance capture in Eq. (15), if the δ_h is properly selected. The wind magnitude must be smaller than the true airspeed.

Figure 2 shows required ranges of $(T-D)$ in various cases of wind gradients to ensure a descending flight with a constant Mach number. If Eq. (37) is violated, a descending aircraft would experience a large decrease in headwind. The aircraft would lose airspeed if it keeps descending with minimum thrust. Mathematically, the aircraft could climb to take advantage of the increasing headwind. In an actual flight, the pilot has to use a larger thrust than idle to maintain a constant Mach number. In contrast, a larger drag is needed if Eq. (38) is violated. For example, spoilers can be used to augment the aerodynamic drag.

Following a specified descent rate or inertial flight-path angle requires a thrust level different from the minimum thrust. If both Eqs. (37) and (38) are satisfied, the computed thrust is less than the drag. When either Eq. (37) or (38) is violated, the computed thrust minus drag follows the trends in Fig. 2. In particular, the required thrust is close to drag if the measured wind gradient is close to the upper bound as in Eq. (39).

D. Constant Calibrated Airspeed Descent Segment

A constant V_{cas} descent segment is similar to a constant Mach descent segment, except that the calibrated airspeed is maintained constant instead of the Mach number. As a result, numerical integrations capture a desired Mach instead of a desired V_{cas}.

In a constant V_{cas} descent segment, altitude and path distance are numerically integrated and true airspeed is solved as a function of altitude from Eqs. (6) and (8) with V_{cas} as a constant. Define

$$A \triangleq \left(\frac{V_{cas}^2}{7R\Theta_s} + 1\right)^{3.5} - 1 > 0 \tag{42}$$

We have

$$V_t(h) = \sqrt{7R\Theta(h)}\sqrt{(A/\bar{p}(h)+1)^{\frac{2}{7}} - 1} \tag{43}$$

and

$$\frac{dV_t}{dh} = \frac{V_t}{2}\left(\frac{d\Theta/dh}{\Theta}\right) - \frac{R\Theta(h)A}{V_t[A/\bar{p}(h)+1]^{\frac{5}{7}}}\left(\frac{d\bar{p}/dh}{\bar{p}^2}\right) \tag{44}$$

Altitude is integrated from Eq. (28). The corresponding air-relative flight-path angle is given in Eq. (29). Possible capture variables are Mach number and altitude. In this case, γ_a^* is given by

$$\gamma_a^*(h; V_{cas}) = \frac{1}{m}\left\{\frac{T(h, M; \kappa_0) - D(h, V_t; L)}{V_t[(dV_t/dh) - (d/dh)(V_w \cos\theta_{rw})] + g}\right\} \tag{45}$$

where the Mach number changes according to Eq. (8). Actually, we have, from Eqs. (6) and (43),

$$M = \sqrt{5}\sqrt{[A/\bar{p}(h)+1]^{\frac{2}{7}} - 1} \tag{46}$$

and

$$\dot{M} = -\frac{5}{7}\left[\frac{A}{M(A/\bar{p}+1)^{\frac{5}{7}}}\right]\left[\frac{d\bar{p}/dh}{\bar{p}^2}\right]\dot{h} \tag{47}$$

Capturing a desired Mach number requires $\dot{M} \leq -\delta_m$, which results in

$$\frac{d\bar{p}}{dh} \leq \delta_m \frac{\bar{p}^2 f_{CD}(h; V_{cas})}{\dot{h}} < 0 \tag{48}$$

or

$$\frac{d}{dh}\left(\frac{1}{\bar{p}}\right) \geq \delta_m \frac{f_{CD}(h; V_{cas})}{-\dot{h}} > 0 \tag{49}$$

where $\dot{h}$ is given in Eq. (28) and

$$f_{CD}(h; V_{cas}) \triangleq (7/A\sqrt{5})(A/\bar{p}+1)^{\frac{5}{7}}\sqrt{(A/\bar{p}+1)^{\frac{2}{7}} - 1} > 0 \tag{50}$$

Again, the altitude gradient of measured pressure profile must be negative and stay within a certain bound. The specific bound depends on which one of κ, $\dot{h}$, or γ_i is set to constant. If this bound is violated, smoothing of the measured pressure data to continue trajectory synthesis is recommended.

Altitude capture requires $\dot{h} \leq -\delta_h$, which results in the same conditions as in the constant Mach segment, that is, Eqs. (37) and (38) for constant engine thrust setting, Eq. (40) for following a constant descent rate, and Eq. (41) for following a desired inertial flight-path angle. Alternative strategies and thrust-minus-drag requirements are similar as in the case of a constant Mach descent. However, the true airspeed gradient in Eq. (44) for a constant V_{cas} descent is different from that in Eq. (27) for a constant Mach descent. As a result, Eqs. (37) and (38) give different bounds in the two cases. Although details are omitted, capture requirements for a constant V_{cas} descent allow for larger wind gradients than those in a constant Mach descent at the same airspeed and altitude.

E. Level-Deceleration Segment

In a level-deceleration segment, true airspeed and path distance are integrated from Eqs. (1) and (3) and altitude is held constant. Engine thrust is set at minimum. A level-deceleration segment can be connected to another segment through the capture of either path distance or speed (Table 1). Usually, Mach number is used for capture at high altitudes whereas calibrated airspeed is used at low altitudes.

Capturing a calibrated airspeed in a level deceleration segment requires $\dot{V}_{cas} \leq -\delta_c$. From Eqs. (6) and (7), we have

$$\dot{V}_{cas} = \frac{\dot{V}_t}{f_{LD}(h_0, V_t)} \tag{51}$$

where

$$f_{LD}(h_0, V_t) \triangleq \frac{\bar{\Theta}(h_0)}{\bar{p}(h_0)}\left(\frac{V_{cas}}{V_t}\right)\left[\frac{V_t^2}{7R\Theta(h_0)} + 1\right]^{-2.5}$$

$$\times\left(\bar{p}(h_0)\left\{\left[\frac{V_t^2}{7R\Theta(h_0)} + 1\right]^{3.5} - 1\right\} + 1\right)^{\frac{5}{7}} > 0 \tag{52}$$

As a result,

$$T(h_0, M; \kappa_0) \leq D(h_0, V_t; L) - m\delta_c f_{LD}(h_0, V_t) \tag{53}$$

Capturing a Mach number in a level-deceleration segment requires $\dot{M} \leq -\delta_m$. From Eq. (6), we obtain

$$T(h_0, M; \kappa_0) \leq D(h_0, V_t; L) - m\delta_m\sqrt{1.4R\Theta_0} \tag{54}$$

Therefore the minimum thrust must be smaller than the aerodynamic drag by a certain margin in a level-deceleration flight. This margin is different in capturing a Mach number from that in capturing a calibrated airspeed. This requirement is not in conflict with Eqs. (19) and (23) for the descent-acceleration segment. Actually, all these requirements together define a feasible interval for the minimum engine thrust. If Eq. (53) or (54) is not satisfied, a larger drag needs to be used.

Effects of Numerical Integration Scheme

In actual implementations of trajectory synthesis, numerical integrations are performed. The current CTAS uses a second-order Runge–Kutta scheme.[8] Based on Eqs. (9) and(10), the numerical integrations calculate a sequence of states from

$$x_{j+1} = x_j + \Delta t F(x_j, t_j; \Delta t) \tag{55}$$

$$y_{j+1} = \upsilon(x_{j+1}) \tag{56}$$

for $j = 1, \ldots, N$, where

$$F(x_j, t_j; \Delta t) = \tfrac{1}{2}\{f(x_j, t_j) + f[x_j + \Delta t f(x_j, t_j), t_{j+1}]\} \tag{57}$$

The capture logic in CTAS at each integration step follows. Again, we assume $y(t_1) < y^*$ for the convenience of discussion.

1) Start with a Δt and use Eqs. (55) through (57) to calculate values of state and capture variables at the next step.

2) If $|y(t_{j+1}) - y^*| \leq \epsilon$, the variable is captured and $t^* = t_{j+1}$. The integration of the current segment is completed.

3) Otherwise, if

$$y^* - y(t_j) > \epsilon \quad \text{and} \quad y(t_{j+1}) - y^* > \epsilon$$

then, Δt is too large. Decrease Δt, reassign initial conditions, and go to step 1.

4) Or, assign $t_j + \Delta t \rightarrow t_j$ and advance one step.

The current capture accuracies in CTAS are 0.0003 for Mach number, 0.1 kn for calibrated airspeed, 20 ft for altitude, and 0.005 n mile for path distance.

Capture requirements derived for Eqs. (55–57) are different from conditions derived above using continuous-time derivatives. Nonetheless, a segment-by-segment analysis shows that the discrete conditions converge to the continuous-time conditions as the integration time step shrinks to zero. In other words, the continuous-time conditions apply directly if the integration time steps are reasonably small. Furthermore, the continuous-time conditions with larger values of δ will guarantee captures in numerical integrations.

The discrete capture conditions are more complicated than the continuous-time conditions, making them difficult to use. Therefore, the continuous-time conditions are recommended for use in these applications.

Applications of Capture Conditions

The capture conditions have direct applications in trajectory syntheses for air traffic control automation. They can be used in the trajectory synthesis process to verify convergence conditions and to suggest alternatives in case a capture fails. They also can be used to smooth measured data of atmospheric conditions. In these two applications, values of δ need to be specified. Specific values of δ depend on time and iteration allowed for convergence. The following expression gives a useful guide:

$$\delta = \frac{y^* - y_0}{N_{max}\Delta t_{min}} \tag{58}$$

where N_{max} is the maximum number of integration steps allowed, and Δt_{min} is the smallest practical integration step size.

In addition, capture conditions can be used for analyzing feasible ranges of flight altitudes and speeds both for actual flight and for trajectory synthesis. In this case, δ can be set to zero for convenience. Equations (1–3) represent approximate point-mass aircraft equations. Derivations of capture conditions are similar even if more accurate equations are used.

The specific limits imposed by the above capture conditions depend on the thrust and aerodynamic models. Therefore, these limits vary from one aircraft to another. On the other hand, the derived capture conditions represent physical requirements on aircraft performance and atmospheric environment. These conditions are valid, independent of specific algorithms used for trajectory synthesis.

The above-derived capture conditions also can be used if two capture variables are simultaneously searched in segment connections. Capture conditions for both variables must be satisfied.

Profile Capture Conditions and Example

Conditions are derived for every segment for all possible capture variables. The defined descent profiles, such as the fast type illustrated in Fig. 1, combine a series of segments in a predetermined order. The capture variable for each segment in these profiles is usually fixed. Therefore, one can group capture conditions for all the segments in a profile to obtain profile capture conditions.

For the fast-type descent profile in Fig. 1, the level-deceleration segment captures a descent V_{cas} in backward integration, the constant V_{cas} descent segment captures the descent Mach number in backward integration, the constant Mach descent segment captures the initial altitude in backward integration, and the descending-acceleration segment captures the descent Mach number in forward integration. Finally, the trajectory calculation is completed by a backward integration of the cruise segment to capture the initial path distance. Other profiles can be analyzed similarly.

We now summarize the capture conditions for a jet airplane flying a fast-descent-profile trajectory. The thrust for a jet engine is controlled by using the engine pressure ratio, which is set at minimum (idle) during the descending-acceleration segment, the constant Mach descent, the constant V_{cas} descent, and the level-deceleration segment. During the descending-acceleration flight, a desired descent rate is followed. Capture requirements with $\delta = 0$ are as follows:

1) The wind magnitude should always be smaller than the true airspeed as required in Eq. (14).

2) During the descending-acceleration segment, the minimum thrust needs to satisfy Eq. (19) to capture a specified Mach number or Eq. (23) to capture a specified calibrated airspeed. Mathematically, capturing a Mach requires

$$\frac{T(h, M; \kappa_0) - D(h, V_t; L)}{m} - \dot{h}\left[\frac{g}{V_t} + \frac{V_t}{2}\left(\frac{\mathrm{d}\Theta/\mathrm{d}h}{\Theta}\right)\right] > 0 \tag{59}$$

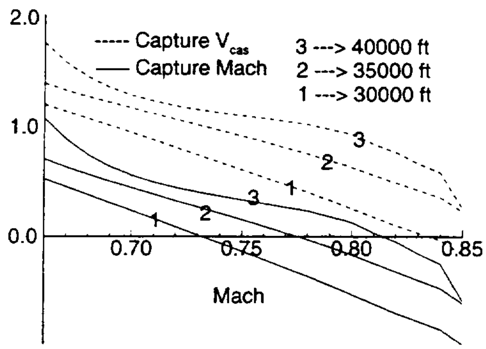

Fig. 3 Capture conditions in descending-acceleration segment.

and capturing a calibrated airspeed requires

$$\frac{T(h, M; \kappa_0) - D(h, V_t; L)}{m} - \dot{h}\left[\frac{g}{V_t} + \frac{V_t}{2}\left(\frac{\mathrm{d}\Theta/\mathrm{d}h}{\Theta}\right) + E(V_t, h)\right] > 0 \tag{60}$$

where $E(V_t, h) > 0$ is defined in Eq. (25).

3) During the constant Mach descent, the wind gradient must stay below the limit defined by Eq. (37) to capture a specified altitude. Otherwise, a higher thrust is required.

4) During the constant V_{cas} descent, the wind gradient must stay below the limit defined by Eq. (37) to capture a specified altitude. Otherwise, a higher thrust is required. To capture a specified Mach number, the altitude gradient of measured pressure needs to be negative, as required by Eq. (48) or (49).

5) During the level-deceleration segment, the minimum thrust needs to be less than the drag as in Eq. (53) to capture a specified calibrated airspeed.

The above conditions are now illustrated using a Boeing 727-200 aircraft in descent. Powered by three Pratt and Whitney JT8D-15 engines, this aircraft has a typical descent weight of 140,000 lb and a default descent calibrated airspeed of 280 kn. The maximum V_{cas} is 350 kn and the minimum V_{cas} is 230 kn. Maximum-cruise Mach number is 0.82. In Figs. 3–6, the standard atmospheric pressure, density, and temperature are assumed. A descent rate of 3000 ft/min is used in segments with specified descent rate.

In the CTAS aircraft models,[9] drag coefficients for the Boeing 727-200 aircraft are given in a two-dimensional table at 6 Mach numbers and, for each Mach number, 11 lift coefficients. The minimum thrust is given in a two-dimensional table at 10 altitudes and 10 Mach numbers. In this paper, the drag coefficient is fitted into a fifth-order polynomial of lift coefficient at each Mach number. A linear interpolation is used for drag coefficient between any two specified Mach numbers. A two-dimensional linear interpolation is used to determine the minimum thrust between specified values of altitude and/or Mach number.

Figure 3 plots the capture requirements in Eqs. (59) and (60) for a descending-acceleration segment at various Mach numbers and three different altitudes. Positive regions of the curves in Fig. 3 define a feasible range of Mach numbers and altitudes within which the descending-acceleration integration can capture. For example, for $h \geq 40{,}000$ ft, the maximum Mach number that can be captured is around 0.814. Clearly, it is easier to capture a calibrated airspeed than a Mach number at the same altitude and speed. These curves will shift up as the aircraft weight or the descent rate increases. If capturing a Mach number becomes difficult, one can either increase the descent rate or switch to capture a calibrated airspeed.

Figures 4 and 5 show the upper bound on wind gradient in Eq. (37) for a constant Mach descent and a constant V_{cas} descent, respectively. These figures show different trends for the constant Mach descent and for the constant V_{cas} descent as altitude increases. The maximum wind gradient that can be tolerated decreases as flight speed increases. Over the altitude range of 10,000–35,000 ft, a wind gradient less than 0.03 s^{-1} meets the requirement at all speeds. This is consistent with the findings of numerical experiments.

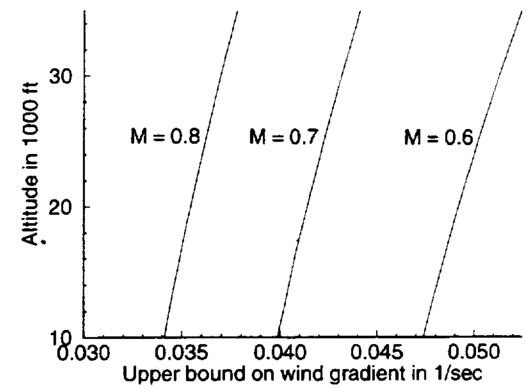

Fig. 4 Upper bound on wind gradient in constant Mach segment.

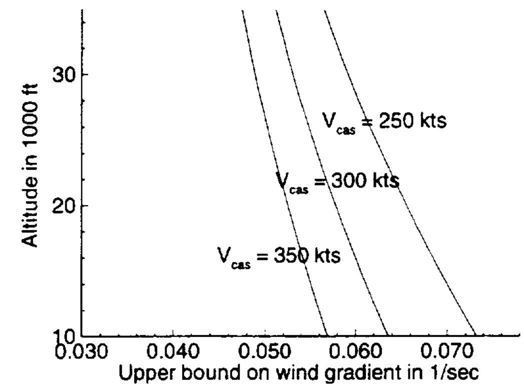

Fig. 5 Upper bound on wind gradient in constant V_{cas} segment.

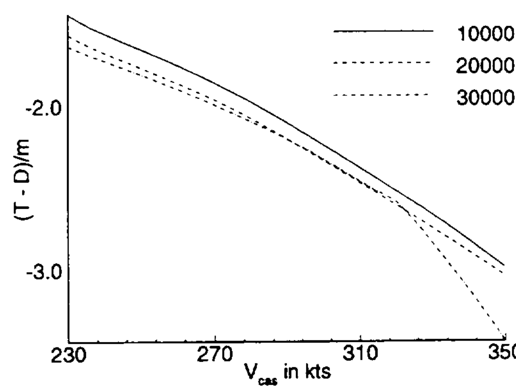

Fig. 6 Thrust minus drag in level-deceleration segment.

Finally, Fig. 6 shows the minimum thrust minus drag at different altitudes and speeds. This quantity is always negative in this example. It is always possible to capture a calibrated airspeed within the range of 230–350 kn.

In summary, this example shows that the capture conditions are satisfied for a Boeing 727-200 in the standard atmospheric conditions, except possibly in capturing a Mach number during the descending-acceleration segment.

Conclusions

This paper presents capture conditions that affect the successful connections of flight segments in numerical integrations for trajectory synthesis. In a commercial descent, an aircraft trajectory is divided into a series of segments according to preferred pilot procedure. A capture variable, either path distance, Mach number, calibrated airspeed, or altitude, is monitored in one segment, and the next segment begins when the capture variable reaches a specified value. Successful connections of segments in trajectory integrations are guaranteed if capture variables are monotonic in the segment for which they are monitored. In this paper, monotonicity conditions for capture variables in different segments of various descent profiles are derived. Alternative flight strategies are suggested when some capture conditions are not satisfied.

These conditions impose bounds on measurements of atmospheric wind, pressure, and temperature, as well as thrust minus drag. These bounds can be used to smooth measurement data for trajectory synthesis. They can be used in trajectory-synthesis algorithms to provide convergence checks and alternative instructions. For given aircraft performance models, these capture conditions define feasible ranges of altitudes and airspeeds within which an aircraft can be directed to fly.

The example of a Boeing 727-200 aircraft descending in a standard atmosphere shows that the capture conditions are satisfied in most flight segments, except possibly the descending-acceleration segment when capturing a Mach number.

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