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LOCATING THE DISCONTINUITIES OF A BOUNDED FUNCTION BY THE PARTIAL SUMS OF ITS FOURIER SERIES I: PERIODICAL CASE

GEORGE KVERNADZE, THOMAS HAGSTROM, AND HENRY SHAPIRO

ABSTRACT. A key step for some methods dealing with the reconstruction of a function with jump discontinuities is the accurate approximation of the jumps and their locations. Various methods have been suggested in the literature to obtain this valuable information.

In the present paper, we develop an algorithm based on identities which determine the jumps of a 2π-periodic bounded not-too-highly oscillating function by the partial sums of its differentiated Fourier series. The algorithm enables one to approximate the locations of discontinuities and the magnitudes of jumps of a bounded function. We study the accuracy of approximation and establish asymptotic expansions for the approximations of a 2π-periodic piecewise smooth function with one discontinuity. By an appropriate linear combination, obtained via derivatives of different order, we significantly improve the accuracy. Next, we use Richardson's extrapolation method to enhance the accuracy even more. For a function with multiple discontinuities we establish simple formulae which "eliminate" all discontinuities of the function but one. Then we treat the function as if it had one singularity following the method described above.

1. INTRODUCTION

It is well known that the main difficulty in applying a Fourier series as a tool for approximating a discontinuous function is the Gibbs phenomenon. Namely, the approximation of a function by the n-th partial sum of its Fourier series is only of order $O(1/n)$ for each point of continuity of the function and oscillations are $O(1)$ in an $O(1/n)$ neighborhood of the discontinuity point.

Two distinct approaches to resolve this difficulty have been suggested in the literature. The first is to reduce the oscillatory behavior by filtering. The second is to use step functions to reconstruct the discontinuous function. The latter approach was first suggested by Gottlieb et. al. [17] and has been further developed in [1], [2], [5], and [16]. The key step in the method of reconstruction suggested in [5] is the accurate approximation of the location and the jumps of a given function.

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Later, Eckhoff [10], [11] considered a different approach to locate the discontinuities using Prony's method. As a result he developed an efficient method of approximating the locations of singularities and the jumps of a piecewise smooth function with multiple discontinuities. The approximations are found as the solution of a system of algebraic equations.

To justify the importance of allocating the discontinuities and the jumps of a function, let us give a brief review of the idea of reconstruction of a function from its truncated Fourier series as developed in the above mentioned papers.

Let g be a 2π-periodic function which is piecewise smooth on the period with a finite number, M, of jump discontinuities. In addition, we assume that the first $2n+1$ Fourier coefficients of the function are known. If $G(\theta) = (\pi - \theta)/2$, $\theta \in (0, 2\pi)$, is the 2π-periodic sawtooth function, then the assumption that the function g is piecewise smooth on $[-\pi, \pi]$ with a finite number of singularities is equivalent to the following representation of the function:

$$g(\theta) = \frac{1}{\pi} \sum_{m=0}^{M-1} [g]_m G(\theta - \theta_m) + \bar{g}(\theta), \tag{1}$$

where θ_m and $[g]_m$, $m = 0, 1, \ldots, M-1$, are the locations of discontinuities and the associated jumps of the function g, and $\bar{g}$ is a 2π-periodic continuous function, which is piecewise smooth on $[-\pi, \pi]$.

Hence, the problem is to find a good approximation for the constants θ_m and $[g]_m$, given the first $2n+1$ Fourier coefficients of the function g. Then $\bar{g}$ can be recovered from the partial sums of its Fourier series based on identity (1) and the undesirable Gibbs phenomenon could be avoided.

Recently another approach to recovery a piecewise smooth function was suggested by Geer and Banerjee. (See [4], [13], and [14].) The authors introduced a family of periodic functions with "built-in" discontinuities to reconstruct a piecewise smooth function with exponential accuracy. The main assumption of the method is knowledge of the jumps and the locations of discontinuity of the given function. To find these, the authors suggested the following: use the well-known formula of symmetric difference of the partial sums of Fourier series which determines the jumps of a bounded function to obtain the first estimate for the location of discontinuities; then utilize the modified least-squares method to improve the accuracy of approximation. It should be mentioned that a method for the recovery of a piecewise smooth function with exponential accuracy, utilizing the Gegenbauer polynomials, was developed in a series of papers by Gottlieb and Shu (see [18] and the indicated references). But again, the authors assume some knowledge of the location of the singularities of the function.

In the present paper, we consider an essentially different approach for the approximation of the points of discontinuity and the jumps of a function based on special formulae determining the jumps of a bounded not-too-highly oscillating function by the partial sums of its differentiated Fourier series. It is shown that the largest local maximum of the absolute value of the differentiated partial sums of the Fourier series occur in the vicinity of the actual points of discontinuity of the function. Furthermore, for a piecewise smooth function with one jump discontinuity, we establish asymptotic expansions for the approximations of the location of the discontinuity and the magnitude of the jump. Utilizing the expansion formulae, we use Richardson's extrapolation method to achieve higher accuracy. For a function with multiple singularities, we establish simple formulae which "eliminate" all discontinuities of the function but one. Then we treat the modified function as if it had only one discontinuity, using the method described above.

2. Definitions

Throughout this paper we use the following general notations: N, Z_+, Z, and R are the sets of positive integers, nonnegative integers, integers, and real numbers, respectively. $L[a,b]$ is the space of integrable functions. $W[a,b]$ is the space of functions on $[a,b]$ which may have discontinuities only of the first kind and are normalized by the condition $g(\theta) = (g(\theta+) + g(\theta-))/2$, $\theta \in (a,b)$. (Here, and elsewhere, $g(\theta+)$ and $g(\theta-)$ mean the right and left hand-side limits of a function g at a point θ, respectively). $C[a,b]$ is the space of continuous functions on $[a,b]$ with uniform norm $\|\cdot\|_{[a,b]}$. By $C^p[a,b]$, $p \in N$, we denote the space of p-times continuously differentiable functions on $[a,b]$.

All functions below are assumed to be 2π-periodic with the obvious exceptions.

If $g \in L[-\pi,\pi]$, then g has a Fourier series with respect to the trigonometric system $\{1, \cos n\theta, \sin n\theta\}_{n=1}^{\infty}$, and we denote the n-th partial sum of the Fourier series of g by $S_n(g;\theta)$, i.e.,

$$S_n(g;\theta) = \frac{a_0(g)}{2} + \sum_{k=1}^{n}(a_k(g)\cos k\theta + b_k(g)\sin k\theta),$$

where

$$a_k(g) = \frac{1}{\pi}\int_{-\pi}^{\pi} g(\tau)\cos k\tau d\tau \quad \text{and} \quad b_k(g) = \frac{1}{\pi}\int_{-\pi}^{\pi} g(\tau)\sin k\tau d\tau$$

are the k-th Fourier coefficients of the function g.

By $\tilde{S}_n(g;\theta)$ we denote the n-th partial sum of the conjugate series, i.e.,

$$\tilde{S}_n(g;\theta) = \sum_{k=1}^{n}(a_k(g)\sin k\theta - b_k(g)\cos k\theta).$$

Correspondingly, by $\tilde{g}$ we denote the conjugate function, i.e.,

$$\tilde{g}(\theta) = \lim_{h\to 0}\{-\frac{1}{\pi}\int_{h}^{\pi}\frac{g(\theta+\tau) - g(\theta-\tau)}{2\tan\frac{\tau}{2}}d\tau\},$$

which exists and is finite almost everywhere for any $g \in L[-\pi,\pi]$ (cf. [19, Theorem, p. 79]).

By K we denote positive constants, possibly depending on some fixed parameters and in general distinct in different formulae. For positive quantities A_n and B_n, possibly depending on some other variables as well, we write $A_n = o(B_n)$, $A_n = O(B_n)$, or

$A_n \simeq B_n$, if $\lim_{n\to\infty} A_n/B_n = 0$, $\sup_{n\in N} A_n/B_n < \infty$, or $K_1 < A_n/B_n < K_2$, respectively, where $K_1 > 0$ and $K_2 > 0$ are some absolute constants.

Definition. Let $\Lambda = (\lambda_k)_{k=1}^{\infty}$ be a nondecreasing sequence of positive numbers such that

$$\sum_{k=1}^{\infty} \frac{1}{\lambda_k} = \infty. \tag{2}$$

A function f is said to have Λ-bounded variation on $[a, b]$, i.e., $f \in \Lambda BV[a, b]$, if

$$\sup_{\Pi} \sum_{k=1}^{n} \frac{|f(a_k) - f(b_k)|}{\lambda_k} < \infty,$$

where Π is an arbitrary system of disjoint intervals $(a_k, b_k) \in [a, b]$.

We say that a function f is of *harmonic bounded variation* on $[a, b]$, i.e., $f \in HBV[a, b]$, if $\lambda_k = k$, $k \in N$.

Remark 1. For a reader unfamiliar with $\Lambda BV[a, b]$ classes of functions we give some basic properties of these classes.

The Λ-variation "measures" the total oscillation of a bounded function. $\Lambda BV[a, b]$ is a generalization of $V[a, b]$, the class of functions of bounded variation (obviously $\Lambda BV[a, b] = V[a, b]$ if $\lambda_k = 1$, $k \in N$).

Waterman [23, p. 108] mentioned that the inclusion

$$\Lambda BV[a, b] \subset W[a, b] \tag{3}$$

holds for any $\Lambda BV[a, b]$ class of functions.

It is known as well [22, Theorem 3, p. 114] that for $\Gamma BV[a, b]$ and $\Lambda BV[a, b]$ Waterman's classes of functions, defined by the sequences $\Gamma = (\gamma_k)_{k=1}^{\infty}$ and $\Lambda = (\lambda_k)_{k=1}^{\infty}$, respectively, the inclusion $\Lambda BV[a, b] \subset \Gamma BV[a, b]$ holds if and only if $\sum_{k=1}^{n} 1/\gamma_k = O(\sum_{k=1}^{n} 1/\lambda_k)$.

The constraint on the sequence Λ is natural, since if series (2) converges, $\Lambda BV[a, b] = B[a, b]$, where $B[a, b]$ is the class of all bounded functions on $[a, b]$. This makes it clear that the $HBV[a, b]$ class is sufficiently wide and "almost" covers $B[a, b]$, since $\Lambda = (k^{1+\epsilon})_{k=1}^{\infty}$ converges for any $\epsilon > 0$. □

If there is no ambiguity, we shall usually suppress the dependence on the domain and simply write C, ΛBV, etc.

3. Main identities

The identity determining the jumps of a function of bounded variation by means of the partial sums of its Fourier series has been known for a long time:

Theorem 1 (**[8] and [12]**). *Let $g \in V$. Then the identity*

$$\lim_{n\to\infty} \frac{S_n'(g;\theta)}{n} = \frac{1}{\pi}(g(\theta+) - g(\theta-)) \tag{4}$$

is valid for each fixed $\theta \in [-\pi, \pi]$.

Golubov [15] generalized identity (4) for Wiener's [24] V_p classes of functions and higher derivatives of the partial sums of Fourier and conjugate series. Further generalizations, extending the results of Golubov to ΛBV classes of functions, have been obtained by one of the authors.

Theorem 2 (**[20]**). *Let $r \in Z_+$ and suppose ΛBV is the class of functions of Λ-bounded variation determined by the sequence $\Lambda = (\lambda_k)_{k=1}^{\infty}$. Then*
a) the identity

$$\lim_{n\to\infty} \frac{S_n^{(2r+1)}(g;\theta)}{n^{2r+1}} = \frac{(-1)^r}{(2r+1)\pi}(g(\theta+) - g(\theta-)) \tag{5}$$

is valid for every $g \in \Lambda BV$ and each fixed $\theta \in [-\pi, \pi]$ if and only if

$$\Lambda BV \subseteq HBV. \tag{6}$$

b) There is no way to determine the jump at the point $\theta \in [-\pi, \pi]$ of an arbitrary function $g \in \Lambda BV$ by means of the sequence $\left(S_n^{(2r)}(g;\theta)\right)_{n=0}^{\infty}$.

Theorem 3 (**[20]**). *Let $r \in N$ and suppose ΛBV is the class of functions of Λ-bounded variation determined by the sequence $\Lambda = (\lambda_k)_{k=1}^{\infty}$. Then*
a) the identity

$$\lim_{n\to\infty} \frac{\tilde{S}_n^{(2r)}(g;\theta)}{n^{2r}} = \frac{(-1)^{r+1}}{2r\pi}(g(\theta+) - g(\theta-)) \tag{7}$$

is valid for every $g \in \Lambda BV$ and each fixed $\theta \in [-\pi, \pi]$ if and only if condition (6) holds.
b) There is no way to determine the jump at the point $\theta \in [-\pi, \pi]$ of an arbitrary function $g \in \Lambda BV$ by means of the sequence $\left(\tilde{S}_n^{(2r-1)}(g;\theta)\right)_{n=1}^{\infty}$.

Remark 2. Theorems 2 and 3 (see [20, Theorems 1 and 4]) implicitly include the following statement: if $g \in C \cap HBV$, then the convergence of (5) and (7) to zero is *uniform* with respect to $\theta \in [-\pi, \pi]$. □

Furthermore, as a simple corollary from Theorems 2 and 3 follow the identities which determine the jumps of the derivatives of a continuous function.

Corollary 1. *Let $r \in N$ and $r-p$ be a positive odd number, and suppose $g \in C^{p-1}$ is such that $g^{(p)} \in HBV$. Then the identity*

$$\lim_{n\to\infty} \frac{S_n^{(r)}(g;\theta)}{n^{r-p}} = \frac{(-1)^{\frac{r-p-1}{2}}}{(r-p)\pi}(g^{(p)}(\theta+) - g^{(p)}(\theta-)) \tag{8}$$

is valid for each fixed $\theta \in [-\pi, \pi]$.

Proof. By virtue of (3), $g^{(p)} \in HBV \subset W$. Hence

$$S_n^{(r)}(g;\theta) = S_n^{(r-p)}(g^{(p)};\theta) \tag{9}$$

for $r \geq p$. Then identity (8) instantly follows from (9) and Theorem 2. □

The following statement is proved similarly.

Corollary 2. *Let $r \in N$ and $r-p$ be a positive even number, and suppose $g \in C^{p-1}$ is such that $g^{(p)} \in HBV$. Then the identity*

$$\lim_{n\to\infty} \frac{\tilde{S}_n^{(r)}(g;\theta)}{n^{r-p}} = \frac{(-1)^{\frac{r-p}{2}-1}}{(r-p)\pi}(g^{(p)}(\theta+) - g^{(p)}(\theta-)) \tag{10}$$

is valid for each fixed $\theta \in [-\pi, \pi]$.

4. Preliminaries

In what follows we need the following additional notations.

By $\theta_m \equiv \theta_m(g)$ and $[g]_m \equiv g(\theta_m+) - g(\theta_m-)$, $m = 0, 1, \ldots, M-1$, we denote the points of discontinuity and the associated jumps of a function $g \in W$. By $M \equiv M(g)$ we denote the number of discontinuities (finite or infinite) of the function $g \in W$.

For a fixed $p \in Z_+$, $r \in N$, and $g \in L$ we set

$$DT_n(p;r;g;\theta) \equiv \frac{(r-p)\pi}{n^{r-p}} \begin{cases} (-1)^{\frac{r-p-1}{2}} S_n^{(r)}(g;\theta) & \text{if } r-p \text{ is odd,} \\ (-1)^{\frac{r-p}{2}-1} \tilde{S}_n^{(r)}(g;\theta) & \text{if } r-p \text{ is even.} \end{cases} \tag{11}$$

For a fixed $p \in Z_+$, $r \in N$, and $M \in N$, the points $\theta_m(p;r;g;n)$, $m = 0, 1, \ldots, M-1$, are defined via the following condition:

$$|DT_n(p;r;g;\theta_m(p;r;g;n))| = \max\{|DT_n(p;r;g;\theta)| : \theta \in B(\theta_m;\Delta(g))\}, \tag{12}$$

where $B(\theta_m;\Delta(g))$ is the closed ball around θ_m with the radius $\Delta(g) = \frac{1}{3}\min\{|\theta_m - \theta_k| \bmod 2\pi : \ m,k = 0,1,\ldots,M-1 \ \text{ and } \ m \neq k\}$.

To simplify notations, we sometimes omit fixed parameters and write $DT_n(\theta)$, $DT_n(g;\theta)$, or $DT_n(r;g;\theta)$. Similarly we simplify the notation for $\theta_m(p;r;g;n)$.

By $G(\theta) \equiv (\pi - \theta)/2$, $\theta \in (0, 2\pi)$, we denote the 2π-periodic extension of the sawtooth function. If $\gamma \in R$, then following the notations in [11] we set

$$G(\gamma;\theta) \equiv G(\theta - \gamma) \quad \text{and} \quad G_{k+1}(\gamma;\theta) \equiv \int G_k(\gamma;\theta)d\theta$$

for $k \in Z_+$, where $G_0 \equiv G$ and the constants of integration are successively determined by the condition

$$\int_{-\pi}^{\pi} G_k(\gamma;\tau)d\tau = 0.$$

It is trivial to check that

$$S_n'(G;\theta) = \left(\sum_{k=1}^{n} \frac{\sin k\theta}{k}\right)' = D_n(\theta) - \frac{1}{2}, \tag{13}$$

where

$$D_n(\theta) = \frac{1}{2} + \sum_{k=1}^{n} \cos k\theta = \begin{cases} \frac{\sin(n+\frac{1}{2})\theta}{2\sin\frac{\theta}{2}} & \text{for } \theta \notin 2\pi Z, \\ n+\frac{1}{2} & \text{for } \theta \in 2\pi Z \end{cases} \tag{14}$$

is the Dirichlet kernel.

Lemma 1. *Let $r \in N$ be fixed. Then*
a) the closed form of the following sum exists:

$$\sum_{k=1}^{n} k^r = \frac{1}{r+1}n^{r+1} + \frac{1}{2}n^r + \frac{r}{12}n^{r-1} + \dots, \tag{15}$$

where the last term contains either n or n^2.
b) The following expansion holds for every $a - 1 \in N$:

$$\begin{aligned}\sum_{k=1}^{n-1} \frac{1}{k^{r+1}} = \zeta(r+1) \quad &- \quad \frac{1}{r}\frac{1}{n^r} - \frac{1}{2}\frac{1}{n^{r+1}} \\ &+ \quad \sum_{s=2}^{a}(-1)^{s-1}\frac{B_s}{(2s)!}\frac{\Gamma(r+s)}{\Gamma(r+1)}\frac{1}{n^{r+s}} + O(\frac{1}{n^{r+a+1}}),\end{aligned} \tag{16}$$

where $\zeta(r) = \sum_{k=1}^{\infty} k^{-r}$, $r > 1$, is the Riemman zeta function, Γ is the Gamma function and B_s, $s \in N$, are Bernoulli numbers.

Statement a) of the lemma can be found in [9, p. 1]. Using the Laplace method, the proof of expansion (16) is a simple corollary of the integral representation of the Hurwitz zeta function [3, Theorem 12.2, p. 251] and Watson's lemma [6, p. 253]. It was generously offered by Prof. E. Coutsias [7].

Lemma 2 (Bernstein's inequality). *If T_n is a trigonometric polynomial of degree $n \in N$, then*

$$\|T_n'\|_{[a,b]} \le \frac{2\pi n}{b-a}\|T_n\|_{[a,b]},$$

where $[a,b] \subset [-\pi,\pi]$.

Lemma 3. *Let a function $g \in C^q$ be such that $g^{(q)} \in V$. Then*
a) $\tilde{g} \in C^{q-1}$ and $\tilde{g}^{(q-1)} \in Lip\ \alpha$ for all $\alpha \in (0,1)$, i.e., $|\tilde{g}^{(q-1)}(\theta) - \tilde{g}^{(q-1)}(\tau)| \leq K|\theta - \tau|^\alpha$ for some $K > 0$ and all $\theta, \tau \in R$.
b) The following estimates hold:

$$R_n(g),\ \tilde{R}_n(g) = o(\frac{1}{n^q}), \tag{17}$$

where $R_n(g) \equiv \|S_n(g;\cdot) - g\|_{[-\pi,\pi]}$ and $\tilde{R}_n(g) \equiv \|\tilde{S}_n(g;\cdot) - \tilde{g}\|_{[-\pi,\pi]}$, $n \in N$.

Proof. Statement a) can be found in [19, exercise 3, p. 81]. As regards statement b), by virtue of Hölder's inequality, since $g \in C^q$, we have:

$$\begin{aligned} R_n(g),\ \tilde{R}_n(g) &\leq \sum_{k=n}^{\infty}(|a_k(g)| + |b_k(g)|) = \sum_{k=n}^{\infty} \frac{|a_k(g^{(q)})| + |b_k(g^{(q)})|}{k^q} \\ &\leq \sqrt{2}\left(\sum_{k=n}^{\infty} \frac{1}{k^{2q}}\right)^{1/2} \left(\sum_{k=n}^{\infty}(a_k(g^{(q)})^2 + b_k(g^{(q)})^2)\right)^{1/2}. \end{aligned} \tag{18}$$

Meanwhile, it is known [21] that if $g \in C \cap V$, then

$$\sum_{k=n}^{\infty}(a_k(g)^2 + b_k(g)^2) = o(\frac{1}{n}). \tag{19}$$

Now (17) follows as a simple combination of (16), (18), and (19). □

The following are some basic properties of the function $D_n^{(r)}(\theta)$, $r \in N$.

Lemma 4. *Let $\varphi_n \equiv \varphi_n(r) > 0$ and $\psi_n \equiv \psi_n(r) > 0$ be the closest nonzero roots to the point zero of the equations $D_n^{(2r)}(\theta) = 0$ and $D_n^{(2r+1)}(\theta) = 0$, respectively. Then for any fixed $r \in Z_+$:*
a) $\varphi_n \in (\frac{\pi}{2n}, \frac{\pi}{n})$.
b) $\psi_n \in (\frac{\pi}{n}, \frac{2\pi}{n})$.
c) $(-1)^{r+1}D_n^{(2r+1)}(\varphi_n) \simeq n^{2r+2}$.
d) $(-1)^{r+1}D_n^{(2r+1)}(\theta)$ is increasing on $[-\varphi_n(r+1), \varphi_n(r+1)]$, concave on $[-\varphi_n(r+1), 0]$ and convex on $[0, \varphi_n(r+1)]$.
e) $(-1)^r D_n^{(2r)}(\theta)$ is a 2π-periodic even and smooth function with the global maximum attained at $\theta = 0$. In addition, the sequence of the absolute values of the local maxima is decreasing as a function of $\theta \in [0,\pi]$ and

$$|D_n^{(2r)}(0)| > K(r)|D_n^{(2r)}(\psi_n)|, \tag{20}$$

where $K(r) > 1$ and $n \in N$.

Proof. a) Let us prove the statement for an even n, i.e., $n \equiv 2n$. By (14) we have

$$\begin{aligned}\operatorname{sign} D_{2n}^{(2r)}(\frac{\pi}{2n}) &= \operatorname{sign}\left((-1)^r\left(\sum_{k=1}^{n-1} k^{2r}\cos\frac{k\pi}{2n} + \sum_{k=n+1}^{2n} k^{2r}\cos\frac{k\pi}{2n}\right)\right) \\ &= \operatorname{sign}\left((-1)^r\left(\sum_{k=1}^{n-1} k^{2r}\cos\frac{k\pi}{2n} + \sum_{k=0}^{n-1}(2n-k)^{2r}\cos\frac{(2n-k)\pi}{2n}\right)\right) \\ &= \operatorname{sign}\left((-1)^r\left(\sum_{k=1}^{n-1}(k^{2r}-(2n-k)^{2r})\cos\frac{k\pi}{2n} - (2n)^{2r}\right)\right) \\ &= \operatorname{sign}(-1)^{r+1}. \qquad (21)\end{aligned}$$

Again by (14), $\operatorname{sign} D_{2n}^{(2r)}(\theta) = \operatorname{sign}(-1)^r$ for $n \in N$ and $\theta \in [0, \pi/4n]$. The latter combined with (21) and the Mean Value Theorem instantly guarantees $\varphi_n \in (\pi/4n, \pi/2n)$. Similarly we treat the case when n is odd.
b) The statement is proved analogously and we omit the details.
c) According to (14) and (15)

$$(-1)^{r+1} D_n^{(2r+1)}(\theta) = \sum_{k=1}^{n} k^{2r+1}\sin k\theta < \sum_{k=1}^{n} k^{2r+1} \simeq n^{2r+2}. \qquad (22)$$

Meanwhile, since $\varphi_n \in [\pi/2n, \pi/n]$ (see a)), taking into account the well-known inequality $2\theta/\pi \le \sin\theta \le \theta$ for $\theta \in [0, \pi/2]$, we have

$$\sum_{k=1}^{n} k^{2r+1}\sin(k\varphi_n) > \frac{2}{\pi}\sum_{k=1}^{[n/2]} k^{2r+2}\varphi_n > \frac{1}{n}\sum_{k=1}^{[n/2]} k^{2r+2} \simeq n^{2r+2}, \qquad (23)$$

where $[a]$ means the integer part of a number a. Combination of (22) and (23) completes the proof of statemant c).
d) Since the function $(-1)^{r+1} D_n^{(2r+2)}(\theta)$ is positive on $[-\varphi_n(r+1), \varphi_n(r+1)]$ (see (14)), $(-1)^{r+1} D_n^{(2r+1)}(\theta)$ is monotonic on the interval. Furthermore, $(-1)^{(r+1)} D_n^{(2r+3)}(\theta)$ is positive and negative on $[-\psi_n(r+1), 0]$ and $[0, \psi_n(r+1)]$, respectively. But $\varphi_n(r+1) < \psi_n(r+1)$ (see a) and b)). Hence $(-1)^{r+1} D_n^{(2r+1)}(\theta)$ is concave and convex on $[-\varphi_n(r+1), 0]$ and $[0, \varphi_n(r+1)]$, respectively.
e) Let us prove inequality (20) as the rest of the statement is trivial. It is clear that

$$q_n \equiv \sum_{k=1}^{n} k^{2r}\left(\sum_{k>n/4}^{n} k^{2r}\right)^{-1} > 1 \qquad (24)$$

for $n > 4$. But by virtue of (15), $\lim_{n\to\infty} q_n > 1$ as well. The last combined with (24) implies the existence of $K(r)$ such that

$$q_n > K(r) > 1 \qquad (25)$$

for $n > 4$.

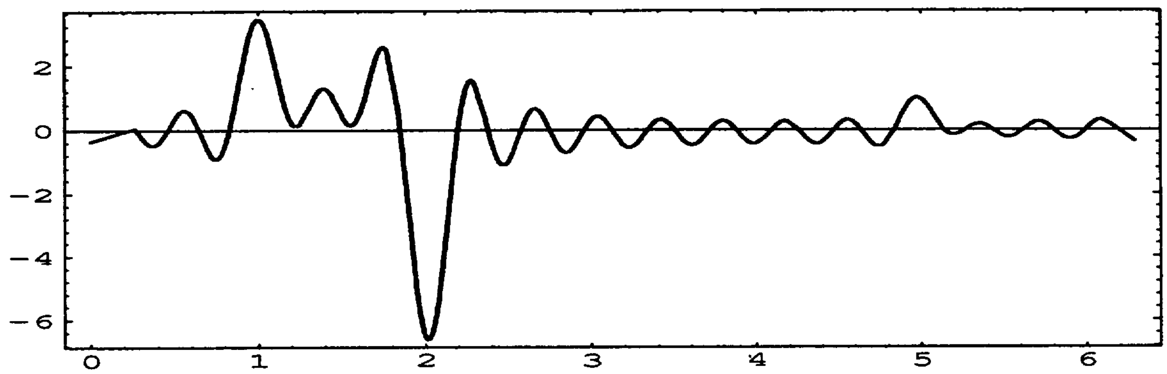

FIGURE 1. $n = 16$.

Besides

$$|D_n^{(2r)}(\psi_n)| = \left| \left(\sum_{k\psi_n \in [\pi/2, 3\pi/2]} + \sum_{k\psi_n \notin [\pi/2, 3\pi/2]} \right) k^{2r} \cos k\psi_n \right|$$

$$< \sum_{k > n/4}^{n} k^{2r}, \tag{26}$$

since ψ_n satisfies the estimate b) and the sums in (26) have different signs. The rest instantly follows from (24)-(26), and the identity $D_n^{(2r)}(0) = (-1)^r \sum_{k=1}^{n} k^{2r}$. Validity of (20) for $n \le 4$ is trivial. □

5. GENERAL IDEA OF ALGORITHMS AND THE ACCURACY OF APPROXIMATIONS

This is the general idea of all the following algorithms: according to identities (5) and (7), if $g \in HBV$, then for a fixed $r \in N$, $p = 0$, and sufficiently large $n \in N$, the function $|DT_n(\theta)|$, $\theta \in [-\pi, \pi]$, (see (11)) must attain the largest local maximum nearby the actual points of discontinuity of the function g, since at the the points of countinuity of g, $DT_n(\theta) = o(1)$ by virtue of Theorems 2 and 3. (The proof of Theorem 4 includes a rigorous proof of this statement.) Hence we search for the singularity locations of a function by locating the largest local spikes of the differentiated partial sums of its Fourier series.

Figures 1-4 represent the graphs of the normalized differentiated partial sums $\frac{1}{n} S_n'(g; \cdot)$ of the function (65) with increasing n. They illustrate the dynamics of creation of sharp spikes in the vicinity of the actual points of discontinuities of the function.

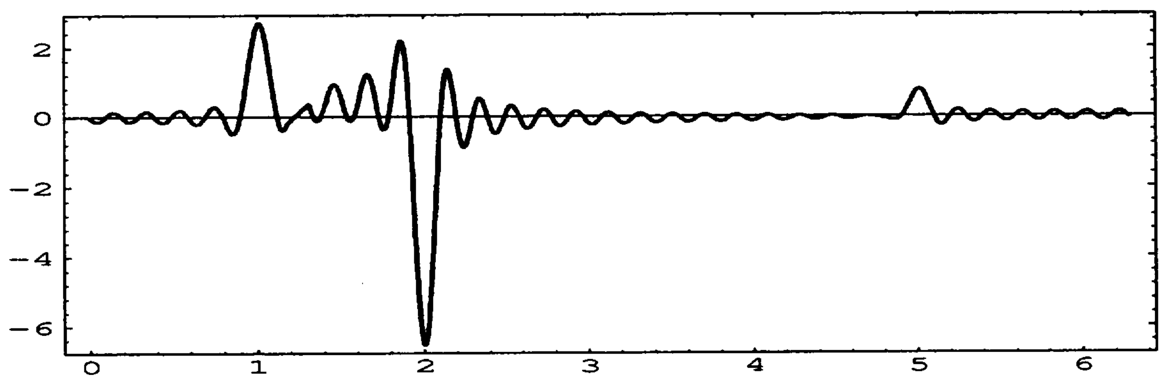

FIGURE 2. $n = 32$.

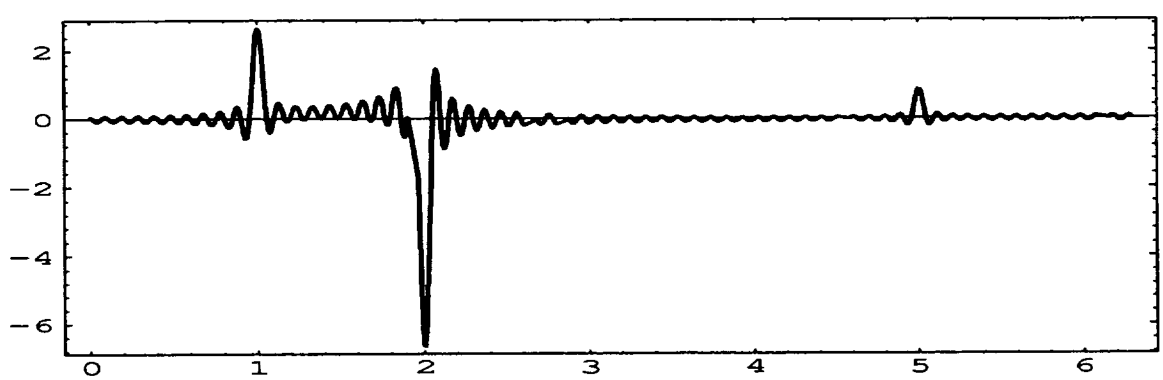

FIGURE 3. $n = 64$.

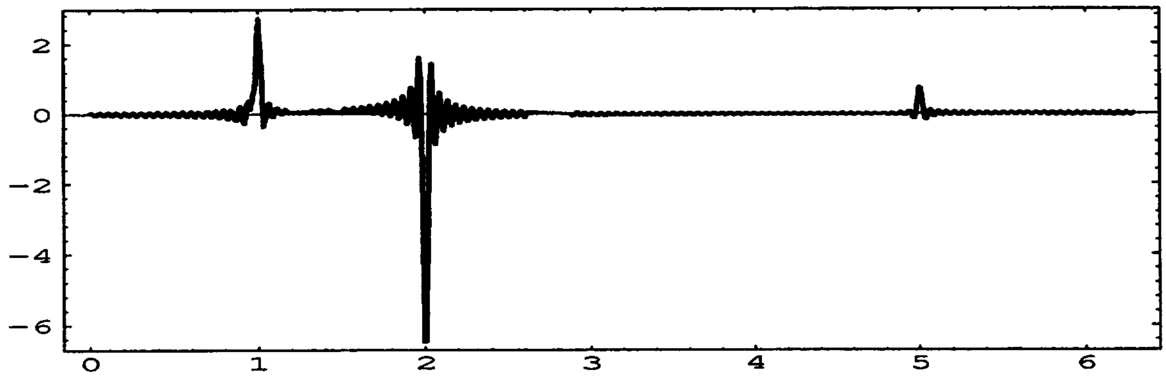

FIGURE 4. $n = 128$.

Now we study how well the points $\theta_m(n)$ and the values $DT_n(\theta_m(n))$ approximate the points of discontinuity θ_m and the jumps $[g]_m$ of a function g.

5.1. Approximation to the points of discontinuity. Let us first consider the worst possible case.

Theorem 4. *Let $p = 0$ and $r \in N$ be fixed, and suppose $g \in HBV$ is a function with a finite number, M, of discontinuities. Then the estimate*

$$\theta_m(n) = \theta_m + \frac{1}{[g]_m \Delta(g)} o(\frac{1}{n}) \tag{27}$$

is valid for each fixed $m = 0, 1, \dots, M-1$.

Proof. Without loss of generality let us make several assumptions. We assume that $M = 2$ and $r \equiv 2r+1$ is an odd number. The points of jump discontinuity of the function g are $\theta_0 = 0$ and θ_1. We shall prove estimate (27) for θ_0 as it is completely analagous for θ_1 by virtue of the periodicity of g.

Now let us set

$$\bar{g}(\theta) \equiv g(\theta) - \frac{[g]_0}{\pi} G(\theta) - \frac{[g]_1}{\pi} G(\theta_1; \theta). \tag{28}$$

It is obvious that

$$\bar{g} \in C \cap HBV, \tag{29}$$

since continuity of $\bar{g}$ follows from (28). Moreover, since $G \in V \subset HBV$ and HBV is a linear vector space (see [23, p. 108]), $\bar{g} \in HBV$ as well.

Besides by virtue of (11), (13), and (28)

$$\begin{aligned} DT_n(g;\theta) &= \frac{[g]_0}{\pi}DT_n(G;\theta) + \frac{[g]_1}{\pi}DT_n(G(\theta_1;\cdot);\theta) + DT_n(\bar{g};\theta) \\ &= \frac{(-1)^r(2r+1)[g]_0}{n^{2r+1}}D_n^{(2r)}(\theta) \\ &\quad + \frac{(-1)^r(2r+1)[g]_1}{n^{2r+1}}D_n^{(2r)}(\theta-\theta_1) + DT_n(\bar{g};\theta) \\ &\equiv I_0(n;\theta) + I_1(n;\theta) + ER(n;\theta). \end{aligned} \tag{30}$$

It is obvious that $|I_0(n;\theta)|$ attains the global maximum at $\theta = 0$ and without $I_1(n;\theta)$ and $ER(n;\theta)$ terms we could *exactly* locate the discontinuity point $\theta_0 = 0$ just searching for the global maximum of $|DT_n(g;\theta)|$ on the period. By virtue of (29) and Remark 2, $ER(n;\theta)$ contributes a small error independent of $\theta \in [-\pi,\pi]$, i.e., $ER(n;\theta) = o(1)$. But according to (14) and (30)

$$\begin{aligned} I_1(n;\theta) &= \frac{(-1)^r(2r+1)[g]_1}{n^{2r+1}}\left(\frac{\sin\left((n+\frac{1}{2})(\theta-\theta_1)\right)}{2\sin\frac{\theta-\theta_1}{2}}\right)^{(2r)} \\ &= \frac{(-1)^r(2r+1)[g]_1}{n^{2r+1}}\Big(\sum_{k=0}^{2r-1} C_{2r}^k(n+\frac{1}{2})^k \sin\left((n+\frac{1}{2})(\theta-\theta_1)+\frac{k\pi}{2}\right) \\ &\quad \times \left(\frac{1}{2\sin\frac{\theta-\theta_1}{2}}\right)^{(2r-k)} + (n+\frac{1}{2})^{2r}\frac{\sin\left((n+\frac{1}{2})(\theta-\theta_1)+r\pi\right)}{2\sin\frac{\theta-\theta_1}{2}}\Big) \\ &= \frac{[g]_1}{\Delta(g)}O(\frac{1}{n}) \end{aligned} \tag{31}$$

as well for $\theta \in B(0;\Delta(g))$. Hence

$$\epsilon_n \equiv \|I_1(n;\cdot) + ER(n;\cdot)\|_{[-\Delta(g),\Delta(g)]} = o(1). \tag{32}$$

Consequently, by virtue of statement e) of Lemma 4 and (32), we have

$$|I_0(n;0)| - \epsilon_n > |I_0(n;\varphi_n)| + \epsilon_n \tag{33}$$

for sufficiently large $n \in N$. But (33) combined with (12), (30), and statements a) and e) of Lemma 4 already guarantees

$$|\theta_0(n)| < \varphi_n < \frac{\pi}{n}$$

for sufficiently large $n \in N$.

Next, to achieve a more accurate estimate, namely (27), we use a simple estimate of a root of an equation.

First, let us mention that since $\theta_0(n)$ is the extremum point, then

$$DT_n'(g;\theta_0(n)) = 0, \tag{34}$$

which itself implies (see (30))

$$I_0'(n;\theta_0(n)) = -I_1'(n;\theta_0(n)) - ER'(n;\theta_0(n)) \equiv T_n(\theta_0(n)), \tag{35}$$

where T_n is an n-th degree trigonometric polynomial.

According to estimate (32), (35), and Lemma 2 we have

$$\|T_n\|_{[-\Delta(g),\Delta(g)]} = \frac{1}{\Delta(g)}o(n). \tag{36}$$

Let us assume for simplicity that $[g]_0 > 0$. Furthermore, we know $I_0'(n;\theta)$ is odd decreasing and convex on $[-\varphi_n(r+1), 0]$ and concave on $[0, \varphi_n(r+1)]$. (See statement d) of Lemma 4.) Hence the line passing through the points $(\pm\varphi_n(r+1), I_0'(n;\pm\varphi_n(r+1)))$ will occur below the positive part of the function $I_0'(n;\theta)$ and above its negative part. So, for sufficiently large $n \in N$, $\theta_0(n)$ will satisfy the inequality

$$|\theta_0(n)| < |\bar{\theta}_0(n)|, \tag{37}$$

where $\bar{\theta}_0(n)$ is the solution of the following equation

$$\frac{I_0'(n;\varphi_n(r+1))}{\varphi_n(r+1)}\theta = T_n(\theta). \tag{38}$$

Here the left hand side of the equation represents the above mentioned line.

Hence, by virtue of (11), (13), (30), (36), and statements a) and c) of Lemma 4, we obtain

$$\bar{\theta}_0(n) = \frac{1}{[g]_0\Delta(g)}o(\frac{1}{n}),$$

which combined with (37) completes the proof. □

Let us now consider a more typical case.

Theorem 5. *Let $p = 0$ and $r \in N$ be fixed, and suppose g is a piecewise continuous function such that $g' \in HBV$. In addition, we assume that $M(g)$ and $M(g')$ are finite. Then the estimate*

$$\theta_m(g;n) = \theta_m(g) + \frac{1}{[g]_m}\left([g']_m O(\frac{1}{n^2}) + \frac{1}{\Delta(g)}\sum_{k\neq m}[g]_k O(\frac{1}{n^2})\right) \tag{39}$$

is valid for each $m = 0, 1, \dots, M(g) - 1$.

Proof. Again for simplicity let us assume that $M(g) = 2$, $M(g') = 1$, $r \equiv 2r+1$ is odd, and $\theta_0(g) = \theta_0(g') = 0$. Furthermore, let us introduce a function $\bar{g}$ now via the following identity:

$$\bar{g}(\theta) = g(\theta) - \frac{1}{\pi}\sum_{m=0}^{1}[g]_m G(\theta_m;\theta) - \frac{1}{\pi}[g']_0 G_1(\theta). \tag{40}$$

Since the conditions of Theorem 5 in particular imply the conditions of Theorem 4, by similar arguments we conclude that $\theta_0(n)$ satisfies the estimate (37), where $\bar{\theta}_0(n)$ is the solution of equation (38) now with

$$\begin{aligned} T_n(\theta) &\equiv -I_1'(n;\theta) - (I_0^{(1)})'(n;\theta) - ER'(n;\theta) \\ &\equiv -\frac{[g]_1}{\pi} DT_n'(G(\theta_1;\cdot);\theta) - \frac{[g']_0}{\pi} DT_n'(G_1;\theta) - DT_n'(\bar{g};\theta). \end{aligned} \tag{41}$$

Hence, to complete the proof it is enough to estimate T_n.

By the construction $\bar{g}' \in C \cap HBV$ (see (29)). Consequently, according to (11), (41), and Remark 2

$$\|ER'(n;\cdot)\|_{[-\pi,\pi]} = o(1). \tag{42}$$

The estimate for I_1 directly follows from (31). Namely,

$$\|I_1'(n;\cdot)\|_{[-\Delta(g),\Delta(g)]} = \frac{[g]_1}{\Delta(g)} O(1). \tag{43}$$

As regards $I_0^{(1)}$, by virtue of (11), (13), (15), and (41) we have

$$\|(I_0^{(1)})'(n;\cdot)\|_{[-\pi,\pi]} = [g']_0 O(1). \tag{44}$$

The combination of (37), (38), (41)-(43), and (44) completes the proof. □

Now we turn our efforts to study probably the most interesting case: a 2π-periodic piecewise smooth function with one jump discontinuity. As expected, the approximation in this case is significantly more regular. Namely, the following statement holds.

Theorem 6. *Let $p = 0$ and $r \in N$ be fixed, and suppose the function g piecewise belongs to C^q, $q \geq 2$, and has a single discontinuity at $\theta_0 \in (-\pi, \pi)$. In addition, we assume that $g^{(q)} \in V$. Then there exist constants $K_1, K_2, \ldots, K_q$ such that*

$$\theta_0(r;n) = \theta_0 + \frac{K_1}{n^2} + \frac{K_2}{n^3} + \ldots \frac{K_q}{n^{q+1}} + o(\frac{1}{n^{q+1}}). \tag{45}$$

Namely

$$K_1 = \frac{r+2}{r}\frac{[g']_0}{[g]_0} \quad \textit{and} \quad K_2 = -\frac{r+2}{r}\frac{[g']_0}{[g]_0} \tag{46}$$

for $r \geq 2$.

In particular, if the derivative of the function g does not have a jump at θ_0, then the approximation has the order $O(1/n^4)$.

Proof. Let us first assume that $r \geq q$. We shall establish an algorithm for computing the constants $K_1, K_2, \dots, K_q$, and perform the actual computation for K_1 and K_2.

Without loss of generality we assume that $\theta_0 = 0$, $r \equiv 2r+1$, and $q \equiv 2q+1$. Now we consider the function $\bar{g}$ defined by

$$\bar{g}(\theta) = g(\theta) - \frac{1}{\pi} \sum_{k=0}^{2q+1} [g^{(k)}]_0 G_k(\theta). \tag{47}$$

Since the function g in particular satisfies the conditions of Theorem 5, by virtue of (39) there exists a constant K_0 such that

$$|\theta_0(n)| < \frac{K_0}{n^2} \tag{48}$$

for $n \in N$.

As we know (see (34) and (47)), $\theta_0(n)$ satisfies the following identity

$$DT_n'(g;\theta_0(n)) = \frac{1}{\pi} \sum_{k=0}^{2q+1} [g^{(k)}]_0 DT_n'(G_k;\theta_0(n)) + DT_n'(\bar{g};\theta_0(n)) = 0. \tag{49}$$

By construction $\bar{g}^{(2q+1)} \in C \cap V \subset C \cap HBV$. Hence by Remark 2 and Lemma 2 we have

$$S_n^{(2r+2)}(\bar{g};\theta) = S_n^{(2r+1-2q)}(\bar{g}^{(2q+1)};\theta) = o(n^{2r+1-2q}) \tag{50}$$

uniformly with respect to $\theta \in [-\pi, \pi]$.

Furthermore, expanding expression (49) into a Taylor series around zero on the interval $[-K_0/n^2, K_0/n^2]$ and taking into consideration (11), (13), (48), and (50), we obtain:

$$\begin{aligned}
&[g]_0\Big(D_n^{(2r+2)}(0)\theta_0(n) + \frac{1}{3!}D_n^{(2r+4)}(0)\theta_0(n)^3 \;+\; \frac{1}{5!}D_n^{(2r+6)}(0)\theta_0(n)^5 + \dots \\
&\qquad\qquad + \frac{1}{(2q+2)!}D_n^{(2q+2r+3)}(\mu_{0,n})\frac{(2K_0)^{2q+2}}{n^{4q+4}}\Big) \\
&+[g']_0\Big(D_n^{(2r)}(0) + \frac{1}{2!}D_n^{(2r+2)}(0)\theta_0(n)^2 \;+\; \frac{1}{4!}D_n^{(2r+4)}(0)\theta_0(n)^4 + \dots \\
&\qquad\qquad + \frac{1}{(2q+1)!}D_n^{(2q+2r+1)}(\mu_{1,n})\frac{(2K_0)^{2q+1}}{n^{4q+2}}\Big) \\
&\vdots \\
&+\; [g^{(2q+1)}]_0\Big(D_n^{(2r-2q)}(0) + D_n^{(2r+1-2q)}(\mu_{2q+1,n})\frac{2K_0}{n^2}\Big) \\
&+\; o(n^{2r+1-2q}) = 0,
\end{aligned} \tag{51}$$

where $|\mu_{k,n}| < K_0/n^2$, $k = 0, 1, \dots, 2q+1$.

It follows from (15) that all error terms in the Taylor expansion have order $O(n^{2r-2q})$.

The expression for $D_n^{(r)}(0)$, $r \in Z_+$, (see (15)) suggests seeking an expression of $\theta_0(n)$ in the form (45).

According to equation (51), since the error term has an order $o(n^{2r-2q+1})$, all coefficients of n^k, $k \geq 2r - 2q + 1$, must equal to 0. This condition generates the set of equations with respect to the yet unknown constants $K_1, K_2, \ldots, K_{2q+1}$.

We set up one by one the equations for powers of n, with decreasing order of degree, starting from n^{2r+1}. It is clear that by (15) and (51), only two terms, namely $[g]_0 D_n^{(2r+2)}(0)\theta_0(n)$ and $[g']_0 D_n^{(2r)}(0)$ contribute n^{2r+1} and n^{2r}. Consequently, the comparision of the coefficient leads to the following system of linear equations with respect to K_1 and K_2 (see (14), (15), and (45)):

$$(-1)^{r+1}[g]_0 \frac{n^{2r+3}}{2r+3}\frac{K_1}{n^2} + (-1)^r [g']_0 \frac{n^{2r+1}}{2r+1} = 0$$

and

$$(-1)^{r+1}[g]_0 \left(\frac{n^{2r+3}}{2r+3}\frac{K_2}{n^3} + \frac{n^{2r+2}}{2}\frac{K_1}{n^2} \right) + (-1)^r [g']_0 \frac{n^{2r}}{2} = 0,$$

which instantly implies (46).

Furthermore, let us observe that the highest degree of n contributed by each term of the sequence $Q_l \equiv (D_n^{(2r+2l-2i)}(0)\theta_0(n)^{2l-i-1})_{i=0}^{2l-1}$, $l = 1, 2, \ldots, q+1$, ignoring the constants of expansion, is $2r - 2l + 3$.

Now we proceed by induction. Let us assume that the constants $K_1, K_2, \ldots, K_{2l-3}$, and K_{2l-2} are already defined by setting up equations with respect to the coefficients of n degree less then $2r - 2(l-1) + 3$. Next, we shall show that a new system of equations for the coefficients of $n^{2r-2l+3}$ and $n^{2r-2l+2}$ represents a system of *linear* equations with respect to K_{2l-1} and K_{2l}. In addition, the determinant of the system is *nonzero*, and hence the system is *consistent*.

Indeed, the only terms which may contribute K_{2l-1} and K_{2l} unknowns are in the sequences Q_j, $j < l$. Hence, by (15) and (45) we have

$$\begin{aligned} D_n^{(2r+2j-2i)}(0)\theta_0(n)^{2j-i-1} &= \left(\frac{n^{2r+2j-2i+1}}{2r+2j-2i+1} + \textit{lower degree terms} \right) \\ &\times \left(\frac{K_1}{n^2} + \ldots + \frac{K_{2l-1}}{n^{2l}} + \frac{K_{2l}}{n^{2l+1}} + O(\frac{1}{n^{2l+2}}) \right)^{2j-i-1}. \end{aligned} \tag{52}$$

Consequently, the highest degree of n contributed by this product with factor K_{2l-1} is

$$\frac{n^{2r+2j-2i+1}}{2r+2j-2i+1} \left(\frac{K_1}{n^2} \right)^{2j-i-2} \frac{K_{2l-1}}{n^{2l}} \simeq n^{2r-2l+3+2(1-j)}.$$

But, $2r - 2l + 3 + 2(1-j) < 2r - 2l + 3$ unless $j = 1$. Hence only the sequence $Q_1 = \{D_n^{(2r+2)}(0)\theta_0(n), D_n^{(2r)}(0)\}$ contributes the constant K_{2l-1} and it clearly appears in the first degree in the expression for $\theta_0(n)$. (We treat the case for K_{2l} similarly.) In addition, the determinant of the linear system with respect to K_{2l-1} and K_{2l} is triangular with nonzero diagonal entries, $(-1)^{r+1}[g]_0/(2r+3) \neq 0$ and $(-1)^{r+1}[g]_0/2 \neq 0$, and that guarantees the solvability of the system.

Finally, the equation for $n^{2r-2l+3}$ defines K_{2l-1}, so the equation for $n^{2r-2q+1}$ will define K_{2q+1}. Let us mention that the coefficients $K_1, K_2, \ldots, K_{2q+1}$ depend only on $[g]_0, [g']_0, \ldots, [g^{(2q+1)}]_0$.

To prove the theorem for the case $r < q$ we need some minor changes in the arguments. First, one has to use expansion (16) for $D_n^{(2r+2j-2i)}(0)$ in estimate (52) as soon as $2r + 2j - 2i < 0$. Second, to estimate the error term in (49), i.e., (50), let us mention the following: since $\bar{g}^{(2r+2)} \in C^{2q-2r-1}$, then by virtue of (17), expanding $\bar{g}$ into Taylor series around zero on the interval $[-K_0/n^2, K_0/n^2]$, we have:

$$
\begin{aligned}
S_n^{(2r+2)}(\bar{g};\theta_0(n)) &= S_n(\bar{g}^{(2r+2)};\theta_0(n)) = \bar{g}^{(2r+2)}(\theta_0(n)) + o(\frac{1}{n^{2q-2r-1}}) \\
&= \bar{g}^{(2r+2)}(0) + \bar{g}^{(2r+3)}(0)\theta_0(n) + \ldots \\
&\quad + \frac{1}{(2q-2r-1)!}\bar{g}^{(2q+1)}(\mu_n)\frac{(2K_0)^{2q-2r-1}}{n^{4q-4r-2}} + o(\frac{1}{n^{2q-2r-1}}) \qquad (53)
\end{aligned}
$$

which, ignoring the constant factor, represents the desired estimate for the error term. The rest of the proof is completely analogous. □

Taking an opportunity of the lucky similarity between the coefficients (46), using a simple linear combination of expansion (45) for $\bar{r}$ and r, $2 \leq \bar{r} < r$, we significantly improve the accuracy of approximation. Namely, the following statement holds.

Corollary 3. *Let $p = 0$ and suppose a function g piecewise belong to C^q, $q > 3$, and has a single discontinuity at $\theta_0 \in (-\pi, \pi)$. In addition, we assume that $g^{(q)} \in V$. Then for each fixed $\bar{r}$ and $r \in N$, $2 \leq \bar{r} < r$, there exist constants $K_1, K_2, \ldots, K_q$ such that*

$$
\frac{r(\bar{r}+2)}{2(r-\bar{r})}\theta_0(r;n) - \frac{\bar{r}(r+2)}{2(r-\bar{r})}\theta_0(\bar{r};n) = \theta_0 + \frac{K_1}{n^4} + \ldots + \frac{K_q}{n^{q+1}} + o(\frac{1}{n^{q+1}}). \qquad (54)
$$

5.2. Approximation to the jumps. Now let us study the approximation to the jumps of a function.

Theorem 7. *Let $p = 0$ and $r \in N$ be fixed, and suppose g is a piecewise continuous function such that $g' \in HBV$. In addition, we assume that $M(g)$ and $M(g')$ are finite. Then the estimate*

$$
DT_n(g;\theta_m(n)) = [g]_m + O(\frac{1}{n})
$$

is valid for each $m = 0, 1, \ldots, M(g) - 1$.

Proof. Again for simplicity we assume that $M(g) = 2$, $M(g') = 1$, $r \equiv 2r + 1$ is an odd number, and $\theta_0(g) = \theta_0(g') = 0$. By virtue of (40) we have

$$\begin{aligned} DT_n(g;\theta_0(n)) &= \frac{[g]_0}{\pi} DT_n(G;\theta_0(n)) + \frac{[g]_1}{\pi} DT_n(G(\theta_1;\cdot);\theta_0(n)) \\ &\quad + \frac{[g']_0}{\pi} DT_n(G_1;\theta_0(n)) + DT_n(\bar{g};\theta_0(n)) \end{aligned} \tag{55}$$

Further analysis is trivial as we take the Taylor expansion of (55) around zero on the interval $[-K_0/n^2, K_0/n^2]$. By virtue of (11) and (13) we get

$$\begin{aligned} DT_n(g;\theta_0(n)) &= \frac{(-1)^r(2r+1)[g]_0}{n^{2r+1}} \left(D_n^{(2r)}(0) + D_n^{(2r+1)}(\nu_n) \frac{2K_0}{n^2} \right) \\ &+ \frac{(-1)^r(2r+1)[g]_1}{n^{2r+1}} D_n^{(2r)}(\theta_0(n) - \theta_1) \\ &+ \frac{(-1)^r(2r+1)[g']_0}{n^{2r+1}} D_n^{(2r-1)}(\theta_0(n)) \\ &+ \frac{(-1)^r(2r+1)\pi}{n^{2r+1}} S_n^{(2r)}(\bar{g}';\theta_0(n)), \end{aligned} \tag{56}$$

where $|\nu_n| < K_0/n^2$.

Taking into account that $\bar{g}' \in C \cap HBV$, the rest of the proof follows from (14), (15), (31), (56), and Remark 2. □

Now, an interested reader will easily fill out the details of proof for the following statement.

Theorem 8. *Let $p = 0$ and $r \in N$ be fixed, and suppose a function g peicewise belong to C^q, $q \geq 2$, and has a single discontinuity at $\theta_0 \in (-\pi, \pi)$. In addition, we assume that $g^{(q)} \in V$. Then there exist constants $K_1, K_2, \ldots, K_q$ such that*

$$DT_n(r;\theta_0(r;n)) = [g]_0 + \frac{K_1}{n} + \frac{K_2}{n^2} + \ldots + \frac{K_q}{n^q} + o(\frac{1}{n^q}). \tag{57}$$

Namely

$$K_1 = \frac{r}{2}[g]_0 \tag{58}$$

and

$$K_2 = \frac{r^2}{12}[g]_0 + \frac{r+2}{r}[g']_0 - \frac{r+2}{r}\frac{[g']_0^2}{[g]_0}$$

for $r \geq 2$.

Extrapolating expansion (57) in r based on identity (58), we improve the accuracy of aproximation. Namely, the following statement holds.

Corollary 4. *Let $p = 0$ and suppose a function g piecewise belong to C^q, $q \geq 2$, and has a single discontinuity at $\theta_0 \in (-\pi, \pi)$. In addition, we assume that $g^{(q)} \in V$. Then for each fixed $\bar{r}$ and $r \in N$, $2 \leq \bar{r} < r$, there exist constants $K_1, K_2, \ldots, K_q$ such that*

$$\frac{r}{r-\bar{r}} DT_n(\bar{r};\theta_0(n)) - \frac{\bar{r}}{r-\bar{r}} DT_n(r;\theta_0(n)) = [g]_0 + \frac{K_1}{n^2} + \ldots + \frac{K_q}{n^q} + o(\frac{1}{n^q}). \tag{59}$$

5.3. Approximation to the discontinuities and the jumps of derivatives of a function. As a simple corollary of (8), (9), (10), and Theorems 6 and 8 we obtain estimates for the location of points of discontinuity of the derivatives of a continuous function and the associated jumps. Below we represent just typical statements.

Theorem 9. *Let a function $g \in C^{p-1}$, $p \in N$, piecewise belong to C^q, $p+2 \leq q$, and suppose $g^{(p)}$ has a single discontinuity $\theta_0 \in (-\pi, \pi)$. In addition, we assume that $g^{(q)} \in V$. Then for each fixed $r \in N$ there exist constants $K_1, K_2, \ldots, K_{q-p}$ such that*

$$\theta_0(p;r;g;n) = \theta_0(g^{(p)}) + \frac{K_1}{n^2} + \ldots + \frac{K_{q-p}}{n^{q-p+1}} + o(\frac{1}{n^{q-p+1}}). \tag{60}$$

Namely,

$$K_1 = \frac{r-p+2}{r-p}\frac{[g^{(p+1)}]_0}{[g^{(p)}]_0} \quad \text{and} \quad K_2 = -\frac{r-p+2}{r-p}\frac{[g^{(p+1)}]_0}{[g^{(p)}]_0},$$

for $r - p \geq 2$.

Theorem 10. *Let a function $g \in C^{p-1}$, $p \in N$, piecewise belong to C^q, $p+2 \leq q$, and suppose $g^{(p)}$ has a single discontinuity $\theta_0 \in (-\pi, \pi)$. In addition, we assume that $g^{(q)} \in V$. Then for each fixed $r \in N$, there exist costants $K_1, K_2, \ldots, K_{q-p}$ such that*

$$DT_n(p;r;g;\theta_0(p;r;g;n)) = [g^{(p)}]_0 + \frac{K_1}{n} + \ldots + \frac{K_{q-p}}{n^{q-p}} + o(\frac{1}{n^{q-p}}).$$

Namely,

$$K_1 = \frac{r-p}{2}[g^{(p)}]_0$$

for $r - p \geq 2$.

6. Description of the algorithm

Now we describe the main idea of the algorithm which we propose to locate the points of discontinuity. For simplicity, we assume that the function is piecewise smooth.

In case the function has a single discontinuity, according to Corollary 3 we search for the global maximum of $|DT_n(\theta)|$ for fixed $\bar{r}$ and r, $2 \leq \bar{r} < r$. Afterwards, utilizing expansion (54) and applying Richardson's method of extrapolation we improve the accuracy.

The situation drastically changes if the function has more than one point of discontinuity. In this case we do not have expansion (54) for the approximation.

To overcome this difficulty we generate, for a fixed $r \in N$, the sequence of partial sums of Fourier series of functions $(g_m)_{m=0}^{M-1}$, defined via the recursion relation

$$g_{m+1}(\theta) = (1 - \cos(\theta - \theta_0(g_m; n)))^d g_m(\theta), \tag{61}$$

where $g_0 \equiv g$, $d \in N$ is fixed, and $\theta_0(g_m; n)$ is the location of $|DT_n(g_m; \theta)|$'s global maximum on the period for sufficiently large $n \in N$.

The idea behind generating this sequence is to "eliminate" (at least numerically) the discontinuities of the function g one by one by simply searching for the global maximum of the function $|DT_n(g_m; \theta)|$ on the period. In other words, "removing" the highest jump of the function, we can see the second largest jump.

A straightforward computation based on simple trigonometric identities generates the Fourier coefficients of g_{m+1} via the Fourier coefficients of g_m. Below we represent the identities for $d = 3$ (see (61)):

$$\begin{aligned} a_k(g_{m+1}) &= \frac{5}{2} a_k(g_m) \\ &- \frac{15}{8} \cos\theta_0(g_m; n)(a_{k-1}(g_m) + a_{k+1}(g_m)) + \frac{15}{8} \sin\theta_0(g_m; n)(b_{k-1}(g_m) - b_{k+1}(g_m)) \\ &+ \frac{3}{4} \cos 2\theta_0(g_m; n)(a_{k-2}(g_m) + a_{k+2}(g_m)) - \frac{3}{4} \sin 2\theta_0(g_m; n)(b_{k-2}(g_m) - b_{k+2}(g_m)) \\ &- \frac{1}{8} \cos 3\theta_0(g_m; n)(a_{k-3}(g_m) + a_{k+3}(g_m)) + \frac{1}{8} \sin 3\theta_0(g_m; n)(b_{k-3}(g_m) - b_{k+3}(g_m)), \end{aligned} \tag{62}$$

and

$$\begin{aligned} b_k(g_{m+1}) &= \frac{5}{2} b_k(g_m) \\ &- \frac{15}{8} \cos\theta_0(g_m; n)(b_{k-1}(g_m) + b_{k+1}(g_m)) - \frac{15}{8} \sin\theta_0(g_m; n)(a_{k-1}(g_m) - a_{k+1}(g_m)) \\ &+ \frac{3}{4} \cos 2\theta_0(g_m; n)(b_{k-2}(g_m) + b_{k+2}(g_m)) + \frac{3}{4} \sin 2\theta_0(g_m; n)(a_{k-2}(g_m) - a_{k+2}(g_m)) \\ &- \frac{1}{8} \cos 3\theta_0(g_m; n)(b_{k-3}(g_m) + b_{k+3}(g_m)) - \frac{1}{8} \sin 3\theta_0(g_m; n)(a_{k-3}(g_m) - a_{k+3}(g_m)), \end{aligned} \tag{63}$$

where $a_k(g_{m+1}) = a_{-k}(g_{m+1})$ and $b_k(g_{m+1}) = -b_{-k}(g_{m+1})$, $k \in N$.

Our approach to eliminate the points of discontinuity could be justified by the following observation: multiplying a function by the factor $(1 - \cos(\theta - \theta_0(g_m; n)))^d$ we are not adding a new point of discontinuity but significantly reducing the jump of the function g_m at θ_m. More precisely, if $\theta_m - \theta(g_m; n) = O(n^{-p})$, then the function g_{m+1} (see (61)) at the point θ_m will have the jump of $(1 - \cos(\theta_m - \theta_0(g_m; n)))^d [g_{m+1}]_0 \simeq n^{-2dp}[g_{m+1}]_0$.

Figures 5-7 illustrate a step by step "elimination" of the points of discontinuity of the function (65) utilizing formulae (62) and (63). They correspond to the graphs of $\frac{1}{n} S_n'(g_m; \cdot)$ for $m = 1, 2, 3$.

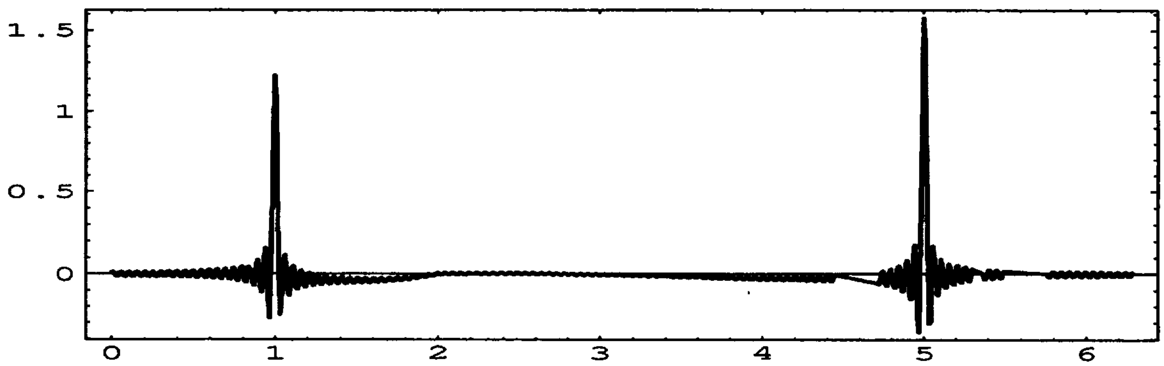

FIGURE 5. $n = 128$.

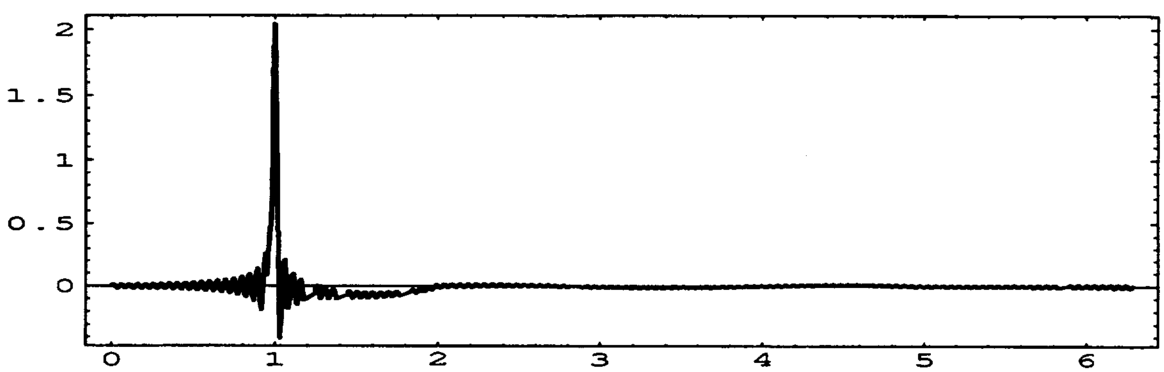

FIGURE 6. $n = 128$.

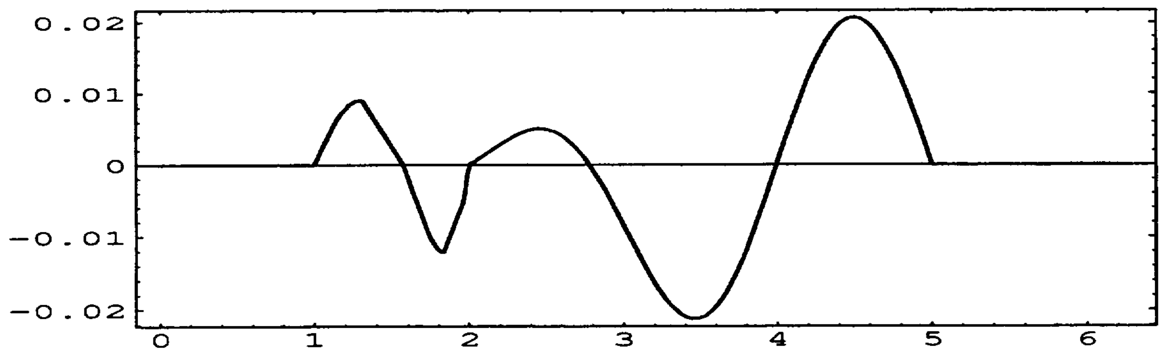

FIGURE 7. $n = 128$.

Summarizing all the above we suggest the following algorithm, which we have implemented using *Mathematica*:

Steps 1–3 find initial approximations for the discontinuities of the function g. These are then refined in Steps 4–5.

Step 1 (Initialization):
Select some fixed $d \in N$, which will be used throughout the entire algorithm. Let $r = 1$; this value is used in defining $DT_n(g_m;\theta)$ in Steps 2 and 3. Let $2n_1 + 1$ coefficients of the function g be given. Select some small value for n_0 (usually $n_0 = 16$); this value should be a power of 2 and is used as the starting subscript of the sequence constructed in Steps 2a and 2b. Let $m = 0$ and $g_0 \equiv g$.

Step 2 (Find a discontinuity, if one exists):

Step 2a (Find the maximum of $|DT_n(g_m;\theta)|$ over the period):

Step 2a(1): Using the adaptive plotting routine internal to *Mathematica*, with the number of initial points set at the maximum of 25 (the default) and n_0^2 (to ensure that we do not miss the maximum), determine the point constructed by the plotting algorithm which has the largest value of $|DT_{n_0}(g_m;\theta)|$.

Step 2a(2): Using the θ-value of the point found in Step 2a(1), apply Newton's method to find the θ-value where the maximum of $|DT_{n_0}(g_m;\theta)|$ occurs as the solution of the equation $DT_{n_0}'(g_m;\theta) = 0$. Let us denote the θ- and corresponding $|DT_{n_0}(g_m;\theta)|$-value by $\theta(n_0)$ and $\tau(n_0)$.

Step 2b (Determine if the θ-value found in Step 2a is a discontinuity):

Form the sequences $\theta(n_0)$, $\theta(2n_0)$, $\theta(4n_0)$, ... and $\tau(n_0)$, $\tau(2n_0)$, $\tau(4n_0)$, ... by successively doubling n_0 and finding the maximum of $|DT_n(g_m;\theta)|$. Find the maximum by using Newton's method, using the previous θ-value in the sequence as the initial approximation.

Stop when one of three conditions arises.

Condition 1: The ratio of two successive τ-values is less than 0.6 (see (57)). The point is not a point of discontinuity. Go to Step 4.

Condition 2: The estimated relative error between two successive θ-values is less than some predefined tolerance. The point is a point of discontinuity. Go to Step 3.

Condition 3: The number of terms in $DT_n(g_m;\theta)$ exceeds $2n_1$. The point is a point of discontinuity. Go to Step 3.

Step 3 (Remove the discontinuity):

Let θ_m denote the final value of θ determined in Step 2b. Increase m by 1, define the Fourier coefficients of the function $g_{m+1}(\theta) = g_m(\theta)(1-\cos(\theta-\theta_m))^d$, utilizing (62) and (63), and return to Step 2.

Step 4 (Refine the estimates of the points of discontinuity):

Two conditions can arise: no discontinuities were found in Steps 2 and 3 (terminate the algorithm) or one or more discontinuities were found (continue). Let M be the number of located discontinuities and θ_0, θ_1, ... , θ_{M-1} their locations.

Select some $2 \le \bar{r} < r$ and some $n > n_0$, $n = 2^p$, to be used in Step 4a.

Do the following for $m = 0$ to $M-1$.

Using the values of θ_m current at the time form the function $g_m^*(\theta) = g(\theta)\prod_{i=0,i\neq m}^{M-1}(1-\cos(\theta-\theta_i))^d$, and apply the extrapolation method of Step 4a.

Step 4a:

Form the sequences $\theta(\bar{r};n_0)$, $\theta(\bar{r};2n_0)$, $\theta(\bar{r};4n_0)$, ... , $\theta(\bar{r};n)$ and $\theta(r;n_0)$, $\theta(r;2n_0)$, $\theta(r;4n_0)$, ... , $\theta(r;n)$, which represent the θ-values where $|DT_n(g_m^*;\theta)|$ takes on its maximum value. This can be done efficiently by using Newton's method with θ_m as the starting point and solving the equation $DT_n'(g_m^*;\theta) = 0$.

Using the extrapolation defined by equation (54), form the sequence

$$\theta(n) = \frac{r(\bar{r}+2)}{2(r-\bar{r})}\theta(r;n) - \frac{\bar{r}(r+2)}{2(r-\bar{r})}\theta(\bar{r};n)$$

for $n = n_0, 2n_0, 4n_0, \dots, n$.

Perform Richardson's extrapolation on the sequence $\theta(n_0)$, $\theta(2n_0)$, $\theta(4n_0)$, ... , $\theta(n)$.

Replace θ_m with the final value obtained.

Step 5 (Additional refinement of the estimates):
If only one discontinuity was detected in Steps 1–3, then stop. Otherwise repeat Step 4 a second time.

Our implementation uses *Mathematica's* capability to do high-precision arithmetic and track the loss of precision that occurs due to accumulating roundoff error. Besides displaying the final answers it displays the number of digits of precision that remain.

7. Some numerical results

In order to illustrate the numerical results obtained by the described algorithm, we will consider several examples.

First we consider a 2π-periodic function g with two discontinuities such that $g \in HBV$, namely

$$g(\theta) = \begin{cases} 0 & \text{if } -\pi < \theta \leq 0 \\ \theta \sin \frac{1}{\theta} + 2 & \text{if } 0 < \theta < \pi \end{cases}$$

It is obvious that the function g is not piecewise absolutely continuous and therefore the methods suggested in [4], [5], and [11] fail. Still according to Theorem 2 we should obtain an $o(1/n)$ approximation. We find the absolute value of the largest error for approximation to the points of discontinuity as follows:

N	32	64	128	256
Location-error	6.1(−2)	2.2(−2)	6.4(−3)	2.0(−3)

The following piecewise smooth function was considered in [5].

(64)
$$g(\theta) = \begin{cases} \sin \frac{\theta}{2} & \text{if } 0 \leq \theta \leq 0.9 \\ -\sin \frac{\theta}{2} & \text{if } 0.9 < \theta < 2\pi \end{cases}$$

Below we present a detailed description of all computations. The first table shows the error in the approximation to the location of the discontinuity by differentiated Fourier partial sums of degree $r = 7$ and $r = 8$, and then their linear combination via formula (54).

N	$r = 7$	$r = 8$	linear combination by (54)
2	2.42(−1)	2.40(−1)	1.82(−1)
4	6.84(−2)	6.70(−2)	1.90(−2)
8	1.86(−2)	1.81(−2)	1.50(−3)
16	4.90(−3)	4.76(−3)	1.06(−4)
32	1.26(−3)	1.22(−3)	7.05(−6)
64	3.19(−4)	3.11(−4)	4.54(−7)
128	8.05(−5)	7.83(−5)	2.88(−8)

Here we present a full table of Richardson's extrapolation started from the last column of previous table.

N	$r = 7,8$						
2	1.8(−1)						
4	1.9(−2)	8.0(−3)					
8	1.5(−3)	3.4(−4)	8.9(−5)				
16	1.0(−4)	1.2(−5)	2.3(−6)	9.2(−7)			
32	7.0(−6)	4.4(−7)	4.6(−8)	1.0(−8)	3.6(−9)		
64	4.5(−7)	1.4(−8)	8.2(−10)	9.6(−11)	1.2(−11)	1.8(−12)	
128	2.8(−8)	4.7(−10)	1.3(−11)	7.7(−13)	2.2(−14)	2.5(−14)	2.1(−14)

The following is the error in the approximation to the jumps of the function using $r = 7$ and $r = 8$, and their combination (59).

N	$r = 7$	$r = 8$	linear combination by (59)
2	2.86(0)	3.35(0)	5.60(−1)
4	1.04(0)	1.22(0)	1.74(−1)
8	4.46(−1)	5.17(−1)	4.55(−2)
16	2.20(−1)	2.37(−1)	1.14(−2)
32	9.90(−2)	1.13(−1)	2.87(−3)
64	4.85(−2)	5.55(−2)	7.18(−4)
128	2.40(−2)	2.74(−2)	1.79(−4)

This table is Richardson's extrapolation applied to data above.

N	$r = 7,8$						
2	5.6(−1)						
4	1.7(−1)	4.5(−2)					
8	4.5(−2)	2.6(−3)	3.5(−3)				
16	1.1(−2)	1.2(−4)	2.2(−4)	1.1(−5)			
32	2.8(−3)	2.9(−6)	1.4(−5)	4.1(−7)	6.1(−8)		
64	7.1(−4)	4.4(−7)	9.3(−7)	1.1(−8)	1.7(−9)	2.7(−9)	
128	1.7(−4)	1.0(−7)	5.8(−8)	3.0(−10)	5.6(−11)	2.9(−11)	8.3(−12)

The next example has been considered in [11].

$$g(\theta) = \begin{cases} 0 & \text{if } 0 < \theta \leq 1, \\ e^{\theta} & \text{if } 1 < \theta \leq 2, \\ \cos\frac{\theta}{2} & \text{if } 2 < \theta \leq 5, \\ 0 & \text{if } 5 < \theta \leq 2\pi. \end{cases} \tag{65}$$

Applying the suggested algorithm the following results have been obtained. Here we simply present the largest error in approximating any of the three discontinuities.

N	32	64	128	256	512
Location-error	1.7(−4)	6.1(−7)	1.4(−8)	3.5(−11)	9.7(−14)

8. Conclusion

Let us give some comments on our results.

As we already mentioned, the formula which determines the jumps of a bounded not-too-highly oscillating function by means of its Fourier series has been known for a long time. But to our best knowledge it has never been utilized for a numerical approximation of the locations of discontinuity points.

Theorems 2 and 3 state that it is possible to detect the locations of discontinuities and the jumps under the condition that a bounded function does not have too high total oscillation (condition (6)). On the other hand, Fourier series fail to distinguish a continuous functions from discontinuous one, if the function is too highly oscillating (the necessity of condition (6)).

It follows from Theorem 4 that identities (5) and (7) represent a powerful tool for the allocation of the points of discontinuity of an almost arbitrary function - excepting the minor restriction on the variation of the function and a finite number of discontinuities we impose no conditions. Still the approximation is of order $o(1/n)$. The factor $([g]_m\Delta(g))^{-1}$ confirms a logical observation: the smaller the jump of a function and the distance between the points of singularity, the more difficult it is to detect its location.

Taking into consideration asymptotic expansions (45), (54), (57), and (59), we think that the method is best suited for a piecewise smooth function with a single discontinuity, although applying the suggested method of "removing" discontinuities leads to good results for a function with multiple discontinuities too.

For piecewise smooth functions, we can obtain very high orders of approximation. The numerical results confirm that high accuracy is indeed attainable with fairly low degree trigonometric polynomials.

Regarding numerical results, applying expansion (54) for different pairs, we observed higher accuracy for larger values of r. For instance, the accuracy of the location of discontinuity is only order of 10^{-9} for the function (64) applying (54) and Richardson's extrapolation for $r = 2, 3$, and $n = 128$. Numerical results confirmed priority of expansion formula (54) over (45): we gained three digits of accuracy. It should be mentioned as well that increasing the order of the partial sums we obtained better results: for the same function (64) using expansion (54), $r = 6, 7$, combined with Richardson's extrapolation for $n = 512$ we achieved an accuracy of 10^{-20} for the location of the point of discontinuity.

The numerical results were obtained from a program written in *Mathematica*, which is available online through the third authors home page `http://www.cs.unm.edu/ shapiro/`.

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13. ABSTRACT (*Maximum 200 words*)

A key step for some methods dealing with the reconstruction of a function with jump discontinuities is the accurate approximation of the jumps and their locations. Various methods have been suggested in the literature to obtain this valuable information. In the present paper, we develop an algorithm based on identities which determine the jumps of a 2π-periodic bounded not-too-highly oscillating function by the partial sums of its differentiated Fourier series. The algorithm enables one to approximate the locations of discontinuities and the magnitudes of jumps of a bounded function. We study the accuracy of approximation and establish asymptotic expansions for the approximations of a 2π-periodic piecewise smooth function with one discontinuity. By an appropriate linear combination, obtained via derivatives of different order, we significantly improve the accuracy. Next, we use Richardson's extrapolation method to enhance the accuracy even more. For a function with multiple discontinuities we establish simple formulae which "eliminate" all discontinuities of the function but one. Then we treat the function as if it had one singularity following the method described above.

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