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Final Report on NASA MSFC contract NCC8-56

Monitored by: **Martin Volz, Ph.D.**
Microgravity Processing Branch, Mail Code ES75, Bldg. 4493, Room 103
NASA Marshall Space Flight Center
Huntsville, AL 35812
phone: (205) 544-5078 FAX: (205) 544-8762

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ELECTROMAGNETIC FIELD EFFECTS IN SEMICONDUCTOR CRYSTAL GROWTH

Principal Investigator: **George S. Dulikravich, Ph.D., P.E.**
Department of Aerospace Engineering, 233 Hammond Building
The Pennsylvania State University, University Park, PA16802
phone: (814) 863-0134 FAX: (814) 865-7092 E-mail: FT7@PSU.EDU

This final report provides information on the work accomplished by the P. I. and his research team that is directly related to the research tasks as specified in the proposal funded by the contract NCC8-56. Any other results and accomplishments of the P. I. and his research team in the areas not directly related to this NASA contract are excluded from this report.

ABSTRACT

This proposed two-year research project was to involve development of an analytical model, a numerical algorithm for its integration, and a software for the analysis of a solidification process under the influence of electric and magnetic fields in microgravity. Due to the complexity of the analytical model that was developed and its boundary conditions, only a preliminary version of the numerical algorithm was developed while the development of the software package was not completed.

SUMMARY OF TECHNICAL ACCOMPLISHMENTS

We have developed and refined a new analytical formulation of combined electro-magneto-hydrodynamics (EMHD) and have shown the inconsistencies and shortcomings of the existing separate electro-hydrodynamic (EHD) and magneto-hydrodynamic (MHD) theories. The new model consists of a system of at least twelve coupled non-linear partial differential equations in case of a three-dimensional flow. The model is valid for multi-dimensional, unsteady, viscous fluid flows involving electrically charged particles and electric linear polarization and magnetization effects. All interaction between externally applied and internally induced electric and magnetic fields have been incorporated in the model. We have also developed a fully conservative vector operator form of the EMHD system that is suitable for a direct discretization and numerical integration. Derivations of characteristic and non-reflecting open boundary conditions were also performed, thus completing the analytical part of the research effort. A numerical algorithm for iterative integration of the discretized fully conservative EMHD system in boundary-conforming non-orthogonal curvilinear coordinate system was incompletely developed that is based on dual time-stepping. Development of an accompanying computer code for the analysis of unsteady three-dimensional EMHD flows is in its initial debugging stages.

SUMMARY OF MEASURABLE ACCOMPLISHMENTS

INDUSTRY SUPPORT: one

ALCOA Research Center

- provided a Faculty Research Fellowship to the P. I. to work on inverse problems

GOVERNMENT SUPPORT: two

NASA Ames Research Center NAS facility

- provided free computing time on a Cray C-90 computer

NASA Marshall Space Flight Center

- provided a three day training for a graduate student working on an experiment

NUMBER OF STUDENTS FUNDED: one

Sean R. Lynn: "A Unified Theory and Simulation of Unsteady Electro-Magneto-Hydrodynamics," M.Sc. degree candidate.
(Student accepted a permanent employment offer from Allison Gas Turbines Company on October 1, 1996. Does not intend to return for completion of his degree.)

NUMBER OF RELATED B.Sc. HONORS THESIS COMPLETED: two

Bates, Craig: "Inverse Determination of Locations and Strengths of Electric Impulses Inside a Human Heart Based on Chest Surface Measurements of Electric Potential", B. Sc. Honors Thesis, Dept. of Eng. Science & Mechanics, The Pennsylvania State University, May 1996.

Gross, Chris: "Feasibility Study of a Magnetohydrodynamic Blood Pump", B. Sc. Honors Thesis, Dept. of Eng. Science & Mechanics, The Pennsylvania State University, May 1996.

NUMBER OF RELATED M.Sc. THESIS COMPLETED: one

Craig Bates: "Forward and Inverse Electro-Cardiographic Calculations on a Multidipole Model of Human Cardiac Electrophysiology", Dept. of Eng. Science & Mechanics, The Pennsylvania State University, August 1997.

NUMBER OF RELATED B.Sc. HONORS THESIS IN PROGRESS: one

James Tao: "Inverse Determination of Magnet Strengths Generating a Desired Magnetic Field Pattern", Dept. of Eng. Science & Mechanics, The Pennsylvania State University, expected May 1998.

NUMBER OF RELATED PAPERS PUBLISHED IN REFEREED JOURNALS: five

Dulikravich, G. S. and Lynn, S. R., "Unified Electro-Magneto-Fluid Dynamics (EMFD): A Survey of Mathematical Models," **International Journal of Non-Linear Mechanics**, Vol. 32, No. 5, September 1997, pp. 923-932.

Dulikravich, G. S. and Lynn, S. R., "Unified Electro-Magneto-Fluid Dynamics (EMFD): Introductory Concepts," **International Journal of Non-Linear Mechanics**, Vol. 32, No. 5, September 1997, pp. 913-922.

Martin, T. J. and Dulikravich, G. S., "Inverse Determination of Boundary Conditions in Steady Heat Conduction with Heat Generation," **ASME Journal of Heat Transfer**, Vol. 118, No. 3, August 1996, pp. 546-554.

Choi, K. -Y. and Dulikravich, G. S., "Acceleration of Iterative Algorithms on Highly Clustered Grids," **AIAA Journal**, Vol. 34, No. 4, April 1996, pp. 691-699.

Martin, T. J. and Dulikravich, G. S., "Finding Unknown Surface Temperatures and Heat Fluxes in Steady Heat Conduction," **IEEE Transactions on Components, Packaging and Manufacturing Technology (CPMT) - Part A**, Vol. 18, No. 3, September 1995, pp. 540-545.

NUMBER OF RELATED JOURNAL ARTICLES IN PRESS: two

Martin, T. J. and Dulikravich, G. S., "Inverse Determination of Steady Convective Local Heat Transfer Coefficients," **ASME Journal of Heat Transfer.**

Martin, T. J. and Dulikravich, G. S., "Non-Iterative Determination of Temperature-Dependent Heat Conductivities," **ASME Journal of Heat Transfer.**

NUMBER OF RELATED BOOK CHAPTERS AND PROCEEDINGS: four

Dulikravich, G. S.: **"Design and Optimization Tools Development"**, Chapters no. 10-15 in ***New Design Concepts for High Speed Air Transport,*** (editor: H. Sobieczky), Springer, Wien/New York, 1997, pp. 159-236.

Dulikravich, G. S. and Martin, T. J.: **"Inverse Shape and Boundary Condition Problems and Optimization in Heat Conduction"**, Chapter no. 10 in ***Advances in Numerical Heat Transfer*** - *Volume I* (editors: W. J. Minkowycz and E. M. Sparrow), Taylor and Francis, November 1996, pp. 381-426.

Proceedings of Symposium on Inverse Design Problems in Heat Transfer and Fluid Flow, Editors: G. S. Dulikravich and K. A. Woodbury, ASME National Heat Transfer Conference, Baltimore, MD, August 10-12, 1997, ASME HTD-Vol. 340, Volume 2.

Proceedings of Symposium on Developments in Electrorheological Flows-1995, Editors: D. A. Siginer and G. S. Dulikravich, ASME WAM'95, San Francisco, CA, November 12-17, 1995, ASME FED-Vol. 235, MD-Vol. 71.

NUMBER OF RELATED BOOK CHAPTERS AND PROCEEDINGS IN PRESS: one

Dulikravich, G. S., **"Electro-Magneto-Hydrodynamics and Solidification,"** Chapter no. 9 in ***Advances in Non-Newtonian Flows and Rheology,*** (editors: D. A. Siginer, D. De Kee and R. P. Chhabra), Elsevier Publishers, Spring 1998.

REFEREED JOURNAL ARTICLES IN PREPARATION: one

Dulikravich, G. S., "Unified Electro-Magneto-Hydrodynamics: A Mathematical Model and Boundary Conditions", **Physics of Fluids.**

PAPERS PRESENTED AT TECHNICAL MEETINGS: twelve

Dulikravich, G. S. and Jing, Y. -H., "Boundary Conditions for Electro-Magneto-Hydrodynamics," **Symposium on Rheology and Fluid Mechanics of Non-Linear Materials,** Editors: S. G. Advani and D. A. Siginer, ASME IMECE'97, Dallas, TX, Nov. 16-21, 1997, ASME FED-Vol. 243/MD-Vol. 78, pp. 101-117.

Dulikravich, G. S. and Martin, T. J., "Inverse Determination of Steady Local Convective Heat Transfer Coefficients," **Symposium on Inverse Design Problems in Heat Transfer and Fluid Flow,** Editors: G. S. Dulikravich and K. A. Woodbury, ASME National Heat Transfer Conference, Baltimore, MD, August 10-12, 1997, ASME HTD-Vol. 340, Volume 2, pp. 151-158.

Dulikravich, G. S. and Martin, T. J., "Non-Iterative Inverse Determination of Temperature-Dependent Heat Conductivities," **Symposium on Inverse Design Problems in Heat Transfer and Fluid Flow,** Editors: G. S. Dulikravich and K. A. Woodbury, ASME National Heat Transfer Conference, Baltimore, MD, August 10-12, 1997, ASME HTD-Vol. 340, Volume 2, pp. 141-150.

Dulikravich, G. S. and Martin, T. J., "Inverse Determination of Boundary Conditions in Field Problems, (Invited Lecture), **Advanced Technology in Experimental Mechanics - ATEM97**, Editor: Y. Morimoto, Wakayama City, Osaka, Japan, July 25-26, 1997, pp. 83-88.

Martin, T. J. and Dulikravich, G. S., "Inverse Determination of Boundary Conditions in Multidomain Heat Conduction Problems," (Invited Lecture), **BETECH '97 - 9th International Conference on Boundary Element Technology**, Editor: J. Frankel, Knoxville, TN, April 9-11, 1997, pp. 99-110.

Dulikravich, G. S. and Martin, T. J., "Inverse Determination of Temperatures and Heat Fluxes on Surfaces of 3-D Objects," **Pan-American Congress of Applied Mechanics (PACAM-V),** San Juan, Puerto Rico, January 2-4, 1997, in ***Applied Mechanics in the Americas***, Editors: M. Rysz, L. A. Godoy and L. E. Suarez, Vol. 5, pp. 133-136.

Dulikravich, G. S. and Jing, Y. -H., "Fully Conservative Forms of a System of Unified Electro-Magneto-Hydrodynamic Equations," **Symposium on Rheology and Fluid Mechanics of Nonlinear Materials,** Editors: D. A. Siginer and S. G. Advani, ASME International Mechanical Engineering Congress and Exposition, Atlanta, GA, November 17-22, 1996, AMD-Vol. 217, pp. 309-314.

Dulikravich, G. S., "An Unified Theory of Electro-Magneto-Gasdynamics", **Hypersonic MAGLEV Group Meeting**, National High Magnetic Field Lab., Tallahassee, FL, December 12, 1995.

Dulikravich, G. S. and Lynn, S. R., "Electro-Magneto-Hydrodynamics: Part 2 - A Survey of Mathematical Models", **Symposium on Developments in Electrorheological Flows-1995**, Editors: D. A. Siginer and G. S. Dulikravich, ASME WAM'95, San Francisco, CA, November 12-17, 1995, ASME FED-Vol. 235, MD-Vol. 71, pp. 59-70.

Dulikravich, G. S. and Lynn, S. R., "Electro-Magneto-Hydrodynamics: Part 1 - Introductory Concepts", **Symposium on Developments in Electrorheological Flows-1995**, Editors: D. A. Siginer and G. S. Dulikravich, ASME WAM'95, San Francisco, CA, Nov. 12-17, 1995, ASME FED-Vol. 235, MD-Vol. 71, pp. 49-58.

Dulikravich, G. S., Choi, K. -Y. and Lee, S., "Magnetic Field Control of Vorticity in Steady Incompressible Laminar Flows," **Symposium on Developments in Electrorheological Flows and Measurement Uncertainty 1994, ASME WAM'94,** Editors: D. A. Siginer, J. H. Kim, S. A. Sheriff and H. W. Colleman, Chicago, IL, November 6-11,1994, ASME FED-Vol. 205/AMD-Vol. 190, pp. 125-142.

Dulikravich, G. S. and Martin, T. J., "Inverse Problems and Design in Heat Conduction," **2nd IUTAM International Symposium on Inverse Problems in Engineering Mechanics**, Editors: H. D. Bui, M. Tanaka, M. Bonnet, H. Maigre, E. Luzzato and M. Reynier, Paris, France, November 2-4, 1994, A. A. Balkema, Rotterdam, 1994, pp. 13-20.

OTHER PERTINENT TECHNICAL PRESENTATIONS: twenty

Invited Lecture, Graduate School of Eng., Kyoto University, Kyoto, Japan, August 1995.: "Inverse Problems and Optimization in Heat Transfer"

Invited Lecture, Mechanical Eng. Dept., Shinshu University, Nagano, Japan, August 1995.: "Inverse Problems and Optimization in Heat Transfer"

Invited Lecture, Ebara Research Company, Ebara Company, Kanagawa, Japan, August 1995.: "Inverse Problems and Optimization in Fluid Mechanics"

Invited Lecture, National Aerospace Laboratory - NAL, Tokyo, Japan, August 1995.: "Multidisciplinary Inverse Problems and Optimization"

Invited Lecture, Mechanical Eng. Dept., Ashikaga Inst. of Tech, Ashikaga, Japan, Aug 1995.: "Electro-Magneto-Hydrodynamics and Solidification"

Invited Lecture, Mechanical Eng. Dept., University of Tokyo, Tokyo, Japan, July 1995.: "Inverse Problems and Optimization in Fluid Mechanics"

Invited Lecture, Ishikawajima-Harima Heavy Industries R & D, Tokyo, Japan, July 1995.: "Multidisciplinary Inverse Problems and Optimization"

Invited Lecture, Fundamental Research Labs, NEC Corp., Tsukuba, Japan, July 1995.: "Electro-Magneto-Hydrodynamics and Solidification"

Invited Lecture, Dept. of Aero. & Space Eng., Tohoku University, Japan, July 1995.: "Inverse Problems and Optimization in Fluid Mechanics"

Invited Lecture, Toshiba Corp. R & D Center, Kawasaki, Japan, July 1995.: "Multidisciplinary Inverse Problems and Optimization"

Invited Lecture, Mechanical Eng. Lab., Hitachi, Ltd., Tsuchiura, Japan, July 1995.: "Multidisciplinary Inverse Problems and Optimization"

Invited Lecture, Mechanical Faculty, Nat. Tech. Univ. of Athens, Athens, Greece, June 1995.: "Multidisciplinary Inverse Problems and Optimization"

Invited Lecture, Mechanical Faculty, Aero. Inst., Univ. of Belgrade, Yugoslavia, May 1995.: "Inverse Problems and Optimization in Fluid Mechanics"

Lecture, Aerospace Eng. Dept., Pennsylvania State Univ., University Park, PA, March 1995.: "Inverse Problems and Optimization in Heat Transfer"

Lecture, Center for Theor. and Comput. Materials Science, NIST, Gaithersburg, Feb. 1995.: "Electro-Magneto-Hydrodynamics and Solidification"

Invited Lecture, Mechanical Eng. Dept., Univ. of Minnesota, Minneapolis, MN Jan. 1995.: "Inverse Problems and Optimization in Heat Transfer"

Invited Lecture, Mechanical Eng. Dept., University of Pittsburgh, Pittsburgh. PA, Jan. 1995.: "Electro-Magneto-Hydrodynamics and Solidification"

Invited Lecture, ALCOA Technical Center, ALCOA Center, PA, August 1996.: "Electro-

Magneto-Hydrodynamics and Solidification"

Invited Lecture, Mech. Eng. Dept., California State University, Fullerton, CA, July 1996.: "Inverse Problems and Optimization in Heat Transfer"

Invited Lecture, Mech. Eng. Dept, The Johns Hopkins Univ., Baltimore, MD, Feb. 1996.: "Inverse Problems and Optimization in Heat Transfer"

PERTINENT PROFESSIONAL MEETINGS AND SESSIONS ORGANIZED: sixteen

"Forum on Functional Fluids", forum co-organizer at 1999 Joint ASME/JSME Fluids Engineering Conference, San Francisco, CA, July 18 - 23, 1999.

"Multidisciplinary Inverse Problems and Optimization in Heat Transfer", co-organizer and co-chairman of the Symposium at ASME IMECE98, Anaheim, CA, November 15-20, 1998.

"Elastic Fluids," co-chairman of the session at ASME IMECE, Dallas, TX, November 16-21, 1997.

"Inverse Design Problems in Heat Transfer and Fluid Flow," co-organizer and co-chairman of the Symposium at ASME National Heat Transfer Conference, Baltimore, MD, August 10-12, 1997.

"Advanced Technology in Experimental Mechanics-ATEM97," invited speaker, session chair, and member of the International Program Committee, Wakayama City, Osaka, Japan, July 25-26, 1997.

"BETECH '97 - 9th International Conference on Boundary Element Technology," member of the Scientific Advisory Committee, Knoxville, TN, April 9-11, 1997.

"Heat Transfer", session chairman at the Pan-American Congress of Applied Mechanics (PACAM-V), San Juan, Puerto Rico, January 2-4, 1997.

"Rheology and Fluid Mechanics of Nonlinear Materials IV: Complex Flows", session vice-chairman at ASME IMECE'96, Atlanta, GA, Nov. 17-22, 1996.

"Second International Conference on Inverse Problems in Engineering: Theory and Practice", member of the Scientific Advisory Committee, Nantes, France, June 1996.

"BETECH '96 - 9th International Conference on Boundary Element Technology," member of the Scientific Advisory Committee, Maui, Hawaii, April 24-26, 1996.

"3rd International Symposium on Magnetic Suspension Technology", session chairman, Tallahassee, FL, December 13-15, 1995.

"Symposium on Electrorheological Flows - III", session co-organizer, ASME WAM'95, San Francisco, CA, November 12-17, 1995.

"The Seventh Inverse Problems in Engineering Seminar", member of the Organizing Committee, Columbus, OH, June 12-13, 1995.

"PACAM IV- Pan-American Congress of Applied Mechanics," member of the Organizing Committee, Buenos Aires, Argentina, January 3-6, 1995.

"Symposium on Inverse Problems in Mechanics - III", session co-chairperson, ASME WAM'94, Chicago, IL, November 6-11, 1994.

"Symposium on Inverse Problems in Engineering Mechanics ISIP'94," member of the International Scientific Committee, November 2-4, 1994, Paris, France.

HONORS AND AWARDS: two

Fellow, American Society of Mechanical Engineers (ASME), Nov. 1996.

ALCOA Foundation Faculty Fellow Research Award (July 1,1996 - June 30, 1998)

ELECTRO-MAGNETO-HYDRODYNAMICS AND SOLIDIFICATION

G. S. Dulikravich

Aerospace Engineering Department, The Pennsylvania State University, University Park, Pennsylvania 16802, USA

1. INTRODUCTION

Fluid flow influenced by electric and magnetic fields has classically been divided into two separate fields of study: electro-hydrodynamics (EHD) studying fluid flows containing electric charges under the influence of an electric field and no magnetic field, and magneto-hydrodynamics (MHD) studying fluid flows containing no free electric charges under the influence of a magnetic field and no electric field. Traditionally, this division was necessary to reduce the extreme complexity of the coupled system of Navier-Stokes, Maxwell's and constitutive equations describing combined electro-magneto-hydrodynamic flows. Recent advances in numerical techniques and computing technology, as well as fully rigorous theoretical treatments, have made analysis of combined electro-magneto-hydrodynamic flows well within reach. A survey of electro-magnetics and the theory describing combined electro-magneto-hydrodynamic (EMHD) flows is presented with an emphasis on describing the intricacies of the mathematical models and the corresponding boundary conditions for fluid flows involving polarization and magnetization. This survey concludes with a presentation of EHD and MHD flow models involving solidification.

NOMENCLATURE

b	= electric charge mobility coefficient, kg A s^2
B	= magnetic flux density vector, kg A^{-1} s^{-2}

$\underline{\underline{d}} = \left(\nabla\underline{v} + \nabla\underline{v}^*\right)/2$	= average rate of deformation tensor, s^{-1}
D_o	= electric charge diffusion coefficient, $m^2\ s^{-1}$
$\underline{D} = \varepsilon_o\underline{E} + \underline{P}$	= electric displacement field vector, $A\ s\ m^{-2}$
$e = c_v T + (\underline{v}\cdot\underline{v})/2$	= total energy per unit mass, $m^2\ s^{-2}$
$\underline{E}$	= electric field vector, $kg\ m\ s^{-3}\ A^{-1}$, or $V\ m^{-1}$
$\underline{\mathcal{E}} = \underline{E} + \underline{v}\times\underline{B}$	= electromotive intensity vector, $kg\ m\ s^{-3}\ A^{-1}$
$\underline{f}$	= mechanical body force vector per unit mass, $m\ s^{-2}$
$\underline{g}$	= acceleration due to gravity, $m\ s^{-2}$
h	= heat source or sink per unit mass, $m^2\ s^{-3}$
$\underline{H} = \underline{B}/\mu_o - \underline{M}$	= magnetic field intensity vector, $A\ m^{-1}$
$\underline{J} = \underline{J}_c + \underline{J}_d$	= electric current density vector, $A\ m^{-2}$
$\underline{J}_c$	= electric conduction current vector, $A\ m^{-2}$
$\underline{J}_d = \underline{v}q_o$	= electric drift current vector, $A\ m^{-2}$
$\underline{M}$	= total magnetization vector per unit volume, $A\ m^{-1}$
$\underline{\mathcal{M}} = \underline{M} + \underline{v}\times\underline{P}$	= magnetomotive intensity vector per unit volume, $A\ m^{-1}$
p	= pressure, $kg\ m^{-1}\ s^{-2}$
$\underline{P}$	= total polarization vector per unit volume, $A\ s\ m^{-2}$
q_o	= total or free electric charge per unit volume, $A\ s\ m^{-3}$
$\underline{\dot{q}}$	= heat flux vector, $kg\ s^{-3}$
s	= entropy per unit mass, $m^2\ kg^{-1}\ K^{-1}\ s^{-2}$
T	= absolute temperature, K
$\underline{v}$	= fluid velocity vector, $m\ s^{-1}$

GREEK SYMBOLS

α	= volumetric thermal expansion coefficient, K^{-1}
β	= Chorin's (1967) artificial compressibility coefficient
ε	=dielectric constant (electric permittivity), $kg^{-1}\ m^{-3}\ s^4\ A^2$
$\varepsilon_o = 8.854\times10^{-12}$	= vacuum electric permittivity, $kg^{-1}\ m^{-3}\ s^4\ A^2$
$\varepsilon_r = \varepsilon/\varepsilon_o$	= relative electric permittivity
κ	= thermal conductivity coefficient, $kg\ m\ s^{-3}\ K^{-1}$
σ	= electric conductivity coefficient, $kg^{-1}\ m^{-3}\ s^3\ A^2$
ρ	= fluid density, $kg\ m^{-3}$
$\underline{\underline{\tau}}^v = 2\mu_v\underline{\underline{d}} + \mu_{v2}\underline{\underline{I}}(\nabla\cdot\underline{v})$	= Newtonian viscous stress tensor, $kg\ m^{-1}\ s^{-2}$
$\underline{\underline{\tau}}^{EM}$	= electromagnetic stress tensor, $kg\ m^{-1}\ s^{-2}$

$\underline{\underline{\tau}} = \underline{\underline{\tau}}^{v} + \underline{\underline{\tau}}^{EM}$	= stress tensor (viscous plus electromagnetic), kg m^{-1} s^{-2}
μ	= magnetic permeability coefficient, kg m A^{-2} s^{-2}
$\mu_o = 4\pi \times 10^{-7}$	= magnetic permeability of vacuum, kg m A^{-2} s^{-2}
$\mu_r = \mu / \mu_o$	= relative magnetic permeability
μ_v	= shear coefficient of viscosity, kg m^{-1} s^{-1}
μ_{v2}	= second coefficient of viscosity, kg m^{-1} s^{-1}
$\chi^E = \varepsilon_r - 1$	= electric susceptibility
$\chi^M = \mu_r - 1$	= magnetic susceptibility
ϕ	= electric potential, V
$\Phi = \underline{\underline{\tau}}^{v} : \underline{\underline{\mathbf{d}}}$	= viscous dissipation function, kg $m^{-1}s^{-3}$

2. BACKGROUND

The scientific field of study that analyzes the ability of electro-magnetic fields to influence fluid flow-field and heat transfer has been investigated for decades. The equations that are most often used to model this phenomena consist of the system of Navier-Stokes equations for fluid motion coupled with Maxwell's equations of electro-magnetics augmented with the material constitutive relations. The field studying these flows is often called electro-magneto-dynamics of fluids [1], electro-magneto-fluid dynamics (EMFD) [2-5], electro-magneto-hydrodynamics [6], magneto-gas-dynamics and plasma dynamics [7], or the electro-dynamics of continua [8-10]. The full system of governing equations has, until recently, been far too difficult to solve because Navier-Stokes system becomes very complex when modeling flows involving turbulence, chemical reactions, multiple phases, non-Newtonian effects, etc. When coupled with Maxwell's equations, the complexity of the combined EMHD system is raised by orders of magnitude. To reduce this complexity, the analytical modeling has traditionally been divided [11] into flows influenced only by externally applied electric fields acting upon electrically charged particles in the fluid, and flows influenced only by externally applied magnetic fields without electric charges in the fluid. The former are called Electro-Hydrodynamic (EHD) flows [12] and the latter Magneto-Hydrodynamic (MHD) flows [13]. More recently, rigorous continuum mechanics treatments of EHD [14] and unified EMHD flows [9,10] have been developed. These continuum mechanics approaches are limited to non-relativistic, quasi-static or relatively low frequency phenomenon [15-17].

This chapter should provide an introductory survey of the background theory to allow implementation of numerical analysis of unified EMHD flows and of classical MHD and EHD flows with addition of liquid/solid phase change. An overview of electro-magnetic theory with concentrated effort placed on descriptions of the electric and magnetic fields and electric charges and currents will be made to provide a physical understanding of the field-material interactions causing polarization and magnetization effects. The system of equations governing the unified EMHD theory and the corresponding boundary conditions will be presented together with its fully conservative form that is ready for numerical discretization.

3. POLARIZATION AND GAUSS' LAW

Charge polarization is created when electric charges of opposite signs are separated by a distance. Although many references define several sources of polarization [18], there are essentially two main sources of polarization: natural and induced [13]. Natural polarization arises from natural dipoles and charged particles. An example of a natural dipole is a water molecule which has a geometry such that the centers of positive charges and negative charges do not coincide. Since the molecules are allowed to move freely and orient randomly, water will not have polarization on a continuum level. Now consider the fluid water as it is frozen with an applied electric field. An induced polarization will be created by the electric field by inducing an initial charge separation in neutral particles [19], by causing greater charge separation within the molecules, and by causing molecular alignment with the applied electric field in case of natural dipoles [19]. Once locked in the ice crystal structure, the water molecules will no longer be able to change their position or orientation. Consequently, even after the electric field is removed, the ice will still have polarization on a continuum level since the polarization caused by the electric field aligning the water molecules was literally frozen into the ice.

From this example it may seem that there is no reason, when dealing with fluids, to consider natural polarization. This, however, would be an erroneous assumption. Though the natural polarization may show no continuum effects without the presence of an electric field, in an electric field the total polarization, $\underline{\mathbf{P}}$, combines both the induced polarization due to the electric field and the natural polarization of the molecules which are now aligned by the electric field [13, p.22].

If polarization is assumed to be a linear function of the steady or relatively low frequency electric field, then it can be defined as

$$\underline{\mathbf{P}} = \varepsilon_o \chi^E (\underline{\mathbf{E}} + \underline{\mathbf{v}} \times \underline{\mathbf{B}}) = \varepsilon_P (\underline{\mathbf{E}} + \underline{\mathbf{v}} \times \underline{\mathbf{B}}) = \varepsilon_P \underline{\underline{\mathbf{E}}} \tag{1}$$

The electric displacement vector then becomes [19, p.164] [9, p.178].

$$\underline{\mathbf{D}} = \varepsilon_o \underline{\mathbf{E}} + \underline{\mathbf{P}} = \varepsilon_o (1 + \chi^E) \underline{\mathbf{E}} + \varepsilon_o \chi^E \underline{\mathbf{v}} \times \underline{\mathbf{B}} = \varepsilon_o \varepsilon_r \underline{\mathbf{E}} + \varepsilon_P \underline{\mathbf{v}} \times \underline{\mathbf{B}} = \varepsilon \underline{\mathbf{E}} + \varepsilon_P \underline{\mathbf{v}} \times \underline{\mathbf{B}} \tag{2}$$

where the material property, χ^E, is the dielectric susceptibility. It is typically obtained experimentally [20, p.86] and could be a function of frequency.

Electric charges come in two types: free and bound. Free charges arise from electrons in the outer or free atomic shells and from ions. Bound charges are those arising from the molecular geometry and displacement of atomic inner electron shells [13, p.21]. Gauss's law for a linearly polarizable medium then becomes [13, p.22]

$$\nabla \cdot \underline{\mathbf{D}} = q_o \tag{3}$$

or

$$\nabla \cdot (\varepsilon_o \underline{\mathbf{E}} + \underline{\mathbf{P}}) = \nabla \cdot (\varepsilon \underline{\mathbf{E}} + \varepsilon_P \underline{\mathbf{v}} \times \underline{\mathbf{B}}) = q_o \tag{4}$$

At this point it is important to note that q_o multiplied with the charged particle drift velocity, $\underline{\mathbf{v}}_d$, creates the convection or drift electric current, $\mathbf{J}_d$ [13, p.67], while polarization current, $\mathbf{J}_P$, is defined as the variation of the total polarization with respect to time [19, p.121 and p.147].

4. MAGNETIZATION AND AMPERE-MAXWELL'S LAW

If the material in question may be considered linear, that is, if the magnetization is a function of one material property and the strength and direction of the applied magnetic field, then the magnetization is defined as [9, p.178] [19, p.164] [20, p.92-96] [21, p.371-377]

$$\underline{\underline{\mathbf{M}}} = \underline{\mathbf{M}} + \underline{\mathbf{v}} \times \underline{\mathbf{P}} = \frac{\chi^M}{\mu_o (1 + \chi^M)} \underline{\mathbf{B}} \tag{5}$$

In addition to the electric currents arising from magnetization and direct charge motion, other phenomenological currents have been observed and must be taken into account when defining the total current, $\underline{\mathbf{J}}$ [9, p.162-163]. Introducing the effects of magnetization and polarization and rearranging constants, the Ampere-Maxwell's law of electrodynamics may be rewritten as [13, p.30]

$$\nabla \times \underline{\mathbf{B}} = \mu_o \left(\nabla \times \underline{\mathbf{M}} + \underline{\mathbf{J}}_d + \underline{\mathbf{J}}_p + \frac{\partial \varepsilon_o \underline{\mathbf{E}}}{\partial t} \right) \tag{6}$$

Magnetization and magnetic field vectors are often combined to form the magnetic field strength vector, $\underline{\mathbf{H}}$, defined as

$$\underline{\mathbf{H}} = \frac{\underline{\mathbf{B}}}{\mu_o} - \underline{\mathbf{M}} \tag{7}$$

The total current, $\underline{\mathbf{J}}$, is defined as the sum of the apparent magnetization current, $\nabla \times \underline{\mathbf{M}}$, charge drift current, $\underline{\mathbf{J}}_d$, and phenomenological polarization currents, $\underline{\mathbf{J}}_p$ [13, p.26] since the contribution to the magnetization current by intrinsic magnetization is zero. The Ampere-Maxwell's law for polarizable, magetizable media can therefore be written as [19, p.132]

$$\frac{\partial \underline{\mathbf{D}}}{\partial t} - \nabla \times \underline{\mathbf{H}} = -\underline{\mathbf{J}} \tag{8}$$

Detailed descriptions of these equations can be found in any number of texts [19, 20, 21].

5. A MODEL OF UNIFIED ELECTRO-MAGNETO-GASDYNAMICS (EMGD)

The full system of equations governing unified EMGD flows consists of the Maxwell's equations governing electro-magnetism, the Navier-Stokes equations governing compressible fluid flow, and constitutive equations describing material behavior. Assuming a single-phase fluid and only one type of charged particles in the fluid, this set has a minimum of 12 partial differential equations that contains 13 unknowns: ρ, q_o, T, p, and the three vector components of $\underline{\mathbf{v}}$, $\underline{\mathbf{E}}$, and $\underline{\mathbf{B}}$, respectively. The thirteenth equation is the equation of state for the fluid. The foundations of the electro-magneto-gasdynamic (EMGD) theory were formulated by Eringen and Maugin [9,10] and are based on continuum mechanics [22-25]. The rigor with which the constitutive, force, and energy terms were derived leads to a model more complete and robust than any of those found in classical literature [8,7,1,11-13,18-21].

Dulikravich and Jing [6,26] have shown that a compact vector form of the unified EMGD system can be written as a combination of the Maxwell's electro-magnetic subsystem and the Navier-Stokes fluid flow subsystem.

The Maxwell's subsystem (consisting of seven PDE's) is composed of Ampere-Maxwell's law for polarizable and magnetizable medium

$$\frac{\partial \underline{\mathbf{D}}}{\partial t} - \nabla \times \underline{\mathbf{H}} = -\underline{\mathbf{J}} \tag{9}$$

which can also be written as

$$\frac{\partial \underline{\mathbf{E}}}{\partial t} - \nabla \times \frac{\underline{\mathbf{H}}}{\varepsilon_o} = -\frac{1}{\varepsilon_o}\left(\underline{\mathbf{J}} + \frac{\partial \underline{\mathbf{P}}}{\partial t}\right) \tag{10}$$

Faraday's law

$$\frac{\partial \underline{\mathbf{B}}}{\partial t} + \nabla \times \underline{\mathbf{E}} = \mathbf{0} \tag{11}$$

and conservation of electric charges

$$\frac{\partial q_o}{\partial t} + \nabla \cdot \underline{\mathbf{J}} = 0 \tag{12}$$

which is a combination of Gauss' law

$$\nabla \cdot \underline{\mathbf{D}} = q_o \tag{13}$$

and the Ampere-Maxwell's law. Conservation of magnetic flux

$$\nabla \cdot \underline{\mathbf{B}} = 0 \tag{14}$$

is also a part of the Maxwell's subsystem, but is not solved for explicitly.

The second part of the unified EMGD is the viscous, compressible flow Navier-Stokes subsystem consisting of five PDE's and an equation of state of a perfect gas. It is composed of conservation of mass equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{v}) = 0 \qquad (15)$$

and a conservation of linear momentum (including electromagnetic effects)

$$\frac{\partial \rho \underline{v}}{\partial t} + \nabla \cdot (\underline{v}\rho\underline{v} - \underline{\underline{\tau}}) - \nabla \cdot (\underline{v}(\underline{P} \times \underline{B})) - \nabla \cdot ((\underline{B} \cdot \underline{M})\underline{\underline{I}} + (\underline{E} \cdot \underline{P})\underline{\underline{I}}) = \underline{S}^v \qquad (16)$$

Here, $\underline{\underline{I}}$ is the identity (unity) tensor and $\underline{S}^v$ is a vector of source terms. The following dyadic identities were used in equation (16)

$$(\nabla \underline{B}) \cdot \underline{M} = \nabla \cdot ((\underline{B} \cdot \underline{M})\underline{\underline{I}}) - (\nabla \underline{M}) \cdot \underline{B} \qquad (17)$$

$$(\nabla \underline{E}) \cdot \underline{P} = \nabla \cdot ((\underline{E} \cdot \underline{P})\underline{\underline{I}}) - (\nabla \underline{P}) \cdot \underline{E} \qquad (18)$$

Conservation of energy equation is also a part of the Navier-Stokes subsystem

$$\frac{\partial (\rho e)}{\partial t} + \nabla \cdot (\rho e \underline{v}) - \nabla \cdot (\underline{\underline{\tau}} \cdot \underline{v}) + \nabla \cdot \underline{\dot{q}} - \rho h - \rho \underline{E} \cdot \frac{D\left(\frac{\underline{P}}{\rho}\right)}{Dt} + \underline{M} \cdot \frac{D\underline{B}}{Dt} - \underline{J}_c \cdot \underline{E} = 0 \qquad (19)$$

It can be replaced by the Clausius-Duheim entropy inequality [9,2,4]

$$\rho \frac{Ds}{Dt} \geq \frac{\rho h + \Phi}{T} - \nabla \cdot \left(\frac{\underline{\dot{q}}}{T}\right) - \frac{\underline{\dot{q}} \cdot \nabla T}{T^2} + \frac{\rho \underline{E} \cdot \frac{D\left(\frac{\underline{P}}{\rho}\right)}{Dt} - \underline{M} \cdot \frac{D\underline{B}}{Dt} + \underline{J}_c \cdot \underline{E}}{T} \qquad (20)$$

The viscous stress tensor for a non-linear fluid is given as

$$\underline{\underline{\tau}}^v = 2\mu_v \underline{\underline{d}} + \mu_{v2} \underline{\underline{I}}(\nabla \cdot \underline{v}) + \alpha_1 \underline{\underline{d}}^2 \qquad (21)$$

In the case of a media with non-linear physical properties, the unified EMGD formulation for the electric conduction current and the heat flux can be expressed as [9, p.161-162].

$$\begin{aligned}\underline{\mathbf{J}}_c &= \sigma_1\underline{\mathbf{E}} + \sigma_2\underline{\underline{\mathbf{d}}}\cdot\underline{\mathbf{E}} + \sigma_3\underline{\underline{\mathbf{d}}}^2\cdot\underline{\mathbf{E}} + \sigma_4\nabla T + \sigma_5\underline{\underline{\mathbf{d}}}\cdot\nabla T + \sigma_6\underline{\underline{\mathbf{d}}}^2\cdot\nabla T + \sigma_7\underline{\mathbf{E}}\times\underline{\mathbf{B}}\\ &+ \sigma_8\left(\underline{\underline{\mathbf{d}}}\cdot(\underline{\mathbf{E}}\times\underline{\mathbf{B}}) - (\underline{\underline{\mathbf{d}}}\cdot\underline{\mathbf{E}})\times\underline{\mathbf{B}}\right) + \sigma_9\nabla T\times\underline{\mathbf{B}}\\ &+ \sigma_{10}\left(\underline{\underline{\mathbf{d}}}\cdot(\nabla T\times\underline{\mathbf{B}}) - (\underline{\underline{\mathbf{d}}}\cdot\nabla T)\times\underline{\mathbf{B}}\right) + \sigma_{11}(\underline{\mathbf{B}}\cdot\underline{\mathbf{E}})\underline{\mathbf{B}} + \sigma_{12}(\underline{\mathbf{B}}\cdot\nabla T)\underline{\mathbf{B}}\end{aligned} \tag{22}$$

$$\begin{aligned}\underline{\dot{\mathbf{q}}} &= \kappa_1\underline{\mathbf{E}} + \kappa_2\underline{\underline{\mathbf{d}}}\cdot\underline{\mathbf{E}} + \kappa_3\underline{\underline{\mathbf{d}}}^2\cdot\underline{\mathbf{E}} + \kappa_4\nabla T + \kappa_5\underline{\underline{\mathbf{d}}}\cdot\nabla T + \kappa_6\underline{\underline{\mathbf{d}}}^2\cdot\nabla T + \kappa_7\underline{\mathbf{E}}\times\underline{\mathbf{B}}\\ &+ \kappa_8\left(\underline{\underline{\mathbf{d}}}\cdot(\underline{\mathbf{E}}\times\underline{\mathbf{B}}) - (\underline{\underline{\mathbf{d}}}\cdot\underline{\mathbf{E}})\times\underline{\mathbf{B}}\right) + \kappa_9\nabla T\times\underline{\mathbf{B}}\\ &+ \kappa_{10}\left(\underline{\underline{\mathbf{d}}}\cdot(\nabla T\times\underline{\mathbf{B}}) - (\underline{\underline{\mathbf{d}}}\cdot\nabla T)\times\underline{\mathbf{B}}\right) + \kappa_{11}(\underline{\mathbf{B}}\cdot\underline{\mathbf{E}})\underline{\mathbf{B}} + \kappa_{12}(\underline{\mathbf{B}}\cdot\nabla T)\underline{\mathbf{B}}\end{aligned} \tag{23}$$

The electro-magneto-thermal stress tensor for a non-linear fluid is [9, p.177-178]

$$\begin{aligned}\underline{\underline{\tau}}^{EM} &= \alpha_2\underline{\mathbf{E}}\otimes\underline{\mathbf{E}} + \alpha_3\underline{\mathbf{B}}\otimes\underline{\mathbf{B}} + \alpha_4\nabla T\otimes\nabla T + \alpha_5\left(\underline{\mathbf{E}}\otimes\underline{\underline{\mathbf{d}}}\cdot\underline{\mathbf{E}}\right)_s\\ &+ \alpha_6\left(\underline{\mathbf{E}}\otimes\underline{\underline{\mathbf{d}}}^2\cdot\underline{\mathbf{E}}\right)_s + \alpha_7(\nabla T\otimes\underline{\underline{\mathbf{d}}}\cdot\nabla T)_s + \alpha_8\left(\nabla T\otimes\underline{\underline{\mathbf{d}}}^2\cdot\nabla T\right)_s\\ &+ \alpha_9\left(\underline{\underline{\mathbf{d}}}\cdot\underline{\underline{\mathbf{W}}} - \underline{\underline{\mathbf{W}}}\cdot\underline{\underline{\mathbf{d}}}\right) + \alpha_{10}\underline{\underline{\mathbf{W}}}\cdot\underline{\underline{\mathbf{d}}}\cdot\underline{\underline{\mathbf{W}}} + \alpha_{11}\left(\underline{\underline{\mathbf{d}}}^2\cdot\underline{\underline{\mathbf{W}}} - \underline{\underline{\mathbf{W}}}\cdot\underline{\underline{\mathbf{d}}}^2\right)\\ &+ \alpha_{12}\left(\underline{\underline{\mathbf{W}}}\cdot\underline{\underline{\mathbf{d}}}\cdot\underline{\underline{\mathbf{W}}}^2 - \underline{\underline{\mathbf{W}}}^2\cdot\underline{\underline{\mathbf{d}}}\cdot\underline{\underline{\mathbf{W}}}\right) + \alpha_{13}(\underline{\mathbf{E}}\otimes\nabla T)_s + \alpha_{14}\left(\underline{\underline{\mathbf{W}}}\cdot\underline{\mathbf{E}}\otimes\underline{\mathbf{E}}\cdot\underline{\underline{\mathbf{W}}}\right)_s\\ &+ \alpha_{15}\left(\underline{\mathbf{E}}\otimes\underline{\underline{\mathbf{W}}}\cdot\underline{\mathbf{E}}\right)_s + \alpha_{16}\left(\underline{\underline{\mathbf{W}}}\cdot\underline{\mathbf{E}}\otimes\underline{\underline{\mathbf{W}}}^2\cdot\underline{\mathbf{E}}\right)_s + \alpha_{17}\left(\underline{\underline{\mathbf{W}}}\cdot(\underline{\mathbf{E}}\otimes\nabla T - \nabla T\otimes\underline{\mathbf{E}})\right)_s\\ &+ \alpha_{18}\underline{\underline{\mathbf{d}}}\cdot(\underline{\mathbf{E}}\otimes\nabla T - \nabla T\otimes\underline{\mathbf{E}}) - \alpha_{18}(\underline{\mathbf{E}}\otimes\nabla T - \nabla T\otimes\underline{\mathbf{E}})\cdot\underline{\underline{\mathbf{d}}}\end{aligned} \tag{24}$$

where $\underline{\underline{\mathbf{W}}} = W_{ij} = \varepsilon_{ijk}B_k$, while the subscript s indicates symmetrization. Expressions for total polarization, $\underline{\mathbf{P}}$, and magnetization, $\underline{\mathbf{M}}$, of non-linear media can be modeled with expressions of similar complexity [9, p.175].

In these formulas, α_i, σ_i and κ_i are the physical properties of the media. Most of these coefficients are still unknown although their exploitation can offer potentially significant benefits in applications involving interacting electric, magnetic, thermal, and stress fields. This theory is valid for the frequencies of the electric and the magnetic fields that are less than approximately 1 kHz and for fluid speeds considerably less than the speed of light [14-17]. For higher frequencies, certain physical properties become functions of the frequencies. For higher speeds, relativistic effects will have to be taken into account.

6. CONSERVATIVE FORMS OF ELECTRO-MAGNETO-HYDRODYNAMIC (EMHD) SYSTEM

A necessary condition that an iterative numerical solution of the EMGD system will converge to the exact solution of the analytical EMGD system as the computational grid is infinitely refined, requires that the EMGD system must be rewritten in a fully conservative (divergence-free) form. This is especially needed if strong gradients of dependent variables are expected to exist in the solution domain. The fully conservative forms can then be used directly in the finite difference, finite volume, or finite element discretization of the EMGD system and its iterative integration process.

In the following derivations, it will be assumed that the fluid is incompressible, homocompositional, that it has linear polarization and linear magnetization properties, and that the frequencies of the applied electric and magnetic fields are less than approximately 1000 Hz for this mathematical model to be realistic. These are the only assumptions to be used in this model which will be referred to as a unified electro-magneto-hydrodynamics (EMHD).

A fully conservative EMHD system in a vector operator form is given as [6]

$$\frac{\partial \underline{\mathbf{E}}}{\partial t} - \nabla \times \frac{\underline{\mathbf{H}}}{\varepsilon_o} = \underline{\mathbf{S}}^E \tag{25}$$

$$\frac{\partial \underline{\mathbf{B}}}{\partial t} + \nabla \times \underline{\mathbf{E}} = \underline{\mathbf{0}} \tag{26}$$

$$\frac{\partial q_o}{\partial t} + \nabla \cdot \underline{\mathbf{J}} = 0 \tag{27}$$

$$\nabla \cdot \underline{\mathbf{v}} = 0 \tag{28}$$

$$\frac{\partial \underline{\mathbf{v}}}{\partial t} + \nabla \cdot \left(\underline{\mathbf{v}}\underline{\mathbf{v}} - \frac{1}{\rho} \underline{\underline{\tau}} \right) - \frac{1}{\rho} \nabla \cdot \left[\underline{\mathbf{v}}(\underline{\mathbf{P}} \times \underline{\mathbf{B}}) + (\underline{\mathbf{B}} \cdot \underline{\mathbf{M}})\underline{\underline{\mathbf{I}}} + (\underline{\mathbf{E}} \cdot \underline{\mathbf{P}})\underline{\underline{\mathbf{I}}} \right] = \underline{\mathbf{S}}^v \tag{29}$$

$$\frac{\partial e}{\partial t} + \frac{1}{\rho} \nabla \cdot \left(\rho e \underline{\mathbf{v}} - \underline{\underline{\tau}} \cdot \underline{\mathbf{v}} + \underline{\dot{\mathbf{q}}} \right) = S^e \tag{30}$$

For simplicity of notation we can define the following terms as [6,26]

$$\overline{\mu} = \frac{1}{\mu_o\left(1+\chi^M\right)} = \frac{1}{\mu} \tag{31}$$

$$\overline{\varepsilon} = \frac{1}{\varepsilon_o\left(1+\chi^E\right)} = \frac{1}{\varepsilon} \tag{32}$$

$$\varepsilon_P = \varepsilon_o \chi^E = \varepsilon - \varepsilon_o \tag{33}$$

$$A = \frac{\rho \varepsilon_P}{\rho\left(1+\chi^E\right) + \varepsilon_P \underline{\mathbf{B}} \cdot \underline{\mathbf{B}}} \tag{34}$$

$$\underline{\mathbf{R}} = \rho \underline{\mathbf{f}} + q_o \underline{\mathbf{E}} + \underline{\mathbf{J}} \times \underline{\mathbf{B}} + (\nabla \underline{\mathbf{E}}) \cdot \underline{\mathbf{P}} + (\nabla \underline{\mathbf{B}}) \cdot \underline{\mathbf{M}} + \nabla \cdot \left(\underline{\mathbf{v}}(\underline{\mathbf{P}} \times \underline{\mathbf{B}})\right) - \nabla \cdot \left(\underline{\mathbf{v}} \rho \underline{\mathbf{v}} - \underline{\underline{\tau}}\right) \tag{35}$$

$$\underline{\mathbf{D}}_t = \nabla \times \left(\frac{\underline{\mathbf{B}}}{\mu_o} - \underline{\mathbf{M}}\right) - \underline{\mathbf{J}} = \nabla \times \underline{\mathbf{H}} - \underline{\mathbf{J}} \tag{36}$$

$$\begin{aligned} \underline{\mathbf{P}}_t = {} & \frac{A}{\varepsilon_o} \underline{\mathbf{D}}_t + A(\nabla \times \underline{\mathbf{E}}) \times \underline{\mathbf{v}} + \frac{A}{\rho}\left[\left(\chi^E \underline{\mathbf{D}}_t + \varepsilon_P (\nabla \times \underline{\mathbf{E}}) \times \underline{\mathbf{v}}\right) \cdot \underline{\mathbf{B}}\right]\underline{\mathbf{B}} \\ & + \frac{A}{\rho}\left[\underline{\mathbf{R}} - \underline{\mathbf{P}} \times (\nabla \times \underline{\mathbf{E}})\right] \times \underline{\mathbf{B}} \end{aligned} \tag{37}$$

If we now assume that the fluid which is subjected to applied electric and magnetic fields is of Newtonian type and if we allow only for linear polarization (equation 1) and linear magnetization (equation 5), the constitutive relations for the electric conduction current and for the heat flux vector become [9, p.173-174]

$$\begin{aligned} \underline{\mathbf{J}}_c = {} & \sigma_1(\underline{\mathbf{E}} + \underline{\mathbf{v}} \times \underline{\mathbf{B}}) + \sigma_4 \nabla T + \sigma_7(\underline{\mathbf{E}} + \underline{\mathbf{v}} \times \underline{\mathbf{B}}) \times \underline{\mathbf{B}} \\ & + \sigma_9 \nabla T \times \underline{\mathbf{B}} + \sigma_{11}\left(\underline{\mathbf{B}} \cdot (\underline{\mathbf{E}} + \underline{\mathbf{v}} \times \underline{\mathbf{B}})\right)\underline{\mathbf{B}} + \sigma_{12}(\underline{\mathbf{B}} \cdot \nabla T)\underline{\mathbf{B}} \end{aligned} \tag{38}$$

$$\underline{\dot{q}} = \kappa_1(\underline{\mathbf{E}} + \underline{\mathbf{v}} \times \underline{\mathbf{B}}) + \kappa_4 \nabla T + \kappa_7(\underline{\mathbf{E}} + \underline{\mathbf{v}} \times \underline{\mathbf{B}}) \times \underline{\mathbf{B}}$$
$$+ \kappa_9 \nabla T \times \underline{\mathbf{B}} + \kappa_{11}(\underline{\mathbf{B}} \cdot (\underline{\mathbf{E}} + \underline{\mathbf{v}} \times \underline{\mathbf{B}}))\underline{\mathbf{B}} + \kappa_{12}(\underline{\mathbf{B}} \cdot \nabla T)\underline{\mathbf{B}} \tag{39}$$

Then, the EMHD source terms can be given in a compact vector form [6,26] as

$$\underline{\mathbf{S}}^E = -\frac{1}{\varepsilon_o}(\underline{\mathbf{J}} + \underline{\mathbf{P}}_t) \tag{40}$$

$$\underline{\mathbf{S}}^v = \underline{\mathbf{f}} + \frac{1}{\rho}\left[q_o \underline{\mathbf{E}} + (\nabla \times \underline{\mathbf{E}}) \times \underline{\mathbf{P}} - (\nabla \underline{\mathbf{M}}) \cdot \underline{\mathbf{B}} - (\nabla \underline{\mathbf{P}}) \cdot \underline{\mathbf{E}} + (\underline{\mathbf{J}} + \underline{\mathbf{P}}_t) \times \underline{\mathbf{B}}\right] \tag{41}$$

$$S^e = h + \frac{1}{\rho}(\underline{\mathbf{E}} + \underline{\mathbf{v}} \times \underline{\mathbf{B}}) \cdot \left[(\underline{\mathbf{v}} \cdot \nabla)\underline{\mathbf{P}} + \underline{\mathbf{J}}_c + \underline{\mathbf{P}}_t\right] - \frac{1}{\rho}\bar{\mu}\chi^M \, \underline{\mathbf{B}} \cdot ((\underline{\mathbf{v}} \cdot \nabla)\underline{\mathbf{B}} - \nabla \times \underline{\mathbf{E}}) \tag{42}$$

Notice that these source terms have been formulated in such a way as not to contain explicit time derivatives [6,26].

6.1 Fully conservative Cartesian form of the EMHD system

The EMHD system of equations (equations 25-30) can now be written in a general conservative form in terms of (x,y,z) orthogonal coordinate system as

$$\frac{\partial \tilde{\mathbf{Q}}}{\partial t} + \frac{\partial \tilde{\mathbf{E}}}{\partial x} + \frac{\partial \tilde{\mathbf{F}}}{\partial y} + \frac{\partial \tilde{\mathbf{G}}}{\partial z} = \tilde{\mathbf{S}} \tag{43}$$

Here, the solution vector of unknown quantities is given as

$$\tilde{\mathbf{Q}} = \left\{E_x,\ E_y,\ E_z,\ B_x,\ B_y,\ B_z,\ q_o,\ \frac{p}{\beta},\ v_x,\ v_y,\ v_z,\ e\right\}^* \tag{44}$$

where the asterisk symbol designates transpose of a vector. The vector of source terms (those terms that do not contain divergence operator) is given as

$$\tilde{\mathbf{S}} = \left\{S_x^E,\ S_y^E,\ S_z^E,\ 0,\ 0,\ 0,\ 0,\ S_x^v,\ S_y^v,\ S_z^v,\ S^e\right\}^* \tag{45}$$

In equation (44), Chorin's [27] artificial compressibility coefficient, β, was used to create the unsteady term in the mass conservation since physical unsteady term does not exist in the mass conservation for incompressible fluids.

By combining equations (5), (7), (1), and (31), the Cartesian components of the magnetic field intensity vector can be defined [6,26] as

$$\begin{aligned}
H_x &= \bar{\mu}\, B_x + \varepsilon_P v_y (E_z + v_x B_y - v_y B_x) - \varepsilon_P v_z (E_y + v_z B_x - v_x B_z) \\
H_y &= \bar{\mu}\, B_y + \varepsilon_P v_z (E_x + v_y B_z - v_z B_y) - \varepsilon_P v_x (E_z + v_x B_y - v_y B_x) \\
H_z &= \bar{\mu}\, B_z + \varepsilon_P v_x (E_y + v_z B_x - v_x B_z) - \varepsilon_P v_y (E_x + v_y B_z - v_z B_y)
\end{aligned} \tag{46}$$

The flux vectors in equation (43) can then be defined as

$$\tilde{\mathbf{E}} = \begin{Bmatrix} 0 \\ H_z/\varepsilon_o \\ -H_y/\varepsilon_o \\ 0 \\ -E_z \\ E_y \\ J_x \\ v_x \\ v_x^2 + \frac{1}{\rho}\left(p - \tau_{xx} - N_{BM}^{EP} - v_x N_x^{PB}\right) \\ v_x v_y - \frac{1}{\rho}\left(\tau_{xy} + v_y N_x^{PB}\right) \\ v_x v_z - \frac{1}{\rho}\left(\tau_{xz} + v_z N_x^{PB}\right) \\ e v_x + \frac{1}{\rho}\left(\dot{q}_x - I_x\right) \end{Bmatrix} \tag{47}$$

$$\tilde{\mathbf{F}} = \begin{Bmatrix} -H_z/\varepsilon_o \\ 0 \\ H_x/\varepsilon_o \\ E_z \\ 0 \\ -E_x \\ J_y \\ v_y \\ v_y v_x - \frac{1}{\rho}\left(\tau_{xy} + v_x N_y^{PB}\right) \\ v_y^2 + \frac{1}{\rho}\left(p - \tau_{yy} - N_{BM}^{EP} - v_y N_y^{PB}\right) \\ v_y v_z - \frac{1}{\rho}\left(\tau_{yz} + v_z N_y^{PB}\right) \\ e v_y + \frac{1}{\rho}\left(\dot{q}_y - I_y\right) \end{Bmatrix} \tag{48}$$

$$\tilde{\mathbf{G}}=\begin{Bmatrix} H_y/\varepsilon_o \\ -H_x/\varepsilon_o \\ 0 \\ -E_y \\ E_x \\ 0 \\ J_z \\ v_z \\ v_z v_x - \frac{1}{\rho}\left(\tau_{xz} + v_x N_z^{PB}\right) \\ v_z v_y - \frac{1}{\rho}\left(\tau_{yz} + v_y N_z^{PB}\right) \\ v_z^2 + \frac{1}{\rho}\left(p - \tau_{zz} - N_{BM}^{EP} - v_z N_z^{PB}\right) \\ e v_z + \frac{1}{\rho}\left(\dot{q}_z - I_z\right) \end{Bmatrix} \tag{49}$$

Here, we have written components of $(\underline{\mathbf{P}} \times \underline{\mathbf{B}})$ as

$$\begin{aligned} N_x^{PB} &= P_y B_z - P_z B_y \\ N_y^{PB} &= P_z B_x - P_x B_z \\ N_z^{PB} &= P_x B_y - P_y B_x \end{aligned} \tag{50}$$

In addition, we have defined the terms

$$N_{BM}^{EP} = E_x P_x + E_y P_y + E_z P_z + B_x (H_x - \frac{B_x}{\mu_o}) + B_y (H_y - \frac{B_y}{\mu_o}) + B_z (H_z - \frac{B_z}{\mu_o}) \tag{51}$$

$$N_{BP} = B_x P_x + B_y P_y + B_z P_z \tag{52}$$

$$N_{BT} = B_x \frac{\partial T}{\partial x} + B_y \frac{\partial T}{\partial y} + B_z \frac{\partial T}{\partial z} \tag{53}$$

$$I_x = v_x(-p + \tau_{xx}) + v_y \tau_{xy} + v_z \tau_{xz}$$
$$I_y = v_x \tau_{xy} + v_y(-p + \tau_{yy}) + v_z \tau_{yz} \tag{54}$$
$$I_z = v_x \tau_{xz} + v_y \tau_{yz} + v_z(-p + \tau_{zz})$$

Components of the electric current vector, **J**, were defined as

$$J_x = v_x q_o + \frac{\sigma_1}{\varepsilon_P} P_x + \sigma_4 \frac{\partial T}{\partial x} + \frac{\sigma_7}{\varepsilon_P} N_x^{PB} + \sigma_9 (\frac{\partial T}{\partial y} B_z - \frac{\partial T}{\partial z} B_y)$$
$$+ \frac{\sigma_{11}}{\varepsilon_P} N_{BP} B_x + \sigma_{12} N_{BT} B_x$$

$$J_y = v_y q_o + \frac{\sigma_1}{\varepsilon_P} P_y + \sigma_4 \frac{\partial T}{\partial y} + \frac{\sigma_7}{\varepsilon_P} N_y^{PB} + \sigma_9 (\frac{\partial T}{\partial z} B_x - \frac{\partial T}{\partial x} B_z)$$
$$+ \frac{\sigma_{11}}{\varepsilon_P} N_{BP} B_y + \sigma_{12} N_{BT} B_y$$

$$J_z = v_z q_o + \frac{\sigma_1}{\varepsilon_P} P_z + \sigma_4 \frac{\partial T}{\partial z} + \frac{\sigma_7}{\varepsilon_P} N_z^{PB} + \sigma_9 (\frac{\partial T}{\partial x} B_y - \frac{\partial T}{\partial y} B_x)$$
$$+ \frac{\sigma_{11}}{\varepsilon_P} N_{BP} B_z + \sigma_{12} N_{BT} B_z \tag{55}$$

and heat flux vector components were defined as

$$\dot{q}_x = \frac{\kappa_1}{\varepsilon_P} P_x + \kappa_4 \frac{\partial T}{\partial x} + \frac{\kappa_7}{\varepsilon_P} N_x^{PB} + \kappa_9 (\frac{\partial T}{\partial y} B_z - \frac{\partial T}{\partial z} B_y) + \frac{\kappa_{11}}{\varepsilon_P} N_{BP} B_x + \kappa_{12} N_{BT} B_x$$

$$\dot{q}_y = \frac{\kappa_1}{\varepsilon_P} P_y + \kappa_4 \frac{\partial T}{\partial y} + \frac{\kappa_7}{\varepsilon_P} N_y^{PB} + \kappa_9 (\frac{\partial T}{\partial z} B_x - \frac{\partial T}{\partial x} B_z) + \frac{\kappa_{11}}{\varepsilon_P} N_{BP} B_y + \kappa_{12} N_{BT} B_y \tag{56}$$

$$\dot{q}_z = \frac{\kappa_1}{\varepsilon_P} P_z + \kappa_4 \frac{\partial T}{\partial z} + \frac{\kappa_7}{\varepsilon_P} N_z^{PB} + \kappa_9 (\frac{\partial T}{\partial x} B_y - \frac{\partial T}{\partial y} B_x) + \frac{\kappa_{11}}{\varepsilon_P} N_{BP} B_z + \kappa_{12} N_{BT} B_z$$

7. CHARACTERISTIC-BASED INFLOW AND OUTFLOW BOUNDARY CONDITIONS

For most boundary value problems of electro-magneto dynamics, jump conditions are exclusively used [9,28] to formulate solid wall boundary conditions where a discontinuity occurs. At the inflow and outflow boundaries where no surface or line discontinuities exist, an alternative approach based on conservation law for continuous surfaces or lines become necessary. Characteristic boundary condition formulation [29,30], which starts from a characteristic form of the EMHD system, will be sketched here since it leads to non-reflecting boundary condition formulation [31-36,26]. To find the characteristic boundary conditions, it is first necessary to determine analytical expressions for all eigenvalues of the characteristic system. The most common approach is to use one of the symbolic programming languages software (LISP, MACSIMA) in order to determine analytical expressions for each eigenvalue. Since these software packages cannot be used for systems that have more than five coupled partial differential equations, in the case of a complete EMHD system which has twelve coupled partial differential equations, it is impossible to find the eigenvalues using available symbolic programming software.

Consequently, we will use an alternative approach in which we will divide the unified EMHD system into a Maxwell's subsystem and the Navier-Stokes subsystem [33]. Each of these two subsystems will then be analyzed separately by finding the analytical expressions for its eigenvalues by hand.

7.1 Characteristic-based boundary conditions for Maxwell's subsystem

For example, characteristic treatment of the Maxwell's subsystem can be formulated by rewriting the fully conservative Maxwell's subsystem

$$\frac{\partial \tilde{\mathbf{Q}}_{EM}}{\partial t} + \frac{\partial \tilde{\mathbf{E}}_{EM}}{\partial x} + \frac{\partial \tilde{\mathbf{F}}_{EM}}{\partial y} + \frac{\partial \tilde{\mathbf{G}}_{EM}}{\partial z} = \tilde{\mathbf{S}}_{EM} \tag{57}$$

in a non-conservative (characteristic) form as

$$\frac{\partial \tilde{\mathbf{Q}}_{EM}}{\partial t} + \underline{\underline{\mathbf{A}}}_{EM} \frac{\partial \tilde{\mathbf{Q}}_{EM}}{\partial x} + \underline{\underline{\mathbf{B}}}_{EM} \frac{\partial \tilde{\mathbf{Q}}_{EM}}{\partial y} + \underline{\underline{\mathbf{C}}}_{EM} \frac{\partial \tilde{\mathbf{Q}}_{EM}}{\partial z} = \tilde{\mathbf{S}}_{EM} \tag{58}$$

In order to perform characteristic analysis for Maxwell's subsystem, care must be exercised to ensure that all the terms appearing in the fluxes $\tilde{\mathbf{E}}_{EM}, \tilde{\mathbf{F}}_{EM}, \tilde{\mathbf{G}}_{EM}$ are expressed as functions of the primitive variables

$$\tilde{\mathbf{Q}}_{EM} = \{E_x, \quad E_y, \quad E_z, \quad B_x, \quad B_y, \quad B_z, \quad q_o\}^* \tag{59}$$

For illustration, the flux vector $\tilde{\mathbf{E}}_{EM}$ can be extracted from equation (47) as

$$\tilde{\mathbf{E}}_{EM} = \begin{Bmatrix} 0 \\ H_z / \varepsilon_o \\ -H_y / \varepsilon_o \\ 0 \\ -E_z \\ E_y \\ J_x \end{Bmatrix} \tag{60}$$

For fluids with linear polarization and magnetization, H_z and H_y are the same as in equations (46), while J_x is given in equation (55). The flux vector Jacobian matrix $\underline{\underline{\mathbf{A}}}_{EM}$ is obtained as

$$\underline{\underline{\mathbf{A}}}_{EM} = \frac{\partial \tilde{\mathbf{E}}_{EM}}{\partial \tilde{\mathbf{Q}}_{EM}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & a_{24} & a_{25} & a_{26} & 0 \\ a_{31} & 0 & a_{33} & a_{34} & a_{35} & a_{36} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ a_{71} & a_{72} & a_{73} & a_{74} & a_{75} & a_{76} & v_x \end{bmatrix} \tag{61}$$

where the coefficients are

$$a_{21} = -\chi^E v_y \qquad a_{31} = -\chi^E v_z \tag{62}$$

$$a_{22} = \chi^E v_x \qquad a_{33} = \chi^E v_x \tag{63}$$

$$a_{24} = \chi^E v_x v_z \qquad a_{34} = -\chi^E v_x v_y \tag{64}$$

$$a_{25} = \chi^E v_y v_z \qquad a_{36} = -\chi^E v_y v_z \tag{65}$$

$$a_{26} = \frac{1}{\mu\varepsilon_o} - \chi^E\left(v_x^2 + v_y^2\right) \qquad a_{35} = -\frac{1}{\mu\varepsilon_o} + \chi^E\left(v_x^2 + v_z^2\right) \tag{66}$$

$$a_{71} = \sigma_1 + \sigma_{11}B_x^2 \qquad a_{72} = \sigma_7 B_z + \sigma_{11}B_x B_y \tag{67}$$

$$a_{73} = -\sigma_7 B_y + \sigma_{11}B_x B_z \tag{68}$$

$$a_{74} = \sigma_7(v_y B_y + v_z B_z) + \sigma_{11}(E_x B_x + \frac{N_{BP}}{\varepsilon_P}) + \sigma_{12}(N_{BT} + B_x \frac{\partial T}{\partial x}) \tag{69}$$

$$a_{75} = -\sigma_1 v_z - \frac{\sigma_7}{\varepsilon_P}(P_z + \varepsilon_P v_x B_y) - \sigma_9 \frac{\partial T}{\partial z} + \sigma_{11}E_y B_x + \sigma_{12}B_x \frac{\partial T}{\partial y} \tag{70}$$

$$a_{76} = \sigma_1 v_y + \frac{\sigma_7}{\varepsilon_P}(P_y - \varepsilon_P v_x B_z) + \sigma_9 \frac{\partial T}{\partial y} + \sigma_{11}E_z B_x + \sigma_{12}B_x \frac{\partial T}{\partial z} \tag{71}$$

Matrices $\underline{\underline{\mathbf{B}}}_{EM}$ and $\underline{\underline{\mathbf{C}}}_{EM}$ may be obtained in the same fashion as equation (61).

After tedious algebraic manipulations [26], the vector of eigenvalues of the flux vector Jacobian matrix $\underline{\underline{\mathbf{A}}}_{EM}$ is found as

$$\tilde{\lambda}_{EM} = \left\{0,\ \lambda_E^+,\ \lambda_E^-,\ 0,\ \lambda_B^+,\ \lambda_B^-,\ v_x\right\}^* \tag{72}$$

This means that the eigenvalues $\lambda_1 = \lambda_4 = 0$, while $\lambda_7 = v_x$. The remaining four eigenvalues can be obtained from the fourth order algebraic equation

$$\lambda^4 + \alpha_{EM}\lambda^3 + v_{EM}\lambda^2 + \gamma_{EM}\lambda + \delta_{EM} = 0 \tag{73}$$

where the coefficients in the fourth order characteristic polynomial are

$$\alpha_{EM} = -a_{22} - a_{33} \tag{74}$$

$$\nu_{EM} = a_{22}a_{33} - a_{26} + a_{35} \quad (75)$$

$$\gamma_{EM} = -a_{26}a_{33} - a_{22}a_{35} \quad (76)$$

$$\delta_{EM} = a_{25}a_{36} - a_{35}a_{26} \quad (77)$$

The four eigenvalues are the analytical roots given as

$$\lambda_E^+ = -\frac{1}{4}\Phi_{EM1} + \sqrt{\frac{1}{4}\Phi_{EM1}^2 - \Omega_{EM1}} \quad (78)$$

$$\lambda_E^- = -\frac{1}{4}\Phi_{EM1} - \sqrt{\frac{1}{4}\Phi_{EM1}^2 - \Omega_{EM1}} \quad (79)$$

$$\lambda_B^+ = -\frac{1}{4}\Phi_{EM2} + \sqrt{\frac{1}{4}\Phi_{EM2}^2 - \Omega_{EM2}} \quad (80)$$

$$\lambda_B^- = -\frac{1}{4}\Phi_{EM2} - \sqrt{\frac{1}{4}\Phi_{EM2}^2 - \Omega_{EM2}} \quad (81)$$

Here, different terms are defined as

$$\Phi_{EM1} = \alpha_{EM} + \sqrt{\alpha_{EM}^2 - 4\nu_{EM} + 4\psi_{EM}} \quad (82)$$

$$\Phi_{EM2} = \alpha_{EM} - \sqrt{\alpha_{EM}^2 - 4\nu_{EM} + 4\psi_{EM}} \quad (83)$$

$$\Omega_{EM1} = 2\left(\psi_{EM} + \sqrt{\psi_{EM}^2 - 4\delta_{EM}}\right) \quad (84)$$

$$\Omega_{EM2} = 2\left(\psi_{EM} - \sqrt{\psi_{EM}^2 - 4\delta_{EM}}\right) \quad (85)$$

$$\psi_{EM} = \sqrt[3]{Y_{EM} + \sqrt{Z_{EM}^3 + Y_{EM}^2}} + \sqrt[3]{Y_{EM} - \sqrt{Z_{EM}^3 + Y_{EM}^2}} + \frac{\nu_{EM}}{3} \quad (86)$$

$$Z_{EM} = \frac{3(\alpha_{EM}\gamma_{EM} - 4\delta_{EM}) - \nu_{EM}^2}{9} \quad (87)$$

$$\Upsilon_{EM} = \frac{v_{EM}(4\delta_{EM} - \alpha_{EM}\gamma_{EM})}{6} + \frac{v_{EM}^3}{27} - \frac{(4v_{EM}\delta_{EM} - \gamma_{EM}^2 - \alpha_{EM}^2\delta_{EM})}{2} \tag{88}$$

For illustrative purposes, the following are the eigenvalues in the case of one-dimensional EMHD flow where $v_y = v_z = 0$ and $a_{22} = a_{33}$ and $a_{25} = 0$. Hence

$$\lambda_E^+ = \lambda_B^+ = \frac{1}{2}\left[\chi^E v_x + \sqrt{\chi^{E^2} v_x + 4\left(\frac{1}{\varepsilon_o\mu_o(1+\chi^M)} - \chi^E v_x^2\right)}\right] \tag{89}$$

$$\lambda_E^- = \lambda_B^- = \frac{1}{2}\left[\chi^E v_x - \sqrt{\chi^{E^2} v_x + 4\left(\frac{1}{\varepsilon_o\mu_o(1+\chi^M)} - \chi^E v_x^2\right)}\right] \tag{90}$$

Since $1/\sqrt{\varepsilon_o\mu_o}$ equals the speed of light in vacuum, it seems that for most practical applications the incoming and the outgoing electromagnetic waves will not be influenced by the fluid except in the situations where the fluid is very highly ionized or when the fluid moves with a speed comparable to the speed of light. In the case of a pure electro-magnetics without any fluid motion, polarization, magnetization, or electric charges ($\underline{\mathbf{v}} = \underline{\mathbf{P}} = \underline{\mathbf{M}} = q_o = 0$), these eigenvalues reduce to the eigenvalues of Maxwell's equations for electro-magnetic fields in vacuum [35]

$$\lambda = \left\{0, \frac{1}{\sqrt{\varepsilon_o\mu_o}}, -\frac{1}{\sqrt{\varepsilon_o\mu_o}}, 0, \frac{1}{\sqrt{\varepsilon_o\mu_o}}, -\frac{1}{\sqrt{\varepsilon_o\mu_o}}\right\}^* \tag{91}$$

After introducing the similarity transformation matrix $\underline{\underline{\mathbf{S}}}_{EM}$ of the flux vector Jacobian matrix $\underline{\underline{\mathbf{A}}}_{EM}$, the eigenmatrix $\underline{\underline{\tilde{\lambda}}}_{EM}$ corresponding to $\underline{\underline{\mathbf{A}}}_{EM}$ becomes

$$\underline{\underline{\tilde{\lambda}}}_{EM} = \mathrm{diag}\left[0, \lambda_E^+, \lambda_E^-, \lambda_B^+, \lambda_B^-, v_x\right] \tag{92}$$

where $\lambda_E^+, \lambda_E^-, \lambda_B^+, \lambda_B^-$ are given by equations (78-81).

For locally one-dimensional problems, wave propagation direction is well defined. For multi-dimensional problems, there is no unique direction of propagation, because the flux vector Jacobian matrices $\underline{\underline{\mathbf{A}}}_{EM}, \underline{\underline{\mathbf{B}}}_{EM}, \underline{\underline{\mathbf{C}}}_{EM}$ cannot be simultaneously diagonalized. Therefore, characteristic boundary condition analysis allows that only one of these matrices (relating to only one coordinate direction) can be diagonalized at a time.

In the case that the x-coordinate is in the main flow direction, premultiplying the equation (58) with the inverse of the similarity matrix, $\underline{\underline{\mathbf{S}}}^{-1}_{EM}$, gives

$$\underline{\underline{\mathbf{S}}}^{-1}_{EM} \frac{\partial \tilde{\mathbf{Q}}_{EM}}{\partial t} + \underline{\tilde{\lambda}}_{EM} \underline{\underline{\mathbf{S}}}^{-1}_{EM} \frac{\partial \tilde{\mathbf{Q}}_{EM}}{\partial x} + \underline{\underline{\mathbf{S}}}^{-1}_{EM} \tilde{\mathbf{H}}_{EM} = 0 \tag{93}$$

Here, vector $\tilde{\mathbf{H}}_{EM}$ is given as

$$\tilde{\mathbf{H}}_{EM} = \underline{\underline{\mathbf{B}}}_{EM} \frac{\partial \tilde{\mathbf{Q}}_{EM}}{\partial y} + \underline{\underline{\mathbf{C}}}_{EM} \frac{\partial \tilde{\mathbf{Q}}_{EM}}{\partial z} - \tilde{\mathbf{S}}_{EM} \tag{94}$$

For the hyperbolic system, time dependent boundary conditions could be derived based on the principle that outgoing waves are described by characteristic equations, while the incoming waves may often be specified by a non-reflecting boundary condition [31,32,36]. Following this approach, the characteristic and non-reflecting boundary conditions at the inlet boundary x = a and at the outlet boundary x = b can be given by the i-th equation of the system (93)

$$\left(\underline{\underline{\mathbf{S}}}^{-1}_{i,EM} \frac{\partial \tilde{\mathbf{Q}}_{EM}}{\partial t} + L_{i,EM} + \underline{\underline{\mathbf{S}}}^{-1}_{i,EM} \tilde{\mathbf{H}}_{EM} \right) \Bigg|_{x=a,b} = 0 \tag{95}$$

Here, the left eigenvector $\underline{\underline{\mathbf{S}}}^{-1}_{i,EM}$ is the i-th row of $\underline{\underline{\mathbf{S}}}^{-1}_{EM}$ and

$$L_{i,EM} = \begin{cases} \underline{\tilde{\lambda}}_{i,EM} \underline{\underline{\mathbf{S}}}^{-1}_{i,EM} \dfrac{\partial \tilde{\mathbf{Q}}_{EM}}{\partial x} & \text{for outgoing waves} \\ 0 & \text{for incoming waves} \end{cases} \tag{96}$$

7.2 Characteristic-based boundary conditions for Navier-Stokes subsystem

Similar derivations can be used to determine analytical expressions for the eigenvalues and the non-reflecting boundary conditions of the Navier-Stokes subsystem of the unified EMHD as shown by Dulikravich and Jing [26].

Characteristic treatment of the Navier-Stokes subsystem of the unified EMHD system can be performed by converting its conservative form

$$\frac{\partial \tilde{\mathbf{Q}}_{NS}}{\partial t} + \frac{\partial \tilde{\mathbf{E}}_{NS}}{\partial x} + \frac{\partial \tilde{\mathbf{F}}_{NS}}{\partial y} + \frac{\partial \tilde{\mathbf{G}}_{NS}}{\partial z} = \tilde{\mathbf{S}}_{NS} \tag{97}$$

into its non-conservative (characteristic) form

$$\frac{\partial \tilde{\mathbf{Q}}_{NS}}{\partial t} + \underline{\underline{\mathbf{A}}}_{NS} \frac{\partial \tilde{\mathbf{Q}}_{NS}}{\partial x} + \underline{\underline{\mathbf{B}}}_{NS} \frac{\partial \tilde{\mathbf{Q}}_{NS}}{\partial y} + \underline{\underline{\mathbf{C}}}_{NS} \frac{\partial \tilde{\mathbf{Q}}_{NS}}{\partial z} = \tilde{\mathbf{S}}_{NS} \tag{98}$$

where the solution vector of unknowns is given as

$$\tilde{\mathbf{Q}}_{NS} = \{p/\beta, \quad v_x, \quad v_y, \quad v_z, \quad e\}^* \tag{99}$$

From equation (47) it can be seen that flux vector $\tilde{\mathbf{E}}_{NS}$ becomes

$$\tilde{\mathbf{E}}_{NS} = \begin{Bmatrix} v_x \\ v_x^2 + \frac{p}{\rho} - \frac{\tau_{xx}}{\rho} - \frac{N_{BM}^{EP}}{\rho} - \frac{v_x N_x^{PB}}{\rho} \\ v_x v_y - \frac{\tau_{xy}}{\rho} - \frac{v_y N_x^{PB}}{\rho} \\ v_x v_z - \frac{\tau_{xz}}{\rho} - \frac{v_z N_x^{PB}}{\rho} \\ e v_x + \frac{\dot{q}_x}{\rho} - \frac{I_x}{\rho} \end{Bmatrix} \tag{100}$$

Terms related to $\underline{\underline{\mathbf{d}}}, \underline{\underline{\mathbf{d}}}^2$ and ∇T will not be considered in the evaluation of coefficients of the flux vector Jacobian matrix $\underline{\underline{\mathbf{A}}}_{NS}$ since they are associated with first derivatives of velocity, $\underline{\mathbf{v}}$, or temperature, T. The flux vector Jacobian matrix $\underline{\underline{\mathbf{A}}}_{NS} = \partial \tilde{\mathbf{E}}_{NS} / \partial \tilde{\mathbf{Q}}_{NS}$ then becomes

$$\underline{\underline{\mathbf{A}}}_{NS} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ \beta/\rho & a_{22} & a_{23} & a_{24} & 0 \\ 0 & a_{32} & a_{33} & a_{34} & 0 \\ 0 & a_{42} & a_{43} & a_{44} & 0 \\ v_x\beta/\rho & a_{52} & a_{53} & a_{54} & v_x \end{bmatrix} \tag{101}$$

The coefficients in this matrix are given in detail by Dulikravich and Jing [26]. Eigenvalue vector of the flux vector Jacobian matrix $\underline{\underline{\mathbf{A}}}_{NS}$ is

$$\tilde{\lambda}_{NS} = \left\{ v_x, \quad \lambda_u^+, \quad \lambda_v^+, \quad \lambda_w^+, \quad \lambda_e^+ \right\}^* \tag{102}$$

which can be written as a diagonal eigenvalue matrix

$$\underline{\underline{\tilde{\lambda}}}_{NS} = \mathrm{diag}[v_x, \ \lambda_u^+, \ \lambda_v^+, \ \lambda_w^+, \ \lambda_e^+] \tag{103}$$

The eigenvalues $\lambda_u^+, \lambda_v^+, \lambda_w^+, \lambda_e^+$ are obtained analytically by solving a fourth order characteristic polynomial (similar to equation 73) where

$$\alpha_{NS} = a_{22} + a_{33} + a_{44} \tag{104}$$

$$v_{NS} = a_{22}a_{33} + a_{22}a_{44} + a_{33}a_{44} - a_{34}a_{43} - a_{24}a_{42} - a_{23}a_{32} - \frac{\beta}{\rho} \tag{105}$$

$$\begin{aligned} \gamma_{NS} = & -a_{34}a_{43}a_{22} - a_{22}a_{33}a_{44} - a_{24}a_{32}a_{43} + a_{24}a_{33}a_{42} + a_{23}a_{32}a_{44} \\ & - a_{23}a_{34}a_{42} + (a_{33} + a_{44})\frac{\beta}{\rho} \end{aligned} \tag{106}$$

$$\delta_{NS} = (a_{34}a_{43} - a_{33}a_{44})\frac{\beta}{\rho} \tag{107}$$

so that the four eigenvalues are

$$\lambda_u^+ = -\frac{1}{4}\Phi_{NS1} + \sqrt{\frac{1}{4}\Phi_{NS1}^2 - \Omega_{NS1}} \tag{108}$$

$$\lambda_v^+ = -\frac{1}{4}\Phi_{NS1} - \sqrt{\frac{1}{4}\Phi_{NS1}^2 - \Omega_{NS1}} \tag{109}$$

$$\lambda_w^+ = -\frac{1}{4}\Phi_{NS2} + \sqrt{\frac{1}{4}\Phi_{NS2}^2 - \Omega_{NS2}} \tag{110}$$

$$\lambda_e^+ = -\frac{1}{4}\Phi_{NS2} - \sqrt{\frac{1}{4}\Phi_{NS2}^2 - \Omega_{NS2}} \tag{111}$$

with the coefficients given by equations of the type similar to equations (82-88).

Characteristic waves defined by the Navier-Stokes equations in the EMHD system have a great dependency on both fluid dynamics and electro-magneto-dynamics, in particular, the electro-magnetic properties of the media and electro-magnetic field quantities. When electric and magnetic fields are absent, these eigenvalues reduce to the well-known eigenvalues of a classical Navier-Stokes system for Newtonian, incompressible flows. These eigenvalues are $\{v_{x,} v_{x,} v_{x,} v_x + c, v_x - c\}$. Here, the equivalent local speed of sound is defined as $c = \sqrt{v_x^2 + (\beta/\rho)}$.

Following Thompson's approach [30,31], non-reflecting boundary conditions for the Navier-Stokes subsystem are hence formulated as follows. The characteristic form of Navier-Stokes subsystem influenced by the electro-magnetic effects is possible to write as

$$\underline{\underline{\mathbf{S}}}_{NS}^{-1}\frac{\partial \tilde{\mathbf{Q}}_{NS}}{\partial t} + \tilde{\underline{\underline{\lambda}}}_{NS}\underline{\underline{\mathbf{S}}}_{NS}^{-1}\frac{\partial \tilde{\mathbf{Q}}_{NS}}{\partial x} + \underline{\underline{\mathbf{S}}}_{NS}^{-1}\tilde{\mathbf{H}}_{NS} = 0 \tag{112}$$

where the i-th equation is

$$\underline{\underline{\mathbf{S}}}_{i,NS}^{-1}\frac{\partial \tilde{\mathbf{Q}}_{NS}}{\partial t} + \tilde{\underline{\underline{\lambda}}}_{i,NS}\underline{\underline{\mathbf{S}}}_{i,NS}^{-1}\frac{\partial \tilde{\mathbf{Q}}_{NS}}{\partial x} + \underline{\underline{\mathbf{S}}}_{i,NS}^{-1}\tilde{\mathbf{H}}_{NS} = 0 \tag{113}$$

and the new source vector is

$$\tilde{\mathbf{H}}_{NS} = \underline{\underline{\mathbf{B}}}_{NS}\frac{\partial \tilde{\mathbf{Q}}_{NS}}{\partial y} + \underline{\underline{\mathbf{C}}}_{NS}\frac{\partial \tilde{\mathbf{Q}}_{NS}}{\partial z} - \tilde{\mathbf{S}}_{NS} \tag{114}$$

Here, the left eigenvector $\underline{\underline{S}}^{-1}_{i,NS}$ is the i-th row of $\underline{\underline{S}}^{-1}_{NS}$.

$$\left(\underline{\underline{S}}^{-1}_{i,NS} \frac{\partial \tilde{Q}_{NS}}{\partial t} + L_{i,NS} + \underline{\underline{S}}^{-1}_{i,NS} \tilde{H}_{NS} \right)_{x=a,b} = 0 \tag{115}$$

where

$$L_{i,NS} = \begin{cases} \tilde{\lambda}_{i,NS} \underline{\underline{S}}^{-1}_{i,NS} \dfrac{\partial \tilde{Q}_{NS}}{\partial x} & \text{for outgoing waves} \\ 0 & \text{for incoming waves} \end{cases} \tag{116}$$

Practical implementation of Thompson-type [31-33,36,26] non-reflecting boundary conditions deserves further comments. The essence of his approach is that one-dimensional characteristic analysis can be performed by considering the transverse terms as a constant source term. In order to provide well-posed non-reflecting boundary conditions in multi-dimensional cases, substantial modifications may be required to take into account the transverse terms at the boundaries [37,38]. It should be emphasized that physically there are cases where flow information propagates back from the outside of the domain into the inside through the boundaries by the incoming waves [39]. This fact makes it possible that building a perfectly non-reflecting (absorbing) boundary condition [40] might lead to an ill-posed problem. Under these circumstances, corrections may be needed to make them partially non-reflecting.

7.3 Numerical integration of EMHD system

It is often highly desirable to have a time-accurate unsteady solution to the governing EMHD equations. One numerical integration algorithm that could be used is an advanced form of the dual time-stepping technique, also called an iterative-implicit technique, originally developed by Jameson [41].

To create an instantaneous picture of the solution of the entire EMHD system at a given physical time, equation (43) must be driven to zero in its entirety, not, as is commonly done in time-marching techniques by driving only the physical time-dependent term to zero. To this end, a pseudo-time derivative is added to the EMHD system (equation 43) which can be rewritten as

$$\frac{\partial \hat{Q}}{\partial \tau} + \frac{\partial \tilde{Q}}{\partial t} + \frac{\partial \tilde{E}}{\partial x} + \frac{\partial \tilde{F}}{\partial y} + \frac{\partial \tilde{G}}{\partial z} = \tilde{S} \tag{117}$$

or as

$$\frac{\partial \hat{\mathbf{Q}}}{\partial \tau} = \hat{\underline{\mathbf{R}}} - \frac{\partial \tilde{\mathbf{Q}}}{\partial t} \tag{118}$$

where $\hat{\underline{\mathbf{R}}}$ is a composite of the spatial and source terms and is called the residual. Thus, given a physical time step the governing equations are time marched in pseudo time, τ. Upon convergence, the right-hand side of equation (118) becomes zero and the solution at the desired physical time level, t, is obtained. Note that the pseudo-time dependent variable vector, $\hat{\mathbf{Q}}$, does not have to be the same as the physical time dependent variable vector, $\tilde{\mathbf{Q}}$.

An additional concern of great importance is that the system of equations develops zero terms in the pseudo-time dependent variable vector, $\hat{\mathbf{Q}}$, for incompressible fluids, fluids without electric charges, or systems in which the electric and magnetic fields are non-interacting. This poses significant problems for time-marching numerical solutions. This problem may be alleviated, however, by proper selection of pseudo-time dependent variable vector, $\hat{\mathbf{Q}}$, and through the use of matrix preconditioning.

By premultiplying $\hat{\mathbf{Q}}$ with a properly selected matrix, it is possible to directly control the system eigenvalues. This prevents development of zeros in the pseudo-time dependent variable vector, $\hat{\mathbf{Q}}$, and vastly improves iterative convergence rates over a wide variety of flow regimes (low and high Mach and Reynolds number combinations). The preconditioning matrix, $\underline{\Gamma}'(\hat{\mathrm{Q}})$, for the EMHD system could be based on one developed by Merkle and Choi [42] for the Navier-Stokes system. The preconditioned EMHD system may be written as

$$\underline{\Gamma}' \frac{\partial \hat{\mathbf{Q}}}{\partial \tau} = \hat{\underline{\mathbf{R}}} - \frac{\partial \tilde{\mathbf{Q}}}{\partial t} \tag{119}$$

Equation (119) can be transformed to a body-conforming non-orthogonal curvilinear time-dependent (ξ, η, ζ; t) coordinate system. A high order of accuracy is desired to properly resolve unsteady motions. A finite difference scheme using fourth order accurate spatial differencing and second order accurate physical time differencing could be used while the solution is advanced in pseudo-time using a four-stage Runge-Kutta scheme which is second order accurate for non-linear problems. Fourth order accuracy should be selected for the spatial derivatives based on extensive research completed by Carpenter et al. [30] which found that a Runge-Kutta advanced fourth order accurate scheme provided the best convergence and stability of higher order schemes at reasonable computational cost. Second order accurate differencing in physical time could be selected based on stability and convergence studies performed by Melson et al. [43] who found that for a Runge-Kutta advanced dual time-stepping scheme second order backward differencing provided the most stable physical time discretization while providing excellent resolution. The new physical time step could be treated implicitly in pseudo-time, while all old physical time steps and spatial derivatives could be treated explicitly. This is unlike Jameson's early method [41] that treats both the physical time and the spatial derivative explicitly and causes a restriction on the maximum physical time step allowed. The discretized preconditioned system may be written as

$$\hat{\mathbf{Q}}^0 = \hat{\mathbf{Q}}^n \tag{120}$$

$$\left(\underline{\Gamma}'\underline{\Gamma}^{-1} + \alpha_i \frac{3}{2}\frac{\Delta\tau}{\Delta t}\right)\left(\tilde{\mathbf{Q}}^i - \tilde{\mathbf{Q}}^0\right)^{m+1} = \alpha_i\,\Delta\tau\,\underline{\mathbf{R}}^{m+1,\,i-1} - \alpha_i\,\Delta\tau\left(\frac{3\tilde{\mathbf{Q}}^{m+1} - 4\tilde{\mathbf{Q}}^m + \tilde{\mathbf{Q}}^{m-1}}{2\Delta t}\right)^{i=0} \tag{121}$$

$$\hat{\mathbf{Q}}^{n+1} = \hat{\mathbf{Q}}^4 \tag{122}$$

where m=1,2,3,... represents the physical time step, n=1,2,3,... represents the pseudo-time step, and i=1,2,3,4 is the Runge-Kutta stage number. Also, $\underline{\Gamma} = \partial\tilde{\mathbf{Q}}/\partial\hat{\mathbf{Q}}$ and α_i are the Runge-Kutta coefficients. Note that the physical time-dependent term on the right hand side of equation (120) is held constant for all four Runge-Kutta stages.

8. SUBMODELS OF EMHD

Until now, the numerical solutions of the unsteady three-dimensional EMHD flows that have been reported in the open literature [34-36] did not account for polarization or magnetization effects and did not involve charge density transport equation. The reason is that the complete unified EMHD system is very large having extremely complicated source terms and two extremely different time scales for the electro-magnetic fields and the flow-field. Consequently, a number of simplified versions of the EMHD system have been traditionally used in practical applications. These submodels can be grouped in two general categories: EHD models and MHD models [11-13,44].

From the unified EMHD model, it can be seen that the electromagnetic field is not the only cause of electric current and that the temperature gradient is not the only source of heat conduction as is commonly assumed. The electric field, magnetic field, and heat conduction may couple to produce charge motion and heat transfer. These couplings are called phenomenological cross effects and may be placed in four general categories: 1) thermoelectric, 2) galvanomagnetic, 3) thermomagnetic, and 4) second order effects [9, p.161-163]. These categories are based on the source of the effect and each will be described in turn, as will be a comparison between classical EHD and MHD models and the unified EMHD theory. The comparison concentrates on similarities and differences between electro-magnetic force and electric current and heat conduction terms in the EHD, MHD, and EMHD models. The inadequacies of simple superpositioning of classical simplified models to fully describe the unified EMHD flows are also noted.

Couplings between the temperature gradient and the electric field cause thermoelectric effects so that a temperature gradient in the material produces an electric current (Thompson effect), while applied electric field produces heat conduction in the material (Peltier effect). These two effects together are known as the Seebeck effect and form the basis for thermocouples. Also note that the σ_1 term in the electric conduction current (equation 22) and the κ_4 term in the heat conduction (equation 23) are the ohmic charge conduction and Fourier heat transfer, respectively.

When the electric and magnetic fields are simultaneously applied but are not parallel, electric current (Hall effect) and heat conduction (Ettingshausen effect) perpendicular to the plane containing the electric and the magnetic fields are induced in the media. These effects are termed galvanomagnetic [9, p.161-163].

When the temperature gradient and the magnetic field are simultaneously applied but are not parallel, electric current (Nernst effect) and heat conduction (Righi-LeDuc effect) perpendicular to the plane containing the temperature gradient and the magnetic field are induced in the material. These effects are termed thermomagnetic.

It should be noticed (equation 22) that the interaction of the average rate of deformation tensor and the electric field can also create the electric current, while the interaction of the material deformation tensor and the electric field can create the temperature gradient (equation 23). These piezo-electric and piezo-magnetic effects can further be enhanced if the material is non-isotropic.

8.1 Classical electro-hydrodynamics (EHD)

As mentioned previously, EHD flows are those in which magnetic effects may be neglected and charged particles are present, while only a quasi-static electric field is applied so that the magnetic field, both applied and induced, may be neglected [11]. One of the implied assumptions is that the flows are at non-relativistic speeds, although in astrophysical flows this assumption cannot be made [1]. Atten and Moreau [44] present a detailed coverage of classical EHD modeling and discuss the relative importance of terms in the force and electric current through stability analysis. With these assumptions, the Maxwell's system reduces to [11]

$$\nabla \cdot \underline{\mathbf{D}} = \nabla \cdot (\varepsilon \underline{\mathbf{E}}) = q_o \tag{123}$$

$$\frac{\partial q_o}{\partial t} + \nabla \cdot \underline{\mathbf{J}} = 0 \tag{124}$$

With classical EHD assumptions, the electro-magnetic force in the unified EMHD theory reduces to:

$$\underline{\mathbf{f}}^{\mathbf{EM}} = q_o \underline{\mathbf{E}} + (\nabla \underline{\mathbf{E}}) \cdot \underline{\mathbf{P}} = q_o \underline{\mathbf{E}} + (\nabla \underline{\mathbf{E}}) \cdot \varepsilon_P \underline{\mathbf{E}} \tag{125}$$

This is not the form of the electro-magnetic force usually seen in classical EHD formulations [11]. Through the use of thermodynamics and the material constitutive equation of state, the electric force per unit volume in EHD is most often used in the following equivalent forms [10, p.505-507][8, p.59-63]

$$\underline{\mathbf{f}}^{\mathbf{EM}} = q_o \underline{\mathbf{E}} - \frac{\underline{\mathbf{E}} \cdot \underline{\mathbf{E}}}{2} \nabla \varepsilon + \frac{1}{2} \left[\underline{\mathbf{E}} \cdot \underline{\mathbf{E}} \rho \left(\frac{\partial \varepsilon}{\partial \rho} \right)_{T=const} \right] \tag{126}$$

$$\underline{\mathbf{f}}^{\mathbf{EM}} = q_o \underline{\mathbf{E}} - \frac{\underline{\mathbf{E}} \cdot \underline{\mathbf{E}}}{2} (\nabla T) \left(\frac{\partial \varepsilon}{\partial T} \right)_{\rho=const} + \frac{\rho}{2} \nabla \left[\underline{\mathbf{E}} \cdot \underline{\mathbf{E}} \left(\frac{\partial \varepsilon}{\partial \rho} \right)_{T=const} \right] \tag{127}$$

The three terms in the equation are the electrophoretic, dielectrophoretic and electrostrictive terms, respectively.

The electrophoretic force or Coulomb force is caused by the electric field acting on free charges in the fluid. It is an irrotational force except when charge gradients are present [45].

The dielectrophoretic force is also a translational force, but is caused by polarization of the fluid and particles in the fluid. The dielectrophoretic force will occur where high gradients of electric permittivity are present. This condition will be true in high temperature gradient flows, multi-constituent flows, particulate flows [18] or any time the electric field must pass through two contacting media of different permittivities [46]. Grassi and DiMarco [47] treat the dielectrophoretic force as it applies to bubbly flows and heat transfer. Poulter and Allen [45] note that the dielectrophoretic force produces greatest circulation when the dielectric permittivity is inhomogeneous and non-parallel with the applied electric field.

The last force, the electrostrictive force, is a distortive force (as opposed to the previous translational forces) associated with fluid compression and shear. The electrostrictive force is usually smaller than the -phoretic forces. It is present in high pressure gradient flows, compressible flows, and flows with a non-uniform applied electric field. Pohl [18] describes this phenomenon in greater detail.

Classical EHD modeling derives directly from the unified EMHD theory. Thus, the electric current density, using EHD assumptions, reduces to

$$\underline{\mathbf{J}} = q_o \underline{\mathbf{v}} + \sigma_1 \underline{\mathbf{E}} + \sigma_4 \nabla T \qquad (128)$$

However, this is not the form seen in classical EHD models [11] which typically define the conduction electric current as only the first term of equation (21). However, more advanced classical EHD models define the current as [9, p.562]

$$\underline{\mathbf{J}} = q_o \underline{\mathbf{v}} + \underline{\mathbf{J}}_c = q_o \underline{\mathbf{v}} + q_o b\, \underline{\mathbf{E}} - D_o \nabla q_o \qquad (129)$$

The last two equations imply that the temperature gradient is directly related to the electric charge gradient. This may be shown to be true based on the Einstein-Fokker relationships, derived from studies of Brownian motion [25, p.264-273], which relate any concentration gradient to a charge mobility and a diffusion. Newman [48] also provides a detailed discussion of the concepts of diffusion and mobility. The electric charge diffusion term is often neglected where only limited amount of free charges are available [49].

By introducing classical EHD assumptions in the unified EMHD theory, the equation (23) for heat flux reduces to

$$\underline{\dot{\mathbf{q}}} = \kappa_1 \underline{\mathbf{E}} + \kappa_4 \nabla T \qquad (130)$$

The classical EHD models neglect the contribution to heat transfer from the electric field so that equation (130) reduces to Fourier's law of heat conduction.

$$\underline{\dot{q}} = -\kappa \nabla T \tag{131}$$

Although classical EHD modeling seems to neglects heat transfer induced by the electric field and electric current, Joule heating effect ($-\underline{\mathbf{J}}_c \cdot \underline{\mathbf{E}}$ term from EMHD equation 19) is usually included in the EHD computations [50,51].

8.2 Classical magneto-hydrodynamics (MHD)

The classical modeling of MHD assumes non-relativistic and quasi-magnetostatic conditions. It implies that electric current comes primarily from conductive means and that there are no free electric charges in the fluid [11]. With these assumptions Maxwell's system becomes

$$\nabla \cdot \underline{\mathbf{B}} = 0 \tag{132}$$

$$\nabla \times \underline{\mathbf{E}} = -\frac{\partial \underline{\mathbf{B}}}{\partial t} \tag{133}$$

$$\nabla \times \underline{\mathbf{H}} = \underline{\mathbf{J}} \tag{134}$$

$$\nabla \cdot \underline{\mathbf{J}} = 0 \tag{135}$$

The modifications to the Navier-Stokes relations come from the electro-magnetic force on the fluid from which all induced electric field terms have been neglected. Using the MHD assumptions, the electro-magnetic force per unit volume in the unified EMHD theory becomes [11]

$$\underline{\mathbf{f}}^{EM} = \underline{\mathbf{J}} \times \underline{\mathbf{B}} + (\nabla \underline{\mathbf{B}}) \cdot \underline{\mathbf{M}} \tag{136}$$

The second term, source of dimagnetophoretic and magnetostrictive forces, is typically neglected in classical MHD [10, p.508]. Thus, the electro-magnetic force per unit volume in the classical MHD is modeled as [11]

$$\underline{\mathbf{f}}^{EM} = \underline{\mathbf{J}} \times \underline{\mathbf{B}} \tag{137}$$

By making MHD assumptions, the conduction current in the EMHD can be expressed with equation (38). However, classical MHD theory usually defines the electric conduction current as [10, p.510]

$$\underline{\mathbf{J}}_c = \sigma_1 \underline{\mathbf{E}} + \sigma_4 \nabla T = \sigma_1 \underline{\mathbf{E}} + \sigma_1 (\underline{\mathbf{v}} \times \underline{\mathbf{B}}) + \sigma_4 \nabla T \tag{138}$$

Here, σ_4 is the Seebeck coefficient [9, p.174] which in some classical MHD formulations is not used [11]. Clearly, the classical MHD formulations neglect a significant number of physical effects [52,53].

Similarly, in classical MHD modeling, Joule heating is often included in the energy relation, but the heat transfer constitutive relation remains the same as in equation (11). In comparison, the unified EMHD model with classical MHD assumptions can be expressed with equation (39).

It could be concluded that classical EHD models include many important effects and correspond to the unified EMHD theory well, while classical MHD formulations need improvements in the force, current and heat transfer terms.

As in classical EHD modeling, it is important to be aware of the fact that many force, current and heat transfer terms can be written in several different forms, each of which is equivalent. It is, therefore, important to recognize the potential danger of simply adding terms from different MHD models.

9. SOLIDIFICATION WITH ELECTRO-MAGNETIC FIELDS

During solidification from a melt, if the control of melt motion is performed exclusively via an externally applied variable temperature field, it will take quite a long time for the thermal front to propagate throughout the melt thus eventually causing local melt density variations and altering the thermal buoyancy forces. It has been well known that an externally applied steady magnetic or electric field can, practically instantaneously, influence the flow-field vorticity and change the flow pattern in an electrically conducting fluid [51-59,33]. Similarly, it is well-known that applying an electric potential difference to a flow-field of a homogeneous mixture will cause fractionation or separation of the homogeneous mixture into regions having high concentration of the constituents. This phenomena, known as free-flow electrophoresis, has been extensively studied experimentally and, to a lesser extent, numerically [50] using classical EHD modeling. Nevertheless, there are no publications yet on actual algorithms for determining the proper variation of intensity and orientation of the externally applied magnetic and electric fields. This is not a trivial problem because we are dealing with a moving electrically conducting fluid within which an electric current is induced as the fluid cuts through the externally applied magnetic field lines [11]. This induced electric current generates heat (Joule effect) as it passes through the fluid that has a finite electrical resistivity. In the case of solidification, the amount of heat generated through the Joule effect due to the externally applied magnetic field is often neglected compared to the latent heat of solidification and the amount of heat transferred in the melt by thermal conduction.

The latent heat released or absorbed per unit mass of mushy region (where $T_{liquidus} > T > T_{solidus}$) is proportional to the local volumetric liquid/(liquid + solid) ratio often modeled [59] as

$$f = \frac{V_\ell}{V_\ell + V_s} = \left(\frac{T - T_{solidus}}{T_{liquidus} - T_{solidus}} \right)^n = \theta^n \tag{139}$$

Here, θ is the non-dimensional temperature, the exponent n is typically $0.2 < n < 5$, subscripts ℓ and s designate liquid and solid phases, respectively, while $f = 1$ for $T \geq T_{liquidus}$ and $f = 0$ for $T \leq T_{solidus}$. Physical properties (density, viscosity, heat conductivity, heat capacity, etc.) are often significantly different in the melt as compared to the solid phase. We will assume linear variation of density as a function of the non-dimensional temperature, θ, in the liquid

$$\rho_\ell = \rho_r \left[1 + \left(\frac{\partial(\rho_\ell/\rho_r)}{\partial\theta} \right)_r \right] (\theta - \theta_r) = \rho_r \left[1 - \alpha_\ell (\theta - \theta_r) \right] \tag{140}$$

with a similar expression for the solid phase where the reference values are designated with the subscript "r". In this work, we assumed that electric conductivity and magnetic permeability do not vary with temperature.

The EHD and the MHD systems of equations including solidification can be non-dimensionalized in a number of ways. The typical non-dimensional numbers are [33,60]:

Reynolds hydrodynamic	Froude	Eckert	
$R_e = \frac{\rho_r V_r \ell_r}{\mu_{vr}}$	$F_R^2 = \frac{V_r^2}{g_r \ell_r}$	$E_C = \frac{V_r^2}{c_r \Delta T_r}$	(141)

Prandtl hydrodynamic

$$P_R = \frac{\mu_{vr} c_r}{\kappa_r}$$

Stefan

$$S_{TE} = \frac{c_r \Delta T_r}{L_r}$$

Grashof

$$G_R = \frac{\rho_r^2 \alpha_r g_r \Delta T_r \ell_r^3}{\mu_{vr}^2} \quad (142)$$

Hartmann

$$H_T = \mu_r \ell_r H_r \left(\frac{\sigma_r}{\mu_{vr}} \right)^{1/2}$$

Prandtl magnetic

$$P_m = \frac{\mu_{vr} \sigma_r \mu_r}{\rho_r}$$

Prandtl electric

$$P_E = \frac{\mu_{vr}}{\rho_r b_r \Delta\phi_r} \quad (143)$$

Coulomb

$$S_E = \frac{q_{or} \Delta\phi_r}{\rho_r V_r^2}$$

Electric field

$$N_E = \frac{q_{or} \ell_r^2}{\varepsilon_r \Delta\phi_r}$$

Charge diffusivity

$$D_E = \frac{\mu_{vr}}{\rho_r D_{or}} \quad (144)$$

where $\mu_{vr}, c_r, \Delta\phi_r, \kappa_r, \mu_r, L_r, \ell_r$ are the reference values of viscosity, specific heat, electric potential difference, heat conductivity, magnetic permeability, latent heat of liquid-solid phase change, and length, respectively. Also, mixture density and modified heat capacity can be defined as

$$\rho_{mix} = f\, \rho_\ell + (1-f)\rho_s \quad (145)$$

$$c_{mix} = f\rho_\ell \frac{\partial (c_\ell\, \theta_\ell)}{\partial \theta} + (1-f)\rho_s \frac{\partial (c_s^{eq}\, \theta_s)}{\partial \theta} \quad (146)$$

An enthalpy method [58,59] can be used to formulate the equivalent specific heat coefficient in the solid phase defined as

$$c_s^{eq} = c_s - \frac{1}{S_{TE}} \frac{\partial L}{\partial \theta} \quad (147)$$

so that latent heat is released in the mushy region according to equation (139).

9.1 EHD and solidification

EHD equations for phase-changing liquid-solid mixtures, where the solid phase is treated as the second liquid with extremely high viscosity, can be derived using Boussinesq approximation for thermal buoyancy [61]. We can also define mixture electric charge mobility

$$b_{mix} = f\, b_\ell + (1-f) b_s \tag{148}$$

and combined hydrodynamic and hydrostatic pressures in liquid and solid

$$\hat{p}_\ell = \frac{p}{\rho_\ell} + \frac{\varphi}{F_R^2} \qquad \text{and} \qquad \hat{p}_s = \frac{p}{\rho_s} + \frac{\varphi}{F_R^2} \tag{149}$$

where φ is the non-dimensional gravity potential defined as $\underline{g} = \nabla\varphi$.

Assuming equal velocities for both phases, the mass conservation is

$$\nabla \cdot \underline{v} = 0 \tag{150}$$

Linear momentum conservation for two-phase EHD flows with thermal buoyancy and Coulomb force

$$\begin{aligned}
&\rho_{mix} \frac{\partial \underline{v}}{\partial t} + f\rho_\ell \nabla \cdot \left(\underline{v}\underline{v} + \hat{p}_\ell \underline{\underline{I}}\right) + (1-f)\rho_s \nabla \cdot \left(\underline{v}\underline{v} + \hat{p}_s \underline{\underline{I}}\right) \\
&= f\left\{\nabla \cdot \left[\frac{\mu_{v\ell}}{R_e}\left(\nabla\underline{v} + (\nabla\underline{v})^*\right)\right] + \frac{G_R}{R_e^2}\rho_\ell \alpha_\ell \theta\, \underline{g}\right\} \\
&+ (1-f)\left\{\nabla \cdot \left[\frac{\mu_{vs}}{R_e}\left(\nabla\underline{v} + (\nabla\underline{v})^*\right)\right] + \frac{G_R}{R_e^2}\rho_s \alpha_s \theta\, \underline{g}\right\} + S_E q_o \underline{E}
\end{aligned} \tag{151}$$

Energy conservation for incompressible two-phase EHD flows including Joule heating can be written as [60]

$$\begin{aligned}
&c_{mix} \frac{\partial \theta}{\partial t} + f\rho_\ell \nabla \cdot \left(c_\ell \theta \underline{v}\right) + (1-f)\rho_s \nabla \cdot \left(c_s^{eq} \theta \underline{v}\right) \\
&= \frac{1}{R_e P_R}\left[f\, \nabla \cdot \left(\kappa_\ell \nabla\theta\right) + (1-f)\, \nabla \cdot \left(\kappa_s \nabla\theta\right)\right] \\
&+ S_E E_c \left(q_o \underline{v} \cdot \underline{E} + q_o b_{mix} \frac{\underline{E}\cdot\underline{E}}{R_e P_E} - b_{mix} \frac{\underline{E}\cdot\nabla q_o}{R_e D_E} \right)
\end{aligned} \tag{152}$$

Electric charge conservation equation including migration and diffusion is

$$\frac{\partial q_o}{\partial t}+\nabla\cdot\left[q_o\left(\underline{v}+\frac{b_{mix}}{R_eP_E}\underline{E}\right)\right]=\frac{1}{R_eD_E}\nabla\cdot\left(b_{mix}\nabla q_o\right) \tag{153}$$

Since $\underline{E}=-\nabla\phi$, the electric potential equation resulting from equation (13)

$$\nabla\cdot\left[\left(f\varepsilon_\ell+(1-f)\varepsilon_s\right)\nabla\phi\right]=-N_Eq_o \tag{154}$$

must be solved simultaneously with the equations (151-154).

9.2 MHD and solidification

MHD two-phase solid-liquid flows can be modeled using a similar approach. The non-dimensional Navier-Stokes equations for phase-changing mixtures of two liquids (solid phase is treated as the second liquid with extremely high viscosity), can be formulated [33] so that the mixture mass conservation is

$$\nabla\cdot\underline{v}=0 \tag{155}$$

Linear momentum conservation for two-phase MHD flows with thermal buoyancy and magnetic force

$$\rho_{mix}\frac{\partial\underline{v}}{\partial t}+f\rho_\ell\nabla\cdot\left(\underline{v}\underline{v}+\hat{p}_\ell\underline{\underline{I}}\right)+(1-f)\rho_s\nabla\cdot\left(\underline{v}\underline{v}+\hat{p}_s\underline{\underline{I}}\right)$$

$$=f\left\{\nabla\cdot\left[\frac{\mu_{v\ell}}{R_e}\left(\nabla\underline{v}+(\nabla\underline{v})^*\right)\right]+\frac{G_R}{R_e^2}\rho_\ell\alpha_\ell\theta\ \underline{g}+\frac{H_T^2}{P_mR_e^2}\mu_\ell(\nabla\times\underline{H})\times\underline{H}\right\} \tag{156}$$

$$+(1-f)\left\{\nabla\cdot\left[\frac{\mu_{vs}}{R_e}\left(\nabla\underline{v}+(\nabla\underline{v})^*\right)\right]+\frac{G_R}{R_e^2}\rho_s\alpha_s\theta\ \underline{g}+\frac{H_T^2}{P_mR_e^2}\mu_s(\nabla\times\underline{H})\times\underline{H}\right\}$$

The non-dimensional hydrodynamic, hydrostatic, and magnetic pressures were combined to give

$$\hat{p}_\ell=\frac{p}{\rho_\ell}+\frac{\varphi}{F_R^2}+\frac{H_T^2}{P_mR_e^2}\mu_\ell\underline{H}\cdot\underline{H}\qquad\text{and}\qquad\hat{p}_s=\frac{p}{\rho_s}+\frac{\varphi}{F_R^2}+\frac{H_T^2}{P_mR_e^2}\mu_s\underline{H}\cdot\underline{H} \tag{157}$$

where φ is the non-dimensional gravity potential defined as $\underline{\mathbf{g}} = \nabla\varphi$. Then, the energy conservation for incompressible two-phase MHD flows including Joule heating can be written as [33]

$$c_{mix}\frac{\partial\theta}{\partial t} + f\rho_\ell\nabla\cdot\left(c_\ell\theta\underline{\mathbf{v}}\right) + (1-f)\rho_s\nabla\cdot\left(c_s^{eq}\theta\underline{\mathbf{v}}\right)$$

$$= f\left[\frac{1}{R_eP_R}\nabla\cdot\left(\kappa_\ell\nabla\theta\right) + \frac{1}{\sigma_\ell}\frac{H_T^2E_c}{P_m^2R_e^3}\left(\nabla\times\underline{\underline{\mathbf{H}}}\right)\cdot\left(\nabla\times\underline{\underline{\mathbf{H}}}\right)\right] \tag{158}$$

$$+(1-f)\left[\frac{1}{R_eP_R}\nabla\cdot\left(\kappa_s\nabla\theta\right) + \frac{1}{\sigma_s}\frac{H_T^2E_c}{P_m^2R_e^3}\left(\nabla\times\underline{\underline{\mathbf{H}}}\right)\cdot\left(\nabla\times\underline{\underline{\mathbf{H}}}\right)\right]$$

The magnetic field transport equation for the two-phase MHD flow in its non-dimensional form becomes [1, p.150]

$$\frac{\partial\underline{\underline{\mathbf{H}}}}{\partial t} - \nabla\times\left(\underline{\mathbf{v}}\times\underline{\underline{\mathbf{H}}}\right) = -\frac{1}{P_mR_e}\nabla\times\left[\left(\frac{f}{\sigma_\ell\mu_\ell} + \frac{1-f}{\sigma_s\mu_s}\right)\nabla\times\underline{\underline{\mathbf{H}}}\right]$$

If electric conductivity and magnetic permeability are assumed constant, then

$$\frac{\partial\underline{\underline{\mathbf{H}}}}{\partial t} - \nabla\times\left(\underline{\mathbf{v}}\times\underline{\underline{\mathbf{H}}}\right) = \frac{f/\left(\sigma_\ell\mu_\ell\right) + (1-f)/\left(\sigma_s\mu_s\right)}{P_mR_e}\nabla^2\underline{\underline{\mathbf{H}}} \tag{159}$$

needs to be solved either simultaneously or intermittently [33] with the equations (155-158).

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