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Modeling of Turbulent Swirling Flows

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Abstract

Aircraft engine combustors generally involve turbulent swirling flows in order to enhance fuel-air mixing and flame stabilization. It has long been recognized that eddy viscosity turbulence models are unable to appropriately model swirling flows. Therefore, it has been suggested that, for the modeling of these flows, a second order closure scheme should be considered because of its ability in the modeling of rotational and curvature effects. However, this scheme will require solution of many complicated second moment transport equations (six Reynolds stresses plus other scalar fluxes and variances), which is a difficult task for any CFD implementations. Also, this scheme will require a large amount of computer resources for a general combustor swirling flow.

This report is devoted to the development of a cubic Reynolds stress-strain model for turbulent swirling flows, and was inspired by the work of Launder's group at UMIST. Using this type of model, one only needs to solve two turbulence equations, one for the turbulent kinetic energy k and the other for the dissipation rate ε. The cubic model developed in this report is based on a general Reynolds stress-strain relationship (Shih and Lumley, 1993). Two flows have been chosen for model evaluation. One is a fully developed rotating pipe flow, and the other is a more complex flow with swirl and recirculation.

Contents

1 Introduction

For better fuel-air mixing and flame stabilization in a combustor, a swirl is generally associated with the flows. Therefore, accurate modeling of turbulent swirling flows is important in engine combustor design. Common turbulence models used in engineering calculations are eddy viscosity models which include zero-equation and two-equation models (e.g., mixing length models and k-ε models). However, it has long been recognized that this type of eddy viscosity model is not appropriate for predicting swirling flows. In fact, the deficiency of eddy viscosity models for swirling flows can be analytically demonstrated by modeling a fully developed rotating pipe flow (Fu, 1995). Measured swirl velocity in the pipe varies approximately as the square of the normalized radius (r^2), however, eddy viscosity models produce an exact linear profile of the swirl velocity, which describes a solid body rotation.

To avoid this kind of deficiency of eddy viscosity models, a second order closure scheme has been suggested for modeling of swirling flows because of its ability to simulate the effects of mean rotation and curvature. However, this requires solving many complicated second moment transport equations, which involve six Reynolds stresses plus other scalar fluxes and variances. Because of this complexity and because of the large computer resources required, second moment transport equation models have not been successfully implemented in combustor swirling flows.

Recent developments in nonlinear Reynolds stress-strain models bring a practical method for combustion flow calculations because of their potential in simulating turbulent swirling flows with only two modeled turbulence transport equations (Craft et al, 1993). Further development and evaluation of these models are of great interest to both CFD development and modern aircraft engine combustor design.

The model developed in this report is based on a general Reynolds stress-strain relationship which is an explicit expression for the Reynolds stresses in terms of a tensorial polynomial of mean velocity gradients. It is derived from a generalized Cayley-Hamilton relation. This general formulation contains terms up to the sixth power of the mean velocity gradient with eleven undetermined coefficients. Obviously, for any practical application, we need to truncate this polynomial. Shih, Zhu and Lumley (1995) suggested a quadratic formulation and determined the three relevant coefficients by using the realizability constraints of Reynolds stresses and a result from rapid distortion theory analysis. This quadratic model works quite successfully for many complex flows including flows with separation. However, our recent calculations of swirling flows show that the swirl velocity is not appropriately predicted, which verifies the finding from Launder's group at UMIST. Launder (1995) pointed out that "the weaknesses of the linear eddy viscosity model can not be rectified by introducing just quadratic terms to the stress-strain relation."

In this report, we retain the cubic terms from a general Reynolds stress-stain formulation and determine the coefficients by using a similar method used in Shih et al's quadratic model and the measured data from rotating pipe flows. Modeled k-ε equations are used together with the cubic Reynolds stress-strain model for mean flow calculations. The first test flow is that of fully developed pipe flow rotating about its own axial axis with various rotation rates (Imao, Itoh and Harada, 1996). The second test flow is a more complex flow with swirl and recirculation (Roback and Johnson, 1983). These two flows both have detailed experimental data on mean velocity components. The comparisons between the experimental data and computational results from models will be reported in detail.

In this report, there are four appendices. In Appendix A, the derivation of the proposed cubic model is described. Appendix B gives the equations in a general coordinate system, which will be useful for studying flows in various curvilinear coordinate systems. For example, axisymmetric flows will be most conveniently studied in a cylindrical coordinate system. Therefore, in Appendix C and Appendix D, we write the equations for a general flow and an axisymmetric flow respectively in a cylindrical coordinate system.

2 Cubic Reynolds stress model

A cubic Reynolds stress model, used in this study for modeling of turbulent swirling flows, is developed in Appendix A. The resultant cubic model can be expressed in terms of mean velocity gradients, $U_{i,j}$, or in terms of mean strain and rotation rates, S_{ij} and Ω_{ij}. Here, we list both forms for convenience of their applications.

In terms of mean velocity gradients, the cubic model for Reynolds stresses is

$$\begin{aligned}
-\rho\overline{u_i u_j} = &-\frac{2}{3}\rho k\ \delta_{ij} + \mu_T\left(U_{i,j} + U_{j,i} - \frac{2}{3}U_{k,k}\ \delta_{ij}\right) \\
&+ A_3\frac{\rho k^3}{2\varepsilon^2}\left(U_{k,i}U_{k,j} - U_{i,k}U_{j,k}\right) \\
&+ A_5\frac{\rho k^4}{\varepsilon^3}\Big[U_{k,i}U_{k,p}U_{p,j} + U_{k,j}U_{k,p}U_{p,i} - \frac{2}{3}\Pi_3\ \delta_{ij} \\
&-\frac{1}{2}I_S\left(U_{i,k}U_{k,j} + U_{j,k}U_{k,i} - \frac{2}{3}\Pi_1\delta_{ij}\right) \\
&-\frac{1}{2}I_S\left(U_{k,i}U_{k,j} + U_{i,k}U_{j,k} - \frac{2}{3}\Pi_2\delta_{ij}\right)\Big]
\end{aligned} \tag{1}$$

where "$_{,j}$" means a tensorial derivative with respect to j. I_S is the first principal invariant of S_{ij}, i.e., S_{kk}. The invarints Π_1, Π_2 and Π_3 (which appear in Eq.(A.1)) are defined as follows

$$\Pi_1 = U_{i,j}U_{j,i}\ , \quad \Pi_2 = U_{i,j}U_{i,j}\ , \quad \Pi_3 = U_{i,k}U_{i,p}U_{p,k} \tag{2}$$

The three coefficients μ_T, A_3 and A_5 are

$$\mu_T = \rho C_\mu f_\mu \frac{k^2}{\varepsilon}\ , \quad \text{or} \quad \mu_T = \rho C_\mu f_\mu \frac{k(k+\sqrt{\nu\varepsilon})}{\varepsilon} \tag{3}$$

$$C_\mu = \frac{1}{4.0 + A_S\dfrac{kU^*}{\varepsilon}}\ , \quad f_\mu = Eq.(22), \quad \text{or} \quad Eq.(26) \tag{4}$$

$$A_3 = \frac{\sqrt{1 - \dfrac{9}{2}C_\mu^2\left(\dfrac{kS^*}{\varepsilon}\right)^2}}{0.5 + \dfrac{3}{2}\dfrac{k^2}{\varepsilon^2}\Omega^*\ S^*} \tag{5}$$

$$A_5 = \frac{1.6\ \mu_T}{\frac{\rho k^4}{\varepsilon^3}\frac{7(S^*)^2 + (\Omega^*)^2}{4}} \tag{6}$$

in which

$$A_S = \sqrt{6}\cos\phi, \quad \phi = \frac{1}{3}\arccos(\sqrt{6}W^*), \quad W^* = \frac{S^*_{ij}S^*_{jk}S^*_{ki}}{(S^*)^3} \tag{7}$$

$$U^* = \sqrt{(S^*)^2 + (\Omega^*)^2}\ , \quad S^* = \sqrt{S^*_{ij}S^*_{ij}}\ , \quad \Omega^* = \sqrt{\Omega_{ij}\Omega_{ij}} \tag{8}$$

The model coefficient C_μ is also constrained by the following conditions:

$$C_\mu \le \frac{\sqrt{2}}{3}\left(\frac{kS^*}{\varepsilon}\right)^{-1}, \quad \text{and} \quad C_\mu \le \left(A_S\frac{kS^*}{\varepsilon}\right)^{-1} + \frac{k^2}{\varepsilon^2}II_S A_5 \tag{9}$$

where II_S is defined in Eq. (12).

In terms of mean strain and rotation rates, Eq. (1) can be written as

$$\begin{aligned} -\rho\overline{u_i u_j} =& -\frac{2}{3}\rho k\delta_{ij} + \mu_T 2S^*_{ij} + A_3\frac{\rho k^3}{\varepsilon^2}\left(S_{ik}\Omega_{kj} - \Omega_{ik}S_{kj}\right) \\ & - 2A_5\frac{\rho k^4}{\varepsilon^3}\left(\Omega_{ik}S^2_{kj} - S^2_{ik}\Omega_{kj} + \Omega_{ik}S_{km}\Omega_{mj} - \frac{1}{3}\Omega_{kl}S_{lm}\Omega_{mk}\delta_{ij} + II_S S^*_{ij}\right) \end{aligned} \tag{10}$$

where

$$S^*_{ij} = S_{ij} - \frac{1}{3}S_{kk}\delta_{ij}\ , \ S^2_{ij} = S_{ik}S_{kj}\ , \ S_{ij} = \frac{1}{2}\left(U_{i,j} + U_{j,i}\right)\ , \ \Omega_{ij} = \frac{1}{2}\left(U_{i,j} - U_{j,i}\right) \tag{11}$$

II_S is the second principal invariant of S_{ij} defined as

$$II_S = \frac{1}{2}\left(S_{kk}S_{mm} - S^2_{kk}\right) \tag{12}$$

Note that in the above equations, S_{kk} means $S_{11} + S_{22} + S_{33}$ and S^2_{kk} means $S_{1p}S_{p1} + S_{2p}S_{p2} + S_{3p}S_{p3}$ in which each term contains a summation operator on the subscript "p".

It should also be mentioned that the eddy viscosity μ_T in Eq. (3) will become the standard form of $\mu_T = \rho C_\mu \frac{k^2}{\varepsilon}$ for high turbulent Reynolds number flows ($\frac{k^2}{\nu\varepsilon} >> 1$).

3 Modeling of turbulent swirling flows

The model proposed in the previous section will be used for modeling of swirling flows in this study. The first flow is a fully developed rotating pipe flow (Imao, Itoh and Harada, 1996). This flow was used for model development; however, a pipe flow with various axial rotating rates is still a critical test case for the model. The second flow is a more complex swirling flow with recirculation and separation (Roback and Johnson, 1983), which is often encountered in an aircraft engine combustor.

3.1 Rotating pipe flow

A fully developed rotating pipe flow provides a very clean test case for checking the turbulence model's ability to model swirling flows. As mentioned previously, commonly used eddy viscosity models fail to predict this flow. In fact, one can show that any eddy viscosity model will produce a solution of solid body rotation for a rotating pipe flow, while experimental data shows that the flow is not a solid body rotation. Experiments further demonstrate that the characteristics of a pipe flow changes significantly with the axial rotation rate. For example, for a fixed mass flux, the axial rotation will strongly reduce the pressure drop. In other words, for a fixed pressure drop, the axial rotation will increase the total mass flux. However, standard eddy viscosity models show no such changes at all.

In a fully developed turbulent pipe flow, all the axial gradients, $\partial/\partial x$, and the azimuthal derivatives, $\partial/\partial\theta$, are zero, and so is the radial velocity $V = 0$. The non-zero velocity components are the axial velocity U and the tangential (or swirl) velocity $W = r\Omega$, where Ω is the angular velocity. Equations for this flow are

$$\frac{\partial r\rho U}{\partial t} = -r\frac{\partial \overline{P}}{\partial x} + \frac{\partial}{\partial r}\left[(\mu+\mu_T)r\frac{\partial U}{\partial r}\right] + \frac{\partial r\tau_{xr}}{\partial r} \tag{13}$$

$$\frac{\partial r^2\rho W}{\partial t} = \frac{\partial}{\partial r}\left[(\mu+\mu_T)r\frac{\partial rW}{\partial r}\right] - 2\frac{\partial}{\partial r}\left[(\mu+\mu_T)rW\right] + \frac{\partial r\tau_{\theta r}}{\partial r} \tag{14}$$

$$\frac{\partial r\rho k}{\partial t} = \frac{\partial}{\partial r}\left[\left(\mu+\frac{\mu_T}{\sigma_k}\right)r\frac{\partial k}{\partial r}\right] + rP_k - r\rho\varepsilon \tag{15}$$

$$\frac{\partial r\rho\varepsilon}{\partial t} = \frac{\partial}{\partial r}\left[\left(\mu+\frac{\mu_T}{\sigma_\varepsilon}\right)r\frac{\partial \varepsilon}{\partial r}\right] + C_{\varepsilon1}f_1\, rP_k\frac{\varepsilon}{k} - C_{\varepsilon2}f_2\,\frac{r\rho\varepsilon^2}{k} + \frac{\mu\mu_T}{\rho}r\left(\frac{\partial S}{\partial r}\right)^2 \tag{16}$$

or

$$\frac{\partial r\rho\varepsilon}{\partial t} = \frac{\partial}{\partial r}\left[\left(\mu+\frac{\mu_T}{\sigma_\varepsilon}\right)r\frac{\partial \varepsilon}{\partial r}\right] + C_1f_1\, r\rho S\,\varepsilon - C_2f_2\,\frac{r\rho\varepsilon^2}{k+\sqrt{\nu\varepsilon}} + \frac{\mu\mu_T}{\rho}r\left(\frac{\partial S}{\partial r}\right)^2 \tag{17}$$

where $S = \sqrt{2S_{ij}S_{ij}} = \sqrt{\left(\frac{\partial U}{\partial r}\right)^2 + \left(\frac{\partial W}{\partial r} - \frac{W}{r}\right)^2}$. The nonlinear parts of turbulent stresses, τ_{xr} and $\tau_{\theta r}$, from the proposed cubic model, Eq. (1) or Eq. (10), are

$$\tau_{xr} = \quad 0 \tag{18}$$

$$\tau_{\theta r} = - A_5 \frac{\rho k^4}{\varepsilon^3}\left[W\left(\frac{\partial U}{\partial r}\right)^2 + W\frac{\partial W}{\partial r}\left(\frac{\partial W}{\partial r} - \frac{W}{r}\right)\right] \tag{19}$$

The production rate of turbulent kinetic energy P_k is

$$\begin{aligned} P_k = \quad & \mu_T\left[\left(\frac{\partial U}{\partial r}\right)^2 + \left(\frac{\partial W}{\partial r} - \frac{W}{r}\right)^2\right] \\ & - A_5\frac{\rho k^4}{\varepsilon^3}\frac{W}{r}\left(\frac{\partial W}{\partial r} - \frac{W}{r}\right)\left[\left(\frac{\partial U}{\partial r}\right)^2 + \frac{\partial W}{\partial r}\left(\frac{\partial W}{\partial r} - \frac{W}{r}\right)\right] \end{aligned} \tag{20}$$

where

$$\mu_T = \rho C_\mu\, f_\mu \frac{k^2}{\varepsilon} \tag{21}$$

$$f_\mu = \left[1 - exp(-a_1 R_k - a_3 R_k^3 - a_5 R_k^5)\right]^{1/2} \tag{22}$$

$$f_1 = 1, \qquad f_2 = 1 - 0.22 exp\left(-Rt^2/36\right) \tag{23}$$

and $a_1 = 1.7 * 10^{-3}$, $a_3 = 1 * 10^{-9}$, $a_1 = 5 * 10^{-10}$, $R_k = \rho\sqrt{k}y/\mu$. Other model constants used in this report are standard: $\sigma_k = 1$, $\sigma_\varepsilon = 1.3$, $C_{\varepsilon 1} = 1.44$ and $C_{\varepsilon 2} = 1.92$. Depending on particular modeled k-ε equations, the model coefficients and damping function f_μ may have different formulations proposed by various researchers. For example, if Eq. (17) (Shih et al, 1995) is used together with

$$\mu_T = \rho C_\mu f_\mu \frac{k(k + \sqrt{\nu\varepsilon})}{\varepsilon} \tag{24}$$

then

$$C_1 = max\left\{0.43,\ \frac{\eta}{5+\eta}\right\}, \quad C_2 = 1.9, \quad \eta = \frac{S\ k}{\varepsilon} \tag{25}$$

and f_μ, f_1 are

$$f_\mu = 1 - exp\left\{-\left(a_1 R + a_2 R^2 + a_3 R^3 + a_4 R^4 + a_5 R^5\right)\right\} \tag{26}$$

$$f_1 = 1 - exp\left\{-\left(a_1' R + a_2' R^2 + a_3' R^3 + a_4' R^4 + a_5' R^5\right)\right\} \tag{27}$$

$$f_2 = Eq.(23) \tag{28}$$

where

$$R = \frac{k^{1/2}\left(k+\sqrt{\nu\varepsilon}\right)^{3/2}}{\nu\varepsilon} \tag{29}$$

and

$$\begin{aligned} &a_1 = 3.3*10^{-3}, \quad a_2 = -6*10^{-5}, \quad a_3 = 6.6*10^{-7}, \\ &a_4 = -3.6*10^{-9}, \quad a_5 = 8.4*10^{-12} \end{aligned} \tag{30}$$

$$\begin{aligned} &a'_1 = 2.53*10^{-3}, \quad a'_2 = -5.7*10^{-5}, \quad a'_3 = 6.55*10^{-7}, \\ &a'_4 = -3.6*10^{-9}, \quad a'_5 = 8.3*10^{-12} \end{aligned} \tag{31}$$

From Eq. (14), it is easy to show that any eddy viscosity model will produce a solution of solid body rotation, i.e., $W/W_{wall} = r/R$, where W_{wall} is the swirl velocity of the wall and R is the radius of the pipe. It can also be shown that any quadratic Reynolds stress models will have no contributions to the component $\tau_{\theta r}$ for a fully develed rotating pipe flow. Therefore, they will also produce a solution of solid body rotation, just like an eddy viscosity model does. Equations (13)-(17) can be easily and accurately solved by a parabolic code. Figures 1 - 3 show the results of the present cubic model with Eqs. (15) and (17) compared with the measurements by Imao, et al (1996). The results from the standard k-ε eddy viscosity model are also included for comparison. In the figures, the rotation parameter N is defined as $N = W_{wall}/U_m$, where U_m is the average velocity of the pipe. The Reynolds number based on U_m and R is 20000. As shown in these figures, the standard k-ε eddy viscosity model has totally missed the effect of axial rotations on the pipe flow. In contrast, the present cubic Reynolds stress model can capture all the effects of the axial rotation on the pipe flow: it increases the centerline velocity and changes the axial velocity profile towards a parabolic shape, it maintains non-solid body swirl velocity profile, and it reduces the relative turbulent kinetic energy k/U_m^2.

3.2 Complex swirling flow with recirculation

A confined swirling coaxial jet was experimentally studied by Roback and Johnson (1983). Figure 4 shows the general features of the flow. At the inlet, an inner jet and an annular jet are ejected into an enlarged duct. Besides an annular separation due to sudden expansion of the duct, a central recirculation bubble is created by the swirling flow. This flow feature is often observed in an aircraft engine combustor. In this figure, calculated velocity vectors in an axisymmetric plane from the cubic model is compared with the one from the standard k-ε eddy viscosity model. Solutions were obtained by two Navier-Stokes codes. One is CORSAIR (Liu et al, 1996) and the other is FAST-2D (Zhu, 1991). Eq. (16) and Eq. (17) are respectively used in this calculation. Numerical results from the two codes are quite close to each other. Figure 5 compares the calculations of the centerline velocity using a standard k-ε eddy viscosity model (SKE) and the present cubic model with the experimental data. The negative velocity indicates the central recirculation. It is seen that both models predict the strength of central recirculation quite well, but the present model predicts the rear stagnation point much better than does the SKE model. This is also reflected in Fig. 4 that the recirculation bubble from the cubic model is larger than that from the standard SKE model. Figure 6 shows the comparison of calculated and measured mean velocity profiles at x=51mm. Both models give reasonably good profiles which are within experimental scatter. However, significant differences in the tangential velocity profile between the two models have been found in the downstream region. For example, Fig. 7 shows the swirl velocity profile at x=305mm. SKE model predicts a nearly solid body rotation, whereas the cubic model shows a non-solid body rotation which is consistent with experimental observation.

4 Conclusion and discussion

This study shows that nonlinear cubic Reynolds stress-strain models with modeled k-ε equations have the potential to simulate turbulent swirling flows encountered in aircraft engine combustors. The model proposed in this report appears simple and numerically robust in CFD applications in which the aircraft engine industry is particularly interested. However, further evaluations against other flows are needed in order to determine the flow range of the model's validity and to seek possible further improvements.

The proposed cubic Reynolds stress model can be combined with existing k-ε model equations, yet the best combination needs further studies and evaluations.

The proposed cubic model appears the simplest among other cubic or higher order models; however it requires about 15% more CPU time than does a linear k-ε eddy viscosity model for a general 2D axisymmetric swirling flow. We expect that if a higher order model (e.g., fourth or fifth order) is used, then the CPU time for calculating Reynolds stresses will significantly increase and the model may become very costly for the calculation of a general 3D swirling flow.

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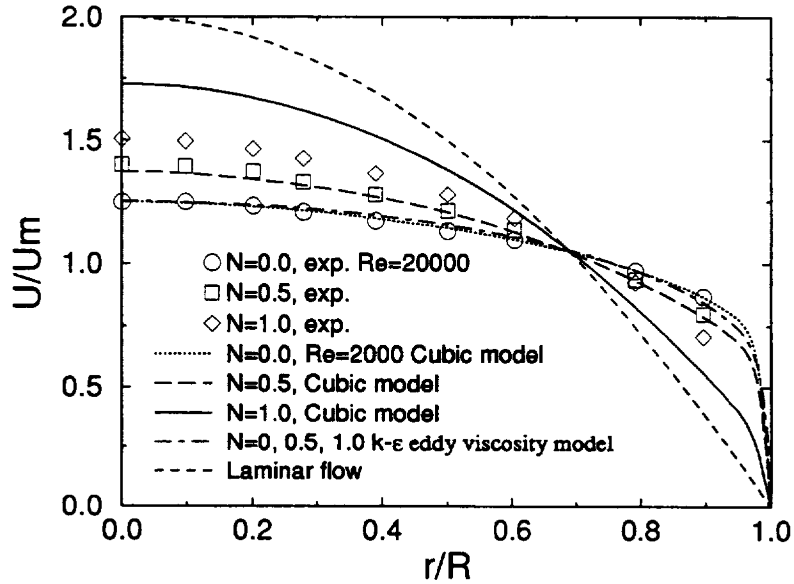

Fig. 1. Axial velocity profile in a rotating pipe

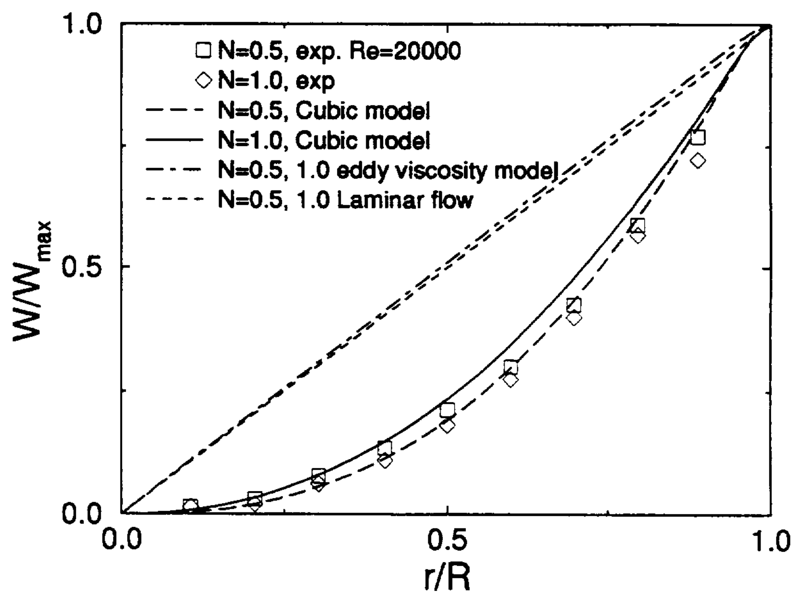

Fig. 2. Tangential velocity profile in a rotating pipe

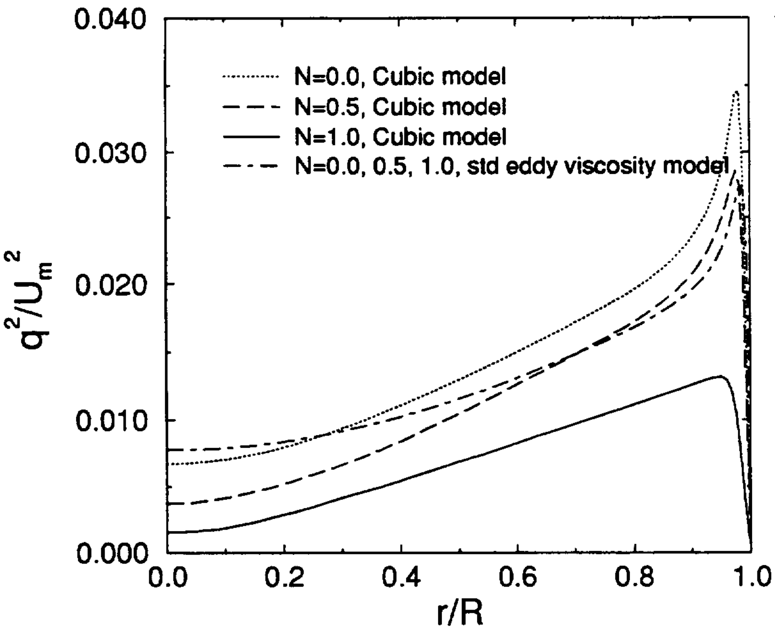

Fig. 3. Effect of rotation on turbulent kinetic energy $\frac{k}{U_m^2}$

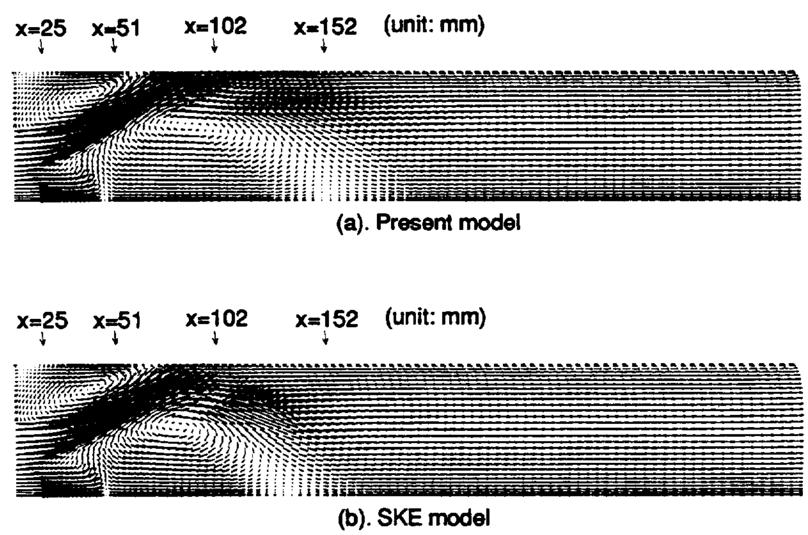

Fig. 4. Velocity vectors in an axisymmetric plane. (a) from present model, (b) from SKE model.

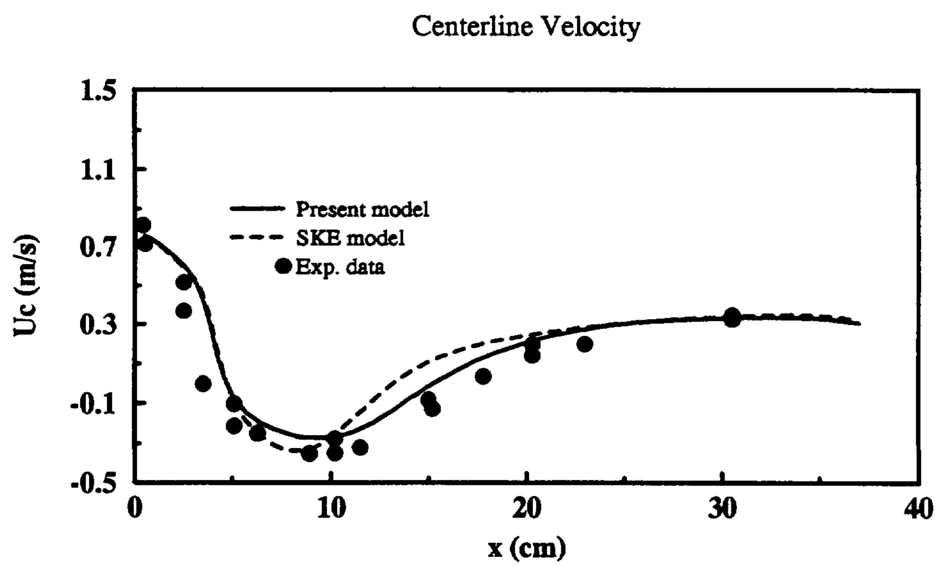

Fig. 5. Centerline velocity in Roback and Johson flow

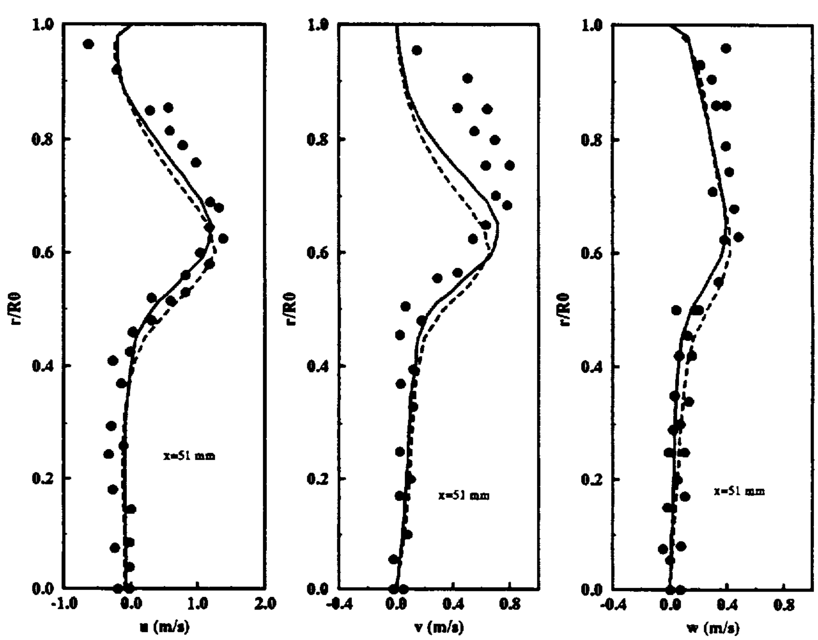

Fig. 6. Mean velocity profiles at $x = 51mm$

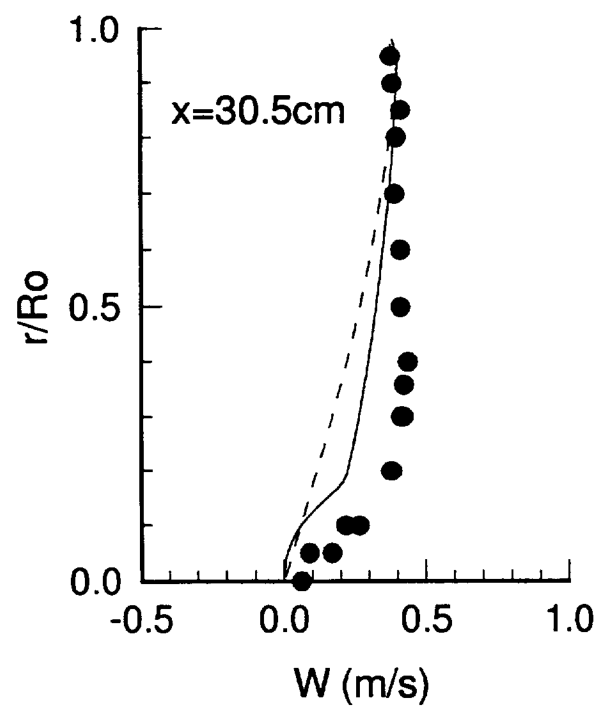

Fig. 7. Tangential velocity profile at x=305mm

A Appendix: Development of a Cubic Turbulent Model

A truncated general cubic turbulent stress-strain relation from Shih and Lumley (1993) can be written as

$$
\begin{aligned}
-\rho\overline{u_i u_j} = & -\frac{2}{3}\rho k\ \delta_{ij} + C_\mu \rho \frac{k^2}{\varepsilon}\left(U_{i,j} + U_{j,i} - \frac{2}{3}U_{k,k}\ \delta_{ij}\right) \\
& + C_1 \frac{\rho k^3}{\varepsilon^2}\left(U_{i,k}U_{k,j} + U_{j,k}U_{k,i} - \frac{2}{3}\Pi_1\ \delta_{ij}\right) \\
& + C_2 \frac{\rho k^3}{\varepsilon^2}\left(U_{i,k}U_{j,k} - \frac{1}{3}\Pi_2\ \delta_{ij}\right) \\
& + C_3 \frac{\rho k^3}{\varepsilon^2}\left(U_{k,i}U_{k,j} - \frac{1}{3}\Pi_2\ \delta_{ij}\right) \\
& + C_4 \frac{\rho k^4}{\varepsilon^3}\left(U_{i,k}U_{j,p}U_{p,k} + U_{i,p}U_{p,k}U_{j,k} - \frac{2}{3}\Pi_3\ \delta_{ij}\right) \\
& + C_5 \frac{\rho k^4}{\varepsilon^3}\left(U_{k,i}U_{k,p}U_{p,j} + U_{k,j}U_{k,p}U_{p,i} - \frac{2}{3}\Pi_3\ \delta_{ij}\right)
\end{aligned}
\tag{A.1}
$$

The six model coefficients in Eq.(A.1) will be determined by the following procedure. First, we consider two extreme cases: a pure strain flow and a pure shear flow, and apply realizability constraints on the Reynolds stresses to ensure positive energy components and Schwarz' inequality. This was suggested by Reynolds (1987) and Shih et al (1995), which will allow us to determine the model coefficients of C_μ, C_1, C_2 and C_3. The second procedure is to determine the model coefficients C_4 and C_5 by using the experimental data of a fully developed rotating pipe flow. To analize the pure strain and pure shear flows, it is more convenient to write Eq.(A.1) in terms of mean strain and rotation rates, as in the following:

$$
\begin{aligned}
-\rho\overline{u_i u_j} = & -\frac{2}{3}\rho k \delta_{ij} + C_\mu \frac{\rho k^2}{\varepsilon} 2S^*_{ij} + C_1 \frac{\rho k^3}{\varepsilon^2} 2(S^{(2*)}_{ij} + \Omega^{(2*)}_{ij}) \\
& + C_2 \frac{\rho k^3}{\varepsilon^2}(S^{(2*)}_{ij} - \Omega^{(2*)}_{ij} - S^*_{ik}\Omega_{kj} + \Omega_{ik}S^*_{kj}) \\
& + C_3 \frac{\rho k^3}{\varepsilon^2}(S^{(2*)}_{ij} - \Omega^{(2*)}_{ij} + S^*_{ik}\Omega_{kj} - \Omega_{ik}S^*_{kj}) \\
& + 2C_4 \frac{\rho k^4}{\varepsilon^3}\left(S^{(3*)}_{ij} - S^{(2*)}_{ik}\Omega_{kj} + \Omega_{ik}S^{(2*)}_{kj} - \Omega_{ik}S_{km}\Omega_{mj} + \frac{1}{3}\Omega_{kl}S_{lm}\Omega_{mk}\delta_{ij}\right) \\
& + 2C_5 \frac{\rho k^4}{\varepsilon^3}\left(S^{(3*)}_{ij} + S^{(2*)}_{ik}\Omega_{kj} - \Omega_{ik}S^{(2*)}_{kj} - \Omega_{ik}S_{km}\Omega_{mj} + \frac{1}{3}\Omega_{kl}S_{lm}\Omega_{mk}\delta_{ij}\right)
\end{aligned}
\tag{A.2}
$$

where

$$
S^*_{ij} = S_{ij} - \frac{1}{3}S_{kk}\delta_{ij}\ , \quad S^{(2*)}_{ij} = S^2_{ij} - \frac{1}{3}S^2_{kk}\delta_{ij}\ , \quad \Omega^{(2*)}_{ij} = \Omega^2_{ij} - \frac{1}{3}\Omega^2_{kk}\delta_{ij}
$$

$$
S^{(3*)}_{ij} = S^3_{ij} - \frac{1}{3}S^3_{kk}\delta_{ij}\ , \quad S^2_{ij} = S_{il}S_{lj}\ , \quad \Omega^2_{ij} = \Omega_{il}\Omega_{lj}
$$

$$S^3_{ij} = S_{im}S_{ml}S_{lj} \ , \quad S_{ij} = \frac{1}{2}(U_{i,j} + U_{j,i}) \ , \quad \Omega_{ij} = \frac{1}{2}(U_{i,j} - U_{j,i}) \tag{A.3}$$

Note that S^*_{ij}, $S^{(2*)}_{ij}$, $\Omega^{(2*)}_{ij}$ and $S^{(3*)}_{ij}$ are all traceless tensors. Using Cayley-Hamilton relation,

$$S^3_{ij} - I_S S^2_{ij} + II_S S_{ij} - III_S \delta_{ij} = 0 \tag{A.4}$$

$S^{(3*)}_{ij}$ can be expressed in terms of quadratic and linear terms as

$$S^{(3*)}_{ij} = I_S S^{(2*)}_{ij} - II_S S^*_{ij} \tag{A.5}$$

where I_S, II_S and III_S are the three principal invariants of S_{ij}:

$$I_S = S_{ii} \ , \quad II_S = \frac{1}{2}\left(S_{kk}S_{mm} - S^2_{kk}\right) \ , \quad III_S = \frac{1}{6}\left(S_{ii}S_{jj}S_{kk} - 3S_{ii}S^2_{jj} + 2S^3_{ii}\right) \tag{A.6}$$

Using Eq.(A.5), we may write Eq.(A.2) as

$$\begin{aligned} -\rho\overline{u_i u_j} = & -\frac{2}{3}\rho k\delta_{ij} + C_\mu \frac{\rho k^2}{\varepsilon} 2S^*_{ij} + 2A_1 \frac{\rho k^3}{\varepsilon^2} S^{(2*)}_{ij} + 2A_2 \frac{\rho k^3}{\varepsilon^2}\Omega^{(2*)}_{ij} \\ & + A_3 \frac{\rho k^3}{\varepsilon^2}\left(S^*_{ik}\Omega_{kj} - \Omega_{ik}S^*_{kj}\right) + 2A_4 \frac{\rho k^4}{\varepsilon^3}\left(S^{(2*)}_{ik}\Omega_{kj} - \Omega_{ik}S^{(2*)}_{kj}\right) \\ & - 2A_5 \frac{\rho k^4}{\varepsilon^3}\left(\Omega_{ik}S_{km}\Omega_{mj} - \frac{1}{3}\Omega_{kl}S_{lm}\Omega_{mk}\delta_{ij} + II_S S^*_{ij} - I_S S^{(2*)}_{ij}\right) \end{aligned} \tag{A.7}$$

where

$$\begin{aligned} & A_1 = \frac{1}{2}(2C_1 + C_2 + C_3) \ , \quad A_2 = \frac{1}{2}(2C_1 - C_2 - C_3) \\ & A_3 = C_3 - C_2 \ , \quad A_4 = (C_5 - C_4) \ , \quad A_5 = (C_4 + C_5) \end{aligned} \tag{A.8}$$

A result from a rapid distortion theory analysis (Reynolds,1987) states that isotropic turbulence should not be affected by a pure mean rotation. To satisfy this result, the simplest way is to eliminate the pure rotation term in Eq.(A.7), i.e., $A_2 = 0$, which indicates that $2C_1 = C_2 + C_3$.

To determine the model coefficients, let us first consider a pure strain flow, in which $\Omega_{ij} = 0$. Under this situation,

$$\rho\overline{u_i u_j} = \frac{2}{3}\rho k\delta_{ij} - (C_\mu - \frac{k^2}{\varepsilon^2} II_S A_5)\frac{\rho k^2}{\varepsilon} 2S^*_{ij} - (A_1 + \frac{k}{\varepsilon} I_S A_5)\frac{\rho k^3}{\varepsilon^2} 2S^{(2*)}_{ij} \tag{A.9}$$

In principal axes of S^*_{ij}, we may write (see Shih, Zhu and Lumley, 1995)

$$S^*_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1+a}{2} & 0 \\ 0 & 0 & -\frac{1-a}{2} \end{pmatrix} S^*_{11}, \quad S^{(2*)}_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1+b}{2} & 0 \\ 0 & 0 & -\frac{1-b}{2} \end{pmatrix} S^{(2*)}_{11} \tag{A.10}$$

where a and b can take on arbitrary values. Then, one may write

$$\rho\overline{u_1^2} = \frac{2}{3}\rho k - (C_\mu - \frac{k^2}{\varepsilon^2}II_S A_5)\frac{\rho k^2}{\varepsilon}2S^*_{11} - (A_1 + \frac{k}{\varepsilon}I_S A_5)\frac{\rho k^3}{\varepsilon^2}2S^{(2*)}_{11} \tag{A.11}$$

If we define

$$S^* = \sqrt{S^*_{ij}S^*_{ij}}\ , \quad S^{(2*)} = \sqrt{S^{(2*)}_{ij}S^{(2*)}_{ij}} \tag{A.12}$$

from Eq.(A.10), we obtain

$$S^* = |S^*_{11}|\sqrt{\frac{3+a^2}{2}}\ , \quad S^{(2*)} = |S^{(2*)}_{11}|\sqrt{\frac{3+b^2}{2}} \tag{A.13}$$

Therefore, Eq.(A.11) may be written as

$$\rho\overline{u_1^2} = \frac{2}{3}\rho k - (C_\mu - \frac{k^2}{\varepsilon^2}II_S A_5)\frac{\rho k^2}{\varepsilon}2S^*\sqrt{\frac{2}{3+a^2}} - (A_1 + \frac{k}{\varepsilon}I_S A_5)\frac{\rho k^3}{\varepsilon^2}2S^{(2*)}\sqrt{\frac{2}{3+b^2}} \tag{A.14}$$

Since $\overline{u_1^2} \geq 0$, we must require the following inequality for any large S^* and $S^{(2*)}$

$$1 - (C_\mu - \frac{k^2}{\varepsilon^2}II_S A_5)\frac{k}{\varepsilon}S^*\sqrt{\frac{18}{3+a^2}} - (A_1 + \frac{k}{\varepsilon}I_S A_5)\frac{k^2}{\varepsilon^2}S^{(2*)}\sqrt{\frac{18}{3+b^2}} \geq 0 \tag{A.15}$$

If we write

$$(C_\mu - \frac{k^2}{\varepsilon^2}II_S A_5)\frac{k}{\varepsilon}S^*\sqrt{\frac{18}{3+a^2}} = \alpha, \quad (A_1 + \frac{k}{\varepsilon}I_S A_5)\frac{k^2}{\varepsilon^2}S^{(2*)}\sqrt{\frac{18}{3+b^2}} = \beta \tag{A.16}$$

then we must require

$$\alpha + \beta \leq 1 \tag{A.17}$$

while we write

$$(C_\mu - \frac{k^2}{\varepsilon^2}II_S A_5) = \frac{\alpha}{\frac{kS^*}{\varepsilon}\sqrt{\frac{18}{3+a^2}}}\ , \quad (A_1 + \frac{k}{\varepsilon}I_S A_5) = \frac{\beta}{\frac{k^2S^{(2*)}}{\varepsilon^2}\sqrt{\frac{18}{3+b^2}}} \tag{A.18}$$

Following Shih, Zhu and Lumley (1995), for simplicity we set $\beta = 0$, i.e., $A_1 = -\frac{k}{\varepsilon} I_S A_5$, which indicates $C_2 + C_3 = -\frac{k}{\varepsilon} I_S A_5$. Then, α must be less than unity, i.e.,

$$(C_\mu - \frac{k^2}{\varepsilon^2} II_S A_5) \leq \frac{1}{A_S \frac{kS^*}{\varepsilon}} \tag{A.19}$$

where A_S equals $\sqrt{\frac{18}{3+a^2}}$ and can be calculated using the following relations [see Shih et al (1995), or Reynolds (1987)]:

$$A_S = \sqrt{6}\cos\phi, \quad \phi = \frac{1}{3}\arccos(\sqrt{6}W^*), \quad W^* = \frac{S^*_{ij} S^*_{jk} S^*_{ki}}{(S^*)^3} \tag{A.20}$$

From Eq.(A.19), C_μ can be written as

$$C_\mu \leq \frac{1}{A_S \frac{kS^*}{\varepsilon}} + \frac{k^2}{\varepsilon^2} II_S A_5 \tag{A.21}$$

Now, let us consider a pure shear flow, in which there is only one non-zero component, $U_{1,2}$, i.e.,

$$U_{i,j} = \begin{pmatrix} 0 & U_{1,2} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

In this case, $S_{12} = \Omega_{12} = \frac{1}{2} U_{1,2}$. Under this situation, we obtain from Eq.(A.7)

$$\rho \overline{u_1^2} = \frac{2}{3}\rho k + 2A_3 \frac{\rho k^3}{\varepsilon^2} S_{12}\Omega_{12} \tag{A.22}$$

$$\rho \overline{u_2^2} = \frac{2}{3}\rho k - 2A_3 \frac{\rho k^3}{\varepsilon^2} S_{12}\Omega_{12} \tag{A.23}$$

$$\rho \overline{u_1 u_2} = -2C_\mu \frac{\rho k^2}{\varepsilon} S_{12} \tag{A.24}$$

Note that in Eqs. (A.22)-(A.24), the condition $(A_1 + \frac{k}{\varepsilon} I_S A_5) = 0$ has been used, and note also that A_3 must be positive since the shear $U_{1,2}$ will make $\overline{u_1^2}$ increase and $\overline{u_2^2}$ decrease. Applying

Schwarz' inequality, $(\overline{u_1 u_2})^2 \le \overline{u_1^2}\ \overline{u_2^2}$, to the above equations, we obtain a constraint for A_3:

$$A_3 \le \frac{\sqrt{1 - 9C_\mu^2 \frac{k^2}{\varepsilon^2} S_{12} S_{12}}}{3\frac{k^2}{\varepsilon^2} S_{12}\Omega_{12}} \tag{A.25}$$

Noting that $(S^*)^2 = 2S_{12}S_{12}$ and $\Omega^* S^* = 2\Omega_{12}S_{12}$ for the pure shear flow, a generalized expression for A_3 may be written as

$$A_3 = \frac{\sqrt{1 - \frac{9}{2}C_\mu^2 \left(\frac{kS^*}{\varepsilon}\right)^2}}{\frac{C_0}{2} + \frac{3}{2}\frac{k^2}{\varepsilon^2}\Omega^* S^*}, \quad C_0 \ge 0 \tag{A.26}$$

where

$$\Omega^* = \sqrt{\Omega_{ij}\Omega_{ij}} \tag{A.27}$$

To ensure a positive real value of A_3, the coefficient C_μ must be also restricted by the following condition for any large values of S^*:

$$C_\mu \le \frac{\sqrt{2}}{3}\left(\frac{kS^*}{\varepsilon}\right)^{-1} \tag{A.28}$$

The formulations for C_μ and A_3, i.e., Eqs. (A.21) and (A.26), will ensure realizability of turbulent stresses. However, A_4 and A_5 are left to be further determined, which are related to the coefficients C_4 and C_5 by Eq.(A.8).

To determine A_4 and A_5, or C_4 and C_5, let us study a fully developed rotating pipe flow. In this case, only two components of the non-linear part of turbulent stresses, τ_{xr} and $\tau_{\theta r}$, appear in the mean flow equations, i.e., Eqs. (13) and (14), which are

$$\tau_{xr} = -\, C_4 \frac{\rho k^4}{\varepsilon^3}\frac{W}{r}\left(\frac{\partial W}{\partial r} - \frac{W}{r}\right)\frac{\partial U}{\partial r} \tag{A.29}$$

$$\begin{aligned} \tau_{\theta r} = &-\, C_4 \frac{\rho k^4}{\varepsilon^3} W \frac{\partial W}{\partial r}\left(\frac{\partial W}{\partial r} - \frac{W}{r}\right) \\ &-\, C_5 \frac{\rho k^4}{\varepsilon^3}\left[W\left(\frac{\partial U}{\partial r}\right)^2 + W\frac{\partial W}{\partial r}\left(\frac{\partial W}{\partial r} - \frac{W}{r}\right)\right] \end{aligned} \tag{A.30}$$

Now integrate the Eq. (14) for the velocity W component at a steady state to obtain

$$(\mu + \mu_T) r \left(\frac{\partial W}{\partial r} - \frac{W}{r} \right) - C_4 \frac{\rho k^4}{\varepsilon^3} W \frac{\partial W}{\partial r} \left(\frac{\partial W}{\partial r} - \frac{W}{r} \right)$$
$$- C_5 \frac{\rho k^4}{\varepsilon^3} \left[W \frac{\partial W}{\partial r} \left(\frac{\partial W}{\partial r} - \frac{W}{r} \right) + W \left(\frac{\partial U}{\partial r} \right)^2 \right] = 0 \tag{A.31}$$

Experimental data show that $\frac{W}{W_{wall}} \approx \left(\frac{r}{R}\right)^2$ for a large range of W_{wall}. Here, R is the radius of the pipe, W_{wall} is the wall swirl velocity. Insert this relation into the above equation, we obtain, for high turbulent Reynolds numbers,

$$\mu_T - C_4 \frac{\rho k^4}{\varepsilon^3} 2 \frac{r^2}{R^4} W_{wall}^2 - C_5 \frac{\rho k^4}{\varepsilon^3} \left[2 \frac{r^2}{R^4} W_{wall}^2 + \left(\frac{\partial U}{\partial r} \right)^2 \right] \approx 0 \tag{A.32}$$

If we write

$$C_4 \frac{\rho k^4}{\varepsilon^3} 2 \frac{r^2}{R^4} W_{wall}^2 = \alpha' \ \mu_T, \quad C_5 \frac{\rho k^4}{\varepsilon^3} \left[2 \frac{r^2}{R^4} W_{wall}^2 + \left(\frac{\partial U}{\partial r} \right)^2 \right] = \beta' \ \mu_T \tag{A.33}$$

then from Eq.(A.32), we must require $\alpha' + \beta' \approx 1$. The coefficients C_4 and C_5 can be expressed as

$$C_4 = \frac{\alpha' \ \mu_T}{\frac{\rho k^4}{\varepsilon^3} 2 \frac{r^2 W_{wall}^2}{R^4}}, \quad C_5 = \frac{\beta' \ \mu_T}{\frac{\rho k^4}{\varepsilon^3} \left[2 \frac{r^2 W_{wall}^2}{R^4} + \left(\frac{\partial U}{\partial r} \right)^2 \right]} \tag{A.34}$$

In a fully developed, rotating, pipe flow, we find that the following relations hold,

$$2 \frac{r^2 W_{wall}^2}{R^4} = \frac{1}{2} \left| S_{ij}^* S_{ij}^* - \Omega_{ij} \Omega_{ij} \right| \tag{A.35}$$

$$\left[2 \frac{r^2 W_{wall}^2}{R^4} + \left(\frac{\partial U}{\partial r} \right)^2 \right] = \frac{1}{4} \left(7 S_{ij}^* S_{ij}^* + \Omega_{ij} \Omega_{ij} \right) \tag{A.36}$$

Finally, we obtain expressions for C_4 and C_5 as follows

$$C_4 = \frac{\alpha' \ \mu_T}{\frac{\rho k^4}{\varepsilon^3} \frac{1}{2} \left| (S^*)^2 - (\Omega^*)^2 \right|} \tag{A.37}$$

$$C_5 = \frac{\beta' \ \mu_T}{\frac{\rho k^4}{\varepsilon^3} \frac{1}{4} \left(7 (S^*)^2 + (\Omega^*)^2 \right)} \tag{A.38}$$

From the calculation of rotating pipe flows, we find that the following coefficients seem appropriate (i.e., we set $\alpha' = 0$, $\beta' = 1.6$):

$$C_\mu = \frac{1}{4.0 + A_S \dfrac{kU^*}{\varepsilon}} \tag{A.39}$$

$$C_4 = 0 \tag{A.40}$$

$$C_5 = \frac{1.6\ \mu_T}{\dfrac{\rho k^4}{\varepsilon^3} \dfrac{7(S^*)^2 + (\Omega^*)^2}{4}} \tag{A.41}$$

where

$$U^* = \sqrt{S^*_{ij}S^*_{ij} + \Omega_{ij}\Omega_{ij}}, \quad S^* = \sqrt{S^*_{ij}S^*_{ij}}, \quad \Omega^* = \sqrt{\Omega_{ij}\Omega_{ij}} \tag{A.42}$$

Equations (A.40) and (A.8) suggest that $A_4 = A_5 = C_5$.

Now, we may summarize the cubic model and its coefficients as follows:

$$\begin{aligned} -\rho\overline{u_i u_j} = & -\frac{2}{3}\rho k \delta_{ij} + \mu_T 2S^*_{ij} + A_3 \frac{\rho k^3}{\varepsilon^2}\left(S_{ik}\Omega_{kj} - \Omega_{ik}S_{kj}\right) \\ & -2A_5 \frac{\rho k^4}{\varepsilon^3}\Big(\Omega_{ik}S^2_{kj} - S^2_{ik}\Omega_{kj} + \Omega_{ik}S_{km}\Omega_{mj} \\ & -\frac{1}{3}\Omega_{kl}S_{lm}\Omega_{mk}\delta_{ij} + II_S S^*_{ij}\Big) \end{aligned} \tag{A.43}$$

where

$$\mu_T = Eq.(3) \tag{A.44}$$

$$C_\mu = \frac{1}{4.0 + A_S \dfrac{kU^*}{\varepsilon}} \tag{A.45}$$

$$A_3 = \frac{\sqrt{1 - \dfrac{9}{2}C^2_\mu \left(\dfrac{kS^*}{\varepsilon}\right)^2}}{0.5 + \dfrac{3}{2}\dfrac{k^2}{\varepsilon^2}\Omega^* S^*} \tag{A.46}$$

$$A_5 = \frac{1.6\mu_T}{\dfrac{\rho k^4}{\varepsilon^3} \dfrac{7(S^*)^2 + (\Omega^*)^2}{4}} \tag{A.47}$$

In Eq. (A.43), we have used the fact that $S^*_{ik}\Omega_{kj} - \Omega_{ik}S^*_{kj} = S_{ik}\Omega_{kj} - \Omega_{ik}S_{kj}$ and $\Omega_{ik}S^{(2*)}_{kj} - S^{(2*)}_{ik}\Omega_{kj} = \Omega_{ik}S^2_{kj} - S^2_{ik}\Omega_{kj}$. In addition, C_μ must also be constrained by the conditions from

Eqs. (A.21) and (A.28), i.e.,

$$C_\mu \le \left(A_S \frac{kS^*}{\varepsilon}\right)^{-1} + \frac{k^2}{\varepsilon^2} II_S A_5 \quad \text{and} \quad C_\mu \le \frac{\sqrt{2}}{3}\left(\frac{kS^*}{\varepsilon}\right)^{-1} \tag{A.48}$$

The cubic model can be directly expressed in terms of mean velocity gradients, i.e., Eq. (A.1). The corresponding coefficients are

$$C_1 = -\frac{1}{2}\frac{k}{\varepsilon} I_S A_5 \tag{A.49}$$

$$C_2 = -\frac{1}{2}\left(A_3 + \frac{k}{\varepsilon} I_S A_5\right) \tag{A.50}$$

$$C_3 = \frac{1}{2}\left(A_3 - \frac{k}{\varepsilon} I_S A_5\right) \tag{A.51}$$

$$C_4 = 0 \tag{A.52}$$

$$C_5 = A_5 \tag{A.53}$$

then the cubic model, Eq.(A.43), becomes

$$\begin{aligned} -\rho\overline{u_i u_j} = &-\frac{2}{3}\rho k\ \delta_{ij} + \mu_T\left(U_{i,j} + U_{j,i} - \frac{2}{3}U_{k,k}\ \delta_{ij}\right) \\ &+ \frac{A_3}{2}\frac{\rho k^3}{\varepsilon^2}\left(U_{k,i}U_{k,j} - U_{i,k}U_{j,k}\right) \\ &+ A_5\frac{\rho k^4}{\varepsilon^3}\Big[U_{k,i}U_{k,p}U_{p,j} + U_{k,j}U_{k,p}U_{p,i} - \frac{2}{3}\Pi_3\ \delta_{ij} \\ &\quad -\frac{1}{2}I_S\left(U_{i,k}U_{k,j} + U_{j,k}U_{k,i} - \frac{2}{3}\Pi_1\delta_{ij}\right) \\ &\quad -\frac{1}{2}I_S\left(U_{k,i}U_{k,j} + U_{i,k}U_{j,k} - \frac{2}{3}\Pi_2\delta_{ij}\right)\Big] \end{aligned} \tag{A.54}$$

B Appendix: Equations in a General Coordinate System

In this appendix, a set of mean flow equations with a general cubic model will be written in a general coordinate system. This appendix will be found useful for studying turbulent flows in a curvilinear coordinate system. We start with the governing equations in general tensorial form. In Appendix C, we will write these equations in a cylindrical coordinate system as an example to show how to write the equations and models for a specific curvilinear coordinate system.

B.1 Equations in tensorial form

$$\rho_{,t} + \left(\rho U^j\right)_{,j} = 0 \tag{B.1}$$

$$(\rho U_i)_{,t} + \left(\rho U_i U^j\right)_{,j} = -P_{,i} + g^{jr}\left[\mu\left(U_{i,j} + U_{j,i} - \frac{2}{3}U^k_{,k}g_{ij}\right) - \rho\overline{u_i u_j}\right]_{,r} \tag{B.2}$$

$$(\rho k)_{,t} + \left(\rho U^i k\right)_{,i} = g^{jr}\left[\left(\mu + \frac{\mu_T}{\sigma_k}\right)k_{,j}\right]_{,r} + P_k - \rho\varepsilon \tag{B.3}$$

$$(\rho\varepsilon)_{,t} + \left(\rho U^i\varepsilon\right)_{,i} = g^{jr}\left[\left(\mu + \frac{\mu_T}{\sigma_\varepsilon}\right)\varepsilon_{,j}\right]_{,r} + C_{\varepsilon 1}f_1\frac{\varepsilon}{k}P_k - C_{\varepsilon 2}f_2\rho\frac{\varepsilon^2}{k} + C_{\varepsilon 3}\frac{\mu\mu_T}{\rho}g^{jr}S_{,j}S_{,r} \tag{B.4}$$

where

$$P_k = g^{kj}\left(-\rho\overline{u_i u_j}\right)U^i_{,k}\ , \qquad S = \sqrt{2S^*_{ij}S^*_{ij}} \tag{B.5}$$

The turbulent stress is written in the following form:

$$-\rho\overline{u_i u_j} = -\frac{2}{3}\rho k g_{ij} + \mu_T\left(U_{i,j} + U_{j,i} - \frac{2}{3}U^k_{,k}g_{ij}\right) + \tau_{ij} \tag{B.6}$$

where the subscript "$_{,i}$" denotes a tensorial derivative, g^{ij} and g_{ij} are the two metric tensors of a coordinate system, which are defined in Eq.(B.16). The nonlinear part of the general cubic model, τ_{ij}, is

$$\begin{aligned}\tau_{ij} = \ & C_1\frac{\rho k^3}{\varepsilon^2}\left(U_{i,k}U^k_{,j} + U_{j,k}U^k_{,i} - \frac{2}{3}\Pi_1\ g_{ij}\right)\\ & + C_2\frac{\rho k^3}{\varepsilon^2}\left(g^{kl}U_{i,k}U_{j,l} - \frac{1}{3}\Pi_2\ g_{ij}\right)\\ & + C_3\frac{\rho k^3}{\varepsilon^2}\left(U_{k,i}U^k_{,j} - \frac{1}{3}\Pi_2\ g_{ij}\right)\end{aligned}$$

$$+ C_4 \frac{\rho k^4}{\varepsilon^3} \left(g^{kl} U_{i,k} U_{j,p} U^p_{,l} + g^{kl} U_{i,p} U^p_{,k} U_{j,l} - \frac{2}{3} \Pi_3 \, g_{ij} \right)$$
$$+ C_5 \frac{\rho k^4}{\varepsilon^3} \left(U^k_{,i} U_{k,p} U^p_{,j} + U^k_{,j} U_{k,p} U^p_{,i} - \frac{2}{3} \Pi_3 \, g_{ij} \right) \tag{B.7}$$

where,

$$\Pi_1 = U^i_{,k} U^k_{,i} \ , \quad \Pi_2 = g^{kl} U^i_{,k} U_{i,l} \ , \quad \Pi_3 = g^{kl} U^i_{,k} U_{i,m} U^m_{,l} \tag{B.8}$$

In addition, the often used scalar parameters S^* and Ω^* defined in Eq.(A.42) and W^* in Eq.(A.20) can be written as

$$(S^*)^2 = \frac{1}{2} \left(g^{ij} U_{k,i} U^k_{,j} + U^i_{,j} U^j_{,i} \right) - \frac{1}{3} (U^i_{,i})^2 \tag{B.9}$$
$$(\Omega^*)^2 = \frac{1}{2} \left(g^{ij} U_{k,i} U^k_{,j} - U^i_{,j} U^j_{,i} \right) \tag{B.10}$$
$$W* = g^{ij} g^{kl} g^{mn} \frac{S^*_{ik} S^*_{lm} S^*_{nj}}{(S^*)^3} \tag{B.11}$$

The nonlinear part of turbulent stress τ_{ij}, Eq. (B.7), can also be expressed in terms of mean strain and rotation rates S_{ij} and Ω_{ij} which will be listed in Eq. (B.33).

B.2 Equations in a general coodinate system

Let x^i represent a general curvilinear coordinate system, then the corresponding contravariant velocity is defined as $U^i = \dfrac{dx^i}{dt}$ and the covariant velocity is defined as $U_i = g_{ij} U^j$. To write Eqs. (B.1)-(B.10) in this general coordinate system, we need the following expressions for various tensorial derivatives:

$$A_{i,j} = \frac{\partial A_i}{\partial x^j} - \Gamma^q_{ij} A_q$$
$$A^i_{,j} = \frac{\partial A^i}{\partial x^j} + \Gamma^i_{qj} A^q$$
$$A_{ij,k} = \frac{\partial A_{ij}}{\partial x^k} - \Gamma^q_{ik} A_{qj} - \Gamma^q_{jk} A_{iq}$$
$$A^j_{i,k} = \frac{\partial A^j_i}{\partial x^k} - \Gamma^q_{ik} A^j_q + \Gamma^j_{qk} A^q_i \tag{B.12}$$

where Γ^i_{jk} is a Cristoffel symbol defined in Eq.(B.17). With the above formulations, Eqs. (B.1)-(B.10) can be written as follows

$$\rho_{,t} + \frac{\partial \rho U^j}{\partial x^j} + \Gamma^j_{jn} \, \rho U^n = 0 \tag{B.13}$$

$$
\begin{aligned}
(\rho U_i)_{,t} + \frac{\partial \rho U_i U^j}{\partial x^j} - \Gamma^n_{ij}\rho U_n U^j + \Gamma^j_{jn}\rho U_i U^n = &-\frac{\partial P}{\partial x^i} \\
&+ g^{jr}\frac{\partial}{\partial x^r}\left[(\mu+\mu_T)\left(\frac{\partial U_i}{\partial x^j} + \frac{\partial U_j}{\partial x^i} - \frac{2}{3}\Theta g_{ij}\right)\right] \\
&- g^{jr}\left\{2(\mu+\mu_T)U_n\frac{\partial}{\partial x^r}\Gamma^n_{ij} + 2\Gamma^n_{ij}\frac{\partial}{\partial x^r}\left[(\mu+\mu_T)U_n\right]\right\} \\
&- g^{jr}\Gamma^p_{jr}(\mu+\mu_T)\left(\frac{\partial U_i}{\partial x^p} + \frac{\partial U_p}{\partial x^i} - 2\Gamma^n_{ip}U_n - \frac{2}{3}\Theta g_{ip}\right) \\
&- g^{jr}\Gamma^p_{ir}(\mu+\mu_T)\left(\frac{\partial U_j}{\partial x^p} + \frac{\partial U_p}{\partial x^j} - 2\Gamma^n_{jp}U_n - \frac{2}{3}\Theta g_{jp}\right) \\
&+ g^{jr}\left(\frac{\partial \tau_{ij}}{\partial x^r} - \Gamma^n_{ir}\tau_{nj} - \Gamma^n_{jr}\tau_{in}\right)
\end{aligned}
\tag{B.14}
$$

where

$$
\Theta = U^k_{,k} = \frac{\partial U^k}{\partial x^k} + \Gamma^k_{kn}U^n \tag{B.15}
$$

and

$$
g^{ij} = \frac{\partial x^i}{\partial \mathcal{X}^k}\frac{\partial x^j}{\partial \mathcal{X}^k}, \qquad g_{ij} = \frac{\partial \mathcal{X}^k}{\partial x^i}\frac{\partial \mathcal{X}^k}{\partial x^j} \tag{B.16}
$$

here $\mathcal{X}^k$ denotes the Cartesian coordinte system while x^i represents a general coordinate system. The symbol Γ^i_{jk}, called the Christoffel symbol, is defined as

$$
\Gamma^i_{jk} = \frac{\partial x^i}{\partial \mathcal{X}^p}\frac{\partial}{\partial \mathcal{X}^j}\left(\frac{\partial \mathcal{X}^p}{\partial x^k}\right) \tag{B.17}
$$

The equations for the turbulent kinetic energy, $k = g^{ij}\overline{u_i u_j} = \overline{u_i u^i}$, and its dissipation rate ε can be written as

$$
\begin{aligned}
(\rho k)_{,t} + \frac{\partial \rho U^i k}{\partial x^i} + \Gamma^i_{in}\rho k U^n = &\, g^{jr}\frac{\partial}{\partial x^r}\left[\left(\mu + \frac{\mu_T}{\sigma_k}\right)\frac{\partial k}{\partial x^j}\right] - g^{jr}\Gamma^n_{jr}\left(\mu + \frac{\mu_T}{\sigma_k}\right)\frac{\partial k}{\partial x^n} \\
&+ P_k - \rho\varepsilon
\end{aligned}
\tag{B.18}
$$

$$
\begin{aligned}
(\rho \varepsilon)_{,t} + \frac{\partial \rho U^i \varepsilon}{\partial x^i} + \Gamma^i_{in}\rho \varepsilon U^n = &\, g^{jr}\frac{\partial}{\partial x^r}\left[\left(\mu + \frac{\mu_T}{\sigma_\varepsilon}\right)\frac{\partial \varepsilon}{\partial x^j}\right] - g^{jr}\Gamma^n_{jr}\left(\mu + \frac{\mu_T}{\sigma_\varepsilon}\right)\frac{\partial \varepsilon}{\partial x^n} \\
&+ C_{\varepsilon 1}f_1\frac{\varepsilon}{k}P_k - C_{\varepsilon 2}f_2\frac{\rho\varepsilon^2}{k} + C_{\varepsilon 3}\frac{\mu\mu_T}{\rho}g^{jr}\frac{\partial S}{\partial x_j}\frac{\partial S}{\partial x_r}
\end{aligned}
\tag{B.19}
$$

where

$$P_k = -g^{kj}\rho\overline{u_i u_j}\left(\frac{\partial U^i}{\partial x^k} + \Gamma^i_{kn}U^n\right) \tag{B.20}$$

$$-\rho\overline{u_i u_j} = -\frac{2}{3}\rho k g_{ij} + \mu_T\left[\frac{\partial U_i}{\partial x^j} + \frac{\partial U_j}{\partial x^i} - 2\Gamma^n_{ij}U_n - \frac{2}{3}\Theta g_{ij}\right] + \tau_{ij} \tag{B.21}$$

If we decompose P_k into two parts, one due to the linear part of $-\rho\overline{u_i u_j}$ and the other due to the nonlinear part, then we may write

$$P_k = P_k^{(1)} + P_k^{(2)} \tag{B.22}$$

where

$$P_k^{(1)} = -\frac{2}{3}(\rho k + \mu_T\Theta)\Theta + g^{kj}\mu_T\left(\frac{\partial U_i}{\partial x^j} + \frac{\partial U_j}{\partial x^i} - 2\Gamma^n_{ij}U_n\right)\left(\frac{\partial U^i}{\partial x^k} + \Gamma^i_{mk}U^m\right) \tag{B.23}$$

$$P_k^{(2)} = g^{kj}\tau_{ij}\left(\frac{\partial U^i}{\partial x^k} + \Gamma^i_{mk}U^m\right) \tag{B.24}$$

The nonlinear part of the cubic model, Eq. (B.7), in a general coordinate system is

$$\begin{aligned}
\tau_{ij} = {} & C_1\frac{\rho k^3}{\varepsilon^2}\left(\frac{\partial U_i}{\partial x^k}\frac{\partial U^k}{\partial x^j} + \frac{\partial U_i}{\partial x^k}\Gamma^k_{qj}U^q - \Gamma^q_{ik}U_q\frac{\partial U^k}{\partial x^j} - \Gamma^p_{ik}\Gamma^k_{qj}U_pU^q\right.\\
& \left. + \frac{\partial U_j}{\partial x^k}\frac{\partial U^k}{\partial x^i} + \frac{\partial U_j}{\partial x^k}\Gamma^k_{qi}U^q - \Gamma^q_{jk}U_q\frac{\partial U^k}{\partial x^i} - \Gamma^p_{jk}\Gamma^k_{qi}U_pU^q - \frac{2}{3}\Pi_1 g_{ij}\right)\\
& + C_2\frac{\rho k^3}{\varepsilon^2}\left[g^{kl}\left(\frac{\partial U_i}{\partial x^k}\frac{\partial U_j}{\partial x^l} - \frac{\partial U_i}{\partial x^k}\Gamma^q_{jl}U_q - \Gamma^p_{ik}U_p\frac{\partial U_j}{\partial x^l} - \Gamma^p_{ik}\Gamma^q_{jl}U_pU_q\right) - \frac{1}{3}\Pi_2 g_{ij}\right]\\
& + C_3\frac{\rho k^3}{\varepsilon^2}\left(\frac{\partial U_k}{\partial x^i}\frac{\partial U^k}{\partial x^j} + \frac{\partial U_k}{\partial x^i}\Gamma^k_{qj}U^q - \Gamma^q_{ik}U_q\frac{\partial U^k}{\partial x^j} - \Gamma^p_{ik}\Gamma^k_{qj}U_pU^q - \frac{1}{3}\Pi_2 g_{ij}\right)\\
& + C_4\frac{\rho k^4}{\varepsilon^3}\left[g^{kl}\left(\frac{\partial U_i}{\partial x^k}\frac{\partial U_j}{\partial x^m}\frac{\partial U^m}{\partial x^l} - \frac{\partial U_i}{\partial x^k}\frac{\partial U^m}{\partial x^l}\Gamma^q_{jm}U_q - \Gamma^p_{ik}U_p\frac{\partial U_j}{\partial x^m}\frac{\partial U^m}{\partial x^l}\right.\right.\\
& + \Gamma^p_{ik}\Gamma^q_{jm}U_pU_q\frac{\partial U^m}{\partial x^l} + \frac{\partial U_j}{\partial x^k}\frac{\partial U_i}{\partial x^m}\frac{\partial U^m}{\partial x^l} - \frac{\partial U_j}{\partial x^k}\frac{\partial U^m}{\partial x^l}\Gamma^q_{im}U_q\\
& - \Gamma^p_{jk}U_p\frac{\partial U_i}{\partial x^m}\frac{\partial U^m}{\partial x^l} + \Gamma^p_{jk}\Gamma^q_{im}U_pU_q\frac{\partial U^m}{\partial x^l} + \frac{\partial U_i}{\partial x^k}\frac{\partial U_j}{\partial x^m}\Gamma^m_{rl}U^r\\
& - \frac{\partial U_i}{\partial x^k}\Gamma^q_{jm}U_q\Gamma^m_{rl}U^r - \frac{\partial U_j}{\partial x^m}\Gamma^q_{ik}U_q\Gamma^m_{rl}U^r + \Gamma^p_{ik}\Gamma^q_{jm}\Gamma^m_{rl}U_pU_qU^r\\
& + \frac{\partial U_j}{\partial x^k}\frac{\partial U_i}{\partial x^m}\Gamma^m_{rl}U^r - \frac{\partial U_j}{\partial x^k}\Gamma^q_{im}U_q\Gamma^m_{rl}U^r - \frac{\partial U_i}{\partial x^m}\Gamma^q_{jk}U_q\Gamma^m_{rl}U^r\\
& \left.\left. + \Gamma^p_{jk}\Gamma^q_{im}\Gamma^m_{rl}U_pU_qU^r\right) - \frac{2}{3}\Pi_3 g_{ij}\right]
\end{aligned}$$

$$
\begin{aligned}
&+ C_5 \frac{\rho k^4}{\varepsilon^3} \left(\frac{\partial U^k}{\partial x^i} \frac{\partial U_k}{\partial x^l} \frac{\partial U^l}{\partial x^j} - \frac{\partial U^k}{\partial x^i} \Gamma^p_{kl} U_p \frac{\partial U^l}{\partial x^j} + \Gamma^k_{qi} U^q \frac{\partial U_k}{\partial x^l} \frac{\partial U^l}{\partial x^j} \right. \\
&\quad - \Gamma^k_{qi} \Gamma^p_{kl} U_p U^q \frac{\partial U^l}{\partial x^j} + \frac{\partial U^k}{\partial x^j} \frac{\partial U_k}{\partial x^l} \frac{\partial U^l}{\partial x^i} - \frac{\partial U^k}{\partial x^j} \Gamma^p_{kl} U_p \frac{\partial U^l}{\partial x^i} \\
&\quad + \Gamma^k_{qj} U^q \frac{\partial U_k}{\partial x^l} \frac{\partial U^l}{\partial x^i} - \Gamma^k_{qj} \Gamma^p_{kl} U_p U^q \frac{\partial U^l}{\partial x^i} + \frac{\partial U^k}{\partial x^i} \frac{\partial U_k}{\partial x^l} \Gamma^l_{rj} U^r \\
&\quad - \frac{\partial U^k}{\partial x^i} \Gamma^p_{kl} \Gamma^l_{rj} U_p U^r + \frac{\partial U_k}{\partial x^l} \Gamma^k_{qi} \Gamma^l_{rj} U^q U^r - \Gamma^k_{qi} \Gamma^p_{kl} \Gamma^l_{rj} U_p U^q U^r \\
&\quad + \frac{\partial U^k}{\partial x^j} \frac{\partial U_k}{\partial x^l} \Gamma^l_{ri} U^r - \frac{\partial U^k}{\partial x^j} \Gamma^p_{kl} \Gamma^l_{ri} U_p U^r + \frac{\partial U_k}{\partial x^l} \Gamma^k_{qj} \Gamma^l_{ri} U^q U^r \\
&\quad \left. - \Gamma^k_{qj} \Gamma^p_{kl} \Gamma^l_{ri} U_p U^q U^r - \frac{2}{3} \Pi_3 g_{ij} \right) \qquad \text{(B.25)}
\end{aligned}
$$

where

$$
\Pi_1 = \left(\frac{\partial U^i}{\partial x^k} + \Gamma^i_{pk} U^p \right) \left(\frac{\partial U^k}{\partial x^i} + \Gamma^k_{qi} U^q \right) \qquad \text{(B.26)}
$$

$$
\Pi_2 = g^{kl} \left(\frac{\partial U^i}{\partial x^k} + \Gamma^i_{pk} U^p \right) \left(\frac{\partial U_i}{\partial x^l} - \Gamma^q_{il} U_q \right) \qquad \text{(B.27)}
$$

$$
\Pi_3 = g^{kl} \left(\frac{\partial U^i}{\partial x^k} + \Gamma^i_{pk} U^p \right) \left(\frac{\partial U_i}{\partial x^m} - \Gamma^q_{im} U_q \right) \left(\frac{\partial U^m}{\partial x^l} + \Gamma^m_{nl} U^n \right) \qquad \text{(B.28)}
$$

The scalar strain and rotation rates are

$$
\begin{aligned}
(S^*)^2 &= \frac{1}{2} \left[g^{ij} \left(\frac{\partial U_k}{\partial x^i} - \Gamma^q_{ki} U_q \right) \left(\frac{\partial U^k}{\partial x^j} + \Gamma^k_{lj} U^l \right) + \left(\frac{\partial U^i}{\partial x^j} + \Gamma^i_{qj} U^q \right) \left(\frac{\partial U^j}{\partial x^i} + \Gamma^j_{li} U^l \right) \right] \\
&\quad - \frac{1}{3} \left(\frac{\partial U^i}{\partial x^i} + \Gamma^i_{li} U^l \right)^2 \qquad \text{(B.29)} \\
(\Omega^*)^2 &= \frac{1}{2} \left[g^{ij} \left(\frac{\partial U_k}{\partial x^i} - \Gamma^q_{ki} U_q \right) \left(\frac{\partial U^k}{\partial x^j} + \Gamma^k_{lj} U^l \right) \right. \\
&\quad \left. - \left(\frac{\partial U^i}{\partial x^j} + \Gamma^i_{qj} U^q \right) \left(\frac{\partial U^j}{\partial x^i} + \Gamma^j_{li} U^l \right) \right] \qquad \text{(B.30)} \\
W^* &= Eq.(B.39) \qquad \text{(B.31)}
\end{aligned}
$$

B.3 Another form of the cubic model

In terms of strain and rotation rates, the cubic Reynolds stress model can be written as

$$
-\rho \overline{u_i u_j} = -\frac{2}{3} \rho\, k\, g_{ij} + C_\mu \frac{\rho k^2}{\varepsilon} 2 \left(S_{ij} - \frac{1}{3} \Theta\, g_{ij} \right) + \tau_{ij} \qquad \text{(B.32)}
$$

where the nonlinear part, τ_{ij}, is

$$
\begin{aligned}
\tau_{ij} = & 2A_1 \frac{\rho k^3}{\varepsilon^2}\left(g^{pq}S_{ip}S_{qj} - \frac{1}{3}S^{(2)}g_{ij}\right) + 2A_2\frac{\rho k^3}{\varepsilon^2}\left(g^{pq}\Omega_{ip}\Omega_{qj} - \frac{1}{3}\Omega^{(2)}g_{ij}\right) \\
& + A_3\frac{\rho k^3}{\varepsilon^2}g^{pq}\left(S_{ip}\Omega_{qj} - \Omega_{ip}S_{qj}\right) + 2A_4\rho\frac{k^4}{\varepsilon^3}g^{pq}g^{rs}\left(S_{ip}S_{qr}\Omega_{sj} - \Omega_{ip}S_{qr}S_{sj}\right) \\
& - 2A_5\frac{\rho k^4}{\varepsilon^3}\left[g^{pq}g^{rs}\Omega_{ip}S_{qr}\Omega_{sj} - \frac{1}{3}\overline{\Omega S\Omega}\, g_{ij} + II_S\left(S_{ij} - \frac{1}{3}\Theta\, g_{ij}\right)\right. \\
& \left. - I_S\left(g^{pq}S_{ip}S_{qj} - \frac{1}{3}S^{(2)}g_{ij}\right)\right] \qquad \text{(B.33)}
\end{aligned}
$$

where

$$
S_{ij} = \frac{1}{2}\left(\frac{\partial U_i}{\partial x^j} + \frac{\partial U_j}{\partial x^i} - 2\Gamma^k_{ij}U_k\right)\ ,\ \Omega_{ij} = \frac{1}{2}\left(\frac{\partial U_i}{\partial x^j} - \frac{\partial U_j}{\partial x^i}\right) \qquad \text{(B.34)}
$$

$$
\Theta = g^{pq}S_{pq}\ ,\ S^{(2)} = g^{pq}g^{rs}S_{pr}S_{sq}\ ,\ \Omega^{(2)} = g^{pq}g^{rs}\Omega_{pr}\Omega_{sq} \qquad \text{(B.35)}
$$

$$
\overline{\Omega S\Omega} = g^{pq}g^{rs}g^{tn}\Omega_{pr}S_{st}\Omega_{nq}\ ,\ I_S = \Theta\ ,\ II_S = \frac{1}{2}\left(\Theta^2 - S^{(2)}\right) \qquad \text{(B.36)}
$$

and

$$
(S^*)^2 = g^{kp}g^{lq}S_{kl}S_{pq} - \frac{1}{3}\Theta^2 \qquad \text{(B.37)}
$$

$$
(\Omega^*)^2 = g^{kp}g^{lq}\Omega_{kl}\Omega_{pq} \qquad \text{(B.38)}
$$

$$
W* = g^{ij}g^{kl}g^{mn}\frac{S^*_{ik}S^*_{lm}S^*_{nj}}{(S^*)^3} \qquad \text{(B.39)}
$$

where,

$$
S^*_{ij} = S_{ij} - \frac{1}{3}g_{ij}
$$

Note that Eq. (B.33) appears to be more compact than Eq. (B.25) and may bring some convenience for the CFD implementation.

C Appendix: Equations in Cylindrical Coordinates

C.1 Mean equations

Now, let us write all equations in a cylindrical coordinate system: $x^i = (x, r, \theta)$. To accomplish this, we need to calculate the metric tensors g^{ij}, g_{ij} and the Christoffel symbol Γ^i_{jk} for cylindrical coordinates. Let $\mathcal{X}^i = (x, y, z)$ be the cartesian system. The relation between the two systems is

$$x = x \; , \qquad y = r\; cos\theta \; , \qquad z = r\; sin\theta \tag{C.1}$$

or

$$x = x \; , \qquad r = \sqrt{y^2 + z^2} \; , \qquad \theta = \arctan{(z/y)} \tag{C.2}$$

We may easily calculate

$$\frac{\partial x^i}{\partial \mathcal{X}^j} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos\theta & sin\theta \\ 0 & -sin\theta/r & cos\theta/r \end{bmatrix} , \quad \frac{\partial \mathcal{X}^i}{\partial x^j} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos\theta & -r\; sin\theta \\ 0 & sin\theta & r\; cos\theta \end{bmatrix} \tag{C.3}$$

The metric tensors g^{ij} and g_{ij} can then be obtained according to Eq. (B.16):

$$g^{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/r^2 \end{bmatrix} , \quad g_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & r^2 \end{bmatrix} \tag{C.4}$$

and the Christoffel symbol Γ^i_{jk} can be obtained from Eq.(B.17)

$$\Gamma^1_{jk} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} , \; \Gamma^2_{jk} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -r \end{bmatrix} , \; \Gamma^3_{jk} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1/r \\ 0 & 1/r & 0 \end{bmatrix} \tag{C.5}$$

The contravariant velocity in the cylindrical coordinates is

$$U^i = (U, V, \Omega) \tag{C.6}$$

where U and V are the axial and radial velocities, Ω is the angular velocity. The corresponding covariant velocity can be obtained from

$$U_i = g_{ij} U^j = (U, V, r^2\Omega) \tag{C.7}$$

With Eqs.(C.1)-(C.7), the equations for turbulent flows in a cylindrical coordinate system become

Continuity equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho U}{\partial x} + \frac{\partial \rho V}{\partial r} + \frac{\partial \rho \Omega}{\partial \theta} + \frac{\rho V}{r} = 0 \tag{C.8}$$

Momentum equations

$$\begin{aligned}
\frac{\partial \rho U}{\partial t} + \frac{\partial \rho U^2}{\partial x} + \frac{\partial \rho UV}{\partial r} + \frac{\partial \rho U\Omega}{\partial \theta} + \frac{\rho UV}{r} = -\frac{\partial \overline{P}}{\partial x} \\
+ \frac{\partial}{\partial x}\left[2(\mu+\mu_T)\left(\frac{\partial U}{\partial x} - \frac{1}{3}\Theta\right)\right] + \frac{\partial}{\partial r}\left[(\mu+\mu_T)\left(\frac{\partial U}{\partial r} + \frac{\partial V}{\partial x}\right)\right] \\
+ \frac{\partial}{r^2\partial \theta}\left[(\mu+\mu_T)\left(\frac{\partial U}{\partial \theta} + \frac{\partial r^2\Omega}{\partial x}\right)\right] + \frac{1}{r}(\mu+\mu_T)\left(\frac{\partial U}{\partial r} + \frac{\partial V}{\partial x}\right) \\
+ \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xr}}{\partial r} + \frac{1}{r^2}\frac{\partial \tau_{x\theta}}{\partial \theta} + \frac{1}{r}\tau_{xr}
\end{aligned} \tag{C.9}$$

$$\begin{aligned}
\frac{\partial \rho V}{\partial t} + \frac{\partial \rho UV}{\partial x} + \frac{\partial \rho V^2}{\partial r} + \frac{\partial \rho V\Omega}{\partial \theta} - r\rho\Omega^2 + \frac{\rho V^2}{r} = -\frac{\partial \overline{P}}{\partial r} \\
+ \frac{\partial}{\partial x}\left[(\mu+\mu_T)\left(\frac{\partial V}{\partial x} + \frac{\partial U}{\partial r}\right)\right] + \frac{\partial}{\partial r}\left[2(\mu+\mu_T)\left(\frac{\partial V}{\partial r} - \frac{1}{3}\Theta\right)\right] \\
+ \frac{\partial}{r^2\partial \theta}\left[(\mu+\mu_T)\left(\frac{\partial V}{\partial \theta} + \frac{\partial r^2\Omega}{\partial r}\right)\right] - \frac{2}{r}\frac{\partial}{\partial \theta}\left[(\mu+\mu_T)\Omega\right] \\
+ \frac{2}{r}(\mu+\mu_T)\left(\frac{\partial V}{\partial r} - \frac{1}{3}\Theta\right) - \frac{2}{r^3}(\mu+\mu_T)\left(\frac{\partial r^2\Omega}{\partial \theta} + rV - \frac{1}{3}r^2\Theta\right) \\
+ \frac{\partial \tau_{rx}}{\partial x} + \frac{\partial \tau_{rr}}{\partial r} + \frac{1}{r^2}\frac{\partial \tau_{r\theta}}{\partial \theta} - \frac{1}{r^3}\tau_{\theta\theta} + \frac{1}{r}\tau_{rr}
\end{aligned} \tag{C.10}$$

$$\begin{aligned}
\frac{\partial \rho r^2\Omega}{\partial t} + \frac{\partial \rho r^2\Omega U}{\partial x} + \frac{\partial \rho r^2\Omega V}{\partial r} + \frac{\partial \rho r^2\Omega^2}{\partial \theta} + r\rho V\Omega = -\frac{\partial \overline{P}}{\partial \theta} \\
+ \frac{\partial}{\partial x}\left[(\mu+\mu_T)\left(\frac{\partial r^2\Omega}{\partial x} + \frac{\partial U}{\partial \theta}\right)\right] + \frac{\partial}{\partial r}\left[(\mu+\mu_T)\left(\frac{\partial r^2\Omega}{\partial r} + \frac{\partial V}{\partial \theta}\right)\right] \\
+ \frac{\partial}{r^2\partial \theta}\left[2(\mu+\mu_T)\left(\frac{\partial r^2\Omega}{\partial \theta} - \frac{1}{3}r^2\Theta\right)\right] + \frac{2}{r}\frac{\partial}{\partial \theta}\left[(\mu+\mu_T)V\right] \\
+ \frac{1}{r}(\mu+\mu_T)\left(\frac{\partial r^2\Omega}{\partial r} + \frac{\partial V}{\partial \theta}\right) - \frac{2}{r}\frac{\partial}{\partial r}\left[(\mu+\mu_T)r^2\Omega\right] \\
+ \frac{\partial \tau_{\theta x}}{\partial x} + \frac{\partial \tau_{\theta r}}{\partial r} + \frac{1}{r^2}\frac{\partial \tau_{\theta\theta}}{\partial \theta} + \frac{1}{r}\tau_{r\theta}
\end{aligned} \tag{C.11}$$

where

$$\Theta = \frac{\partial U}{\partial x} + \frac{\partial V}{\partial r} + \frac{\partial \Omega}{\partial \theta} + \frac{V}{r} \tag{C.12}$$

$$\overline{P} = P + \frac{2}{3}k \tag{C.13}$$

<u>*k*-ε equations in Cylindrical coordinates</u>

$$\begin{aligned}
\frac{\partial \rho k}{\partial t} + \frac{\partial \rho U k}{\partial x} + \frac{\partial \rho V k}{\partial r} + \frac{\partial \rho \Omega k}{\partial \theta} + \frac{V}{r}\rho k &= \frac{\partial}{\partial x}\left[\left(\mu + \frac{\mu_T}{\sigma_k}\right)\frac{\partial k}{\partial x}\right] \\
&+ \frac{\partial}{\partial r}\left[\left(\mu + \frac{\mu_T}{\sigma_k}\right)\frac{\partial k}{\partial r}\right] + \frac{1}{r^2}\frac{\partial}{\partial \theta}\left[\left(\mu + \frac{\mu_T}{\sigma_k}\right)\frac{\partial k}{\partial \theta}\right] \\
&+ \frac{1}{r}\left(\mu + \frac{\mu_T}{\sigma_k}\right)\frac{\partial k}{\partial r} + P_k - \rho\varepsilon
\end{aligned} \tag{C.14}$$

$$\begin{aligned}
\frac{\partial \rho \varepsilon}{\partial t} + \frac{\partial \rho U \varepsilon}{\partial x} + \frac{\partial \rho V \varepsilon}{\partial r} + \frac{\partial \rho \Omega \varepsilon}{\partial \theta} + \frac{V}{r}\rho \varepsilon &= \frac{\partial}{\partial x}\left[\left(\mu + \frac{\mu_T}{\sigma_\varepsilon}\right)\frac{\partial \varepsilon}{\partial x}\right] \\
&+ \frac{\partial}{\partial r}\left[\left(\mu + \frac{\mu_T}{\sigma_\varepsilon}\right)\frac{\partial \varepsilon}{\partial r}\right] + \frac{1}{r^2}\frac{\partial}{\partial \theta}\left[\left(\mu + \frac{\mu_T}{\sigma_\varepsilon}\right)\frac{\partial \varepsilon}{\partial \theta}\right] \\
&+ \frac{1}{r}\left(\mu + \frac{\mu_T}{\sigma_\varepsilon}\right)\frac{\partial \varepsilon}{\partial r} + C_{\varepsilon 1} f_1 \frac{\varepsilon}{k} P_k - C_{\varepsilon 2} f_2 \frac{\rho \varepsilon^2}{k} \\
&+ C_{\varepsilon 3}\frac{\mu \mu_T}{\rho}\left[\left(\frac{\partial S}{\partial r}\right)^2 + \left(\frac{\partial S}{\partial x}\right)^2 + \left(\frac{\partial S}{r \partial \theta}\right)^2\right]
\end{aligned} \tag{C.15}$$

where

$$P_k = P_k^{(1)} + P_k^{(2)} \tag{C.16}$$

$$\begin{aligned}
P_k^{(1)} &= -\frac{2}{3}(\rho k + \mu_T \Theta)\Theta \\
&+ \mu_T\left[2\left(\frac{\partial U}{\partial x}\right)^2 + \frac{\partial V}{\partial x}\left(\frac{\partial V}{\partial x} + \frac{\partial U}{\partial r}\right) + \frac{\partial \Omega}{\partial x}\left(r^2\frac{\partial \Omega}{\partial x} + \frac{\partial U}{\partial \theta}\right)\right] \\
&+ \mu_T\left[\frac{\partial U}{\partial r}\left(\frac{\partial U}{\partial r} + \frac{\partial V}{\partial x}\right) + 2\left(\frac{\partial V}{\partial r}\right)^2 + \frac{\partial \Omega}{\partial r}\left(r^2\frac{\partial \Omega}{\partial r} + \frac{\partial V}{\partial \theta}\right)\right] \\
&+ \frac{\mu_T}{r^2}\left[\frac{\partial U}{\partial \theta}\left(\frac{\partial U}{\partial \theta} + r^2\frac{\partial \Omega}{\partial x}\right) + \frac{\partial V}{\partial \theta}\left(\frac{\partial V}{\partial \theta} + r^2\frac{\partial \Omega}{\partial r}\right)\right. \\
&\left. + 2\left(\frac{V}{r} + \frac{\partial \Omega}{\partial \theta}\right)\left(r^2\frac{\partial \Omega}{\partial \theta} + rV\right)\right]
\end{aligned} \tag{C.17}$$

$$\begin{aligned}
P_k^{(2)} &= \tau_{xx}\frac{\partial U}{\partial x} + \tau_{rx}\frac{\partial V}{\partial x} + \tau_{\theta x}\frac{\partial \Omega}{\partial x} + \tau_{xr}\frac{\partial U}{\partial r} + \tau_{rr}\frac{\partial V}{\partial r} + \tau_{\theta r}\left(\frac{\partial \Omega}{\partial r} + \frac{\Omega}{r}\right) \\
&+ \frac{1}{r^2}\left[\tau_{x\theta}\frac{\partial U}{\partial \theta} + \tau_{r\theta}\left(\frac{\partial V}{\partial \theta} - r\Omega\right) + \tau_{\theta\theta}\left(\frac{\partial \Omega}{\partial \theta} + \frac{V}{r}\right)\right]
\end{aligned} \tag{C.18}$$

After g^{ij}, g_{ij} and Γ^k_{ij} for the cylindrical coordinate system are calculated, we may use Eq. (B.25) or Eq. (B.33) to calculate all the turbulent stresses automatically through a computer program. However, in the cylindrical coordinate system, most components of g^{ij}, g_{ij} and Γ^k_{ij} are zero, therefore it is possiple to manually write down all the turbulent stresses to avoid many unnecessary null operations in the computer code. We write them here in a general form for the cubic model, so that model users can use their particular model coefficients for their applications. Note that with Eq. (A.8), the coefficients C_i can be easily obtained from A_i, or vice versa.

$$
\begin{aligned}
\tau_{xx} &= C_1\frac{\rho k^3}{\varepsilon^2}\left[2\left(\frac{\partial U}{\partial x}\right)^2 + 2\frac{\partial U}{\partial r}\frac{\partial V}{\partial x} + 2\frac{\partial U}{\partial \theta}\frac{\partial \Omega}{\partial x} - \frac{2}{3}\Pi_1\right] \\
&+ C_2\frac{\rho k^3}{\varepsilon^2}\left[\left(\frac{\partial U}{\partial x}\right)^2 + \left(\frac{\partial U}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial U}{\partial \theta}\right)^2 - \frac{1}{3}\Pi_2\right] \\
&+ C_3\frac{\rho k^3}{\varepsilon^2}\left[\left(\frac{\partial U}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial x}\right)^2 + r^2\left(\frac{\partial \Omega}{\partial x}\right)^2 - \frac{1}{3}\Pi_2\right] \\
&+ C_4\frac{\rho k^4}{\varepsilon^3}\left[T^{xx}_{19} + T^{xx}_{20} + \ldots + T^{xx}_{33} + T^{xx}_{34} - \frac{2}{3}\Pi_3\right] \\
&+ C_5\frac{\rho k^4}{\varepsilon^3}\left[T^{xx}_{36} + T^{xx}_{37} + \ldots + T^{xx}_{50} + T^{xx}_{51} - \frac{2}{3}\Pi_3\right] \\
\tau_{xr} &= C_1\frac{\rho k^3}{\varepsilon^2}\left[\frac{\partial U}{\partial x}\frac{\partial U}{\partial r} + \frac{\partial U}{\partial r}\frac{\partial V}{\partial r} + \frac{\partial U}{\partial \theta}\frac{\partial \Omega}{\partial r} + \frac{\partial U}{\partial \theta}\frac{\Omega}{r}\right. \\
&\left. + \frac{\partial U}{\partial x}\frac{\partial V}{\partial x} + \frac{\partial V}{\partial r}\frac{\partial V}{\partial x} + \frac{\partial V}{\partial \theta}\frac{\partial \Omega}{\partial x} - r\Omega\frac{\partial \Omega}{\partial x}\right] \\
&+ C_2\frac{\rho k^3}{\varepsilon^2}\left[\frac{\partial U}{\partial x}\frac{\partial V}{\partial x} + \frac{\partial U}{\partial r}\frac{\partial V}{\partial r} + \frac{1}{r^2}\frac{\partial U}{\partial \theta}\frac{\partial V}{\partial \theta} - \frac{\partial U}{\partial \theta}\frac{\Omega}{r}\right] \\
&+ C_3\frac{\rho k^3}{\varepsilon^2}\left[\frac{\partial U}{\partial x}\frac{\partial U}{\partial r} + \frac{\partial U}{\partial x}\frac{\partial V}{\partial r} + r\Omega\frac{\partial \Omega}{\partial x} + r^2\frac{\partial \Omega}{\partial x}\frac{\partial \Omega}{\partial r}\right] \\
&+ C_4\frac{\rho k^4}{\varepsilon^3}\left[T^{xr}_{19} + T^{xr}_{20} + \ldots + T^{xr}_{33} + T^{xr}_{34}\right] \\
&+ C_5\frac{\rho k^4}{\varepsilon^3}\left[T^{xr}_{36} + T^{xr}_{37} + \ldots + T^{xr}_{50} + T^{xr}_{51}\right] \\
\tau_{x\theta} &= C_1\frac{\rho k^3}{\varepsilon^2}\left[\frac{\partial U}{\partial x}\frac{\partial U}{\partial \theta} + \frac{\partial U}{\partial r}\frac{\partial V}{\partial \theta} + \frac{\partial U}{\partial \theta}\frac{\partial \Omega}{\partial \theta} + \frac{\partial U}{\partial \theta}\frac{V}{r} - r\Omega\left(\frac{\partial U}{\partial r} + \frac{\partial V}{\partial x}\right)\right. \\
&\left. + rV\frac{\partial \Omega}{\partial x} + r^2\frac{\partial \Omega}{\partial x}\left(\frac{\partial U}{\partial x} + \frac{\partial \Omega}{\partial \theta}\right) + \frac{\partial r^2\Omega}{\partial r}\frac{\partial V}{\partial x}\right] \\
&+ C_2\frac{\rho k^3}{\varepsilon^2}\left[r^2\frac{\partial U}{\partial x}\frac{\partial \Omega}{\partial x} + \frac{\partial U}{\partial \theta}\frac{\partial \Omega}{\partial \theta} + \frac{\partial U}{\partial r}\frac{\partial r^2\Omega}{\partial r} - r\Omega\frac{\partial U}{\partial r} + \frac{V}{r}\frac{\partial U}{\partial \theta}\right]
\end{aligned}
$$

$$
\begin{aligned}
&+ && C_3\frac{\rho k^3}{\varepsilon^2}\left[\frac{\partial U}{\partial x}\frac{\partial U}{\partial \theta}+\frac{\partial V}{\partial x}\frac{\partial V}{\partial \theta}+r^2\frac{\partial \Omega}{\partial x}\frac{\partial \Omega}{\partial \theta}+rV\frac{\partial \Omega}{\partial x}-r\Omega\frac{\partial V}{\partial x}\right]\\
&+ && C_4\frac{\rho k^4}{\varepsilon^3}\left[T^{x\theta}_{19}+T^{x\theta}_{20}+\ldots+T^{x\theta}_{33}+T^{x\theta}_{34}\right]\\
&+ && C_5\frac{\rho k^4}{\varepsilon^3}\left[T^{x\theta}_{36}+T^{x\theta}_{37}+\ldots+T^{x\theta}_{50}+T^{x\theta}_{51}\right]\\
\tau_{rr} &= && C_1\frac{\rho k^3}{\varepsilon^2}\left[2\left(\frac{\partial V}{\partial x}\frac{\partial U}{\partial r}+\frac{\partial V}{\partial r}\frac{\partial V}{\partial r}+\frac{\partial V}{\partial \theta}\frac{\partial \Omega}{\partial r}\right)+2\frac{\Omega}{r}\frac{\partial V}{\partial \theta}-2r\Omega\frac{\partial \Omega}{\partial r}-2\Omega^2-\frac{2}{3}\Pi_1\right]\\
&+ && C_2\frac{\rho k^3}{\varepsilon^2}\left[\left(\frac{\partial V}{\partial x}\right)^2+\left(\frac{\partial V}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial V}{\partial \theta}\right)^2-2\frac{\Omega}{r}\frac{\partial V}{\partial \theta}-\Omega^2-\frac{1}{3}\Pi_2\right]\\
&+ && C_3\frac{\rho k^3}{\varepsilon^2}\left[\left(\frac{\partial U}{\partial r}\right)^2+\left(\frac{\partial V}{\partial r}\right)^2+\frac{\partial r^2\Omega}{\partial r}\frac{\partial \Omega}{\partial r}+\Omega^2-\frac{1}{3}\Pi_2\right]\\
&+ && C_4\frac{\rho k^4}{\varepsilon^3}\left[T^{rr}_{19}+T^{rr}_{20}+\ldots+T^{rr}_{33}+T^{rr}_{34}-\frac{2}{3}\Pi_3\right]\\
&+ && C_5\frac{\rho k^4}{\varepsilon^3}\left[T^{rr}_{36}+T^{rr}_{37}+\ldots+T^{rr}_{50}+T^{rr}_{51}-\frac{2}{3}\Pi_3\right]\\
\tau_{r\theta} &= && C_1\frac{\rho k^3}{\varepsilon^2}\left[\frac{\partial V}{\partial x}\frac{\partial U}{\partial \theta}+\frac{\partial V}{\partial r}\frac{\partial V}{\partial \theta}+\frac{\partial V}{\partial \theta}\frac{\partial \Omega}{\partial \theta}-2r\Omega\frac{\partial V}{\partial r}+\frac{V}{r}\frac{\partial V}{\partial \theta}\right.\\
& && \left.+r^2\frac{\partial \Omega}{\partial x}\frac{\partial U}{\partial r}+\frac{\partial r^2\Omega}{\partial r}\frac{\partial V}{\partial r}+r^2\frac{\partial \Omega}{\partial \theta}\frac{\partial \Omega}{\partial r}+rV\frac{\partial \Omega}{\partial r}\right]\\
&+ && C_2\frac{\rho k^3}{\varepsilon^2}\left[r^2\frac{\partial V}{\partial x}\frac{\partial \Omega}{\partial x}+\frac{\partial V}{\partial r}\frac{\partial r^2\Omega}{\partial r}+\frac{\partial V}{\partial \theta}\frac{\partial \Omega}{\partial \theta}-r\Omega\left(\frac{\partial V}{\partial r}+\frac{\partial \Omega}{\partial \theta}\right)+\frac{V}{r}\frac{\partial V}{\partial \theta}+\Omega V\right]\\
&+ && C_3\frac{\rho k^3}{\varepsilon^2}\left[\frac{\partial U}{\partial r}\frac{\partial U}{\partial \theta}+\frac{\partial V}{\partial r}\frac{\partial V}{\partial \theta}+\frac{\partial r^2\Omega}{\partial r}\frac{\partial \Omega}{\partial \theta}+rV\frac{\partial \Omega}{\partial r}-r\Omega\left(\frac{\partial V}{\partial r}+\frac{\partial \Omega}{\partial \theta}\right)+\Omega V\right]\\
&+ && C_4\frac{\rho k^4}{\varepsilon^3}\left[T^{r\theta}_{19}+T^{r\theta}_{20}+\ldots+T^{r\theta}_{33}+T^{r\theta}_{34}\right]\\
&+ && C_5\frac{\rho k^4}{\varepsilon^3}\left[T^{r\theta}_{36}+T^{r\theta}_{37}+\ldots+T^{r\theta}_{50}+T^{r\theta}_{51}\right]\\
\tau_{\theta\theta} &= && C_1\frac{\rho k^3}{\varepsilon^2}\left[2\left(r^2\frac{\partial \Omega}{\partial x}\frac{\partial U}{\partial \theta}+\frac{\partial r^2\Omega}{\partial r}\frac{\partial V}{\partial \theta}+r^2\frac{\partial \Omega}{\partial \theta}\frac{\partial \Omega}{\partial \theta}\right)+2r^2\frac{\partial \Omega}{\partial \theta}\frac{V}{r}\right.\\
& && \left.-2r\Omega\left(\frac{\partial r^2\Omega}{\partial r}+\frac{\partial V}{\partial \theta}\right)+2rV\frac{\partial \Omega}{\partial \theta}+2V^2+2r^2\Omega^2-\frac{2}{3}r^2\Pi_1\right]\\
&+ && C_2\frac{\rho k^3}{\varepsilon^2}\left[\left(r^2\frac{\partial \Omega}{\partial x}\right)^2+\left(\frac{\partial r^2\Omega}{\partial r}\right)^2+r^2\left(\frac{\partial \Omega}{\partial \theta}\right)^2-2r\Omega\frac{\partial r^2\Omega}{\partial r}\right.\\
& && \left.+2rV\frac{\partial \Omega}{\partial \theta}-r^2\Omega^2-V^2-\frac{1}{3}r^2\Pi_2\right]\\
&+ && C_3\frac{\rho k^3}{\varepsilon^2}\left[\left(\frac{\partial U}{\partial \theta}\right)^2+\left(\frac{\partial V}{\partial \theta}\right)^2+r^2\left(\frac{\partial \Omega}{\partial \theta}\right)^2+2rV\frac{\partial \Omega}{\partial \theta}\right.
\end{aligned}
$$

$$-2r\Omega\frac{\partial V}{\partial \theta}+V^2+r^2\Omega^2-\frac{1}{3}r^2\Pi_2\Big]$$

$$+\quad C_4\frac{\rho k^4}{\varepsilon^3}\left[T^{\theta\theta}_{19}+T^{\theta\theta}_{20}+\ldots+T^{\theta\theta}_{33}+T^{\theta\theta}_{34}-\frac{2}{3}r^2\Pi_3\right]$$

$$+\quad C_5\frac{\rho k^4}{\varepsilon^3}\left[T^{\theta\theta}_{36}+T^{\theta\theta}_{37}+\ldots+T^{\theta\theta}_{50}+T^{\theta\theta}_{51}-\frac{2}{3}r^2\Pi_3\right]$$

(C.19)

The terms $T^{ij}_{19}\ldots T^{ij}_{51}$ in the above equations are listed below:

<u>The terms in τ_{xx}:</u>

$$T^{xx}_{19}=\frac{\partial U}{\partial x}\left(\frac{\partial U}{\partial x}\frac{\partial U}{\partial x}+\frac{\partial U}{\partial r}\frac{\partial V}{\partial x}+\frac{\partial U}{\partial \theta}\frac{\partial \Omega}{\partial x}\right)+\frac{\partial U}{\partial r}\left(\frac{\partial U}{\partial x}\frac{\partial U}{\partial r}+\frac{\partial U}{\partial r}\frac{\partial V}{\partial r}+\frac{\partial U}{\partial \theta}\frac{\partial \Omega}{\partial r}\right)$$
$$+\frac{1}{r^2}\frac{\partial U}{\partial \theta}\left(\frac{\partial U}{\partial x}\frac{\partial U}{\partial \theta}+\frac{\partial U}{\partial r}\frac{\partial V}{\partial \theta}+\frac{\partial U}{\partial \theta}\frac{\partial \Omega}{\partial \theta}\right)$$

$T^{xx}_{20}=0$
$T^{xx}_{21}=0$
$T^{xx}_{22}=0$
$T^{xx}_{23}=T^{xx}_{19}$
$T^{xx}_{24}=0$
$T^{xx}_{25}=0$
$T^{xx}_{26}=0$

$$T^{xx}_{27}=\frac{V}{r^3}\left(\frac{\partial U}{\partial \theta}\right)^2$$

$T^{xx}_{28}=0$
$T^{xx}_{29}=0$
$T^{xx}_{30}=0$
$T^{xx}_{31}=T^{xx}_{27}$
$T^{xx}_{32}=0$
$T^{xx}_{33}=0$
$T^{xx}_{34}=0$

$$T^{xx}_{36}=\frac{\partial U}{\partial x}\left(\frac{\partial U}{\partial x}\frac{\partial U}{\partial x}+\frac{\partial U}{\partial r}\frac{\partial V}{\partial x}+\frac{\partial U}{\partial \theta}\frac{\partial \Omega}{\partial x}\right)+\frac{\partial V}{\partial x}\left(\frac{\partial V}{\partial x}\frac{\partial U}{\partial x}+\frac{\partial V}{\partial r}\frac{\partial V}{\partial x}+\frac{\partial V}{\partial \theta}\frac{\partial \Omega}{\partial x}\right)$$
$$+\frac{\partial \Omega}{\partial x}\left(r^2\frac{\partial \Omega}{\partial x}\frac{\partial U}{\partial x}+\frac{\partial r^2\Omega}{\partial r}\frac{\partial V}{\partial x}+r^2\frac{\partial \Omega}{\partial \theta}\frac{\partial \Omega}{\partial x}\right)$$

$$T^{xx}_{37}=-2r\Omega\frac{\partial V}{\partial x}\frac{\partial \Omega}{\partial x}+rV\left(\frac{\partial \Omega}{\partial x}\right)^2$$

$T^{xx}_{38}=0$
$T^{xx}_{39}=0$
$T^{xx}_{40}=T^{xx}_{36}$
$T^{xx}_{41}=T^{xx}_{37}$
$T^{xx}_{42}=0$

$T_{43}^{xx} = 0$
$T_{44}^{xx} = 0$
$T_{45}^{xx} = 0$
$T_{46}^{xx} = 0$
$T_{47}^{xx} = 0$
$T_{48}^{xx} = 0$
$T_{49}^{xx} = 0$
$T_{50}^{xx} = 0$
$T_{51}^{xx} = 0$

The terms in τ_{xr}:

$$T_{19}^{xr} = \frac{\partial U}{\partial x}\left(\frac{\partial V}{\partial x}\frac{\partial U}{\partial x} + \frac{\partial V}{\partial r}\frac{\partial V}{\partial x} + \frac{\partial V}{\partial \theta}\frac{\partial \Omega}{\partial x}\right) + \frac{\partial U}{\partial r}\left(\frac{\partial V}{\partial x}\frac{\partial U}{\partial r} + \frac{\partial V}{\partial r}\frac{\partial V}{\partial r} + \frac{\partial V}{\partial \theta}\frac{\partial \Omega}{\partial r}\right)$$
$$+ \frac{1}{r^2}\frac{\partial U}{\partial \theta}\left(\frac{\partial V}{\partial x}\frac{\partial U}{\partial \theta} + \frac{\partial V}{\partial r}\frac{\partial V}{\partial \theta} + \frac{\partial V}{\partial \theta}\frac{\partial \Omega}{\partial \theta}\right)$$
$$T_{20}^{xr} = -r\Omega\left(\frac{\partial U}{\partial x}\frac{\partial \Omega}{\partial x} + \frac{\partial U}{\partial r}\frac{\partial \Omega}{\partial r} + \frac{1}{r^2}\frac{\partial U}{\partial \theta}\frac{\partial \Omega}{\partial \theta}\right)$$
$T_{21}^{xr} = 0$
$T_{22}^{xr} = 0$
$$T_{23}^{xr} = \frac{\partial V}{\partial x}\left(\frac{\partial U}{\partial x}\frac{\partial U}{\partial x} + \frac{\partial U}{\partial r}\frac{\partial V}{\partial x} + \frac{\partial U}{\partial \theta}\frac{\partial \Omega}{\partial x}\right) + \frac{\partial V}{\partial r}\left(\frac{\partial U}{\partial x}\frac{\partial U}{\partial r} + \frac{\partial U}{\partial r}\frac{\partial V}{\partial r} + \frac{\partial U}{\partial \theta}\frac{\partial \Omega}{\partial r}\right)$$
$$+ \frac{1}{r^2}\frac{\partial V}{\partial \theta}\left(\frac{\partial U}{\partial x}\frac{\partial U}{\partial \theta} + \frac{\partial U}{\partial r}\frac{\partial V}{\partial \theta} + \frac{\partial U}{\partial \theta}\frac{\partial \Omega}{\partial \theta}\right)$$
$T_{24}^{xr} = 0$
$$T_{25}^{xr} = -\frac{\Omega}{r}\left(\frac{\partial U}{\partial x}\frac{\partial U}{\partial \theta} + \frac{\partial U}{\partial r}\frac{\partial V}{\partial \theta} + \frac{\partial U}{\partial \theta}\frac{\partial \Omega}{\partial \theta}\right)$$
$T_{26}^{xr} = 0$
$$T_{27}^{xr} = \frac{\Omega}{r}\left(\frac{\partial U}{\partial r}\frac{\partial V}{\partial \theta} - \frac{\partial U}{\partial \theta}\frac{\partial V}{\partial r}\right) + \frac{V}{r^3}\frac{\partial U}{\partial \theta}\frac{\partial V}{\partial \theta}$$
$$T_{28}^{xr} = -\Omega^2\frac{\partial U}{\partial r} - \frac{\Omega V}{r^2}\frac{\partial U}{\partial \theta}$$
$T_{29}^{xr} = 0$
$T_{30}^{xr} = 0$
$$T_{31}^{xr} = \frac{\Omega}{r}\left(\frac{\partial V}{\partial r}\frac{\partial U}{\partial \theta} - \frac{\partial V}{\partial \theta}\frac{\partial U}{\partial r}\right) + \frac{V}{r^3}\frac{\partial U}{\partial \theta}\frac{\partial V}{\partial \theta}$$
$T_{32}^{xr} = 0$
$$T_{33}^{xr} = \Omega^2\frac{\partial U}{\partial r} - \frac{\Omega V}{r^2}\frac{\partial U}{\partial \theta}$$
$T_{34}^{xr} = 0$
$$T_{36}^{xr} = \frac{\partial U}{\partial x}\left(\frac{\partial U}{\partial x}\frac{\partial U}{\partial r} + \frac{\partial U}{\partial r}\frac{\partial V}{\partial r} + \frac{\partial U}{\partial \theta}\frac{\partial \Omega}{\partial r}\right) + \frac{\partial V}{\partial x}\left(\frac{\partial V}{\partial x}\frac{\partial U}{\partial r} + \frac{\partial V}{\partial r}\frac{\partial V}{\partial r} + \frac{\partial V}{\partial \theta}\frac{\partial \Omega}{\partial r}\right)$$

$$+\frac{\partial\Omega}{\partial x}\left(r^2\frac{\partial\Omega}{\partial x}\frac{\partial U}{\partial r}+\frac{\partial r^2\Omega}{\partial r}\frac{\partial V}{\partial r}+r^2\frac{\partial\Omega}{\partial\theta}\frac{\partial\Omega}{\partial r}\right)$$

$$T_{37}^{xr}=-r\Omega\left(\frac{\partial V}{\partial x}\frac{\partial\Omega}{\partial r}+\frac{\partial V}{\partial r}\frac{\partial\Omega}{\partial x}\right)+rV\frac{\partial\Omega}{\partial r}$$

$$T_{38}^{xr}=0$$

$$T_{39}^{xr}=0$$

$$T_{40}^{xr}=\frac{\partial U}{\partial r}\left(\frac{\partial U}{\partial x}\frac{\partial U}{\partial x}+\frac{\partial U}{\partial r}\frac{\partial V}{\partial x}+\frac{\partial U}{\partial\theta}\frac{\partial\Omega}{\partial x}\right)+\frac{\partial V}{\partial r}\left(\frac{\partial V}{\partial x}\frac{\partial U}{\partial x}+\frac{\partial V}{\partial r}\frac{\partial V}{\partial x}+\frac{\partial V}{\partial\theta}\frac{\partial\Omega}{\partial x}\right)$$
$$+\frac{\partial\Omega}{\partial r}\left(r^2\frac{\partial\Omega}{\partial x}\frac{\partial U}{\partial x}+\frac{\partial r^2\Omega}{\partial r}\frac{\partial V}{\partial x}+r^2\frac{\partial\Omega}{\partial\theta}\frac{\partial\Omega}{\partial x}\right)$$

$$T_{41}^{xr}=T_{37}^{xx}$$

$$T_{42}^{xr}=\frac{\Omega}{r}\left(r^2\frac{\partial\Omega}{\partial x}\frac{\partial U}{\partial x}+\frac{\partial r^2\Omega}{\partial r}\frac{\partial V}{\partial x}+r^2\frac{\partial\Omega}{\partial\theta}\frac{\partial\Omega}{\partial x}\right)$$

$$T_{43}^{xr}=\Omega V\frac{\partial\Omega}{\partial x}-\Omega^2\frac{\partial V}{\partial x}$$

$$T_{44}^{xr}=\frac{\Omega}{r}\left(\frac{\partial U}{\partial x}\frac{\partial U}{\partial\theta}+\frac{\partial V}{\partial x}\frac{\partial V}{\partial\theta}+r^2\frac{\partial\Omega}{\partial\theta}\frac{\partial\Omega}{\partial x}\right)$$

$$T_{45}^{xr}=T_{43}^{xr}$$

$$T_{46}^{xr}=0$$

$$T_{47}^{xr}=0$$

$$T_{48}^{xr}=0$$

$$T_{49}^{xr}=0$$

$$T_{50}^{xr}=0$$

$$T_{51}^{xr}=0$$

The terms in $\tau_{x\theta}$:

$$T_{19}^{x\theta}=\frac{\partial U}{\partial x}\left(r^2\frac{\partial\Omega}{\partial x}\frac{\partial U}{\partial x}+\frac{\partial r^2\Omega}{\partial r}\frac{\partial V}{\partial x}+r^2\frac{\partial\Omega}{\partial\theta}\frac{\partial\Omega}{\partial x}\right)+\frac{\partial U}{\partial r}\left(r^2\frac{\partial\Omega}{\partial x}\frac{\partial U}{\partial r}+\frac{\partial r^2\Omega}{\partial r}\frac{\partial V}{\partial r}+r^2\frac{\partial\Omega}{\partial\theta}\frac{\partial\Omega}{\partial r}\right)$$
$$+\frac{1}{r^2}\frac{\partial U}{\partial\theta}\left(r^2\frac{\partial\Omega}{\partial x}\frac{\partial U}{\partial\theta}+\frac{\partial r^2\Omega}{\partial r}\frac{\partial V}{\partial\theta}+r^2\frac{\partial\Omega}{\partial\theta}\frac{\partial\Omega}{\partial\theta}\right)$$

$$T_{20}^{x\theta}=rV\left(\frac{\partial U}{\partial x}\frac{\partial\Omega}{\partial x}+\frac{\partial U}{\partial r}\frac{\partial\Omega}{\partial r}+\frac{1}{r^2}\frac{\partial U}{\partial\theta}\frac{\partial\Omega}{\partial\theta}\right)-r\Omega\left(\frac{\partial U}{\partial x}\frac{\partial V}{\partial x}+\frac{\partial U}{\partial r}\frac{\partial V}{\partial r}+\frac{1}{r^2}\frac{\partial U}{\partial\theta}\frac{\partial V}{\partial\theta}\right)$$

$$T_{21}^{x\theta}=0$$

$$T_{22}^{x\theta}=0$$

$$T_{23}^{x\theta}=r^2\frac{\partial\Omega}{\partial x}\left(\frac{\partial U}{\partial x}\frac{\partial U}{\partial x}+\frac{\partial U}{\partial r}\frac{\partial V}{\partial x}+\frac{\partial U}{\partial\theta}\frac{\partial\Omega}{\partial x}\right)+\frac{\partial r^2\Omega}{\partial r}\left(\frac{\partial U}{\partial x}\frac{\partial U}{\partial r}+\frac{\partial U}{\partial r}\frac{\partial V}{\partial r}+\frac{\partial U}{\partial\theta}\frac{\partial\Omega}{\partial r}\right)$$
$$+\frac{\partial\Omega}{\partial\theta}\left(\frac{\partial U}{\partial x}\frac{\partial U}{\partial\theta}+\frac{\partial U}{\partial r}\frac{\partial V}{\partial\theta}+\frac{\partial U}{\partial\theta}\frac{\partial\Omega}{\partial\theta}\right)$$

$$T_{24}^{x\theta}=0$$

$$T_{25}^{x\theta}=-r\Omega\left(\frac{\partial U}{\partial x}\frac{\partial U}{\partial r}+\frac{\partial U}{\partial r}\frac{\partial V}{\partial r}+\frac{\partial U}{\partial\theta}\frac{\partial\Omega}{\partial r}\right)+\frac{V}{r}\left(\frac{\partial U}{\partial x}\frac{\partial U}{\partial\theta}+\frac{\partial U}{\partial r}\frac{\partial V}{\partial\theta}+\frac{\partial U}{\partial\theta}\frac{\partial\Omega}{\partial\theta}\right)$$

$$T^{x\theta}_{26} = 0$$
$$T^{x\theta}_{27} = \frac{\Omega}{r}\left(r^2\frac{\partial U}{\partial r}\frac{\partial \Omega}{\partial \theta} - \frac{\partial U}{\partial \theta}\frac{\partial r^2\Omega}{\partial r}\right) + \frac{V}{r}\frac{\partial U}{\partial \theta}\frac{\partial \Omega}{\partial \theta}$$
$$T^{x\theta}_{28} = \Omega V\frac{\partial U}{\partial r} + \frac{V^2}{r^2}\frac{\partial U}{\partial \theta} + \Omega^2\frac{\partial U}{\partial \theta}$$
$$T^{x\theta}_{29} = 0$$
$$T^{x\theta}_{30} = 0$$
$$T^{x\theta}_{31} = \frac{\Omega}{r}\left(\frac{\partial r^2\Omega}{\partial r}\frac{\partial U}{\partial \theta} - r^2\frac{\partial \Omega}{\partial \theta}\frac{\partial U}{\partial r}\right) + \frac{V}{r}\frac{\partial U}{\partial \theta}\frac{\partial \Omega}{\partial \theta}$$
$$T^{x\theta}_{32} = 0$$
$$T^{x\theta}_{33} = -\Omega V\frac{\partial U}{\partial r} + \frac{V^2}{r^2}\frac{\partial U}{\partial \theta} - \Omega^2\frac{\partial U}{\partial \theta}$$
$$T^{x\theta}_{34} = 0$$
$$T^{x\theta}_{36} = \frac{\partial U}{\partial \theta}\left(\frac{\partial U}{\partial x}\frac{\partial U}{\partial x} + \frac{\partial V}{\partial x}\frac{\partial V}{\partial x} + r^2\frac{\partial \Omega}{\partial x}\frac{\partial \Omega}{\partial x}\right) + \frac{\partial V}{\partial \theta}\left(\frac{\partial U}{\partial x}\frac{\partial U}{\partial r} + \frac{\partial V}{\partial x}\frac{\partial V}{\partial r} + r^2\frac{\partial \Omega}{\partial x}\frac{\partial \Omega}{\partial r}\right)$$
$$+ \frac{\partial \Omega}{\partial \theta}\left(\frac{\partial U}{\partial x}\frac{\partial U}{\partial \theta} + \frac{\partial V}{\partial x}\frac{\partial V}{\partial \theta} + r^2\frac{\partial \Omega}{\partial x}\frac{\partial \Omega}{\partial \theta}\right)$$
$$T^{x\theta}_{37} = -r\Omega\left(\frac{\partial V}{\partial x}\frac{\partial \Omega}{\partial \theta} + \frac{\partial V}{\partial \theta}\frac{\partial \Omega}{\partial x}\right) + rV\frac{\partial \Omega}{\partial x}\frac{\partial \Omega}{\partial \theta}$$
$$T^{x\theta}_{38} = 0$$
$$T^{x\theta}_{39} = 0$$
$$T^{x\theta}_{40} = \frac{\partial U}{\partial x}\left(\frac{\partial U}{\partial \theta}\frac{\partial U}{\partial x} + \frac{\partial V}{\partial \theta}\frac{\partial V}{\partial x} + r^2\frac{\partial \Omega}{\partial \theta}\frac{\partial \Omega}{\partial x}\right) + \frac{\partial V}{\partial x}\left(\frac{\partial U}{\partial \theta}\frac{\partial U}{\partial r} + \frac{\partial V}{\partial \theta}\frac{\partial V}{\partial r} + \frac{\partial \Omega}{\partial \theta}\frac{\partial r^2\Omega}{\partial r}\right)$$
$$+ \frac{\partial \Omega}{\partial x}\left(\frac{\partial U}{\partial \theta}\frac{\partial U}{\partial \theta} + \frac{\partial V}{\partial \theta}\frac{\partial V}{\partial \theta} + r^2\frac{\partial \Omega}{\partial \theta}\frac{\partial \Omega}{\partial \theta}\right)$$
$$T^{x\theta}_{41} = T^{x\theta}_{37}$$
$$T^{x\theta}_{42} = \frac{V}{r}\left(r^2\frac{\partial \Omega}{\partial x}\frac{\partial U}{\partial x} + \frac{\partial r^2\Omega}{\partial r}\frac{\partial V}{\partial x} + r^2\frac{\partial \Omega}{\partial \theta}\frac{\partial \Omega}{\partial x}\right) - r\Omega\left(\frac{\partial V}{\partial x}\frac{\partial U}{\partial x} + \frac{\partial V}{\partial r}\frac{\partial V}{\partial x} + \frac{\partial V}{\partial \theta}\frac{\partial \Omega}{\partial x}\right)$$
$$T^{x\theta}_{43} = -\Omega V\frac{\partial V}{\partial x} + r^2\Omega^2\frac{\partial \Omega}{\partial x} + V^2\frac{\partial \Omega}{\partial x}$$
$$T^{x\theta}_{44} = \frac{V}{r}\left(\frac{\partial U}{\partial x}\frac{\partial U}{\partial \theta} + \frac{\partial V}{\partial x}\frac{\partial V}{\partial \theta} + r^2\frac{\partial \Omega}{\partial \theta}\frac{\partial \Omega}{\partial x}\right)$$
$$- r\Omega\left(\frac{\partial U}{\partial x}\frac{\partial U}{\partial r} + \frac{\partial V}{\partial x}\frac{\partial V}{\partial r} + \frac{\partial \Omega}{\partial x}\frac{\partial r^2\Omega}{\partial r}\right)$$
$$T^{x\theta}_{45} = T^{x\theta}_{43}$$
$$T^{x\theta}_{46} = 0$$
$$T^{x\theta}_{47} = 0$$
$$T^{x\theta}_{48} = 0$$
$$T^{x\theta}_{49} = 0$$
$$T^{x\theta}_{50} = 0$$
$$T^{x\theta}_{51} = 0$$

The terms in τ_{rr}:

$$
\begin{aligned}
T_{19}^{rr} &= \frac{\partial V}{\partial x}\left(\frac{\partial U}{\partial x}\frac{\partial V}{\partial x}+\frac{\partial V}{\partial r}\frac{\partial V}{\partial x}+\frac{\partial V}{\partial \theta}\frac{\partial \Omega}{\partial x}\right)+\frac{\partial V}{\partial r}\left(\frac{\partial V}{\partial x}\frac{\partial U}{\partial r}+\frac{\partial V}{\partial r}\frac{\partial V}{\partial r}+\frac{\partial V}{\partial \theta}\frac{\partial \Omega}{\partial r}\right)\\
&\quad+\frac{1}{r^2}\frac{\partial V}{\partial \theta}\left(\frac{\partial V}{\partial x}\frac{\partial U}{\partial \theta}+\frac{\partial V}{\partial r}\frac{\partial V}{\partial \theta}+\frac{\partial V}{\partial \theta}\frac{\partial \Omega}{\partial \theta}\right)\\
T_{20}^{rr} &= -r\Omega\left(\frac{\partial V}{\partial x}\frac{\partial \Omega}{\partial x}+\frac{\partial V}{\partial r}\frac{\partial \Omega}{\partial r}+\frac{1}{r^2}\frac{\partial V}{\partial \theta}\frac{\partial \Omega}{\partial \theta}\right)\\
T_{21}^{rr} &= -\frac{\Omega}{r}\left(\frac{\partial V}{\partial x}\frac{\partial U}{\partial \theta}+\frac{\partial V}{\partial r}\frac{\partial V}{\partial \theta}+\frac{\partial V}{\partial \theta}\frac{\partial \Omega}{\partial \theta}\right)\\
T_{22}^{rr} &= \Omega^2\frac{\partial \Omega}{\partial \theta}\\
T_{23}^{rr} &= T_{19}^{rr}\\
T_{24}^{rr} &= T_{20}^{rr}\\
T_{25}^{rr} &= T_{21}^{rr}\\
T_{26}^{rr} &= T_{22}^{rr}\\
T_{27}^{rr} &= \frac{V}{r^3}\left(\frac{\partial V}{\partial \theta}\right)^2\\
T_{28}^{rr} &= -\frac{\Omega V}{r^2}\frac{\partial V}{\partial \theta}-\Omega^2\frac{\partial V}{\partial r}\\
T_{29}^{rr} &= -\frac{\Omega V}{r^2}\frac{\partial V}{\partial \theta}+\Omega^2\frac{\partial V}{\partial r}\\
T_{30}^{rr} &= \frac{\Omega^2 V}{r}\\
T_{31}^{rr} &= T_{27}^{rr}\\
T_{32}^{rr} &= T_{28}^{rr}\\
T_{33}^{rr} &= T_{29}^{rr}\\
T_{34}^{rr} &= T_{30}^{rr}\\
T_{36}^{rr} &= \frac{\partial U}{\partial r}\left(\frac{\partial U}{\partial r}\frac{\partial U}{\partial x}+\frac{\partial V}{\partial r}\frac{\partial V}{\partial x}+r^2\frac{\partial \Omega}{\partial r}\frac{\partial \Omega}{\partial x}\right)+\frac{\partial V}{\partial r}\left(\frac{\partial U}{\partial r}\frac{\partial U}{\partial r}+\frac{\partial V}{\partial r}\frac{\partial V}{\partial r}+\frac{\partial \Omega}{\partial r}\frac{\partial r^2\Omega}{\partial r}\right)\\
&\quad+\frac{\partial \Omega}{\partial r}\left(\frac{\partial U}{\partial r}\frac{\partial U}{\partial \theta}+\frac{\partial V}{\partial r}\frac{\partial V}{\partial \theta}+r^2\frac{\partial \Omega}{\partial r}\frac{\partial \Omega}{\partial \theta}\right)\\
T_{37}^{rr} &= -2r\Omega\frac{\partial V}{\partial r}\frac{\partial \Omega}{\partial r}+rV\left(\frac{\partial \Omega}{\partial r}\right)^2\\
T_{38}^{rr} &= \frac{\Omega}{r}\left(r^2\frac{\partial \Omega}{\partial x}\frac{\partial U}{\partial r}+\frac{\partial r^2\Omega}{\partial r}\frac{\partial V}{\partial r}+r^2\frac{\partial \Omega}{\partial r}\frac{\partial \Omega}{\partial \theta}\right)\\
T_{39}^{rr} &= \Omega V\frac{\partial \Omega}{\partial r}-\Omega^2\frac{\partial V}{\partial r}\\
T_{40}^{rr} &= T_{36}^{rr}\\
T_{41}^{rr} &= T_{37}^{rr}\\
T_{42}^{rr} &= T_{38}^{rr}\\
T_{43}^{rr} &= T_{39}^{rr}
\end{aligned}
$$

$$T^{rr}_{44} = \frac{\Omega}{r}\left(\frac{\partial U}{\partial r}\frac{\partial U}{\partial \theta} + \frac{\partial V}{\partial r}\frac{\partial V}{\partial \theta} + r^2\frac{\partial \Omega}{\partial r}\frac{\partial \Omega}{\partial \theta}\right)$$
$$T^{rr}_{45} = T^{rr}_{39}$$
$$T^{rr}_{46} = \Omega^2\frac{\partial \Omega}{\partial \theta}$$
$$T^{rr}_{47} = \frac{\Omega^2}{r}V$$
$$T^{rr}_{48} = T^{rr}_{44}$$
$$T^{rr}_{49} = T^{rr}_{45}$$
$$T^{rr}_{50} = T^{rr}_{46}$$
$$T^{rr}_{51} = T^{rr}_{47}$$

The terms in $\tau_{r\theta}$:

$$T^{r\theta}_{19} = \frac{\partial V}{\partial x}\left(r^2\frac{\partial \Omega}{\partial x}\frac{\partial U}{\partial x} + \frac{\partial r^2\Omega}{\partial r}\frac{\partial V}{\partial x} + r^2\frac{\partial \Omega}{\partial \theta}\frac{\partial \Omega}{\partial x}\right) + \frac{\partial V}{\partial r}\left(r^2\frac{\partial \Omega}{\partial x}\frac{\partial U}{\partial r} + \frac{\partial r^2\Omega}{\partial r}\frac{\partial V}{\partial r} + r^2\frac{\partial \Omega}{\partial \theta}\frac{\partial \Omega}{\partial r}\right)$$
$$+ \frac{1}{r^2}\frac{\partial V}{\partial \theta}\left(r^2\frac{\partial \Omega}{\partial x}\frac{\partial U}{\partial \theta} + \frac{\partial r^2\Omega}{\partial r}\frac{\partial V}{\partial \theta} + r^2\frac{\partial \Omega}{\partial \theta}\frac{\partial \Omega}{\partial \theta}\right)$$
$$T^{r\theta}_{20} = rV\left(\frac{\partial V}{\partial x}\frac{\partial \Omega}{\partial x} + \frac{\partial V}{\partial r}\frac{\partial \Omega}{\partial r} + \frac{1}{r^2}\frac{\partial V}{\partial \theta}\frac{\partial \Omega}{\partial \theta}\right) - r\Omega\left(\frac{\partial V}{\partial x}\frac{\partial V}{\partial x} + \frac{\partial V}{\partial r}\frac{\partial V}{\partial r} + \frac{1}{r^2}\frac{\partial V}{\partial \theta}\frac{\partial V}{\partial \theta}\right)$$
$$T^{r\theta}_{21} = -\frac{\Omega}{r}\left(r^2\frac{\partial \Omega}{\partial x}\frac{\partial U}{\partial \theta} + \frac{\partial r^2\Omega}{\partial r}\frac{\partial V}{\partial \theta} + r^2\frac{\partial \Omega}{\partial \theta}\frac{\partial \Omega}{\partial \theta}\right)$$
$$T^{r\theta}_{22} = \Omega^2\frac{\partial V}{\partial \theta} - \Omega V\frac{\partial \Omega}{\partial \theta}$$
$$T^{r\theta}_{23} = r^2\frac{\partial \Omega}{\partial x}\left(\frac{\partial V}{\partial x}\frac{\partial U}{\partial x} + \frac{\partial V}{\partial r}\frac{\partial V}{\partial x} + \frac{\partial V}{\partial \theta}\frac{\partial \Omega}{\partial x}\right) + \frac{\partial r^2\Omega}{\partial r}\left(\frac{\partial V}{\partial x}\frac{\partial U}{\partial r} + \frac{\partial V}{\partial r}\frac{\partial V}{\partial r} + \frac{\partial V}{\partial \theta}\frac{\partial \Omega}{\partial r}\right)$$
$$+ \frac{\partial \Omega}{\partial \theta}\left(\frac{\partial V}{\partial x}\frac{\partial U}{\partial \theta} + \frac{\partial V}{\partial r}\frac{\partial V}{\partial \theta} + \frac{\partial V}{\partial \theta}\frac{\partial \Omega}{\partial \theta}\right)$$
$$T^{r\theta}_{24} = -r\Omega\left(r^2\frac{\partial \Omega}{\partial x}\frac{\partial \Omega}{\partial x} + \frac{\partial r^2\Omega}{\partial r}\frac{\partial \Omega}{\partial r} + \frac{\partial \Omega}{\partial \theta}\frac{\partial \Omega}{\partial \theta}\right)$$
$$T^{r\theta}_{25} = -r\Omega\left(\frac{\partial V}{\partial x}\frac{\partial U}{\partial r} + \frac{\partial V}{\partial r}\frac{\partial V}{\partial r} + \frac{\partial V}{\partial \theta}\frac{\partial \Omega}{\partial r}\right) + \frac{V}{r}\left(\frac{\partial V}{\partial x}\frac{\partial U}{\partial \theta} + \frac{\partial V}{\partial r}\frac{\partial V}{\partial \theta} + \frac{\partial V}{\partial \theta}\frac{\partial \Omega}{\partial \theta}\right)$$
$$T^{r\theta}_{26} = r^2\Omega^2\frac{\partial \Omega}{\partial r} - \Omega V\frac{\partial \Omega}{\partial \theta}$$
$$T^{r\theta}_{27} = r\Omega\left(\frac{\partial V}{\partial r}\frac{\partial \Omega}{\partial \theta} - \frac{\partial V}{\partial \theta}\frac{\partial \Omega}{\partial x}\right) + \frac{V}{r}\frac{\partial V}{\partial \theta}\frac{\partial \Omega}{\partial \theta}$$
$$T^{r\theta}_{28} = \Omega V\frac{\partial V}{\partial r} + \frac{V^2}{r^2}\frac{\partial V}{\partial \theta} + \Omega^2\frac{\partial V}{\partial \theta}$$
$$T^{r\theta}_{29} = \Omega^2\frac{\partial r^2\Omega}{\partial r} - \Omega V\frac{\partial \Omega}{\partial \theta}$$
$$T^{r\theta}_{30} = -r\Omega^3 - \frac{\Omega}{r}V^2$$

$$T_{31}^{r\theta} = \frac{\Omega}{r}\left(\frac{\partial r^2\Omega}{\partial r}\frac{\partial V}{\partial \theta} - r^2\frac{\partial \Omega}{\partial \theta}\frac{\partial V}{\partial r}\right) + \frac{V}{r}\frac{\partial V}{\partial \theta}\frac{\partial \Omega}{\partial \theta}$$

$$T_{32}^{r\theta} = -\Omega^2\frac{\partial r^2\Omega}{\partial r} - \Omega V\frac{\partial \Omega}{\partial \theta}$$

$$T_{33}^{r\theta} = -\Omega V\frac{\partial V}{\partial r} + \frac{V^2}{r^2}\frac{\partial V}{\partial \theta} - \Omega^2\frac{\partial V}{\partial \theta}$$

$$T_{34}^{r\theta} = r\Omega^3 - \frac{\Omega}{r}V^2$$

$$T_{36}^{r\theta} = \frac{\partial U}{\partial \theta}\left(\frac{\partial U}{\partial r}\frac{\partial U}{\partial x} + \frac{\partial V}{\partial r}\frac{\partial V}{\partial x} + r^2\frac{\partial \Omega}{\partial r}\frac{\partial \Omega}{\partial x}\right) + \frac{\partial V}{\partial \theta}\left(\frac{\partial U}{\partial r}\frac{\partial U}{\partial r} + \frac{\partial V}{\partial r}\frac{\partial V}{\partial r} + \frac{\partial \Omega}{\partial r}\frac{\partial r^2\Omega}{\partial r}\right)$$
$$+ \frac{\partial \Omega}{\partial \theta}\left(\frac{\partial U}{\partial r}\frac{\partial U}{\partial \theta} + \frac{\partial V}{\partial r}\frac{\partial V}{\partial \theta} + r^2\frac{\partial \Omega}{\partial r}\frac{\partial \Omega}{\partial \theta}\right)$$

$$T_{37}^{r\theta} = -r\Omega\left(\frac{\partial V}{\partial r}\frac{\partial \Omega}{\partial \theta} + \frac{\partial V}{\partial \theta}\frac{\partial \Omega}{\partial r}\right) + rV\frac{\partial \Omega}{\partial r}\frac{\partial \Omega}{\partial \theta}$$

$$T_{38}^{r\theta} = \frac{\Omega}{r}\left(r^2\frac{\partial \Omega}{\partial x}\frac{\partial U}{\partial \theta} + \frac{\partial r^2\Omega}{\partial r}\frac{\partial V}{\partial \theta} + r^2\frac{\partial \Omega}{\partial \theta}\frac{\partial \Omega}{\partial \theta}\right)$$

$$T_{39}^{r\theta} = \Omega V\frac{\partial \Omega}{\partial \theta} - \Omega^2\frac{\partial V}{\partial \theta}$$

$$T_{40}^{r\theta} = \frac{\partial U}{\partial r}\left(\frac{\partial U}{\partial \theta}\frac{\partial U}{\partial x} + \frac{\partial V}{\partial \theta}\frac{\partial V}{\partial x} + r^2\frac{\partial \Omega}{\partial \theta}\frac{\partial \Omega}{\partial x}\right) + \frac{\partial V}{\partial r}\left(\frac{\partial U}{\partial \theta}\frac{\partial U}{\partial r} + \frac{\partial V}{\partial \theta}\frac{\partial V}{\partial r} + \frac{\partial \Omega}{\partial \theta}\frac{\partial r^2\Omega}{\partial r}\right)$$
$$+ \frac{\partial \Omega}{\partial r}\left(\frac{\partial U}{\partial \theta}\frac{\partial U}{\partial \theta} + \frac{\partial V}{\partial \theta}\frac{\partial V}{\partial \theta} + r^2\frac{\partial \Omega}{\partial \theta}\frac{\partial \Omega}{\partial \theta}\right)$$

$$T_{41}^{r\theta} = T_{37}^{x\theta}$$

$$T_{42}^{r\theta} = \frac{V}{r}\left(r^2\frac{\partial \Omega}{\partial x}\frac{\partial U}{\partial r} + \frac{\partial r^2\Omega}{\partial r}\frac{\partial V}{\partial r} + r^2\frac{\partial \Omega}{\partial \theta}\frac{\partial \Omega}{\partial r}\right)$$
$$-r\Omega\left(\frac{\partial V}{\partial x}\frac{\partial U}{\partial r} + \frac{\partial V}{\partial r}\frac{\partial V}{\partial r} + \frac{\partial V}{\partial \theta}\frac{\partial \Omega}{\partial r}\right)$$

$$T_{43}^{r\theta} = -\Omega V\frac{\partial V}{\partial r} + r^2\Omega^2\frac{\partial \Omega}{\partial r} + V^2\frac{\partial \Omega}{\partial r}$$

$$T_{44}^{r\theta} = \frac{V}{r}\left(\frac{\partial U}{\partial r}\frac{\partial U}{\partial \theta} + \frac{\partial V}{\partial r}\frac{\partial V}{\partial \theta} + r^2\frac{\partial \Omega}{\partial r}\frac{\partial \Omega}{\partial \theta}\right) - r\Omega\left(\frac{\partial U}{\partial r}\frac{\partial U}{\partial r} + \frac{\partial V}{\partial r}\frac{\partial V}{\partial r} + \frac{\partial \Omega}{\partial r}\frac{\partial r^2\Omega}{\partial r}\right)$$

$$T_{45}^{r\theta} = T_{43}^{x\theta}$$

$$T_{46}^{r\theta} = \Omega V\frac{\partial \Omega}{\partial \theta} - \Omega^2\frac{\partial r^2\Omega}{\partial r}$$

$$T_{47}^{r\theta} = r\Omega^3 + \frac{\Omega}{r}V^2$$

$$T_{48}^{r\theta} = \frac{\Omega}{r}\left(\frac{\partial U}{\partial \theta}\frac{\partial U}{\partial \theta} + \frac{\partial V}{\partial \theta}\frac{\partial V}{\partial \theta} + r^2\frac{\partial \Omega}{\partial \theta}\frac{\partial \Omega}{\partial \theta}\right)$$

$$T_{49}^{r\theta} = T_{39}^{r\theta}$$
$$T_{50}^{r\theta} = T_{39}^{r\theta}$$
$$T_{51}^{r\theta} = T_{47}^{r\theta}$$

The terms in $\tau_{\theta\theta}$:

$$T_{19}^{\theta\theta} = r^2\frac{\partial\Omega}{\partial x}\left(r^2\frac{\partial\Omega}{\partial x}\frac{\partial U}{\partial x} + \frac{\partial r^2\Omega}{\partial r}\frac{\partial V}{\partial x} + r^2\frac{\partial\Omega}{\partial\theta}\frac{\partial\Omega}{\partial x}\right) + \frac{\partial r^2\Omega}{\partial r}\left(r^2\frac{\partial\Omega}{\partial x}\frac{\partial U}{\partial r} + \frac{\partial r^2\Omega}{\partial r}\frac{\partial V}{\partial r} + r^2\frac{\partial\Omega}{\partial\theta}\frac{\partial\Omega}{\partial r}\right)$$
$$+ \frac{\partial\Omega}{\partial\theta}\left(r^2\frac{\partial\Omega}{\partial x}\frac{\partial U}{\partial\theta} + \frac{\partial r^2\Omega}{\partial r}\frac{\partial V}{\partial\theta} + r^2\frac{\partial\Omega}{\partial\theta}\frac{\partial\Omega}{\partial\theta}\right)$$

$$T_{20}^{\theta\theta} = rV\left(r^2\frac{\partial\Omega}{\partial x}\frac{\partial\Omega}{\partial x} + \frac{\partial r^2\Omega}{\partial r}\frac{\partial\Omega}{\partial r} + r^2\frac{\partial\Omega}{\partial\theta}\frac{\partial\Omega}{\partial\theta}\right) - r\Omega\left(r^2\frac{\partial\Omega}{\partial x}\frac{\partial V}{\partial x} + \frac{\partial r^2\Omega}{\partial r}\frac{\partial V}{\partial r} + r^2\frac{\partial\Omega}{\partial\theta}\frac{\partial V}{\partial\theta}\right)$$

$$T_{21}^{\theta\theta} = -r\Omega\left(r^2\frac{\partial\Omega}{\partial x}\frac{\partial U}{\partial r} + \frac{\partial r^2\Omega}{\partial r}\frac{\partial V}{\partial r} + r^2\frac{\partial\Omega}{\partial\theta}\frac{\partial\Omega}{\partial r}\right) + \frac{V}{r}\left(r^2\frac{\partial\Omega}{\partial x}\frac{\partial U}{\partial\theta} + \frac{\partial r^2\Omega}{\partial r}\frac{\partial V}{\partial\theta} + r^2\frac{\partial\Omega}{\partial\theta}\frac{\partial\Omega}{\partial\theta}\right)$$

$$T_{22}^{\theta\theta} = -r^2\Omega V\frac{\partial\Omega}{\partial r} + r^2\Omega^2\frac{\partial V}{\partial r} + V^2\frac{\partial\Omega}{\partial\theta} - \Omega V\frac{\partial V}{\partial\theta}$$

$$T_{23}^{\theta\theta} = T_{19}^{\theta\theta}$$
$$T_{24}^{\theta\theta} = T_{20}^{\theta\theta}$$
$$T_{25}^{\theta\theta} = T_{21}^{\theta\theta}$$
$$T_{26}^{\theta\theta} = T_{22}^{\theta\theta}$$

$$T_{27}^{\theta\theta} = rV\left(\frac{\partial\Omega}{\partial\theta}\right)^2$$

$$T_{28}^{\theta\theta} = \Omega V\frac{\partial r^2\Omega}{\partial r} + \left(V^2 + r^2\Omega^2\right)\frac{\partial\Omega}{\partial\theta}$$

$$T_{29}^{\theta\theta} = -\Omega V\frac{\partial r^2\Omega}{\partial r} + \left(V^2 - r^2\Omega^2\right)\frac{\partial\Omega}{\partial\theta}$$

$$T_{30}^{\theta\theta} = \frac{V^3}{r}$$

$$T_{31}^{\theta\theta} = T_{27}^{\theta\theta}$$
$$T_{32}^{\theta\theta} = T_{28}^{\theta\theta}$$
$$T_{33}^{\theta\theta} = T_{29}^{\theta\theta}$$
$$T_{34}^{\theta\theta} = T_{30}^{\theta\theta}$$

$$T_{36}^{\theta\theta} = \frac{\partial U}{\partial\theta}\left(\frac{\partial U}{\partial\theta}\frac{\partial U}{\partial x} + \frac{\partial V}{\partial\theta}\frac{\partial V}{\partial x} + r^2\frac{\partial\Omega}{\partial\theta}\frac{\partial\Omega}{\partial x}\right)$$
$$+ \frac{\partial V}{\partial\theta}\left(\frac{\partial U}{\partial\theta}\frac{\partial U}{\partial r} + \frac{\partial V}{\partial\theta}\frac{\partial V}{\partial r} + \frac{\partial\Omega}{\partial\theta}\frac{\partial r^2\Omega}{\partial r}\right) + \frac{\partial\Omega}{\partial\theta}\left(\frac{\partial U}{\partial\theta}\frac{\partial U}{\partial\theta} + \frac{\partial V}{\partial\theta}\frac{\partial V}{\partial\theta} + r^2\frac{\partial\Omega}{\partial\theta}\frac{\partial\Omega}{\partial\theta}\right)$$

$$T_{37}^{\theta\theta} = -2r\Omega\frac{\partial V}{\partial\theta}\frac{\partial\Omega}{\partial\theta} + rV\left(\frac{\partial\Omega}{\partial\theta}\right)^2$$

$$T_{38}^{\theta\theta} = -r\Omega\left(\frac{\partial V}{\partial x}\frac{\partial U}{\partial\theta} + \frac{\partial V}{\partial r}\frac{\partial V}{\partial\theta} + \frac{\partial V}{\partial\theta}\frac{\partial\Omega}{\partial\theta}\right) + \frac{V}{r}\left(r^2\frac{\partial\Omega}{\partial x}\frac{\partial U}{\partial\theta} + \frac{\partial r^2\Omega}{\partial r}\frac{\partial V}{\partial\theta} + r^2\frac{\partial\Omega}{\partial\theta}\frac{\partial\Omega}{\partial\theta}\right)$$

$$T_{39}^{\theta\theta} = -\Omega V\frac{\partial V}{\partial\theta} + r^2\Omega^2\frac{\partial\Omega}{\partial\theta} + V^2\frac{\partial\Omega}{\partial\theta}$$

$$T_{40}^{\theta\theta} = T_{36}^{\theta\theta}$$
$$T_{41}^{\theta\theta} = T_{37}^{\theta\theta}$$
$$T_{42}^{\theta\theta} = T_{38}^{\theta\theta}$$
$$T_{43}^{\theta\theta} = T_{39}^{\theta\theta}$$

$$T_{44}^{\theta\theta} = \frac{V}{r}\left(\frac{\partial U}{\partial\theta}\frac{\partial U}{\partial\theta} + \frac{\partial V}{\partial\theta}\frac{\partial V}{\partial\theta} + r^2\frac{\partial\Omega}{\partial\theta}\frac{\partial\Omega}{\partial\theta}\right)$$
$$-r\Omega\left(\frac{\partial U}{\partial\theta}\frac{\partial U}{\partial r} + \frac{\partial V}{\partial\theta}\frac{\partial V}{\partial r} + \frac{\partial\Omega}{\partial\theta}\frac{\partial r^2\Omega}{\partial r}\right)$$
$$T_{45}^{\theta\theta} = T_{39}^{\theta\theta}$$
$$T_{46}^{\theta\theta} = -\Omega V\left(\frac{\partial r^2\Omega}{\partial r} + \frac{\partial V}{\partial\theta}\right) + r^2\Omega^2\frac{\partial V}{\partial r} + V^2\frac{\partial\Omega}{\partial\theta}$$
$$T_{47}^{\theta\theta} = \frac{V^3}{r} + 2r\Omega^2 V$$
$$T_{48}^{\theta\theta} = T_{44}^{\theta\theta}$$
$$T_{49}^{\theta\theta} = T_{39}^{\theta\theta}$$
$$T_{50}^{\theta\theta} = T_{46}^{\theta\theta}$$
$$T_{51}^{\theta\theta} = T_{47}^{\theta\theta}$$

Other scalar quantities:

$$\Pi_1 = 2\left(-r\Omega\frac{\partial\Omega}{\partial r} + \frac{\Omega}{r}\frac{\partial V}{\partial\theta} + \frac{V}{r}\frac{\partial\Omega}{\partial\theta}\right) - 2\Omega^2 + \frac{V^2}{r^2} \tag{C.20}$$

$$\Pi_2 = \left(\frac{\partial U}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial x}\right)^2 + r^2\left(\frac{\partial\Omega}{\partial x}\right)^2 + \left(\frac{\partial U}{\partial r}\right)^2 + \left(\frac{\partial V}{\partial r}\right)^2 + \frac{\partial\Omega}{\partial r}\frac{\partial r^2\Omega}{\partial r} + 2\Omega^2$$
$$+\frac{1}{r^2}\left[\left(\frac{\partial U}{\partial\theta}\right)^2 + \left(\frac{\partial V}{\partial\theta}\right)^2 + r^2\left(\frac{\partial\Omega}{\partial\theta}\right)^2 - 2r\Omega\frac{\partial V}{\partial\theta} + 2rV\frac{\partial\Omega}{\partial\theta} + V^2\right] \tag{C.21}$$

$$\Pi_3 = \Pi_3^{(1)} + \Pi_3^{(2)} + \Pi_3^{(3)} \tag{C.22}$$

$$\Pi_3^{(1)} = \left(\frac{\partial U}{\partial x}\right)^3 + \frac{\partial U}{\partial x}\left(\frac{\partial U}{\partial r}\frac{\partial V}{\partial x} + \frac{\partial U}{\partial\theta}\frac{\partial\Omega}{\partial x}\right) + \frac{\partial V}{\partial x}\left(\frac{\partial V}{\partial x}\frac{\partial U}{\partial x} + \frac{\partial V}{\partial r}\frac{\partial V}{\partial x} + \frac{\partial V}{\partial\theta}\frac{\partial\Omega}{\partial x}\right)$$
$$+\frac{\partial\Omega}{\partial x}r^2\left(\frac{\partial\Omega}{\partial x}\frac{\partial U}{\partial x} + \frac{\partial\Omega}{\partial r}\frac{\partial V}{\partial x} + \frac{\partial\Omega}{\partial\theta}\frac{\partial\Omega}{\partial x}\right) + rV\left(\frac{\partial\Omega}{\partial x}\right)^2 \tag{C.23}$$

$$\Pi_3^{(2)} = \frac{\partial U}{\partial r}\left(\frac{\partial U}{\partial x}\frac{\partial U}{\partial r} + \frac{\partial U}{\partial r}\frac{\partial V}{\partial r} + \frac{\partial U}{\partial\theta}\frac{\partial\Omega}{\partial r}\right) + \frac{\partial V}{\partial r}\left(\frac{\partial V}{\partial x}\frac{\partial U}{\partial r} + \frac{\partial V}{\partial r}\frac{\partial V}{\partial r} + \frac{\partial V}{\partial\theta}\frac{\partial\Omega}{\partial r}\right)$$
$$+\frac{\partial\Omega}{\partial r}r^2\left(\frac{\partial\Omega}{\partial x}\frac{\partial U}{\partial r} + \frac{\partial\Omega}{\partial r}\frac{\partial V}{\partial r} + \frac{\partial\Omega}{\partial\theta}\frac{\partial\Omega}{\partial r}\right) + \frac{\Omega}{r}\left(\frac{\partial U}{\partial r}\frac{\partial U}{\partial\theta} + \frac{\partial V}{\partial r}\frac{\partial V}{\partial\theta} + r^2\frac{\partial\Omega}{\partial r}\frac{\partial\Omega}{\partial\theta}\right)$$
$$+rV\left(\frac{\partial\Omega}{\partial r}\right)^2 + 2V\Omega\frac{\partial\Omega}{\partial r} + r\Omega\left(\frac{\partial\Omega}{\partial x}\frac{\partial U}{\partial r} + \frac{\partial\Omega}{\partial r}\frac{\partial V}{\partial r} + \frac{\partial\Omega}{\partial\theta}\frac{\partial\Omega}{\partial r}\right)$$
$$+\Omega^2\frac{\partial\Omega}{\partial\theta} + \frac{V}{r}\Omega^2 \tag{C.24}$$

$$\Pi_3^{(3)}r^2 = \frac{\partial U}{\partial\theta}\left(\frac{\partial U}{\partial x}\frac{\partial U}{\partial\theta} + \frac{\partial U}{\partial r}\frac{\partial V}{\partial\theta} + \frac{\partial U}{\partial\theta}\frac{\partial\Omega}{\partial\theta}\right) + \frac{\partial V}{\partial\theta}\left(\frac{\partial V}{\partial x}\frac{\partial U}{\partial\theta} + \frac{\partial V}{\partial r}\frac{\partial V}{\partial\theta} + \frac{\partial V}{\partial\theta}\frac{\partial\Omega}{\partial\theta}\right)$$

$$
\begin{aligned}
&+\frac{\partial\Omega}{\partial\theta}r^2\left(\frac{\partial\Omega}{\partial x}\frac{\partial U}{\partial\theta}+\frac{\partial\Omega}{\partial r}\frac{\partial V}{\partial\theta}+\frac{\partial\Omega}{\partial\theta}\frac{\partial\Omega}{\partial\theta}\right)-r\Omega\left(\frac{\partial U}{\partial\theta}\frac{\partial U}{\partial r}+\frac{\partial V}{\partial\theta}\frac{\partial V}{\partial r}+r^2\frac{\partial\Omega}{\partial\theta}\frac{\partial\Omega}{\partial r}\right)\\
&+\frac{V}{r}\left(\frac{\partial U}{\partial\theta}\frac{\partial U}{\partial\theta}+\frac{\partial V}{\partial\theta}\frac{\partial V}{\partial\theta}+r^2\frac{\partial\Omega}{\partial\theta}\frac{\partial\Omega}{\partial\theta}\right)+2rV\left(\frac{\partial\Omega}{\partial\theta}\right)^2-V\Omega\frac{\partial V}{\partial\theta}+3V^2\frac{\partial\Omega}{\partial\theta}\\
&+rV\left(\frac{\partial\Omega}{\partial x}\frac{\partial U}{\partial\theta}+\frac{\partial\Omega}{\partial r}\frac{\partial V}{\partial\theta}\right)-r\Omega\left(\frac{\partial V}{\partial x}\frac{\partial U}{\partial\theta}+\frac{\partial V}{\partial r}\frac{\partial V}{\partial\theta}\right)\\
&-r^2V\Omega\frac{\partial\Omega}{\partial r}+r^2\Omega^2\frac{\partial V}{\partial r}+\frac{V^3}{r} \qquad\qquad \text{(C.25)}
\end{aligned}
$$

$$
\begin{aligned}
(S^*)^2 = &\frac{1}{2}\left[2\left(\frac{\partial U}{\partial x}\right)^2+\left(\frac{\partial V}{\partial x}\right)^2+r^2\left(\frac{\partial\Omega}{\partial x}\right)^2+\left(\frac{\partial U}{\partial r}\right)^2+2\left(\frac{\partial V}{\partial r}\right)^2\right.\\
&+r^2\left(\frac{\partial\Omega}{\partial r}\right)^2+2\frac{\partial U}{\partial r}\frac{\partial V}{\partial x}+2\frac{\partial U}{\partial\theta}\frac{\partial\Omega}{\partial x}+2\frac{\partial V}{\partial\theta}\frac{\partial\Omega}{\partial r}+\frac{1}{r^2}\left(\frac{\partial U}{\partial\theta}\right)^2+\frac{1}{r^2}\left(\frac{\partial V}{\partial\theta}\right)^2\\
&\left.+2\left(\frac{\partial\Omega}{\partial\theta}\right)^2+4\frac{V}{r}\frac{\partial\Omega}{\partial\theta}+2\frac{V^2}{r^2}\right]-\frac{1}{3}\Theta^2 \qquad\qquad \text{(C.26)}\\
(\Omega^*)^2 = &\frac{1}{2}\left[\left(\frac{\partial V}{\partial x}\right)^2+r^2\frac{\partial\Omega}{\partial x}\frac{\partial\Omega}{\partial x}+\left(\frac{\partial U}{\partial r}\right)^2+r^2\left(\frac{\partial\Omega}{\partial r}\right)^2+4r\Omega\frac{\partial\Omega}{\partial r}+4\Omega^2-2\frac{\partial U}{\partial r}\frac{\partial V}{\partial x}\right.\\
&\left.-2\frac{\partial U}{\partial\theta}\frac{\partial\Omega}{\partial x}-2\frac{\partial V}{\partial\theta}\frac{\partial\Omega}{\partial r}+\frac{1}{r^2}\left(\frac{\partial U}{\partial\theta}\right)^2+\frac{1}{r^2}\left(\frac{\partial V}{\partial\theta}\right)^2-4\frac{\Omega}{r}\frac{\partial V}{\partial\theta}\right] \qquad\qquad \text{(C.27)}\\
W^* = &Eq.(C.41) \qquad\qquad \text{(C.28)}
\end{aligned}
$$

C.3 Another form of τ_{ij}

In terms of S_{ij} and Ω_{ij}, the components of τ_{ij} can be written as

$$
\begin{aligned}
\tau_{xx} = &2A_1\frac{\rho k^3}{\varepsilon^2}\left(S_{11}S_{11}+S_{12}S_{21}+\frac{1}{r^2}S_{13}S_{31}-\frac{1}{3}S^{(2)}\right)\\
&+2A_2\frac{\rho k^3}{\varepsilon^2}\left(\Omega_{12}\Omega_{21}+\frac{1}{r^2}\Omega_{13}\Omega_{31}-\frac{1}{3}\Omega^{(2)}\right)+2A_3\frac{\rho k^3}{\varepsilon^2}\left(S_{12}\Omega_{21}+\frac{1}{r^2}S_{13}\Omega_{31}\right)\\
&+2A_4\frac{\rho k^4}{\varepsilon^3}\left[2S_{11}\left(S_{12}\Omega_{21}+\frac{1}{r^2}S_{13}\Omega_{31}\right)+2S_{12}\left(S_{22}\Omega_{21}+\frac{1}{r^2}S_{23}\Omega_{31}\right)\right.\\
&\left.\quad+\frac{2}{r^2}S_{13}\left(S_{32}\Omega_{21}+\frac{1}{r^2}S_{33}\Omega_{31}\right)\right]\\
&-2A_5\frac{\rho k^4}{\varepsilon^3}\left[\Omega_{12}\left(S_{22}\Omega_{21}+\frac{1}{r^2}S_{23}\Omega_{31}\right)+\frac{1}{r^2}\Omega_{13}\left(S_{32}\Omega_{21}+\frac{1}{r^2}S_{33}\Omega_{31}\right)\right.\\
&\left.\quad-\frac{1}{3}\overline{\Omega S\Omega}+II_S\left(S_{11}-\frac{1}{3}\Theta\right)-I_S\left(S_{11}S_{11}+S_{12}S_{21}+\frac{1}{r^2}S_{13}S_{31}-\frac{1}{3}S^{(2)}\right)\right] \qquad \text{(C.29)}
\end{aligned}
$$

$$
\begin{aligned}
\tau_{xr} = {} & 2A_1\frac{\rho k^3}{\varepsilon^2}\left(S_{11}S_{12}+S_{12}S_{22}+\frac{1}{r^2}S_{13}S_{32}\right)+2A_2\frac{\rho k^3}{\varepsilon^2}\frac{1}{r^2}\Omega_{13}\Omega_{32} \\
& +A_3\frac{\rho k^3}{\varepsilon^2}\left[\Omega_{12}\left(S_{11}-S_{22}\right)+\frac{1}{r^2}\left(S_{13}\Omega_{32}-\Omega_{13}S_{32}\right)\right] \\
& +2A_4\frac{\rho k^4}{\varepsilon^3}\left[S_{11}\left(S_{11}\Omega_{12}+\frac{1}{r^2}S_{13}\Omega_{32}\right)+S_{12}\frac{1}{r^2}S_{23}\Omega_{32}\right. \\
& \quad +\frac{1}{r^2}S_{13}\left(S_{31}\Omega_{12}+\frac{1}{r^2}S_{33}\Omega_{32}\right)-\Omega_{12}\left(S_{22}S_{22}+\frac{1}{r^2}S_{23}S_{32}\right) \\
& \quad \left.-\frac{1}{r^2}\Omega_{13}\left(S_{31}S_{12}+S_{32}S_{22}+\frac{1}{r^2}S_{33}S_{32}\right)\right] \\
& -2A_5\frac{\rho k^4}{\varepsilon^3}\left[\Omega_{12}\left(S_{21}\Omega_{12}+\frac{1}{r^2}S_{23}\Omega_{32}\right)+\frac{1}{r^2}\Omega_{13}\left(S_{31}\Omega_{12}+\frac{1}{r^2}S_{33}\Omega_{32}\right)\right. \\
& \quad \left.+II_S S_{12}-I_S\left(S_{11}S_{12}+S_{12}S_{22}+\frac{1}{r^2}S_{13}S_{32}\right)\right]
\end{aligned} \tag{C.30}
$$

$$
\begin{aligned}
\tau_{x\theta} = {} & 2A_1\frac{\rho k^3}{\varepsilon^2}\left(S_{11}S_{13}+S_{12}S_{23}+\frac{1}{r^2}S_{13}S_{33}\right)+2A_2\frac{\rho k^3}{\varepsilon^2}\Omega_{12}\Omega_{23} \\
& +A_3\frac{\rho k^3}{\varepsilon^2}\left[S_{11}\Omega_{13}-\Omega_{12}S_{23}+\frac{1}{r^2}\Omega_{23}\left(S_{12}-S_{32}\right)\right] \\
& +2A_4\frac{\rho k^4}{\varepsilon^3}\left[S_{11}\left(S_{11}\Omega_{13}+S_{12}\Omega_{23}\right)+S_{12}(S_{21}\Omega_{13}+S_{22}\Omega_{23})\right. \\
& \quad +\frac{1}{r^2}S_{13}S_{32}\Omega_{23}-\Omega_{12}\left(S_{21}S_{13}+S_{22}S_{23}\frac{1}{r^2}S_{23}S_{33}\right) \\
& \quad \left.-\frac{1}{r^2}\Omega_{13}\left(S_{32}S_{23}+\frac{1}{r^2}S_{33}S_{33}\right)\right] \\
& -2A_5\frac{\rho k^4}{\varepsilon^3}\left[\Omega_{12}\left(S_{21}\Omega_{13}+S_{22}\Omega_{23}\right)+\frac{1}{r^2}S_{13}\left(S_{31}\Omega_{13}+S_{32}\Omega_{23}\right)\right. \\
& \quad \left.+II_S S_{13}-I_S\left(S_{11}S_{13}+S_{12}S_{23}+\frac{1}{r^2}S_{13}S_{33}\right)\right]
\end{aligned} \tag{C.31}
$$

$$
\begin{aligned}
\tau_{rr} = {} & 2A_1\frac{\rho k^3}{\varepsilon^2}\left(S_{21}S_{12}+S_{22}S_{22}+\frac{1}{r^2}S_{23}S_{32}-\frac{1}{3}S^{(2)}\right) \\
& +2A_2\frac{\rho k^3}{\varepsilon^2}\left(\Omega_{12}\Omega_{21}+\frac{1}{r^2}\Omega_{23}\Omega_{32}-\frac{1}{3}\Omega^{(2)}\right)+2A_3\frac{\rho k^3}{\varepsilon^2}\left(S_{21}\Omega_{12}+\frac{1}{r^2}S_{23}\Omega_{32}\right) \\
& +2A_4\frac{\rho k^4}{\varepsilon^3}\left[2S_{21}\left(S_{11}\Omega_{12}+\frac{1}{r^2}S_{13}\Omega_{32}\right)+2S_{22}\left(S_{21}\Omega_{12}+\frac{1}{r^2}S_{23}\Omega_{32}\right)\right. \\
& \quad \left.+\frac{2}{r^2}S_{23}\left(S_{31}\Omega_{12}+\frac{1}{r^2}S_{33}\Omega_{32}\right)\right] \\
& -2A_5\frac{\rho k^4}{\varepsilon^3}\left[\Omega_{12}\left(S_{11}\Omega_{12}+\frac{1}{r^2}S_{13}\Omega_{32}\right)+\frac{1}{r^2}\Omega_{23}\left(S_{31}\Omega_{12}+\frac{1}{r^2}S_{33}\Omega_{32}\right)\right. \\
& \quad \left.-\frac{1}{3}\overline{\Omega S\Omega}+II_S\left(S_{22}-\frac{1}{3}\Theta\right)-I_S\left(S_{21}S_{12}+S_{22}S_{22}+\frac{1}{r^2}S_{23}S_{32}-\frac{1}{3}S^{(2)}\right)\right]
\end{aligned} \tag{C.32}
$$

$$
\begin{aligned}
\tau_{r\theta} = {} & 2A_1\frac{\rho k^3}{\varepsilon^2}\left(S_{21}S_{13} + S_{22}S_{23} + \frac{1}{r^2}S_{23}S_{33}\right) + 2A_2\frac{\rho k^3}{\varepsilon^2}\Omega_{21}\Omega_{13} \\
& + A_3\frac{\rho k^3}{\varepsilon^2}\left(S_{21}\Omega_{13} + S_{22}\Omega_{23} - \Omega_{21}S_{13} - \frac{1}{r^2}\Omega_{23}S_{33}\right) \\
& + 2A_4\frac{\rho k^4}{\varepsilon^3}\Big[S_{21}\left(S_{11}\Omega_{13} + S_{12}\Omega_{23}\right) + S_{22}(S_{21}\Omega_{13} + S_{22}\Omega_{23}) \\
& \quad + \frac{1}{r^2}S_{23}S_{31}\Omega_{13} - \Omega_{21}\left(S_{11}S_{13} + S_{12}S_{23}\frac{1}{r^2}S_{13}S_{33}\right) \\
& \quad - \frac{1}{r^2}\Omega_{23}\left(S_{31}S_{13} + \frac{1}{r^2}S_{33}S_{33}\right)\Big] \\
& - 2A_5\frac{\rho k^4}{\varepsilon^3}\Big[\Omega_{12}\left(S_{11}\Omega_{13} + S_{12}\Omega_{23}\right) + \frac{1}{r^2}S_{23}\left(S_{31}\Omega_{13} + S_{32}\Omega_{23}\right) \\
& \quad + II_S S_{23} - I_S\left(S_{21}S_{13} + S_{22}S_{23} + \frac{1}{r^2}S_{23}S_{33}\right)\Big]
\end{aligned}
\tag{C.33}
$$

$$
\begin{aligned}
\tau_{\theta\theta} = {} & 2A_1\frac{\rho k^3}{\varepsilon^2}\left(S_{31}S_{13} + S_{32}S_{23} + \frac{1}{r^2}S_{33}S_{33} - \frac{1}{3}r^2 S^{(2)}\right) \\
& + 2A_2\frac{\rho k^3}{\varepsilon^2}\left(\Omega_{31}\Omega_{13} + \Omega_{32}\Omega_{23} - \frac{1}{3}r^2\Omega^{(2)}\right) + 2A_3\frac{\rho k^3}{\varepsilon^2}\left(S_{31}\Omega_{13} + S_{32}\Omega_{23}\right) \\
& + 2A_4\frac{\rho k^4}{\varepsilon^3}\Big[2S_{31}\left(S_{11}\Omega_{13} + S_{12}\Omega_{23}\right) + 2S_{32}\left(S_{21}\Omega_{13} + S_{22}\Omega_{23}\right) \\
& \quad - \frac{1}{r^2}S_{33}\left(S_{13}\Omega_{31} + S_{23}\Omega_{32}\right)\Big] \\
& - 2A_5\frac{\rho k^4}{\varepsilon^3}\Big[\Omega_{31}\left(S_{11}\Omega_{13} + S_{12}\Omega_{23}\right) + \Omega_{32}\left(S_{21}\Omega_{13} + S_{22}\Omega_{23}\right) - \frac{1}{3}r^2\overline{\Omega S \Omega} \\
& \quad + II_S\left(S_{33} - \frac{1}{3}r^2\Theta\right) - I_S\left(S_{31}S_{13} + S_{32}S_{23} + \frac{1}{r^2}S_{33}S_{33} - \frac{1}{3}r^2 S^{(2)}\right)\Big]
\end{aligned}
\tag{C.34}
$$

The scalars that appear in the above equations are as follows

$$\Theta = S_{11} + S_{22} + \frac{1}{r^2}S_{33} \tag{C.35}$$

$$S^{(2)} = S_{11}S_{11} + S_{22}S_{22} + \frac{1}{r^4}S_{33}S_{33} + 2S_{12}S_{12} + \frac{2}{r^2}(S_{13}S_{13} + S_{23}S_{23}) \tag{C.36}$$

$$\Omega^{(2)} = 2\Omega_{12}\Omega_{12} + \frac{2}{r^2}(\Omega_{13}\Omega_{31} + \Omega_{23}\Omega_{32}) \tag{C.37}$$

$$
\begin{aligned}
\overline{\Omega S \Omega} = {} & \Omega_{12}\Omega_{21}(S_{11} + S_{22}) + \Omega_{12}\Omega_{31}\frac{1}{r^2}(S_{23} + S_{32}) + \Omega_{21}\Omega_{32}\frac{1}{r^2}(S_{13} + S_{31}) \\
& + \frac{1}{r^2}\Omega_{31}\Omega_{23}(S_{12} + S_{21}) + \frac{1}{r^2}\Omega_{13}\Omega_{31}(S_{11} + \frac{1}{r^2}S_{33}) + \frac{1}{r^2}\Omega_{23}\Omega_{32}(S_{22} + \frac{1}{r^2}S_{33})
\end{aligned}
\tag{C.38}
$$

$$I_S = \Theta\,, \quad II_S = \frac{1}{2}(\Theta^2 - S^{(2)}) \tag{C.39}$$

Two other scalars $(S^*)^2$ and $(\Omega^*)^2$ can be expressed as

$$(S^*)^2 = S^{(2)} - \frac{1}{3}\Theta^2 \,, \quad (\Omega^*)^2 = \Omega^{(2)} \tag{C.40}$$

$$\begin{aligned}
W^* = \frac{1}{(S*)^3} \Big[& S^*_{11}\left(S^*_{12}S^*_{12} + S^*_{11}S^*_{11} + \frac{1}{r^2}S^*_{13}S^*_{13}\right) \\
& + S^*_{12}\left(S^*_{21}S^*_{12} + S^*_{22}S^*_{21} + \frac{1}{r^2}S^*_{23}S^*_{31}\right) \\
& + \frac{1}{r^2}S^*_{13}\left(S^*_{31}S^*_{11} + S^*_{32}S^*_{21} + \frac{1}{r^2}S^*_{33}S^*_{31}\right) \\
& + S^*_{21}\left(S^*_{11}S^*_{12} + S^*_{12}S^*_{22} + \frac{1}{r^2}S^*_{13}S^*_{32}\right) \\
& + S^*_{22}\left(S^*_{21}S^*_{12} + S^*_{22}S^*_{22} + \frac{1}{r^2}S^*_{23}S^*_{32}\right) \\
& + \frac{1}{r^2}S^*_{23}\left(S^*_{31}S^*_{12} + S^*_{32}S^*_{22} + \frac{1}{r^2}S^*_{33}S^*_{32}\right) \\
& + \frac{1}{r^2}S^*_{31}\left(S^*_{11}S^*_{13} + S^*_{12}S^*_{23} + \frac{1}{r^2}S^*_{13}S^*_{33}\right) \\
& + \frac{1}{r^2}S^*_{32}\left(S^*_{21}S^*_{13} + S^*_{22}S^*_{23} + \frac{1}{r^2}S^*_{23}S^*_{33}\right) \\
& + \frac{1}{r^4}S^*_{33}\left(S^*_{31}S^*_{13} + S^*_{32}S^*_{23} + \frac{1}{r^2}S^*_{33}S^*_{33}\right) \Big]
\end{aligned} \tag{C.41}$$

where

$$\begin{aligned}
& S^*_{11} = S_{11} - \frac{1}{3}\Theta, \quad S^*_{22} = S_{22} - \frac{1}{3}\Theta, \quad S^*_{33} = S_{33} - \frac{1}{3}r^2\Theta, \\
& S^*_{12} = S_{12}, \quad S^*_{13} = S_{13}, \quad S^*_{23} = S_{23}
\end{aligned} \tag{C.42}$$

Finally, the six components of S_{ij} and the three components of Ω_{ij} (note that $S_{ij} = S_{ji}$ and $\Omega_{ij} = -\Omega_{ji}$) are

$$\begin{aligned}
& S_{11} = \frac{\partial U}{\partial x}, \quad S_{12} = \frac{1}{2}\left(\frac{\partial U}{\partial r} + \frac{\partial V}{\partial x}\right), \quad S_{13} = \frac{1}{2}\left(\frac{\partial U}{\partial \theta} + r^2\frac{\partial \Omega}{\partial x}\right) \\
& S_{22} = \frac{\partial V}{\partial r}, \quad S_{23} = \frac{1}{2}\left(\frac{\partial V}{\partial \theta} + r^2\frac{\partial \Omega}{\partial r}\right), \quad S_{33} = r^2\frac{\partial \Omega}{\partial \theta} + rV \\
& \Omega_{12} = \frac{1}{2}\left(\frac{\partial U}{\partial r} - \frac{\partial V}{\partial x}\right), \quad \Omega_{13} = \frac{1}{2}\left(\frac{\partial U}{\partial \theta} - r^2\frac{\partial \Omega}{\partial x}\right), \quad \Omega_{23} = \frac{1}{2}\left(\frac{\partial V}{\partial \theta} - \frac{\partial r^2\Omega}{\partial r}\right)
\end{aligned} \tag{C.43}$$

D Appendix: Equations for Axisymmetric Flows

Continuity equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho U}{\partial x} + \frac{\partial \rho V}{\partial r} + \frac{\rho V}{r} = 0 \tag{D.1}$$

Momentum equations

$$\begin{aligned}
\frac{\partial \rho U}{\partial t} + \frac{\partial \rho U^2}{\partial x} + \frac{\partial \rho UV}{\partial r} + \frac{\rho UV}{r} &= -\frac{\partial \overline{P}}{\partial x} \\
&+ \frac{\partial}{\partial x}\left[2(\mu+\mu_T)\left(\frac{\partial U}{\partial x} - \frac{1}{3}\Theta\right)\right] + \frac{\partial}{\partial r}\left[(\mu+\mu_T)\left(\frac{\partial U}{\partial r} + \frac{\partial V}{\partial x}\right)\right] \\
&+ \frac{1}{r}(\mu+\mu_T)\left(\frac{\partial U}{\partial r} + \frac{\partial V}{\partial x}\right) + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xr}}{\partial r} + \frac{1}{r}\tau_{xr}
\end{aligned} \tag{D.2}$$

$$\begin{aligned}
\frac{\partial \rho V}{\partial t} + \frac{\partial \rho UV}{\partial x} + \frac{\partial \rho V^2}{\partial r} - r\rho\Omega^2 + \frac{\rho V^2}{r} &= -\frac{\partial \overline{P}}{\partial r} \\
&+ \frac{\partial}{\partial x}\left[(\mu+\mu_T)\left(\frac{\partial V}{\partial x} + \frac{\partial U}{\partial r}\right)\right] + \frac{\partial}{\partial r}\left[2(\mu+\mu_T)\left(\frac{\partial V}{\partial r} - \frac{1}{3}\Theta\right)\right] \\
&+ \frac{2}{r}(\mu+\mu_T)\left(\frac{\partial V}{\partial r} - \frac{1}{3}\Theta\right) - \frac{2}{r^3}(\mu+\mu_T)\left(rV - \frac{1}{3}r^2\Theta\right) \\
&+ \frac{\partial \tau_{rx}}{\partial x} + \frac{\partial \tau_{rr}}{\partial r} - \frac{1}{r^3}\tau_{\theta\theta} + \frac{1}{r}\tau_{rr}
\end{aligned} \tag{D.3}$$

$$\begin{aligned}
\frac{\partial \rho r^2\Omega}{\partial t} + \frac{\partial \rho r^2\Omega U}{\partial x} + \frac{\partial \rho r^2\Omega V}{\partial r} + r\rho V\Omega &= \frac{\partial}{\partial x}\left[(\mu+\mu_T)\frac{\partial r^2\Omega}{\partial x}\right] \\
&+ \frac{\partial}{\partial r}\left[(\mu+\mu_T)\frac{\partial r^2\Omega}{\partial r}\right] + \frac{1}{r}(\mu+\mu_T)\frac{\partial r^2\Omega}{\partial r} \\
&- \frac{2}{r}\frac{\partial}{\partial r}\left[(\mu+\mu_T)r^2\Omega\right] + \frac{\partial \tau_{\theta x}}{\partial x} + \frac{\partial \tau_{\theta r}}{\partial r} + \frac{1}{r}\tau_{r\theta}
\end{aligned} \tag{D.4}$$

where

$$\Theta = \frac{\partial U}{\partial x} + \frac{\partial V}{\partial r} + \frac{V}{r} \tag{D.5}$$

$$\overline{P} = P + \frac{2}{3}k \tag{D.6}$$

k-ε equations

$$\frac{\partial \rho k}{\partial t} + \frac{\partial \rho Uk}{\partial x} + \frac{\partial \rho Vk}{\partial r} + \frac{V}{r}\rho k = \frac{\partial}{\partial x}\left[\left(\mu + \frac{\mu_T}{\sigma_k}\right)\frac{\partial k}{\partial x}\right]$$

$$+\frac{\partial}{\partial r}\left[\left(\mu+\frac{\mu_T}{\sigma_k}\right)\frac{\partial k}{\partial r}\right]+\frac{1}{r}\left(\mu+\frac{\mu_T}{\sigma_k}\right)\frac{\partial k}{\partial r}+P_k-\rho\varepsilon \tag{D.7}$$

$$\frac{\partial\rho\varepsilon}{\partial t}+\frac{\partial\rho U\varepsilon}{\partial x}+\frac{\partial\rho V\varepsilon}{\partial r}+\frac{V}{r}\rho\varepsilon=\frac{\partial}{\partial x}\left[\left(\mu+\frac{\mu_T}{\sigma_\varepsilon}\right)\frac{\partial\varepsilon}{\partial x}\right]$$
$$+\frac{\partial}{\partial r}\left[\left(\mu+\frac{\mu_T}{\sigma_\varepsilon}\right)\frac{\partial\varepsilon}{\partial r}\right]+\frac{1}{r}\left(\mu+\frac{\mu_T}{\sigma_\varepsilon}\right)\frac{\partial\varepsilon}{\partial r}+C_{\varepsilon1}f_1\frac{\varepsilon}{k}P_k-C_{\varepsilon2}f_2\frac{\rho\varepsilon^2}{k}$$
$$+C_{\varepsilon3}\frac{\mu\mu_T}{\rho}\left[\left(\frac{\partial S}{\partial r}\right)^2+\left(\frac{\partial S}{\partial x}\right)^2\right] \tag{D.8}$$

where

$$P_k=P_k^{(1)}+P_k^{(2)} \tag{D.9}$$

$$P_k^{(1)}=-\frac{2}{3}(\rho k+\mu_T\Theta)\Theta$$
$$+\mu_T\left[2\left(\frac{\partial U}{\partial x}\right)^2+\frac{\partial V}{\partial x}\left(\frac{\partial V}{\partial x}+\frac{\partial U}{\partial r}\right)+r^2\left(\frac{\partial\Omega}{\partial x}\right)^2\right]$$
$$+\mu_T\left[\frac{\partial U}{\partial r}\left(\frac{\partial U}{\partial r}+\frac{\partial V}{\partial x}\right)+2\left(\frac{\partial V}{\partial r}\right)^2+r^2\left(\frac{\partial\Omega}{\partial r}\right)^2+2\frac{V^2}{r^2}\right] \tag{D.10}$$

$$P_k^{(2)}=\tau_{xx}\frac{\partial U}{\partial x}+\tau_{rx}\frac{\partial V}{\partial x}+\tau_{\theta x}\frac{\partial\Omega}{\partial x}+\tau_{xr}\frac{\partial U}{\partial r}+\tau_{rr}\frac{\partial V}{\partial r}$$
$$+\tau_{\theta r}\left(\frac{\partial\Omega}{\partial r}+\frac{\Omega}{r}\right)+\frac{1}{r^2}\left(-\tau_{r\theta}\,r\Omega+\tau_{\theta\theta}\frac{V}{r}\right) \tag{D.11}$$

<u>τ_{ij} in axisymmetric flows</u>

$$\tau_{xx}=C_1\frac{\rho k^3}{\varepsilon^2}\left[2\left(\frac{\partial U}{\partial x}\right)^2+2\frac{\partial U}{\partial r}\frac{\partial V}{\partial x}-\frac{2}{3}\Pi_1\right]$$
$$+C_2\frac{\rho k^3}{\varepsilon^2}\left[\left(\frac{\partial U}{\partial x}\right)^2+\left(\frac{\partial U}{\partial r}\right)^2-\frac{1}{3}\Pi_2\right]$$
$$+C_3\frac{\rho k^3}{\varepsilon^2}\left[\left(\frac{\partial U}{\partial x}\right)^2+\left(\frac{\partial V}{\partial x}\right)^2+r^2\left(\frac{\partial\Omega}{\partial x}\right)^2-\frac{1}{3}\Pi_2\right]$$
$$+C_4\frac{\rho k^4}{\varepsilon^3}\left[2\frac{\partial U}{\partial x}\left(\frac{\partial U}{\partial x}\frac{\partial U}{\partial x}+\frac{\partial U}{\partial r}\frac{\partial V}{\partial x}\right)+2\frac{\partial U}{\partial r}\left(\frac{\partial U}{\partial x}\frac{\partial U}{\partial r}+\frac{\partial U}{\partial r}\frac{\partial V}{\partial r}\right)-\frac{2}{3}\Pi_3\right]$$
$$+C_5\frac{\rho k^4}{\varepsilon^3}\left[2\frac{\partial U}{\partial x}\left(\frac{\partial U}{\partial x}\frac{\partial U}{\partial x}+\frac{\partial U}{\partial r}\frac{\partial V}{\partial x}\right)+2\frac{\partial V}{\partial x}\left(\frac{\partial V}{\partial x}\frac{\partial U}{\partial x}+\frac{\partial V}{\partial r}\frac{\partial V}{\partial x}\right)\right.$$
$$\left.+2r^2\frac{\partial\Omega}{\partial x}\left(\frac{\partial\Omega}{\partial x}\frac{\partial U}{\partial x}+\frac{\partial\Omega}{\partial r}\frac{\partial V}{\partial x}\right)+2rV\left(\frac{\partial\Omega}{\partial x}\right)^2-\frac{2}{3}\Pi_3\right] \tag{D.12}$$

$$\tau_{xr} = C_1\frac{\rho k^3}{\varepsilon^2}\left[\frac{\partial U}{\partial x}\frac{\partial U}{\partial r} + \frac{\partial U}{\partial r}\frac{\partial V}{\partial r} + \frac{\partial U}{\partial x}\frac{\partial V}{\partial x} + \frac{\partial V}{\partial r}\frac{\partial V}{\partial x} - r\Omega\frac{\partial \Omega}{\partial x}\right]$$
$$+ C_2\frac{\rho k^3}{\varepsilon^2}\left[\frac{\partial U}{\partial x}\frac{\partial V}{\partial x} + \frac{\partial U}{\partial r}\frac{\partial V}{\partial r}\right]$$
$$+ C_3\frac{\rho k^3}{\varepsilon^2}\left[\frac{\partial U}{\partial x}\frac{\partial U}{\partial r} + \frac{\partial U}{\partial x}\frac{\partial V}{\partial r} + r\Omega\frac{\partial \Omega}{\partial x} + r^2\frac{\partial \Omega}{\partial x}\frac{\partial \Omega}{\partial r}\right]$$
$$+ C_4\frac{\rho k^4}{\varepsilon^3}\left[\frac{\partial U}{\partial x}\left(2\frac{\partial V}{\partial x}\frac{\partial U}{\partial x} + \frac{\partial V}{\partial r}\frac{\partial V}{\partial x}\right) + \frac{\partial U}{\partial r}\left(\frac{\partial V}{\partial x}\frac{\partial U}{\partial r} + 2\frac{\partial V}{\partial r}\frac{\partial V}{\partial r}\right)\right.$$
$$\left. -r\Omega\left(\frac{\partial U}{\partial x}\frac{\partial \Omega}{\partial x} + \frac{\partial U}{\partial r}\frac{\partial \Omega}{\partial r}\right) + \frac{\partial V}{\partial x}\frac{\partial V}{\partial x}\frac{\partial U}{\partial r} + \frac{\partial V}{\partial r}\frac{\partial U}{\partial x}\frac{\partial U}{\partial r}\right]$$
$$+ C_5\frac{\rho k^4}{\varepsilon^3}\left[\frac{\partial U}{\partial x}\left(2\frac{\partial U}{\partial x}\frac{\partial U}{\partial r} + \frac{\partial U}{\partial r}\frac{\partial V}{\partial r}\right) + \frac{\partial V}{\partial x}\left(\frac{\partial V}{\partial x}\frac{\partial U}{\partial r} + 2\frac{\partial V}{\partial r}\frac{\partial V}{\partial r}\right)\right.$$
$$+\frac{\partial \Omega}{\partial x}\left(r^2\frac{\partial \Omega}{\partial x}\frac{\partial U}{\partial r} + \frac{\partial r^2\Omega}{\partial r}\frac{\partial V}{\partial r}\right) - 2r\Omega\left(\frac{\partial V}{\partial x}\frac{\partial \Omega}{\partial r} + \frac{\partial V}{\partial r}\frac{\partial \Omega}{\partial x}\right)$$
$$+2rV\frac{\partial \Omega}{\partial r} + \frac{\partial U}{\partial r}\frac{\partial U}{\partial r}\frac{\partial V}{\partial x} + \frac{\partial V}{\partial r}\frac{\partial V}{\partial x}\frac{\partial U}{\partial x} + \frac{\partial \Omega}{\partial r}\left(r^2\frac{\partial \Omega}{\partial x}\frac{\partial U}{\partial x} + \frac{\partial r^2\Omega}{\partial r}\frac{\partial V}{\partial x}\right)$$
$$\left.+\frac{\Omega}{r}\left(r^2\frac{\partial \Omega}{\partial x}\frac{\partial U}{\partial x} + \frac{\partial r^2\Omega}{\partial r}\frac{\partial V}{\partial x}\right) + 2\Omega V\frac{\partial \Omega}{\partial x} - 2\Omega^2\frac{\partial V}{\partial x}\right] \tag{D.13}$$

$$\tau_{x\theta} = C_1\frac{\rho k^3}{\varepsilon^2}\left[-r\Omega\left(\frac{\partial U}{\partial r} + \frac{\partial V}{\partial x}\right) + rV\frac{\partial \Omega}{\partial x} + r^2\frac{\partial \Omega}{\partial x}\frac{\partial U}{\partial x} + \frac{\partial r^2\Omega}{\partial r}\frac{\partial V}{\partial x}\right]$$
$$+ C_2\frac{\rho k^3}{\varepsilon^2}\left[r^2\frac{\partial U}{\partial x}\frac{\partial \Omega}{\partial x} + \frac{\partial U}{\partial r}\frac{\partial r^2\Omega}{\partial r} - r\Omega\frac{\partial U}{\partial r}\right]$$
$$+ C_3\frac{\rho k^3}{\varepsilon^2}\left[rV\frac{\partial \Omega}{\partial x} - r\Omega\frac{\partial V}{\partial x}\right]$$
$$+ C_4\frac{\rho k^4}{\varepsilon^3}\left[\frac{\partial U}{\partial x}\left(2r^2\frac{\partial \Omega}{\partial x}\frac{\partial U}{\partial x} + \frac{\partial r^2\Omega}{\partial r}\frac{\partial V}{\partial x}\right) + \frac{\partial U}{\partial r}\left(r^2\frac{\partial \Omega}{\partial x}\frac{\partial U}{\partial r} + 2\frac{\partial r^2\Omega}{\partial r}\frac{\partial V}{\partial r}\right)\right.$$
$$+rV\left(\frac{\partial U}{\partial x}\frac{\partial \Omega}{\partial x} + \frac{\partial U}{\partial r}\frac{\partial \Omega}{\partial r}\right) - r\Omega\left(\frac{\partial U}{\partial x}\frac{\partial V}{\partial x} + 2\frac{\partial U}{\partial r}\frac{\partial V}{\partial r}\right) + r^2\frac{\partial \Omega}{\partial x}\frac{\partial U}{\partial r}\frac{\partial V}{\partial x}$$
$$\left.+\frac{\partial r^2\Omega}{\partial r}\frac{\partial U}{\partial x}\frac{\partial U}{\partial r} - r\Omega\frac{\partial U}{\partial x}\frac{\partial U}{\partial r}\right]$$
$$+ C_5\frac{\rho k^4}{\varepsilon^3}\left[\frac{V}{r}\left(r^2\frac{\partial \Omega}{\partial x}\frac{\partial U}{\partial x} + \frac{\partial r^2\Omega}{\partial r}\frac{\partial V}{\partial x}\right) - r\Omega\left(\frac{\partial V}{\partial x}\frac{\partial U}{\partial x} + 2\frac{\partial V}{\partial r}\frac{\partial V}{\partial x}\right)\right.$$
$$\left.-2\Omega V\frac{\partial V}{\partial x} + 2r^2\Omega^2\frac{\partial \Omega}{\partial x} + 2V^2\frac{\partial \Omega}{\partial x} - r\Omega\left(\frac{\partial U}{\partial x}\frac{\partial U}{\partial r} + \frac{\partial \Omega}{\partial x}\frac{\partial r^2\Omega}{\partial r}\right)\right] \tag{D.14}$$

$$\tau_{rr} = C_1\frac{\rho k^3}{\varepsilon^2}\left[2\left(\frac{\partial V}{\partial x}\frac{\partial U}{\partial r} + \frac{\partial V}{\partial r}\frac{\partial V}{\partial r}\right) - 2r\Omega\frac{\partial \Omega}{\partial r} - 2\Omega^2 - \frac{2}{3}\Pi_1\right]$$

$$+ C_2 \frac{\rho k^3}{\varepsilon^2} \left[\left(\frac{\partial V}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial r}\right)^2 - \Omega^2 - \frac{1}{3}\Pi_2 \right]$$
$$+ C_3 \frac{\rho k^3}{\varepsilon^2} \left[\left(\frac{\partial U}{\partial r}\right)^2 + \left(\frac{\partial V}{\partial r}\right)^2 + \frac{\partial r^2\Omega}{\partial r}\frac{\partial \Omega}{\partial r} + \Omega^2 - \frac{1}{3}\Pi_2 \right]$$
$$+ C_4 \frac{\rho k^4}{\varepsilon^3} \left[2\frac{\partial V}{\partial x}\left(\frac{\partial U}{\partial x}\frac{\partial V}{\partial x} + \frac{\partial V}{\partial r}\frac{\partial V}{\partial x}\right) + 2\frac{\partial V}{\partial r}\left(\frac{\partial V}{\partial x}\frac{\partial U}{\partial r} + \frac{\partial V}{\partial r}\frac{\partial V}{\partial r}\right) \right.$$
$$\left. -2r\Omega\left(\frac{\partial V}{\partial x}\frac{\partial \Omega}{\partial x} + \frac{\partial V}{\partial r}\frac{\partial \Omega}{\partial r}\right) + 2\frac{\Omega^2 V}{r} - \frac{2}{3}\Pi_3 \right]$$
$$+ C_5 \frac{\rho k^4}{\varepsilon^3} \left[2\frac{\partial U}{\partial r}\left(\frac{\partial U}{\partial r}\frac{\partial U}{\partial x} + \frac{\partial V}{\partial r}\frac{\partial V}{\partial x} + r^2\frac{\partial \Omega}{\partial r}\frac{\partial \Omega}{\partial x}\right) \right.$$
$$+2\frac{\partial V}{\partial r}\left(\frac{\partial U}{\partial r}\frac{\partial U}{\partial r} + \frac{\partial V}{\partial r}\frac{\partial V}{\partial r} + \frac{\partial \Omega}{\partial r}\frac{\partial r^2\Omega}{\partial r}\right) - 4r\Omega\frac{\partial V}{\partial r}\frac{\partial \Omega}{\partial r} + 2rV\left(\frac{\partial \Omega}{\partial r}\right)^2$$
$$\left. +2\frac{\Omega}{r}\left(r^2\frac{\partial \Omega}{\partial x}\frac{\partial U}{\partial r} + \frac{\partial r^2\Omega}{\partial r}\frac{\partial V}{\partial r}\right) + 4\Omega V\frac{\partial \Omega}{\partial r} - 4\Omega^2\frac{\partial V}{\partial r} + 2\frac{\Omega^2}{r}V - \frac{2}{3}\Pi_3 \right] \quad \text{(D.15)}$$

$$\tau_{r\theta} = C_1 \frac{\rho k^3}{\varepsilon^2} \left[-2r\Omega\frac{\partial V}{\partial r} + r^2\frac{\partial \Omega}{\partial x}\frac{\partial U}{\partial r} + \frac{\partial r^2\Omega}{\partial r}\frac{\partial V}{\partial r} + rV\frac{\partial \Omega}{\partial r} \right]$$
$$+ C_2 \frac{\rho k^3}{\varepsilon^2} \left[r^2\frac{\partial V}{\partial x}\frac{\partial \Omega}{\partial x} + \frac{\partial V}{\partial r}\frac{\partial r^2\Omega}{\partial r} - r\Omega\frac{\partial V}{\partial r} + \Omega V \right]$$
$$+ C_3 \frac{\rho k^3}{\varepsilon^2} \left[rV\frac{\partial \Omega}{\partial r} - r\Omega\frac{\partial V}{\partial r} + \Omega V \right]$$
$$+ C_4 \frac{\rho k^4}{\varepsilon^3} \left[\frac{\partial V}{\partial x}\left(2r^2\frac{\partial \Omega}{\partial x}\frac{\partial U}{\partial x} + \frac{\partial r^2\Omega}{\partial r}\frac{\partial V}{\partial x}\right) + \frac{\partial V}{\partial r}\left(r^2\frac{\partial \Omega}{\partial x}\frac{\partial U}{\partial r} + 2\frac{\partial r^2\Omega}{\partial r}\frac{\partial V}{\partial r}\right) \right.$$
$$+rV\left(\frac{\partial V}{\partial x}\frac{\partial \Omega}{\partial x} + \frac{\partial V}{\partial r}\frac{\partial \Omega}{\partial r}\right) - r\Omega\left(\frac{\partial V}{\partial x}\frac{\partial V}{\partial x} + \frac{\partial V}{\partial r}\frac{\partial V}{\partial r}\right) + r^2\frac{\partial \Omega}{\partial x}\frac{\partial V}{\partial r}\frac{\partial V}{\partial x}$$
$$+\frac{\partial r^2\Omega}{\partial r}\frac{\partial V}{\partial x}\frac{\partial U}{\partial r} - r\Omega\left(r^2\frac{\partial \Omega}{\partial x}\frac{\partial \Omega}{\partial x} + \frac{\partial r^2\Omega}{\partial r}\frac{\partial \Omega}{\partial r}\right)$$
$$\left. -r\Omega\left(\frac{\partial V}{\partial x}\frac{\partial U}{\partial r} + \frac{\partial V}{\partial r}\frac{\partial V}{\partial r}\right) + r^2\Omega^2\frac{\partial \Omega}{\partial r} - 2\frac{\Omega}{r}V^2 \right]$$
$$+ C_5 \frac{\rho k^4}{\varepsilon^3} \left[\frac{V}{r}\left(r^2\frac{\partial \Omega}{\partial x}\frac{\partial U}{\partial r} + \frac{\partial r^2\Omega}{\partial r}\frac{\partial V}{\partial r}\right) - r\Omega\left(\frac{\partial V}{\partial x}\frac{\partial U}{\partial r} + 2\frac{\partial V}{\partial r}\frac{\partial V}{\partial r}\right) \right.$$
$$\left. -2\Omega V\frac{\partial V}{\partial r} + r^2\Omega^2\frac{\partial \Omega}{\partial r} + 2V^2\frac{\partial \Omega}{\partial r} - r\Omega\left(\frac{\partial U}{\partial r}\frac{\partial U}{\partial r} + \frac{\partial \Omega}{\partial r}\frac{\partial r^2\Omega}{\partial r}\right) + 2\frac{\Omega}{r}V^2 \right] \quad \text{(D.16)}$$

$$\tau_{\theta\theta} = C_1 \frac{\rho k^3}{\varepsilon^2} \left[-2r\Omega\frac{\partial r^2\Omega}{\partial r} + 2V^2 + 2r^2\Omega^2 - \frac{2}{3}r^2\Pi_1 \right]$$

$$+ C_2\frac{\rho k^3}{\varepsilon^2}\left[r^4\left(\frac{\partial\Omega}{\partial x}\right)^2 + \left(\frac{\partial r^2\Omega}{\partial r}\right)^2 - 2r\Omega\frac{\partial r^2\Omega}{\partial r} - r^2\Omega^2 - V^2 - \frac{1}{3}r^2\Pi_2\right]$$
$$+ C_3\frac{\rho k^3}{\varepsilon^2}\left[V^2 + r^2\Omega^2 - \frac{1}{3}r^2\Pi_2\right]$$
$$+ C_4\frac{\rho k^4}{\varepsilon^3}\left[2r^2\frac{\partial\Omega}{\partial x}\left(r^2\frac{\partial\Omega}{\partial x}\frac{\partial U}{\partial x} + \frac{\partial r^2\Omega}{\partial r}\frac{\partial V}{\partial x}\right) + 2\frac{\partial r^2\Omega}{\partial r}\left(r^2\frac{\partial\Omega}{\partial x}\frac{\partial U}{\partial r} + \frac{\partial r^2\Omega}{\partial r}\frac{\partial V}{\partial r}\right)\right.$$
$$+2rV\left(r^2\frac{\partial\Omega}{\partial x}\frac{\partial\Omega}{\partial x} + \frac{\partial r^2\Omega}{\partial r}\frac{\partial\Omega}{\partial r}\right) - 2r\Omega\left(r^2\frac{\partial\Omega}{\partial x}\frac{\partial V}{\partial x} + 2\frac{\partial r^2\Omega}{\partial r}\frac{\partial V}{\partial r}\right)$$
$$\left.-2r^3\Omega\frac{\partial\Omega}{\partial x}\frac{\partial U}{\partial r} - 2r^2\Omega V\frac{\partial\Omega}{\partial r} + 2r^2\Omega^2\frac{\partial V}{\partial r} + 2\frac{V^3}{r} - \frac{2}{3}r^2\Pi_3\right]$$
$$+ C_5\frac{\rho k^4}{\varepsilon^3}\left[-2\Omega V\frac{\partial r^2\Omega}{\partial r} + 2r^2\Omega^2\frac{\partial V}{\partial r} + 2\frac{V^3}{r} + 4r\Omega^2 V - \frac{2}{3}r^2\Pi_3\right] \tag{D.17}$$

where

$$\Pi_1 = -2r\Omega\frac{\partial\Omega}{\partial r} - 2\Omega^2 + \frac{V^2}{r^2} \tag{D.18}$$
$$\Pi_2 = \left(\frac{\partial U}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial x}\right)^2 + r^2\left(\frac{\partial\Omega}{\partial x}\right)^2 + \left(\frac{\partial U}{\partial r}\right)^2 + \left(\frac{\partial V}{\partial r}\right)^2$$
$$+\frac{\partial\Omega}{\partial r}\frac{\partial r^2\Omega}{\partial r} + \frac{V^2}{r^2} + 2\Omega^2 \tag{D.19}$$
$$\Pi_3 = \Pi_3^{(1)} + \Pi_3^{(2)} + \Pi_3^{(3)} \tag{D.20}$$

$$\Pi_3^{(1)} = \left(\frac{\partial U}{\partial x}\right)^3 + \frac{\partial U}{\partial x}\frac{\partial U}{\partial r}\frac{\partial V}{\partial x} + \frac{\partial V}{\partial x}\left(\frac{\partial V}{\partial x}\frac{\partial U}{\partial x} + \frac{\partial V}{\partial r}\frac{\partial V}{\partial x}\right)$$
$$+r^2\frac{\partial\Omega}{\partial x}\left(\frac{\partial\Omega}{\partial x}\frac{\partial U}{\partial x} + \frac{\partial\Omega}{\partial r}\frac{\partial V}{\partial x}\right) + rV\left(\frac{\partial\Omega}{\partial x}\right)^2 \tag{D.21}$$
$$\Pi_3^{(2)} = \frac{\partial U}{\partial r}\left(\frac{\partial U}{\partial x}\frac{\partial U}{\partial r} + \frac{\partial U}{\partial r}\frac{\partial V}{\partial r}\right) + \frac{\partial V}{\partial r}\left(\frac{\partial V}{\partial x}\frac{\partial U}{\partial r} + \frac{\partial V}{\partial r}\frac{\partial V}{\partial r}\right)$$
$$+\frac{\partial\Omega}{\partial r}\left(r^2\frac{\partial\Omega}{\partial x}\frac{\partial U}{\partial r} + \frac{\partial r^2\Omega}{\partial r}\frac{\partial V}{\partial r}\right) - r\Omega\frac{\partial V}{\partial r}\frac{\partial\Omega}{\partial r} + rV\left(\frac{\partial\Omega}{\partial r}\right)^2$$
$$+2V\Omega\frac{\partial\Omega}{\partial r} + r\Omega\frac{\partial\Omega}{\partial x}\frac{\partial U}{\partial r} + \frac{V}{r}\Omega^2 \tag{D.22}$$
$$\Pi_3^{(3)} = -V\Omega\frac{\partial\Omega}{\partial r} + \Omega^2\frac{\partial V}{\partial r} + \frac{V^3}{r^3} \tag{D.23}$$

$$(S^*)^2 = \frac{1}{2}\left[2\left(\frac{\partial U}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial x}\right)^2 + r^2\left(\frac{\partial\Omega}{\partial x}\right)^2 + \left(\frac{\partial U}{\partial r}\right)^2 + 2\left(\frac{\partial V}{\partial r}\right)^2\right.$$

$$+r^2\left(\frac{\partial\Omega}{\partial r}\right)^2 + 2\frac{\partial U}{\partial r}\frac{\partial V}{\partial x} + 2\frac{V^2}{r^2}\Bigg] - \frac{1}{3}\Theta^2 \tag{D.24}$$

$$(\Omega^*)^2 = \frac{1}{2}\Bigg[\left(\frac{\partial V}{\partial x}\right)^2 + r^2\left(\frac{\partial\Omega}{\partial x}\right)^2 + \left(\frac{\partial U}{\partial r}\right)^2$$

$$+r^2\left(\frac{\partial\Omega}{\partial r}\right)^2 + 4r\Omega\frac{\partial\Omega}{\partial r} + 4\Omega^2 - 2\frac{\partial U}{\partial r}\frac{\partial V}{\partial x}\Bigg] \tag{D.25}$$

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13. ABSTRACT ***(Maximum 200 words)***
Aircraft engine combustors generally involve turbulent swirling flows in order to enhance fuel-air mixing and flame stabilization. It has long been recognized that eddy viscosity turbulence models are unable to appropriately model swirling flows. Therefore, it has been suggested that, for the modeling of these flows, a second order closure scheme should be considered because of its ability in the modeling of rotational and curvature effects. However, this scheme will require solution of many complicated second moment transport equations (six Reynolds stresses plus other scalar fluxes and variances), which is a difficult task for any CFD implementations. Also, this scheme will require a large amount of computer resources for a general combustor swirling flow. This report is devoted to the development of a cubic Reynolds stress-strain model for turbulent swirling flows, and was inspired by the work of Launder's group at UMIST. Using this type of model, one only needs to solve two turbulence equations, one for the turbulent kinetic energy k and the other for the dissipation rate ε. The cubic model developed in this report is based on a general Reynolds stress-strain relationship. Two flows have been chosen for model evaluation. One is a fully developed rotating pipe flow, and the other is a more complex flow with swirl and recirculation.

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