TECHNICAL NOTE

D-324

THE DESIGN OF FEEDBACK CONTROL SYSTEMS CONTAINING A SATURATION TYPE NONLINEARITY

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SUMMARY

This report is an extension of NASA TN D-20. The extension includes a derivation of the optimum response for a step input for plant transfer functions which have an unstable pole and further data on plants with a single zero in the left half of the s plane. The calculated data are presented tabulated in normalized form.

Optimum control systems are considered. The optimum system is defined as one which keeps the error as small as possible regardless of the input, under the constraint that the input to the plant (or controlled system) is limited. Intuitive arguments show that in the case where only the error can be sensed directly, the optimum system is obtained from the optimum relay or on-off solution. References to known solutions are presented. For the case when the system is of the sampled-data type, arguments are presented which indicate the optimum sampled-data system may be extremely difficult if not impossible to realize practically except for very simple plant transfer functions.

Two examples of aircraft attitude autopilots are presented, one for a statically stable and the other for a statically unstable airframe. The rate of change of elevator motion is assumed limited for these examples. It is shown that by use of nonlinear design techniques described in NASA TN D-20 one can obtain near optimum response for step inputs and reasonable response to sine wave inputs for either case. Also, the nonlinear design prevents inputs from driving the system unstable for either case.

INTRODUCTION

Saturation (or limiting) in a feedback control system can be analyzed and compensated for (if required) by an appropriate application of the root-locus and switch-time methods described in reference 1. The principles of these two methods are reviewed here for the reader's convenience.

To apply the root-locus method one draws the loci of the roots of the characteristic equation as a function of the equivalent[1] gain of the saturating device (equivalent limiter gain). One then views the closed-loop poles as moving on these loci as a function of the input magnitude. If the poles stay in well-damped regions of the s plane for all expected values of equivalent gain, one is assured that saturation causes no instability of the closed-loop system. Conversely, if the closed-loop poles move to lightly damped or unstable regions of the s plane for low values of equivalent limiter gain (large input magnitudes), the system response will be oscillatory or unstable when it is subjected to very large transients. In order to obtain qualitative ideas of how to compensate for the instability one studies how various changes in feedback parameters shift the zeros of the root-locus diagram. Shifts which are suitable are those that allow the equivalent limiter gain to decrease while the closed-loop poles stay in well-damped regions of the s plane. The feedback parameters (or zeros) are made to shift with one or more functions which are similar to the limiter input (such as the error).

The switch-time method is based on the fact that the existence of saturation allows one to define an optimum response. The optimum response uses the saturated variable in such a manner as to restore the error and its derivatives to zero in a minimum time. It has been shown (ref. 1) that the first reversal time (after the application of a step) of the saturated variable can be used to obtain a quantitative measure for comparing the particular system to the optimum. If the actual first reversal time is longer than the optimum, then overshoot exists; conversely, if it is smaller, the actual response time is longer than the optimum. By designing nonlinear functions to make the actual first reversal time equal to the optimum for large step inputs, one obtains a design which has a stable nonoscillatory response for all input magnitudes.

As can be seen, the application of the switch-time method requires a knowledge of the optimum response for the particular "plant"[2] under consideration. A method for obtaining this type data was described in reference 1 where it was shown that if the plant transfer function has a zero in the left half of the s plane, then a considerable reduction in the response time to a step input can be achieved. The solution derived, however, requires some of the variables of the plant to change in a certain manner while the error and its derivatives remain zero.

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[1]Describing function analysis shows that saturation (or limiting) can be treated as an equivalent gain whose value decreases as the input magnitude increases (see, e.g., ref. 2).

[2]The word "plant" refers to the controlled system. The plant transfer function could, for example, be the mathematical relationship between the output of an amplifier (which saturates for large inputs) and a motor shaft position in a servo position controller.

One limitation of reference 1 was that no systems were derived to demonstrate that the above solution, obtained for a plant with a zero in its transfer function, was practically realizable.

A second limitation was that the limiter input equations were not derived for the optimum system. The optimum system is defined here as one which maintains the error as small as possible regardless of the input or initial condition under the constraint of finite limits on the plant input.

A third limitation was that no optimum curves or examples were derived which considered the problem of controlling an unstable plant (i.e., the transfer function has a pole in the right half of the s plane). This problem has considerable practical importance in certain autopilot problems where the aircraft is statically unstable in certain areas of its flight envelope.

It is the purpose of this investigation to remove these limitations. In particular, the following subjects are investigated and some practical solutions studied or suggested.

1. The optimum control system for plants with a saturation-type nonlinearity. The plant in this case is stable and has no zeros.

2. The influence of a zero in the plant transfer function on the optimum control system. The examples considered are for a type 1[3] second-order plant and a type 2 third-order plant.

3. The influence of an unstable pole in the plant transfer function on the optimum system for a type 1 second-order plant.

4. The determination of the optimum response of plant transfer functions of the type encountered in pitch-rate and pitch-attitude autopilots where the control surface rate is limited. This amounts to considering plant transfer functions of the form

(a) $$\frac{K(\tau_1 s + 1)}{s\left(\frac{s^2}{\omega_n^2} + \frac{2\zeta}{\omega_n} s + 1\right)}$$ (b) $$\frac{K(\tau_1 s + 1)}{s^2\left(\frac{s^2}{\omega_n^2} + \frac{2\zeta}{\omega_n} s + 1\right)}$$

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[3]Type in this usage was defined in reference 1 as the number of poles at the origin in the s plane. A type 1 second-order plant (with one zero) has a transfer function, $G(s) = \frac{K(\tau_1 s + 1)}{s(\tau_2 s + 1)}$; similarly a type 2 third-order plant (with zero) has a transfer function, $G(s) = \frac{K(\tau_1 s + 1)}{s^2(\tau_2 s + 1)}$.

where ζ can take on all positive values and

(c) $$\frac{K(\tau_1 s + 1)}{s(\tau_2 s + 1)(\tau_3 s - 1)}$$ (d) $$\frac{K(\tau_1 s + 1)}{s^2(\tau_2 s + 1)(\tau_3 s - 1)}$$

The (c) and (d) cases are for statically unstable airframes.

5. Practical examples of an attitude autopilot. For this, two airframes are considered to demonstrate the difference in control systems for stable and unstable airframes.

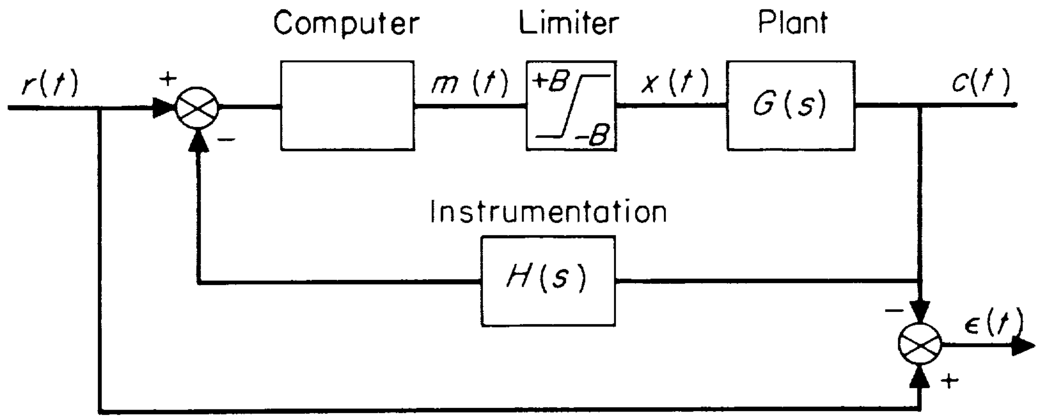
NOTATION

B	limit level
$c(t)$	controlled quantity
$C(s)$	Laplace transform of controlled quantity
$D(s)$	controller transfer function
$D(z)$	digital controller transfer function
e	2.7183
$E(s)$	Laplace transform of error
$G(s)$	plant transfer function
h_ρ	density altitude, ft
$H(s)$	feedback transfer function
K	gain
$L[f(t)]$	Laplace transform of $f(t)$
$m(t)$	controller output
M	Mach number
$r(t)$	input reference quantity
R_o	input step magnitude
$R(s)$	Laplace transform of reference quantity

s	Laplace operator
t	time, sec
T	sampling period, sec
T_1	first reversal time, sec
T_m	minimum response time, sec
$u(t)$	unit step function
$U(s)$	Laplace transform of unit step function
$x(t)$	limiter output (saturated variable)
$X(s)$	Laplace transform of saturated variable
z	$e^{-sT} = Z$ transform operator
δ	control surface deflection, radians
$\epsilon(t)$	error function, $r(t) - c(t)$
ζ	damping ratio
θ	pitch angle, radians
τ	time constant, sec
ω_n	natural frequency, radians/sec

DESCRIPTION OF THE PROBLEM

The problem is best described with the help of the block diagram of sketch (a). The computer in this diagram will be assumed capable of providing any linear or nonlinear relationship desired between its input and output. The output of the computer, $m(t)$, will be assumed to be continuous if an analog computer is used or a staircase signal (a zero-order hold circuit on the output of a digital computer) if a digital computer is used.



Sketch (a)

The output of the computer, m(t), is fed to a limiter. The output of the limiter, x(t), obeys the following equations

for

$$|m(t)| \leq B$$
$$x(t) = m(t) \tag{1}$$

for

$$|m(t)| > B$$
$$x(t) = B \operatorname{sgn}[m(t)] \tag{2}$$

The plant is assumed to be linear and representable by a transfer function, G(s). The plant transfer functions in table I are considered here. The first seven cases were also treated in reference 1.

The feedback transfer function is H(s). In general, one desires the error, $\epsilon(t) = r(t) - c(t)$, to be as small as possible for all the inputs, r(t), the system receives. As will be shown later, this implies that $H(s) = 1$ and all compensation (linear or nonlinear) be accomplished by the computer.

The general problem is how to design the computer operations in order to maintain the error as small as possible under the constraint that $|x(t)| \leq B$. This, as will be shown, generally implies very complicated computer equations for the plant transfer functions for which solutions have been found. At the present time, the solution is not known for most plants whose characteristic equation is of third or higher order. Thus, in this investigation we are principally interested in finding approximate solutions.

The problem will be investigated in the following order:

1. Determination of the optimum switching times for the plants given in table I. This will be done in the section entitled "Optimum Responses."

2. Derivation (based on intuitive argument) of the optimum system. This subject is covered in the section entitled "Optimum Systems." Reference data to known previous work as well as new work which includes simple plants with an unstable pole or stable zero are contained in this section.

3. Approximate optimum systems for

   a. A pitch attitude autopilot for a stable airframe.

   b. A pitch attitude autopilot for a statically unstable airframe.

These systems are derived in the section entitled "Approximate Optimum Systems."

OPTIMUM RESPONSES

A general theorem which was partially proved in reference 1 is as follows: If a Laplace transformable function of time, $f(t)$, is truncated (i.e., $f_T(t) = 0$ for $a > t > b$ where $f_T(t)$ is the truncated time signal), then the Laplace transform of $f_T(t)$, $L[f_T(t)]$, is an entire function.

This theorem can be used to advantage in deriving the optimum response if further information regarding the shape of $x(t)$, the bounded variable, is attainable. Consider sketch (a) for $r(t) = R_o u(t)$ and zero initial conditions. Then

$$L[\epsilon(t)] = E(s) = \frac{R_o}{s} - X(s)\ G(s) \tag{3}$$

A necessary but not a sufficient condition if we are to attain the optimum is that $E(s)$ must be an entire function because $\epsilon(t) = 0$ for $t < 0$ and we desire $\epsilon(t) = 0$ for $t > T_m$. In other words, the desired $\epsilon(t)$ is a truncated time signal. In order to meet the sufficiency condition one must know the optimum shape of $x(t)$ in general terms of unknown reversal times. One then derives $X(s)$ and determines the relationship between these unknown reversal times and R_o by forcing equation (3) to be an entire function.

The general shape of $x(t)$ for the minimum response time can be found by either inspection, or by use of a theorem proved by Bellman, Glicksberg, and Gross (ref. 3) for the conditions for which it applies. The theorem applied to this problem states that, in order to have a minimum response time to an input step, $x(t)$ should be at its maximum value plus or minus throughout the response, the maximum number of reversals being equal to $n-1$ where n is the order of the system. The proof, however, is only for $G(s)$ having real, distinct, and negative roots. It is shown later that if one confines his interest to plants having only poles in the left half plane or at $s = 0$ and is interested only in responses to step inputs, the theorem gives a sufficient number of reversals of $x(t)$ in order to restore the error and its derivatives to zero in a finite time, that is, to allow $E(s)$ given by equation (3) to be an entire function. As will be demonstrated, the response time obtained by using this number of reversals may not be the minimum for plants with lightly damped complex poles. It will also be shown here that if zeros exist in $G(s)$ then one does not want the bounded variable, $x(t)$, to be at its maximum throughout the entire response, that is, until the time when all the states of the system are restored to zero or a constant.

The steps involved in obtaining the optimum responses are summarized below.

(1) Determine $x(t)$ in terms of the unknown switch times for the optimum response. This function will be the limit level times a sum of delayed steps for $G(s)$ having only poles. The delay times will be written as undetermined coefficients.

(2) Determine $X(s)$ from $x(t)$ with the switching times as parameters.

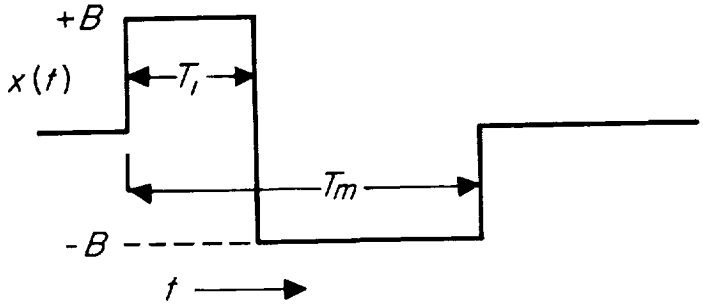
(3) Use the fact that the error transform must be an entire function to obtain algebraic or transcendental equations relating R_o and the unknown switching times and exponential decays (if any) of $x(t)$.

(4) Solve the equations of step (3) to obtain the optimum first reversal time and the minimum response time as functions of the input step magnitude, R_o.

In reference 1 these steps were followed in deriving the optimum responses for the first 7 cases of table I. The optimum responses for cases 8 to 12 will be derived here.

A Type 1 Second-Order Unstable Plant

The plant transfer function for this example (case 8, table I) is $G(s) = K/s(\tau s - 1)$. Arguments similar to those presented in reference 1 show that for $r(t) = R_o u(t)$ where $u(t)$ is the unit step function, $x(t)$ must be of the form shown in sketch (b).



Sketch (b)

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The Laplace transform of $x(t)$ is

$$X(s) = B \frac{1 - 2e^{-T_1 s} + e^{-T_m s}}{s} \tag{4}$$

The Laplace transform of the error is (see eq. (3))

$$E(s) = \frac{R_o}{s} - BK \frac{1 - 2e^{-T_1 s} + e^{-T_m s}}{s^2(s\tau - 1)} \tag{5}$$

The relationships which must be satisfied in order that equation (5) be an entire function are

$$1 - 2e^{-T_1/\tau} + e^{-T_m/\tau} = 0 \tag{6}$$

$$\frac{R_o}{BK} = T_m - 2T_1 \tag{7}$$

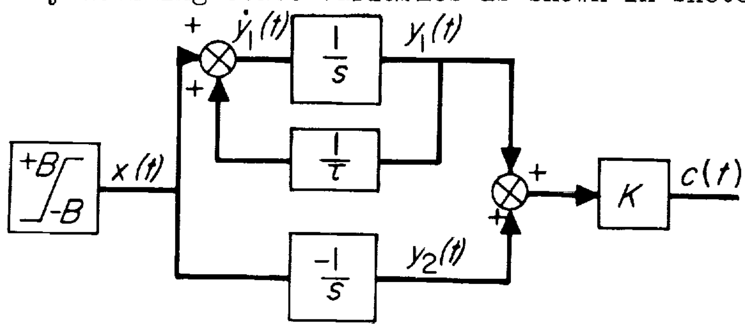
from which one can find

$$\frac{R_o}{BK} = \tau \, ln \frac{1}{2e^{-T_1/\tau} - 1} - 2T_1 \tag{8}$$

$$\frac{R_o}{BK} = T_m - 2\tau \, ln \frac{2}{1 + e^{-T_m/\tau}} \tag{9}$$

Equations (8) and (9) are normalized by dividing by τ and the curves showing T_1/τ and T_m/τ versus $R_o/BK\tau$ are presented in figure 1.

As can be seen from figure 1(a), for the unstable plant the optimum first reversal time approaches a constant for R_o approaching infinity. Since this condition will exist for all the unstable plants considered here, the reason for its existence will be derived. The derivation is accomplished by expanding the plant transfer function by means of partial fractions and by defining state variables as shown in sketch (c).



Sketch (c)

As can be seen from sketch (c), the transfer function, $Y_1(s)/X(s)$, contains the unstable mode (pole in the right half plane). Thus, if $x(t)$ is a step function and $y_1(0) = 0$, then $y_1(t)$ is a growing exponential.

From sketch (c) it is seen that the input to the integrator, $\dot{y}_1(t)$, is $\dot{y}_1(t) = x(t) + (1/\tau)y_1(t)$. If $y_1(0) = 0$ and $x(t) = Bu(t)$, then the input to the integrator grows with time. If control is to be maintained of $y_1(t)$ by $x(t)$, then $x(t)$ must be able to reverse the sign of $\dot{y}_1(t)$. This is impossible if $y_1(t) > \tau B$ when $x(t)$ is limited to a magnitude of B. Thus if we use the optimum motion of $x(t)$ shown in sketch (b), T_1 must never exceed a certain value. This value is, in general, (from eq. (8)) $T_1 = \tau \ln 2$.

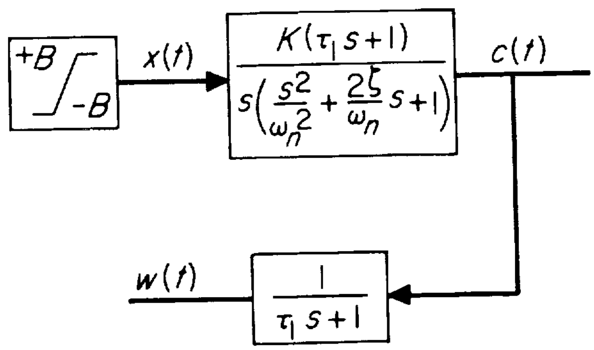
This simple derivation is easily shown for the second-order example by phase plane analysis. It amounts to the fact that $y_1(t)$ must be held within certain limits. If $y_1(t)$ is ever allowed out of these limits, then $c(t)$ grows indefinitely, regardless of $x(t)$. This is the principal difference between stable and unstable plants; that is, with an unstable plant the phase space must be limited if one is to control the plant with a limited input.

A Type 1 Third-Order Stable Plant (With Zero)

For this example (case 9 of table I)

$$G(s) = \frac{K(\tau_1 s + 1)}{s[(s^2/\omega_n^2) + (2\zeta/\omega_n)s + 1]}$$

The problem is to determine the optimum motion of $x(t)$. There are several possibilities. One scheme is to use a pole to cancel the zero, as illustrated in sketch (d). If this solution is used, w(t) will be a zero overshoot response for a step input to the system. The output, c(t), will overshoot, however. The amount of overshoot can be extremely large if τ_1 is large. It therefore would seem desirable to find a solution which has zero overshoot.



Sketch (d)

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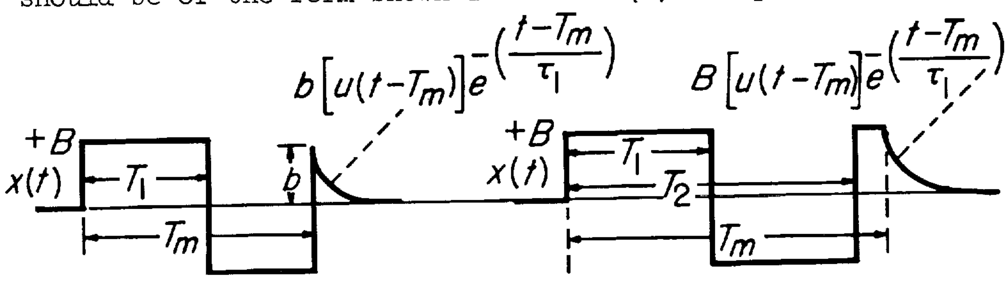
It was shown in reference 1 for a simple plant transfer function that if x(t) is a bang-bang motion followed by a tailing off exponential of time constant τ_1, then c(t) has no overshoot. With this solution all of the states of the plant are not zero when the error and its derivatives are zero and there are practical questions as to how such a system can be implemented. The questions will be resolved later in this report.

For the more complicated plant considered here one questions whether a solution of the type desired can be found by simply taking more switching times for the bang-bang part of x(t). This actually turns out to be the case. The solution which is derived here was actually used in an example in reference 4 and thus one has experimental evidence of its existence.

From these considerations one concludes that the optimum motion of x(t) should be of the form shown in sketch (e). A partial proof that these



Sketch (e)

motions are optimum is obtained if we are successful in finding solutions to the equations which force E(s) to be an entire function; these solutions must have a shorter minimum response time than the one obtained by the scheme shown in sketch (d).

If $x(t)$ is of the form shown in sketch (e 1), the equations which force $E(s)$ to be an entire function are

$$\frac{R_o\omega_n}{BK} = 2\omega_n T_1 - \omega_n T_m + \frac{b}{B}\,\omega_n \tau_1 \tag{10}$$

$$(1 - 2e^{\alpha\omega_n T_1} + e^{\alpha\omega_n T_m})(1 - \alpha\omega_n \tau_1) - \frac{b}{B}\,\alpha\omega_n \tau_1 e^{\alpha\omega_n T_m} = 0 \tag{11}$$

$$(1 - 2e^{\bar{\alpha}\omega_n T_1} + e^{\bar{\alpha}\omega_n T_m})(1 - \bar{\alpha}\omega_n \tau_1) - \frac{b}{B}\,\bar{\alpha}\omega_n \tau_1 e^{\bar{\alpha}\omega_n T_m} = 0 \tag{12}$$

where

$$\alpha = \zeta + \sqrt{\zeta^2 - 1}$$

$$\bar{\alpha} = \zeta - \sqrt{\zeta^2 - 1}$$

If $|b/B|$ from this solution is greater than unity, one must use the $x(t)$ motion given in sketch (e 2). If this motion is used, the equations which force $E(s)$ to be an entire function are

$$\frac{R_o\omega_n}{BK} = 2\omega_n T_1 - 2\omega_n T_2 + \omega_n T_m + \omega_n \tau_1 \tag{13}$$

$$(1 - 2e^{\alpha\omega_n T_1} + 2e^{\alpha\omega_n T_2} - e^{\alpha\omega_n T_m})(1 - \alpha\omega_n \tau_1) - \alpha\omega_n \tau_1 e^{\alpha\omega_n T_m} = 0 \tag{14}$$

$$(1 - 2e^{\bar{\alpha}\omega_n T_1} + 2e^{\bar{\alpha}\omega_n T_2} - e^{\bar{\alpha}\omega_n T_m})(1 - \bar{\alpha}\omega_n \tau_1) - \bar{\alpha}\omega_n \tau_1 e^{\bar{\alpha}\omega_n T_m} = 0 \tag{15}$$

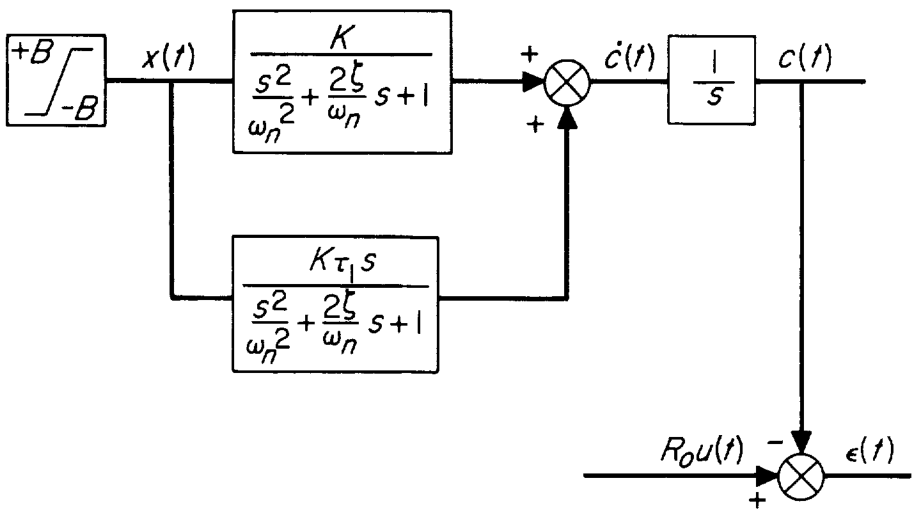
where

$$\alpha = \zeta + \sqrt{\zeta^2 - 1}$$

$$\bar{\alpha} = \zeta - \sqrt{\zeta^2 - 1}$$

As was the case for the type 1 third-order example considered in reference 1, these equations cannot be solved analytically. It is therefore necessary to resort to an iterative procedure and use a digital computer. The digital computer was programmed to solve equations (10), (11), and (12). If the solution obtained has a value of $|b/B| > 1$, this solution is discarded and equations (13), (14), and (15) are used.

As it turns out, for $\zeta > 1$, a sensible solution to these equations always exists. For $\zeta < 1$, however, a sensible solution does not exist for all values of the variables. One can give a physical argument which substantiates this fact. Consider the block diagram of the plant shown in sketch (f).



Sketch (f)

In order for $c(t)$ to reach R_0 (for R_0+) in the minimum time, $\dot{c}(t)$ should have the highest value permissible in the interval. The quantity, $x(t)$, during a large portion of this interval is equal to the positive limit level, B. The motion of $\dot{c}(t)$, until T_1, is seen from sketch (f) to be the sum of two components. Since $x(t) = Bu(t)$ for $t < T_1$, $\dot{c}(t)$ can be expressed as (for $t < T_1$)

$$\dot{c}(t) = KB\left[\begin{pmatrix}\text{unit step response}\\ \text{of a second-order}\\ \text{system}\end{pmatrix} + \tau_1\begin{pmatrix}\text{unit impulse response}\\ \text{of a second-order}\\ \text{system}\end{pmatrix}\right] \qquad (16)$$

The unit step and impulse response of a second-order system for various values of ζ is given in figure 2.

As can be seen from figure 2 (or proved mathematically) for $\zeta > 1$, the impulse response is positive for all time. Since the unit step response is positive (for $\zeta > 0$), we can say that $\dot{c}(t)$ will be positive until T_1 if $\zeta > 1$, regardless of the value of τ_1.

If $\zeta < 1$, however, the unit impulse is negative part of the time. This means that $\dot{c}(t)$, given by equation (16), can reverse sign for certain values of t, ζ, and τ_1. For small values of T_1, however, $\dot{c}(t)$ will be positive so a sensible solution using the $x(t)$ motion given in sketch (e) exists. These facts generally imply that the $x(t)$ motion chosen is non-optimum for all values of the variables. Rather than attempting to find other motions of $x(t)$, we have chosen to present the data where it is a sensible solution. Since the possible size step input or transient to most physical systems is bounded and since sensible solutions exist for a certain range of the step input (starting at zero), this restriction of scope is perhaps not as serious as it could be.

The data summarizing the switch times, b/B, and input step magnitude for various values of ζ and τ_1 are presented in table II. It is believed that the range of the parameters chosen, ζ and τ_1, should be sufficient for a large number of control systems.

A typical plot of $\omega_n T_1$ and $\omega_n T_m$ versus $R_o \omega_n / KB$ for $\zeta = 0.5$ which shows the effects of $\omega_n \tau_1$ is presented in figure 3. As can be seen from this figure a sensible solution exists for all the plotted values of $\omega_n \tau_1$ except $\omega_n \tau_1 = 16$. Here, the solution does not exist for $R_o \omega_n / KB \gtrsim 18$.

As can be seen also, the response time is much shorter for large values of $\omega_n \tau_1$ than for $\omega_n \tau_1 = 0$. Thus a large benefit in speed of response of the system is possible in cases where the plant has a zero close to the origin in comparison with other poles.

A Type 1 Third-Order Unstable Plant (With Zero)

The plant transfer function for this example (case 10, table I) is

$$G(s) = \frac{K(\tau_1 s + 1)}{s(\tau_2 s + 1)(\tau_3 s - 1)}$$

Arguments identical to the ones given for the previous example indicate the $x(t)$ motion should be one of the forms shown in sketch (e). Note also that the results of the second-order unstable plant presented earlier indicate T_1 should approach a finite limit as T_m and R_o approach infinity.

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If the motion of $x(t)$ shown in sketch (e) is assumed, the equations which must be satisfied in order that $E(s)$ be an entire function are

$$\frac{R_o}{BK\tau_2} = \frac{T_m}{\tau_2} - \frac{2T_1}{\tau_2} - \frac{b}{B}\frac{\tau_1}{\tau_2} \tag{17}$$

$$\left(1 - 2e^{T_1/\tau_2} + e^{T_m/\tau_2}\right)\left(1 - \frac{\tau_1}{\tau_2}\right) - \frac{b}{B}\frac{\tau_1}{\tau_2} e^{T_m/\tau_2} = 0 \tag{18}$$

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$$\left[1 - 2e^{-\frac{T_1}{\tau_2}\left(\frac{\tau_2}{\tau_3}\right)} + e^{-\frac{T_m}{\tau_2}\left(\frac{\tau_2}{\tau_3}\right)}\right]\left[1 + \frac{\tau_1}{\tau_2}\left(\frac{\tau_2}{\tau_3}\right)\right] + \frac{b}{B}\frac{\tau_1}{\tau_2}\left(\frac{\tau_2}{\tau_3}\right) e^{-\frac{T_m}{\tau_2}\left(\frac{\tau_2}{\tau_3}\right)} = 0 \tag{19}$$

Equations (17) to (19) are normalized in a manner such that solutions for $R_o/BK\tau_2$, T_m/τ_2, T_1/τ_2, and b/B are desired for various values of the ratios τ_1/τ_2 and τ_2/τ_3. The ratio τ_1/τ_2 can be expressed as

$$\frac{\tau_1}{\tau_2} = \frac{\text{distance of stable pole from the origin of the } s \text{ plane}}{\text{distance of stable zero from the origin of the } s \text{ plane}}$$

similarly

$$\frac{\tau_2}{\tau_3} = \frac{\text{distance of unstable pole from the origin of the } s \text{ plane}}{\text{distance of stable pole from the origin of the } s \text{ plane}}$$

Expressing the ratios in this form gives an understanding of their meaning in terms of a pole-zero plot of the plant transfer function. One can also see that if $\tau_1/\tau_2 \gg 1$ that one might eliminate τ_2 as an approximation and use data for a simpler plant. Also note that for $\tau_1/\tau_2 = 1$ the pole and zero cancel and the data for the second-order unstable plant presented earlier is the solution.

If the solution of these equations gives a value of $|b/B|$ greater than unity, one must discard the answer and derive the solution using the motion of $x(t)$ given in sketch (e 2). If $x(t)$ is of this form, the equations which must be satisfied in order that $E(s)$ be an entire function are

$$\frac{R_o}{BK\tau_2} = \frac{2T_2}{\tau_2} - \frac{2T_1}{\tau_2} - \frac{T_m}{\tau_2} - \frac{\tau_1}{\tau_2} \tag{20}$$

$$\left(1 - 2e^{\frac{T_1}{\tau_2}} + 2e^{\frac{T_2}{\tau_2}} - e^{\frac{T_m}{\tau_2}}\right)\left(1 - \frac{\tau_1}{\tau_2}\right) - \frac{\tau_1}{\tau_2} e^{\frac{T_m}{\tau_2}} = 0 \tag{21}$$

$$\left[1 - 2e^{-\frac{T_1}{\tau_2}\left(\frac{\tau_2}{\tau_3}\right)} + 2e^{-\frac{T_2}{\tau_2}\left(\frac{\tau_2}{\tau_3}\right)} - e^{-\frac{T_m}{\tau_2}\left(\frac{\tau_2}{\tau_3}\right)}\right]\left(1 + \frac{\tau_1}{\tau_3}\right) + \frac{\tau_1}{\tau_2}\left(\frac{\tau_2}{\tau_3}\right) e^{-\frac{T_m}{\tau_2}\left(\frac{\tau_2}{\tau_3}\right)} = 0 \tag{22}$$

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As was the case for the previous example, these two sets of equations, (17) to (19) and (20) to (22) must be solved by an iterative procedure. However, this example is somewhat simpler than the previous one since one can derive solutions which are valid for $T_m \rightarrow \infty$. In equation (19) for example, if $T_m \rightarrow \infty$ we find

$$T_1 = \tau_3 \ln 2 \tag{23}$$

Similarly, one can divide equation (18) by e^{T_m/τ_2} and then let $T_m \rightarrow \infty$ and find

$$\frac{b}{B} = \frac{\tau_2}{\tau_1} - 1 \tag{24}$$

Equation (23) shows that T_1 approaches the same limit for $T_m \rightarrow \infty$ as for the second-order example considered earlier. For the reasons given earlier, this is the maximum value of T_1.

Equation (24) shows that if $\tau_2/\tau_1 \leq 2$ (or $\tau_1/\tau_2 \geq 0.5$) then as $T_m \rightarrow \infty$, $|b/B| \leq 1$. As it turns out, b/B is very small for T_m small and the value given by equation (24) is the maximum regardless of the value of T_m.

Equations (17) to (19) were solved by means of a digital computer for the quantities $R_o/BK\tau_2$, T_1/τ_2, T_m/τ_2, and b/B for various values of the ratios τ_1/τ_2 and τ_2/τ_3. Only one value of $\tau_2/\tau_1 > 2$ (i.e., $\tau_1/\tau_2 = 0$) was computed; therefore equations (20) to (22) were needed for this case only.

The results of the computations are presented in table III. A typical plot of the tabulated results showing the effects of the ratio, τ_1/τ_2, for one value of the ratio, τ_2/τ_3, is presented in figure 4. As can be seen from figure 4(b), a zero in the left half plane much closer to the origin than a pole (i.e., $\tau_1/\tau_2 \gg 1$) provides a considerably smaller response time than if $\tau_1/\tau_2 = 0$. Thus, as was the case for the previous example, a left half plane zero of the plant can be used to advantage if a control system can be designed which provides the bang-bang followed by a tailing off exponential motion required of the saturated variable, x(t).

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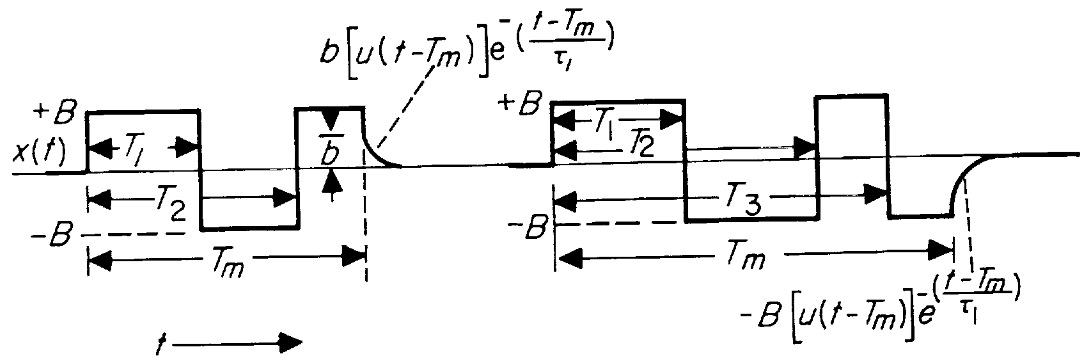
A Type 2 Fourth-Order Plant (With Zero)

The plant transfer function for this example (case 11, table I) is

$$G(s) = \frac{K(\tau_1 s + 1)}{s^2[(s^2/\omega_n^2) + (2\zeta/\omega_n)s + 1]} \tag{25}$$

Arguments similar to the ones given previously show the optimum motion of x(t) should be one of the two forms shown in sketch (g).



(g1) Sketch (g) (g2)

Note that these two motions are identical to sketch (e) except that an additional switch time has been added to the bang-bang part of the motion.

If the x(t) motion is as shown in sketch (g 1), the equations which must be satisfied in order that E(s) be an entire function are

$$2\omega_n T_1 - 2\omega_n T_2 + \omega_n T_m + \frac{b}{B}\,\omega_n \tau_1 = 0 \tag{26}$$

$$\left(1 - 2e^{\alpha\omega_n T_1} + 2e^{\alpha\omega_n T_2} - e^{\alpha\omega_n T_m}\right)\left(1 - \alpha\omega_n \tau_1\right) - \frac{b}{B}\,\alpha\omega_n \tau_1 e^{\alpha\omega_n T_m} = 0 \tag{27}$$

$$\left(1 - 2e^{\bar{\alpha}\omega_n T_1} + 2e^{\bar{\alpha}\omega_n T_2} - e^{\bar{\alpha}\omega_n T_m}\right)\left(1 - \bar{\alpha}\omega_n\tau_1\right) - \frac{b}{B}\,\bar{\alpha}\omega_n\tau_1 e^{\bar{\alpha}\omega_n T_m} = 0 \tag{28}$$

$$\frac{2R_o\omega_n^2}{BK} = 2\omega_n\tau_1(2\omega_n T_1 - 2\omega_n T_2 + \omega_n T_m) - 2(\omega_n T_1)^2 + 2(\omega_n T_2)^2$$

$$- (\omega_n T_m)^2 - 2.\frac{b}{B}\,(\omega_n\tau_1)(\omega_n T_m) \tag{29}$$

where

$$\left.\begin{aligned} \alpha &= \zeta + \sqrt{\zeta^2 - 1} \\ \bar{\alpha} &= \zeta - \sqrt{\zeta^2 - 1} \end{aligned}\right\} \tag{30}$$

If the motion of $x(t)$ is as shown in sketch (g 2), the equations which must be satisfied in order that $E(s)$ be an entire function are

$$2\omega_n T_1 - 2\omega_n T_2 + 2\omega_n T_3 - \omega_n T_m - \omega_n\tau_1 = 0 \tag{31}$$

$$\left(1 - 2e^{\alpha\omega_n T_1} + 2e^{\alpha\omega_n T_2} - 2e^{\alpha\omega_n T_3} + e^{\alpha\omega_n T_m}\right)\left(1 - \alpha\omega_n\tau_1\right) + \alpha\omega_n\tau_1 e^{\alpha\omega_n T_m} = 0 \tag{32}$$

$$\left(1 - 2e^{\bar{\alpha}\omega_n T_1} + 2e^{\bar{\alpha}\omega_n T_2} - 2e^{\bar{\alpha}\omega_n T_3} + e^{\bar{\alpha}\omega_n T_m}\right)\left(1 - \bar{\alpha}\omega_n\tau_1\right) + \bar{\alpha}\omega_n\tau_1 e^{\bar{\alpha}\omega_n T_m} = 0 \tag{33}$$

$$\frac{2R_o\omega_n^2}{BK} = 2(\omega_n\tau_1)^2 - 2(\omega_n T_1)^2 + 2(\omega_n T_2)^2 - 2(\omega_n T_3)^2$$

$$+ (\omega_n T_m)^2 + 2(\omega_n\tau_1)(\omega_n T_m) \tag{34}$$

Equations (26), (27), and (28) are solved for $\omega_n T_1$, $\omega_n T_2$, and b/B for assumed values of $\omega_n T_m$, ζ, and $\omega_n \tau_1$. If the value of $|b/B|$ obtained is less than unity, equation (29) is used to obtain $R_o \omega_n^2/BK$. If the value of $|b/B| > 1$, the solution is discarded and equations (31), (32), and (33) are solved for $\omega_n T_1$, $\omega_n T_2$, and $\omega_n T_3$ for the same values of $\omega_n T_m$, ζ, and $\omega_n \tau_1$. Equation (34) is used for the correct value of $R_o \omega_n^2/BK$. The equations were solved by an iterative method on a digital computer. The results are presented in table IV. A typical plot of the results is given in figure 5. As was the case for the previous examples, it can be seen from figure 5(b) that the minimum response time is reduced as τ_1 is increased.

As was the case for the type 1 third-order stable plant (with zero) considered earlier, a sensible solution to the equations does not exist for $\zeta < 1$ for certain values of $\omega_n \tau_1$ and $\omega_n T_m$. The reason for this is similar to the one presented earlier except, in this case, the output acceleration reverses sign for certain values of $\omega_n \tau_1$ and $\omega_n T_m$ in the interval $t < T_1$ rather than the output velocity as for the previous case. The $x(t)$ motion of sketch (g) is nonoptimum for these ranges of parameters. Rather than attempting to find the optimum motion of $x(t)$ for all ranges of parameters we have chosen to present the data where it is a sensible solution.

A Type 2 Fourth-Order Unstable Plant (With Zero)

The plant transfer function for this example (case 12, table I) is

$$G(s) = \frac{K(\tau_1 s + 1)}{s^2(\tau_2 s + 1)(\tau_3 s - 1)} \tag{35}$$

The optimum motion of $x(t)$ is one of the two forms shown in sketch (g). As was the case for the previous unstable example T_1 must approach a limit as $T_m \to \infty$.

If the motion of $x(t)$ is as shown in sketch (g 1), the equations which must be satisfied in order that $E(s)$ be entire are

$$2\frac{T_1}{\tau_2} - 2\frac{T_2}{\tau_2} + \frac{T_m}{\tau_2} + \frac{b}{B}\frac{\tau_1}{\tau_2} = 0 \tag{36}$$

$$\left(1 - 2e^{\frac{T_1}{\tau_2}} + 2e^{\frac{T_2}{\tau_2}} - e^{\frac{T_m}{\tau_2}}\right)\left(1 - \frac{\tau_1}{\tau_2}\right) - \frac{b}{B}\frac{\tau_1}{\tau_2}e^{\frac{T_m}{\tau_2}} = 0 \tag{37}$$

$$\left[1 - 2e^{-\frac{T_1}{\tau_2}\left(\frac{\tau_2}{\tau_3}\right)} + 2e^{-\frac{T_2}{\tau_2}\left(\frac{\tau_2}{\tau_3}\right)} - e^{-\frac{T_m}{\tau_2}\left(\frac{\tau_2}{\tau_3}\right)}\right]\left[1 + \frac{\tau_1}{\tau_2}\left(\frac{\tau_2}{\tau_3}\right)\right] + \frac{b}{B}\left(\frac{\tau_1}{\tau_2}\right)\left(\frac{\tau_2}{\tau_3}\right) e^{-\frac{T_m}{\tau_2}\left(\frac{\tau_2}{\tau_3}\right)} = 0 \tag{38}$$

$$\frac{2R_o}{BK\tau_2^{\,2}} = -2\frac{\tau_1}{\tau_2}\left(2\frac{T_1}{\tau_2} - 2\frac{T_2}{\tau_2} + \frac{T_m}{\tau_2}\right) + 2\left(\frac{T_1}{\tau_2}\right)^2 - 2\left(\frac{T_2}{\tau_2}\right)^2 + \left(\frac{T_m}{\tau_2}\right)^2 + 2\frac{b}{B}\left(\frac{\tau_1}{\tau_2}\right)\left(\frac{T_m}{\tau_2}\right) \tag{39}$$

If the motion of $x(t)$ is as shown in sketch (g 2), the equations which must be satisfied in order that $E(s)$ be entire are

$$2\frac{T_1}{\tau_2} - 2\frac{T_2}{\tau_2} + 2\frac{T_3}{\tau_2} - \frac{T_m}{\tau_2} - \frac{\tau_1}{\tau_2} = 0 \tag{40}$$

$$\left(1 - 2e^{\frac{T_1}{\tau_2}} + 2e^{\frac{T_2}{\tau_2}} - 2e^{\frac{T_3}{\tau_2}} + e^{\frac{T_m}{\tau_2}}\right)\left(1 - \frac{\tau_1}{\tau_2}\right) + \frac{\tau_1}{\tau_2} e^{\frac{T_m}{\tau_2}} = 0 \tag{41}$$

$$\left[1 - 2e^{-\frac{T_1}{\tau_2}\left(\frac{\tau_2}{\tau_3}\right)} + 2e^{-\frac{T_2}{\tau_2}\left(\frac{\tau_2}{\tau_3}\right)} - 2e^{-\frac{T_3}{\tau_2}\left(\frac{\tau_2}{\tau_3}\right)} + e^{-\frac{T_m}{\tau_2}\left(\frac{\tau_2}{\tau_3}\right)}\right]\left[1 + \left(\frac{\tau_1}{\tau_2}\right)\left(\frac{\tau_2}{\tau_3}\right)\right] - \frac{\tau_1}{\tau_2}\left(\frac{\tau_2}{\tau_3}\right) e^{-\frac{T_m}{\tau_2}\frac{\tau_2}{\tau_3}} = 0 \tag{42}$$

$$\frac{2R_o}{BK\tau_2^{\,2}} = -\frac{2\tau_1^{\,2}}{\tau_2^{\,2}} + 2\left(\frac{T_1}{\tau_2}\right)^2 - 2\left(\frac{T_2}{\tau_2}\right)^2 + 2\left(\frac{T_3}{\tau_2}\right)^2 - \left(\frac{T_m}{\tau_2}\right)^2 - 2\left(\frac{\tau_1}{\tau_2}\right)\left(\frac{T_m}{\tau_2}\right) \tag{43}$$

The solution of these equations is by an iterative procedure so similar to previous examples that no further explanation is considered necessary. The solutions are presented in table V for the same values of τ_2/τ_3 and τ_1/τ_2 as were given for the type 1 third-order plant presented earlier.

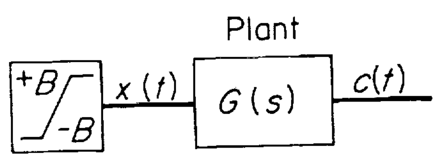
Typical plots of the data are shown in figure 6. As can be seen from figure 6(a), T_1/τ_2 approaches the limit $T_1/\tau_2 = (\tau_3/\tau_2)\ln 2$ as $R_o/BK\tau_2^2 \to \infty$. This is the same limit value as for the previous unstable case. As can be seen from figure 6(b), the minimum response time for a particular input decreases as τ_1 increases which has been true for all the plants considered.

A Method for Determining Approximate Switch Time Data

The data presented in this report and reference 1 cover a large number of plant transfer functions. The examples given here (in later sections) and those of reference 1 serve to show the switch time method has fairly general application if one designs to (a) force the first reversal time to be optimum and (b) force the zeros to shift in the s plane in such a manner that as the equivalent gain of the limiter reduces, the closed-loop poles remain in reasonably damped regions. In the examples considered it is only necessary to change the zero positions as a function of the error. Other examples may require further complications or additions; however, it is believed the general concepts are still applicable.

One of the limitations in applying the concept is the limited optimum switching time data available. The plant may be nonlinear or have a transfer function for which data are not available. It is the purpose of this section to show an approximate method for obtaining these data. The method is well suited to analog computer computation or tests on the actual hardware.

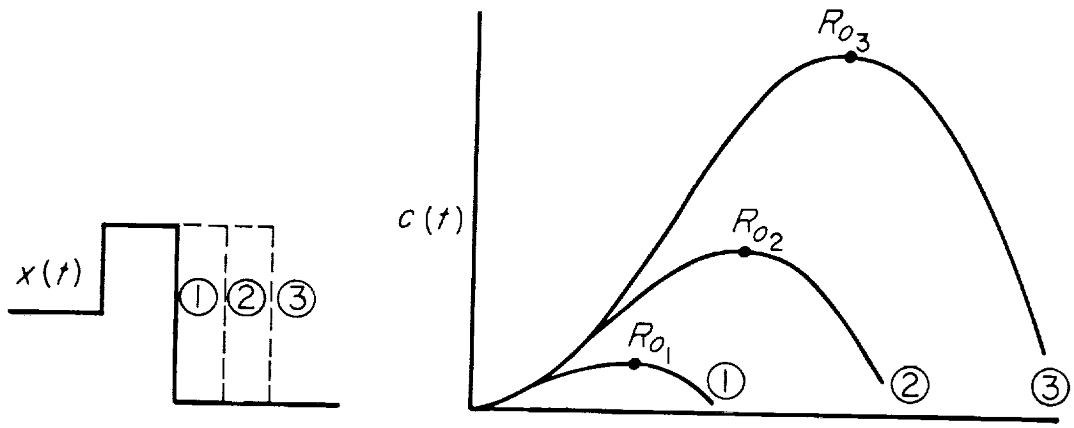
Consider the system of sketch (h). It is desired to know the optimum



Sketch (h)

first reversal time, T_1, as a function of input step magnitude, R_o. It has generally been shown that the optimum motion of $x(t)$ is to remain at the maximum in one direction (+B) until T_1 and then reverses to (-B), the maximum in the opposite direction. As the order of the system is increased, further switchings take place at later times.

The $c(t)$ motions for several $x(t)$ motions which are maximum one way and reverse at T_1 will generally be as shown in sketch (i).



Sketch (i)

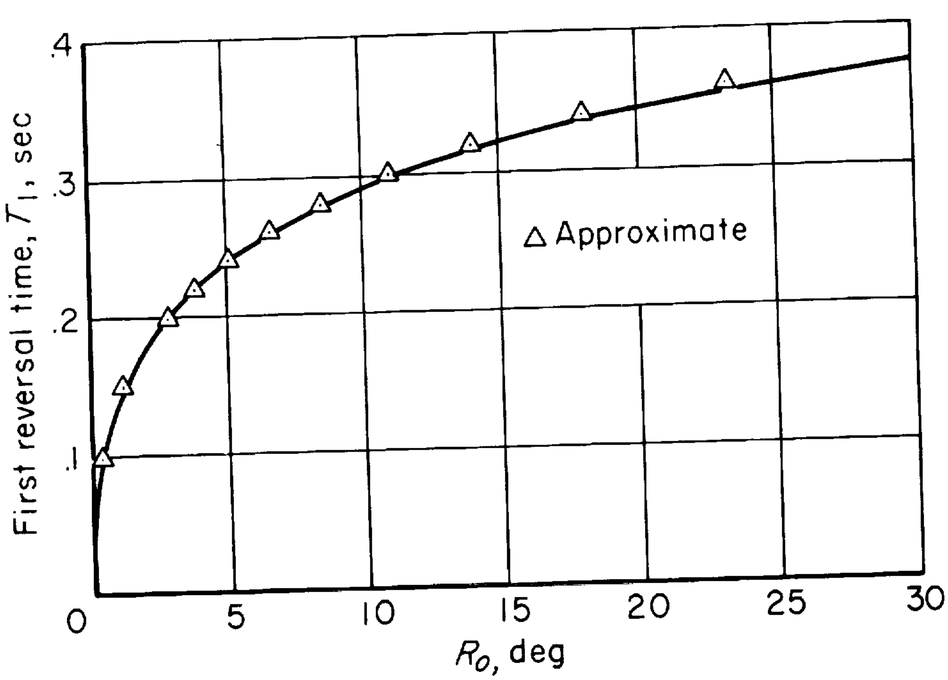
To obtain an approximate curve one could plot the peak value of $c(t)$, (R_{o1}, R_{o2}, R_{o3}), versus the switching times (1, 2, 3). If the plant is simulated on an analog computer or if actual hardware is tested such plots are easily accomplished since the $x(t)$ motion of sketch (i) can be readily generated by electronic devices such as interval timers. This simple scheme was originally suggested in reference 4. It can be seen to be exact for second-order plants since the output velocity is zero at $c(t)$ maximum and returning $x(t)$ to zero at this time (or using a tailing off exponential for plants with a zero in the left half plane) will cause no further change in $c(t)$.

For higher order plants one questions the accuracy of such a simple technique. To obtain an idea of the relative accuracy the relationships between R_o and T_1 used in this scheme can be compared to the exact relationship. This has been done for type 3 third-order and type 4 fourth-order plants. The results are

Plant transfer function	Exact	Approximate
$\frac{K}{s^3}$	$T_1{}^3 = \frac{R_o}{2KB}$	$T_1{}^3 = \frac{R_o}{1.91\ KB}$
$\frac{K}{s^4}$	$T_1{}^4 = \frac{R_o(0.1766)}{KB}$	$T_1{}^4 = \frac{R_o(0.2107)}{KB}$

As can be seen, the error is about 5 percent for the third-order plant and 15 percent for the fourth-order plant. If some of the roots of the plant denominator are in the left half plane, it is believed the error will be less than that given by the above cases. The reason for this belief is that some of the output derivatives are bounded (by the fact that $x(t)$ is bounded) when there are poles in the left half plane. The result is that the ensuing motions required to force all derivatives of $c(t)$ to be zero have less effect than in the cases shown above. To obtain an idea of the relative accuracy for unstable plants approximate data are compared in sketch (j) with exact data (from table V) for a plant transfer function

$$G(s) = \frac{3.125(s + 1)}{s^2(0.183\ s + 1)(0.683\ s - 1)}$$



Sketch (j)

As can be seen, the approximate data, obtained from an analog computer simulation, are in close agreement with the exact curve.

One point should be mentioned with regard to the use of approximate first reversal time data. That is, the approximate T_1 is slightly larger than the exact T_1. The result is that one will obtain a design which has some overshoot. This overshoot is usually readily removed by slight adjustments in the nonlinear function.

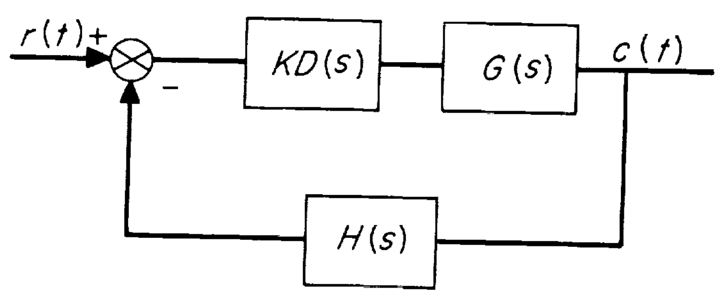
OPTIMUM SYSTEMS

It is the purpose of this section to consider the problem of designing an optimum feedback control system. This type system[4] will be defined as one which maintains the error as small as possible under the constraint that the plant input variable is bounded. Unfortunately, very little is known about such systems at the present time. If the system is continuous of the type shown in sketch (a), and it is subjected to initial conditions only ($r(t) = 0$), it is generally accepted that the relay or bang-bang system is the optimum.[5] If the system is of the sampled-data type ($m(t)$ of sketch (a) is a staircase signal), little is known about the design of the optimum system.

In this section the discussion will be restricted to the plant transfer functions given in table I.

The Optimum Continuous Linear System

Prior to discussing cases where saturation is present, it is pertinent to discuss what the previous definition of an optimum system means in terms of a linear system. To have the optimum (i.e., zero error to any input) it is quite obvious that the closed-loop transfer function relating $C(s)/R(s)$ of sketch (k) must be unity.



Sketch (k)

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[4]Another definition of optimum is made when the input is contaminated with noise. The optimum system is then defined as one which minimizes the quantity $H_d r(t) - c(t)$. The input (without noise) is $r(t)$; $c(t)$ is the output; and H_d is the mathematical representation of the desired response which may or may not be physically realizable (see, e.g., ref. 5).

[5]Optimum in the sense that the error is restored to zero in the minimum time.

Since the closed-loop transfer function must be unity then

$$\frac{C(s)}{R(s)} = 1 = \frac{KD(s)G(s)}{1 + KD(s)G(s)H(s)} \tag{44}$$

Thus, one obtains the following two conditions if equation (44) is to be satisfied:

$$H(s) = 1 \tag{45}$$

$$K = \infty \tag{46}$$

A third condition is obtained from stability considerations:

$$D(s)G(s) = \frac{a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \cdot\cdot\cdot + a_o}{b_n s^n + b_{n-1}s^{n-1} + \cdot\cdot\cdot} \tag{47}$$

The roots of the numerator of equation (47) must have real parts which are negative and the order of the numerator can only be 1 lower than the order of the denominator. The following arguments based on root-locus rules verify this.

a. For $K \to \infty$ all poles of the root locus equation move to zeros; therefore, all zeros must be in the left half of the s plane or the system will be unstable. The roots of the numerator must, therefore, have negative real parts.

b. If the order of the numerator is 3 (or higher) less than the denominator, some of the closed-loop poles for $K \to \infty$ are in the right half plane; therefore, the system will be unstable.

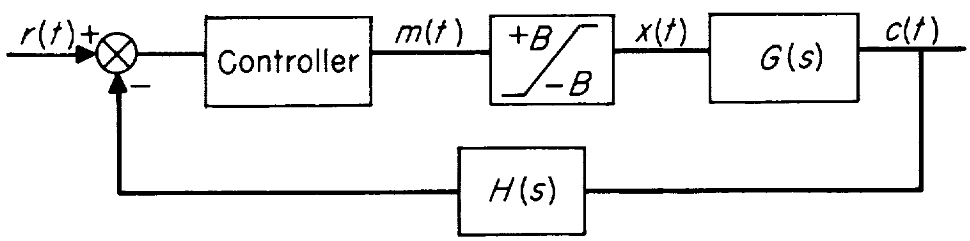
c. If the order of the numerator is 2 lower than the denominator, two of the poles move to infinity (as $K \to \infty$) with asymptotes which have a finite real part. This real part can be negative; however, since it is finite, the impulse response time is finite. Therefore, the closed-loop transfer function cannot be unity.

d. If the order of the numerator is 1 less than the denominator, one pole moves to infinity (as $K \to \infty$) along the negative real axis. With $K = \infty$ this provides a closed-loop transfer function of first order with a time constant equal to zero which is essentially a transfer function of unity.

The important point gained from this consideration of linear systems is that if G(s) has n poles and no zeros, n-1 derivatives of the error must be available in order to design the optimum linear system.[6]

The Optimum Continuous Control System With Saturation

Consider the system shown in sketch (1).



Sketch (1)

If the controller's characteristics are linearized about zero input, it can be seen that the same conditions given for the optimum linear system must be true for this system; that is,

$$H(s) = 1$$

$$\frac{M(s)}{E(s)}_{K\to\infty} = K(1 + a_1 s + a_2 s^2 + a_3 s^3 + \ldots)$$

The roots of s of the equation determined by setting

$$\frac{M(s)}{E(s)} = 0$$

must be in the left half of the s plane. Furthermore, since x(t) is bounded and X(s)G(s) = C(s), certain output derivatives or sums of output derivatives are bounded. For example if

[6]Practically speaking, the optimum linear system cannot exist, since it is impossible to differentiate. Any practical differentiating circuits also introduce poles which, when included in the system, cause the system to show an instability at some finite value of gain, K. If the poles introduced by the differentiating circuits are sufficiently far from the origin in the s plane compared to the plant poles and to the zeros introduced by the differentiating circuits, then one can obtain a system design which approximates the optimum (with a finite gain) over a low-frequency band width.

$$G(s) = \frac{K}{s(\tau_2 s + 1)} \tag{48}$$

then (see sketch (1))

$$|x| = \left| \frac{\tau_2 \ddot{c} + \dot{c}}{K} \right| \leq B \tag{49}$$

If these same bounds are placed on the input which for this example gives

$$\left| \frac{\tau_2 \ddot{r} + \dot{r}}{K} \right| \leq B \tag{50}$$

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then the plant has the capability of following the input exactly. Thus, bounding $x(t)$ forces one to put certain bounds on derivatives of the input (depending on $G(s)$) if zero error is to be obtained.

The question therefore is what can be said about the optimum system when equation (50) is not satisfied. One answer could be that, regardless of the present error and its derivatives, we would like $x(t)$ to take the optimum motion to reduce the error to zero in a minimum time.[7]

If this answer is correct, then the characteristics of the controller[8] would be (with $H(s) = 1$)

$$m(t) = Kf(\epsilon, \dot{\epsilon}, \ddot{\epsilon}, \ldots) \tag{51}$$

where

$$K = \infty$$

The function of error and its derivatives, $f(\epsilon, \dot{\epsilon}, \ddot{\epsilon}, \ldots)$, is the equation for the optimum switching surface in the phase space $\epsilon, \dot{\epsilon}, \ddot{\epsilon}, \ldots$.

A second answer to the question has been pursued by Hopkin and Wang (ref. 6). Their answer amounts to making the error zero at some predicted time in the future. For this answer $m(t)$ becomes

$$m(t) = Kf(\epsilon, \dot{\epsilon}, \ddot{\epsilon}, \ldots, \dot{r}, \ddot{r}, \ldots) \tag{52}$$

where

$$K = \infty$$

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[7]The input could be such that it was impossible for the error to ever be zero. For example, the plant given by equation (48) could not follow a constant acceleration input in a system of the type given in sketch (1), since the limit on $x(t)$ forces the output velocity to be bounded.

[8]The input to the plant $x(t)$ can (for $K = \infty$) be written as $B \operatorname{sgn} f(\epsilon, \dot{\epsilon}, \ddot{\epsilon}, \ldots)$. There is a perfect equivalence between considering the controller to be an infinite gain device and the more customary way of lumping the controller and limiter together and calling the combination an ideal relay.

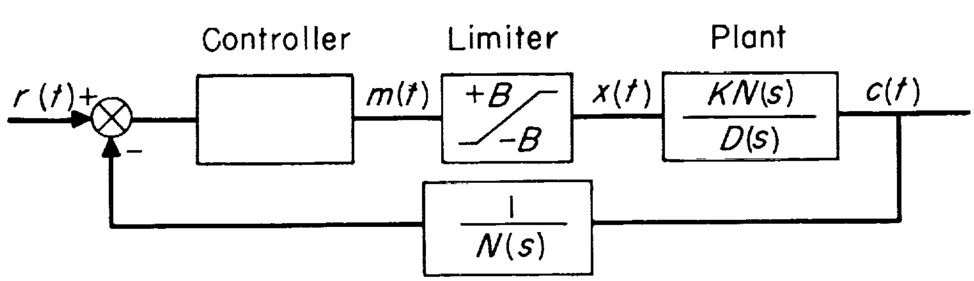
The derivatives of r occurring in equation (52) are a result of prediction of the future error. For the second-order example in which they had run simulation tests their system worked considerably better than a linear function of ϵ and $\dot{\epsilon}$ alone; however, no data were presented for the system of equation (52) where nonlinear functions are required.

In many cases it is not possible to measure the input; that is, the error is sensed directly (for example radar or infrared seekers). For these cases the only choice for an optimum would be given by equation (51).

There is an apparent need for more work on the problem of the optimum system where saturation is present; however, we will assume the system of the type given by equation (51) is the optimum for many situations where the input is not sensed directly. If this assumption is valid, there are previous results available. Table VI lists previous work in which the optimum controllers of some of the plants (given by case number) can be found. The expressions are written out for the examples where this is possible. As can be seen from the table, to the authors' knowledge, only the simplest plants have been studied sufficiently by previous investigators to define the optimum switching surface. For these cases, a study of the reference for the particular plant involved allows one to determine the complexity of the control equation required if one desires the optimum system. The minimum response data of reference 1 and the previous section demonstrate what can be gained (compared to some linear controller) by the use of this usually complicated control equation.

The Influence of Plant Zeros on the Optimum System

It was shown in sections 3.5 and 3.8 of reference 1 and the previous section of this report that zeros in the plant transfer function make the minimum response time considerably lower than the value obtained when the zeros are neglected. A system which gives the relay, or bang-bang, solution can always be obtained with zeros in the plant transfer function (if they are in the left half plane) by the simple artifice of adding a lag network in the feedback path as was shown in sketch (d) and repeated in sketch (m) for the convenience of the reader.



Sketch (m)

The terms N(s) and D(s) represent the numerator and denominator of G(s), respectively, written in polynomial form. The controller equation is then that given or referred to in table VI for the particular plant (if such data are available).

It has already been shown that this system is nonoptimum for step inputs and it is quite easy to show that this system is not the optimum for slow changing inputs. To show this, one linearizes the controller equations around zero input. For a very high gain controller then

$$\frac{C(s)}{R(s)} \approx N(s) \tag{53}$$

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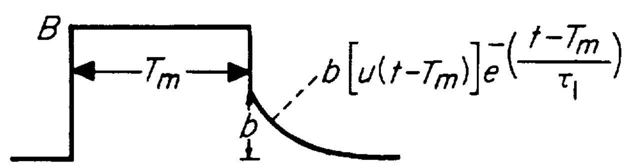
or, the output "leads" the input. Two cases will be investigated in this section to show how one can use the knowledge gained thus far in deriving an optimum system.

A type 1 second-order plant.- For this case

$$G(s) = \frac{K(\tau_1 s + 1)}{s(\tau_2 s + 1)}$$

It was shown in chapter III of reference 1 that if one restricts his interest to plants where $\tau_1/\tau_2 > 1/2$, then the optimum motion for x(t) for step inputs is as shown in sketch (n). The error to step inputs is



Sketch (n)

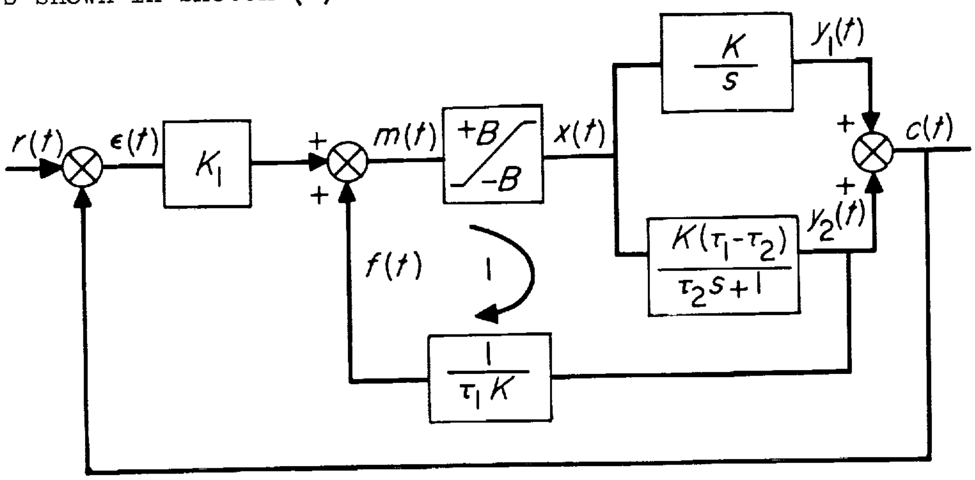
zero when $t = T_m$. Thus, the control system desired is one that holds x(t) on its limits until the error is zero and then causes the motion of x(t) to have an exponential decay of time constant, τ_1.

The amplitude of the tailing off exponential, $x(T_m) = b$, is

$$b = B\left(1 - e^{-\frac{T_m}{\tau_2}}\right)\left(1 - \frac{\tau_2}{\tau_1}\right) \tag{54}$$

(See ref. 1, pp. 44-50 for the derivation of this equation.)

A control system which provides the optimum motion of $x(t)$ of sketch (n) is shown in sketch (o).



Sketch (o)

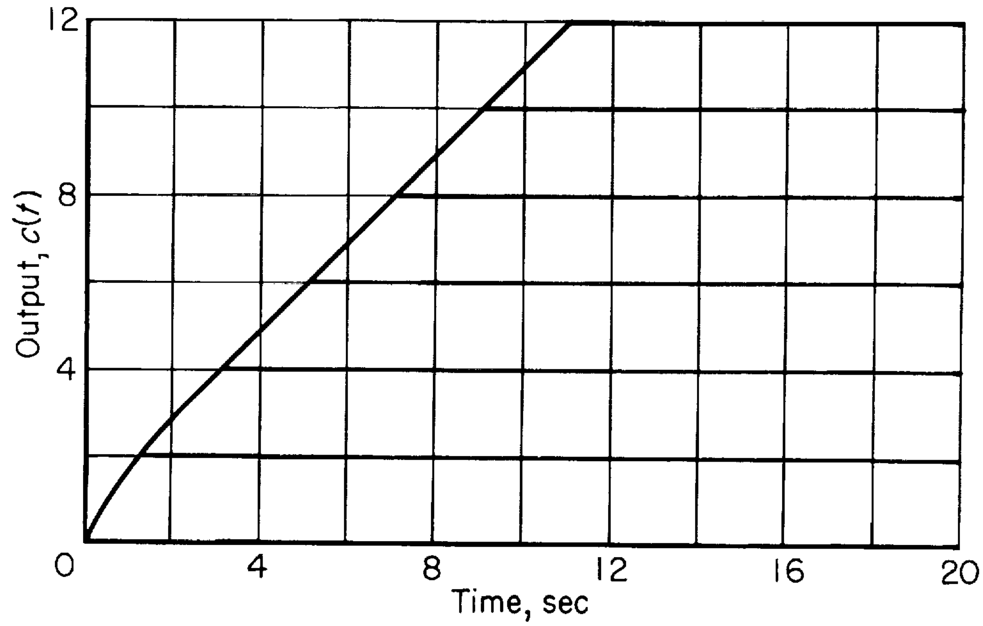
The quantity K_1 of sketch (o) approaches infinity with the restriction that $\epsilon K_1 = \infty$ if ϵ has any value and $\epsilon K_1 = 0$ if $\epsilon = 0$.

A study of sketch (o) shows that the main control of $x(t)$ comes from the error signal $\epsilon(t)$. In other words, if the error has any value whatsoever, $c(t)$ will be moving in such a direction to reduce the error as fast as possible; that is, $\dot{c}(t)$ will be at its maximum. Only when the error is zero does the signal $f(t)$ have any effect. The time constant of loop number 1 when $x(t)$ is not at its limit can be readily calculated to be equal to τ_1. The value of $x(T_m)$ can be seen from sketch (o) to be equal to $y_2(T_m)/\tau_1 K$. Since $y_2(0) = 0$, the value of $y_2(T_m)$ is readily calculated by the Laplace transform to be

$$y_2(T_m) = L^{-1}\left\{\left(\frac{B}{s}\right)\left[\frac{K(\tau_1 - \tau_2)}{(\tau_2 s + 1)}\right]\right\}\Bigg|_{T_m} = BK\tau_1\left(1 - \frac{\tau_2}{\tau_1}\right)\left(1 - e^{-\frac{T_m}{\tau_2}}\right) \qquad (55)$$

It can be seen that $x(T_m) = y_2(T_m)/K\tau_1$ agrees with equation (54).

No system has been derived for the case of $\tau_1/\tau_2 < 1/2$; however, since one reversal of $x(t)$ is required before the error is zero, it is probable that a more complicated system is required than that shown in sketch (o). Sketch (p) shows some typical time histories of the output response, $c(t)$, for various step input magnitudes. For this example $\tau_1 = 2$, $K = 1$, $\tau_2 = 1$, and $|B| = 1$.

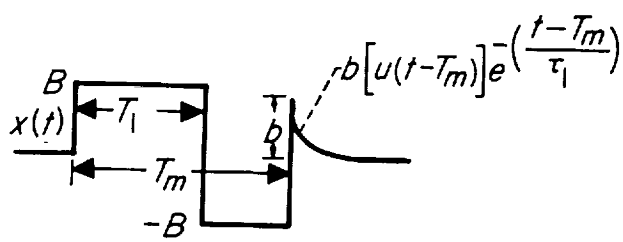


Sketch (p)

A type 2 third-order plant.- For this example

$$G(s) = \frac{K(\tau_1 s + 1)}{s^2(\tau_2 s + 1)}$$

If one restricts his interest to examples where $\tau_1/\tau_2 > 1/2$, then the optimum motion of $x(t)$ is as given in sketch (q).

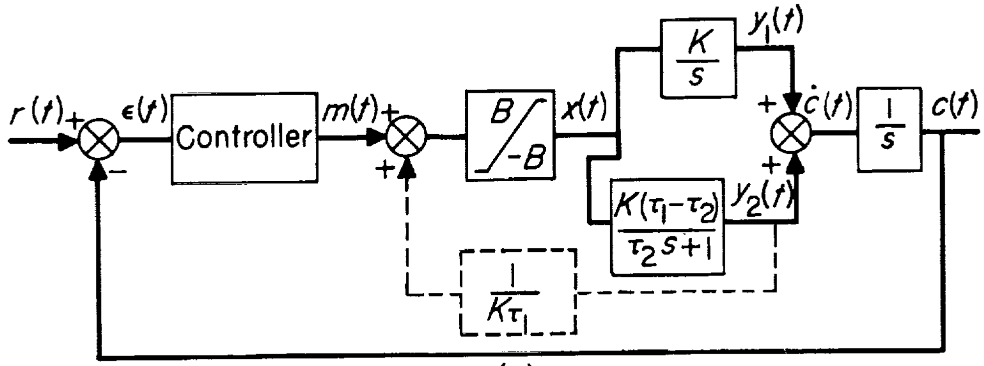


Sketch (q)

One can expand G(s) and define state variables as follows:

$$G(s) = \frac{K(\tau_1 s + 1)}{s^2(\tau_2 s + 1)} = \frac{1}{s}\left[\frac{K}{s} + \frac{K(\tau_1 - \tau_2)}{\tau_2 s + 1}\right] = \frac{1}{s}\left[\frac{Y_1(s)}{X(s)} + \frac{Y_2(s)}{X(s)}\right] \tag{56}$$

This system is shown in the following block diagram.



Sketch (r)

The dotted feedback in sketch (r) has been added here for the same reason as the previous example; that is, if we get the error and error rate to zero, this feedback allows $y_1(t)$ and $y_2(t)$ to change while $\dot{c}(t)$ remains zero and $c(t)$ remains a constant (or zero). The fact that the solution following T_m of sketch (r) will be correct can be seen by checking the time constant of the dotted loop as well as reviewing section 3.8 of reference 1.

The completion of the design requires that the controller equations be determined and realized. In order to overcome the dotted feedback the gain of the controller must be infinite. Consider step inputs only; the motion shown in sketch (q) shows that only one reversal need take place since, when $t = T_m$, both error and error rate are zero. This suggests that an equation which is a function of ϵ and $\dot{\epsilon}$ of form, for example

$$m(t) = [\epsilon + f(\dot{\epsilon})]K_1 \tag{57}$$

$$K_1 \to \infty$$

should be sufficient.

The nonlinear function, $f(\dot{\epsilon})$, given in equation (57) was calculated using the method presented in chapter II of reference 1 to make T_1 equal to the optimum, and it was found that this scheme will not work since

$m(t)$ given by equation (57) does not reverse sign at T_1 if $x(t)$ is the shape of sketch (q).[9] The reason for this is best explained if the switching curve in the phase plane $(\epsilon, \dot{\epsilon})$ is determined and an optimum trajectory for one input magnitude is shown. We know all the necessary information to compute both the trajectory and the optimum switching curve. For example

$$\epsilon(t) = R_o u(t) - L^{-1}\left[X(s)G(s)\right] \tag{58}$$

$X(s)$ is the Laplace transform of $x(t)$ shown in sketch (q). The relationship between R_o, T_1, and T_m is determined from the optimum curves of chapter III of reference 1. The error velocity is

$$\dot{\epsilon}(t) = R_o \delta(t) - L^{-1}\left[sX(s)G(s)\right] \tag{59}$$

The δ function may be omitted since it is only an impulse at the beginning of the trajectory. For any assumed value of R_o, one can therefore compute and plot ϵ versus $\dot{\epsilon}$.

The optimum switching curve is found by plotting $\epsilon(T_1)$ versus $\dot{\epsilon}(T_1)$ as obtained from equations (58) and (59); R_o in equation (58) is obtained from the curves of chapter III (ref. 1) for each value of T_1.

The results of these computations for the example where $B = 1$ and $G(s) = (2s + 1)/s^2(s + 1)$ are shown in sketch (s). An examination of the optimum trajectory after it meets the optimum switching curve shows the reason why the switching function of equation (57) is unsatisfactory. The optimum trajectory stays in the $+x$ region; therefore, a system which uses this scheme cannot be expected to be optimum. The simulation results showed the system to have a reasonably large first overshoot, but no undershoot to the response. The amount of overshoot was smaller than that which would be given by a system of the type shown in sketch (m); however, any overshoot at all means we do not have the optimum system.

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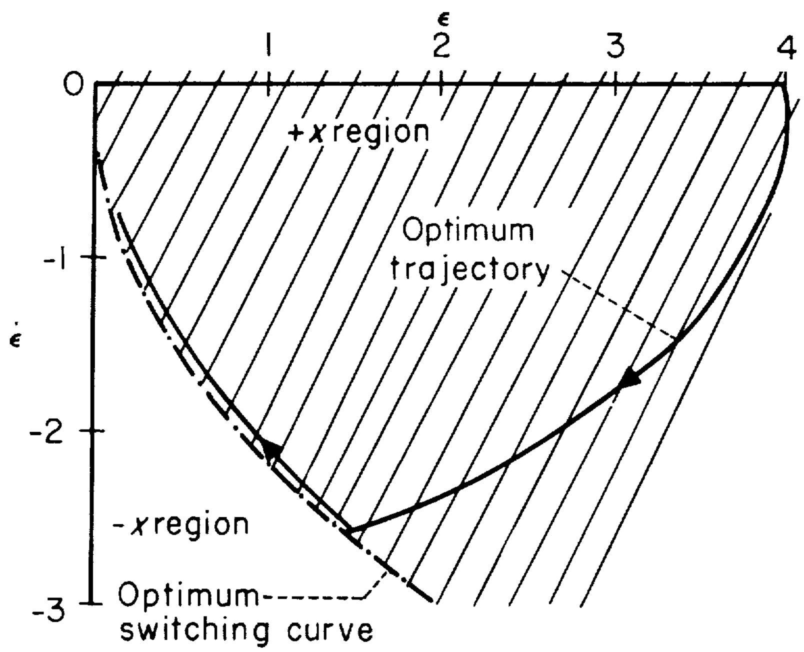
[9]A note should be added that this is not a limitation of the switch-time method. This method is based on the fact that we have a linear region and the switch-time method simply provides techniques for determining nonlinear functions which provide near-optimum response for large step inputs. For the case at hand, we are attempting to use the switch-time method for determining an optimum switching surface in the phase space, since this is a third-order problem. We therefore cannot use the switch-time method directly and expect to have an optimum system.



Sketch (s)

What is really needed to obtain the optimum system is the equation for the switching surface in the phase space. This equation is not known at this time, so we shall, instead, design the system which will have optimum response to steps but nonoptimum response to other inputs. We must introduce a new variable in the system so that when the trajectory hits the switching surface, x will change sign. The particular system derived here will, in reality, be an approximate optimum system and should perhaps have been introduced in the next section. This was not done since a somewhat different design viewpoint will be introduced then.

It may be noted from sketch (s) that if a straight line is drawn from the origin to the desired switching point on the optimum switching curve, then the optimum trajectory will always be to the left of this line. This suggests that a switching equation of the type

$$m(t) = K_1[\epsilon + \dot{\epsilon}f(y_3)] \qquad (60)$$

$$K_1 \to \infty$$

will be satisfactory if the variable y_3 is "slow" compared with $\dot{\epsilon}$. There are probably a number of choices which could be made; however, for this example, the choice was that

$$y_3(t) = y_1(t) - y_2(t) \qquad (61)$$

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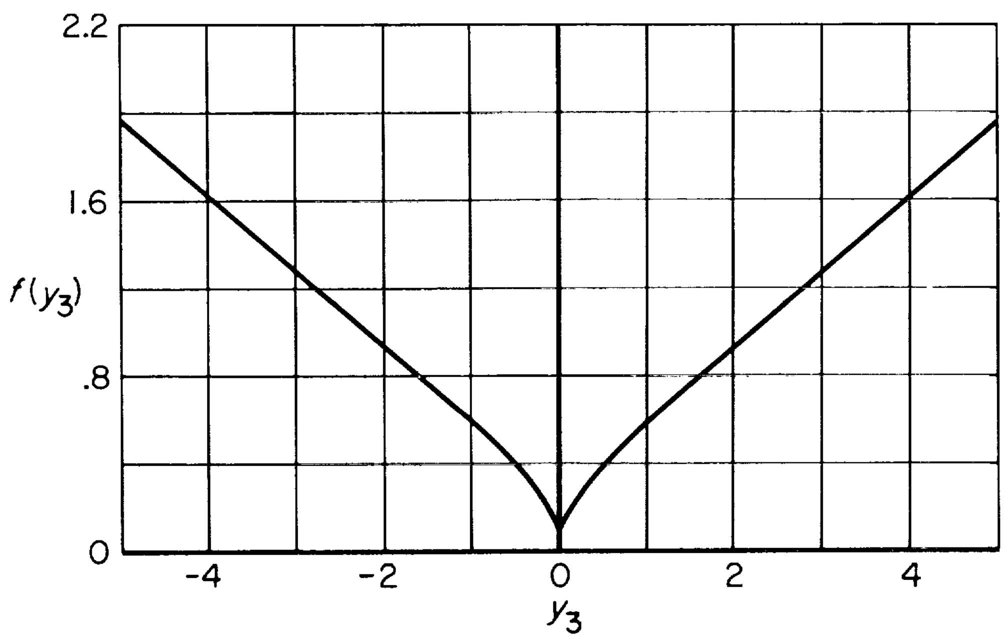
For the particular values of τ_1 and τ_2 chosen for the example,

$$Y_3(s) = \frac{sC(s)}{2s + 1}$$

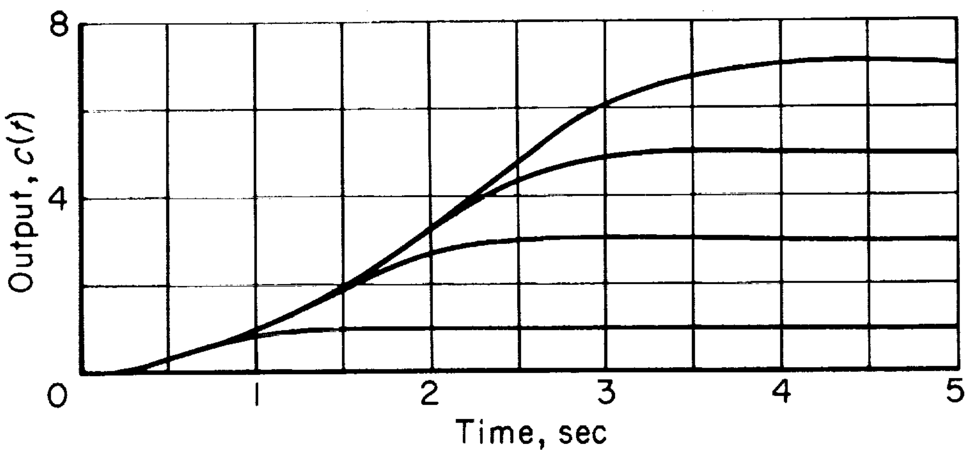
Thus $y_3(t)$ is nothing more than the output velocity fed through a time lag. Since the choice was not based on phase space considerations, it, therefore, is not surprising that the results of tests conducted on the analog computer showed optimum response to steps and certain initial conditions but showed nonoptimum response to other initial conditions. In all cases where nonoptimum response was found, however, it was due to excessive damping rather than instabilities or low damping.

The function $f(y_3)$ can be obtained for any example by forcing the first reversal time to be correct by the switch time method given in chapter II of reference 1. For the example case, the function $f(y_3)$ is shown in sketch (t). The step responses obtained from the analog computer



Sketch (t)

are shown in sketch (u). As was mentioned, good response was obtained for a number of initial conditions; however, it was not the optimum for all conditions. The question remains then as to what the optimum controller is for this example. The need for further research for the case where the plant has zeros is certainly indicated.



Sketch (u)

The Influence of an Unstable Pole on the Optimum System

As has been shown earlier, if the plant has an unstable pole, the optimum first reversal time of the saturated variable must approach a finite limit as $R_o \rightarrow \infty$; otherwise, control over the plant is impossible. To demonstrate what this implies in terms of the optimum system it is convenient to use a type 1 second-order plant. The optimum switching times for this example are presented in figure 1.

The plant transfer function is

$$G(s) = \frac{K}{s(\tau s - 1)}$$

Although one can obtain the optimum switching curve by either the switch-time method or phase plane for second-order problems (as was shown in ref. 1), it is believed the phase plane derivation is the simplest.

By use of the fact that $x(t)$ is either $\pm B$ one can derive the trajectory equations in the phase plane $c(t)$, $\dot{c}(t)$. For initial conditions (or steps) one can substitute ϵ for $-c$ and $\dot{\epsilon}$ for $-\dot{c}$. For this example

$$\tau\ddot{c}(t) - \dot{c}(t) = \pm KB \tag{62}$$

Let

$$c(t) = x \qquad \dot{c}(t) = y$$

then

$$\ddot{c}(t) = \frac{dy}{dt} = \frac{dy}{dt}\left(\frac{dt}{dx}\right) y = y\,\frac{dy}{dx} \tag{63}$$

Substituting the relationships of equation (63) in (62) one obtains

$$\frac{dy}{dx} = \frac{1}{\tau}\left(\frac{y \pm KB}{y}\right) \tag{64}$$

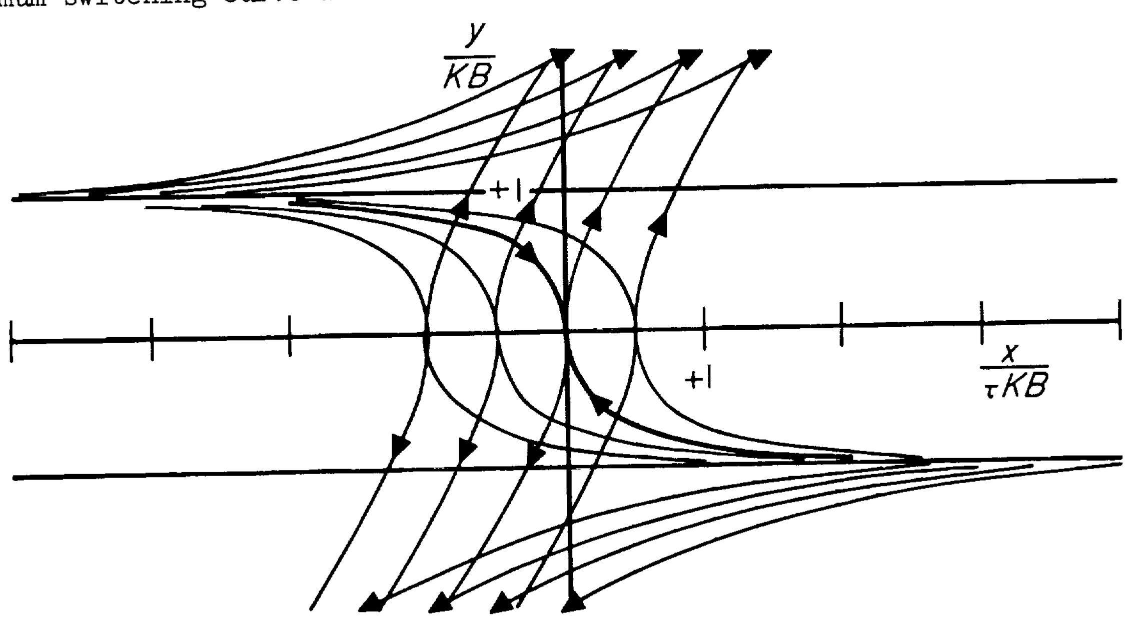
The solution of equation (64) is

$$\frac{x}{\tau} = y \mp KB\; ln(y \pm KB) + C \tag{65}$$

If equation (65) is divided by KB one obtains

$$\frac{x}{\tau KB} = \frac{y}{KB} \mp ln(\pm KB) \mp ln\left(1 + \frac{y}{\pm KB}\right) + C_1 \tag{66}$$

If C_1 is chosen to be $\pm ln(\pm KB)$ then x will be zero when y is zero. With this choice of constant, equation (66) represents the two branches of the optimum switching curve. Typical trajectories along with the optimum switching curve are shown in sketch (v). As is seen in sketch (v),



Sketch (v)

if $|y/KB| < 1$, there are trajectories which lead to the origin. If $|y/KB| > 1$, there are no trajectories which lead to the origin. Thus, $|y/KB|$ must be kept less than unity if one is to maintain control over the unstable plant.

It is apparent that the optimum system for step inputs or initial conditions will use an ideal relay whose switching equation could be of the form

$$f(\epsilon) + \dot{\epsilon} = 0 \tag{67}$$

or

$$\epsilon + f(\dot{\epsilon}) = 0 \tag{68}$$

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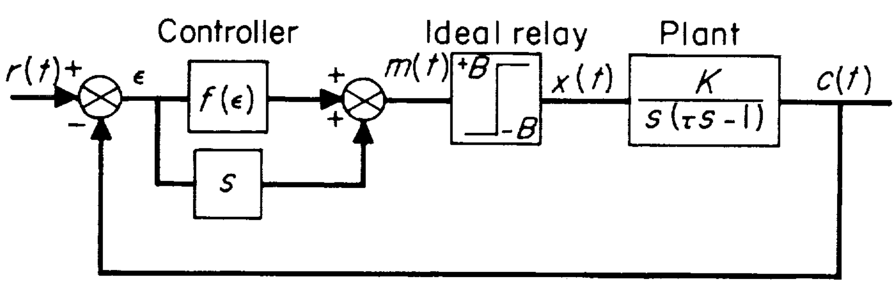
Equations (67) and (68) are the only possibilities if only functions of a single variable are considered.

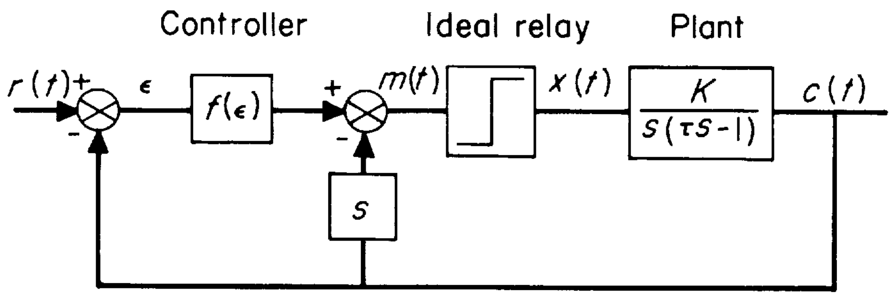
The function $f(\epsilon)$ in normalized form is the optimum curve shown in sketch (v) if one views $x/KB\tau$ as ϵ, the input to the function generator. Similarly, $f(\dot{\epsilon})$ is the optimum switching curve shown in normalized form in sketch (v) if one views y/KB as $\dot{\epsilon}$, the input to the function generator. From practical standpoints it is believed that equation (67) is easier to fabricate since the function has the characteristic of a saturating device.

The optimum system for steps and initial conditions will therefore be of the form shown in sketch (w).



Sketch (w)

This system is probably near optimum for slowly changing inputs (i.e., $\dot{r}(t) \ll KB$); however, it certainly is nonoptimum for $\dot{r}(t) > KB$ since this condition (if held for any length of time) could drive the system into instability. One way of preventing the input from being able to drive the system into instability is to use output rate feedback rather than error rate feedback. This alternative is shown in sketch (x).



Sketch (x)

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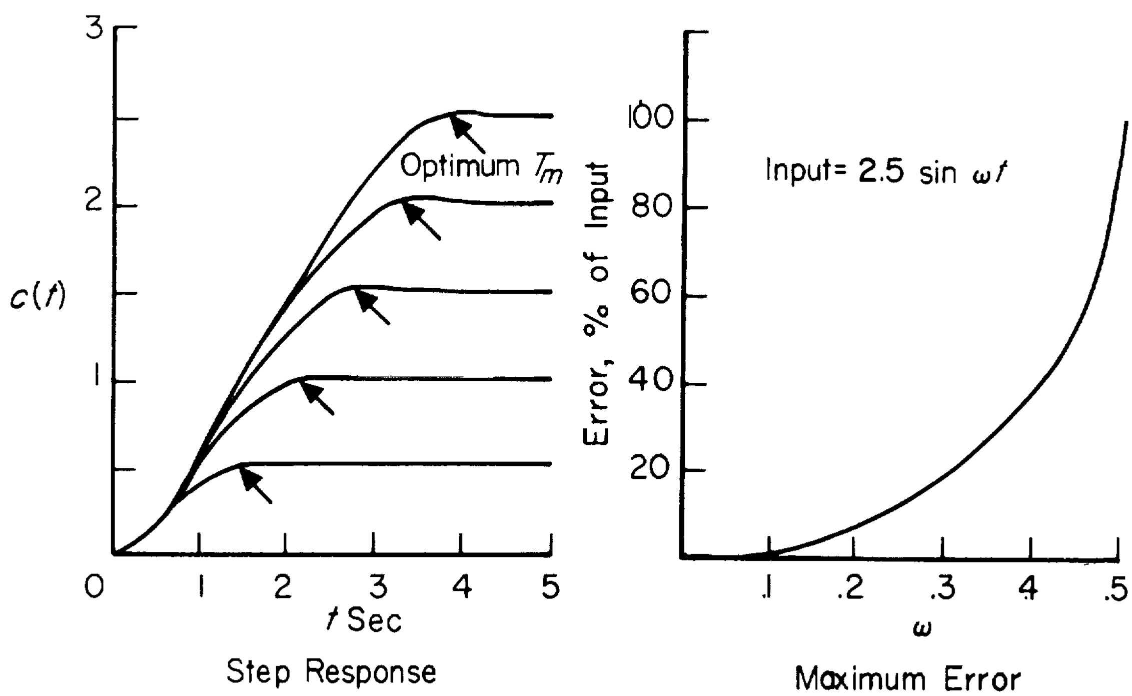
The problem with this alternative is that zero error is not possible for changing inputs. The f(ε) function shown in sketch (w) is very high gain for small errors, however, so the error for slowly changing inputs will be small. It shall be assumed that the system shown in sketch (x) is a near optimum system. The optimum system probably requires sensing of input derivatives and will not be determined here.

Some simulation tests were made of the system shown in sketch (x) for B = K = τ = 1. The step response for several input magnitudes and the maximum error versus frequency for a constant amplitude sine wave input are shown in sketch (y). The step responses show the characteristic



Sketch (y)

of velocity limiting for the large inputs. This velocity limiting is a result of the nonlinear function of sketch (v). It must exist in order to prevent the system from becoming unstable. The maximum error versus frequency curve shows the characteristic one would expect from sketch (v). That is, for $\dot{r}(max) = 2.5\omega < 1/4$, the error is quite small. As the maximum rate of the input approaches unity and above, however, the maximum error increases rapidly.

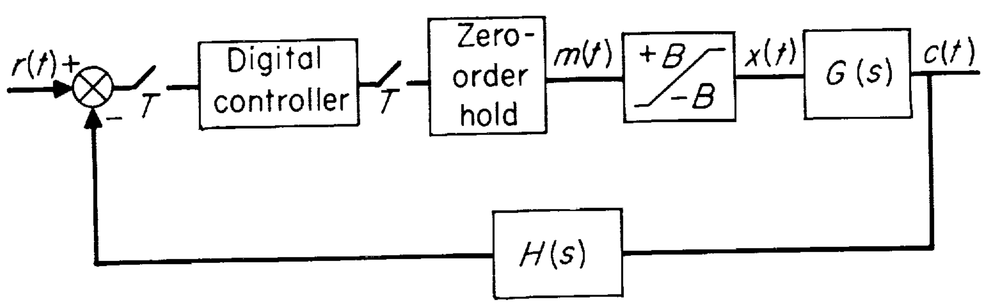
Optimum Sampled-Data Systems

As has been shown, the optimum, continuous, saturated control system is directly obtained from the relay, or bang-bang, solution (at least, for stable plants). There still remain questions as to whether these systems are optimum for all inputs and certainly practical questions as to how one differentiates the error a number of times. The problem, at least, is very well understood in comparison to saturated sampled-data control systems. Perhaps because of the newness of sampled-data systems and the additional complexity caused by the introduction of sampling, very little is known about optimum, saturated, sampled-data systems.

It is the purpose of this section to consider the questions: (1) What is a definition of the optimum, saturated, sampled-data system and what are its qualitative characteristics? (2) What is the optimum sampled-data system for a type 1 first-order plant? and (3) What are some of the pitfalls of trying to derive optimum systems for higher order plants? These three questions are considered in sequence.

An optimum, saturated, sampled-data system shall be defined as one which, regardless of the input or initial conditions, will maintain the error as small as possible with the operating sampling period, T, and with a limited input to the plant. The addition of the T is what makes the problem difficult. As $T \to 0$ the sampled-data system approaches the continuous system. As T becomes very large, the sampled-data system performance with respect to varying inputs and response time to step inputs becomes very poor.

Consider the system shown in sketch (z).



Sketch (z)

Although there are many other sampled-data configurations possible, we shall restrict ourselves to this system where there is one sampler and a digital controller in the loop. The following information regarding the systems can be obtained from linear considerations. To have an optimum, one needs a transfer function of unity. Therefore

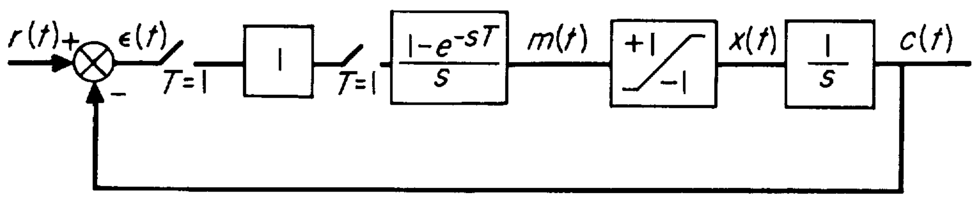
$$\left.\begin{aligned} H(s) &= 1 \\ T &\rightarrow 0 \\ \text{Gain of digital controller} &\rightarrow \infty \end{aligned}\right\} \qquad (69)$$

If T is any finite value, the gain can never be infinite, so perhaps the optimum sampled-data system becomes a "finite settling time design" (refs. 7, 8, 9) with $H(s) = 1$. We must for relatively obvious reasons introduce the constraint of "zero ripple" (refs. 7, 9). The finite settling time design provides:

(1) The minimum number of sampling instants to restore the error to zero from an input transient. (Steps, ramps, etc., are permitted, depending upon the number of poles of $G(s)$ at $s = 0$.) See, for example, reference 7 or 9.

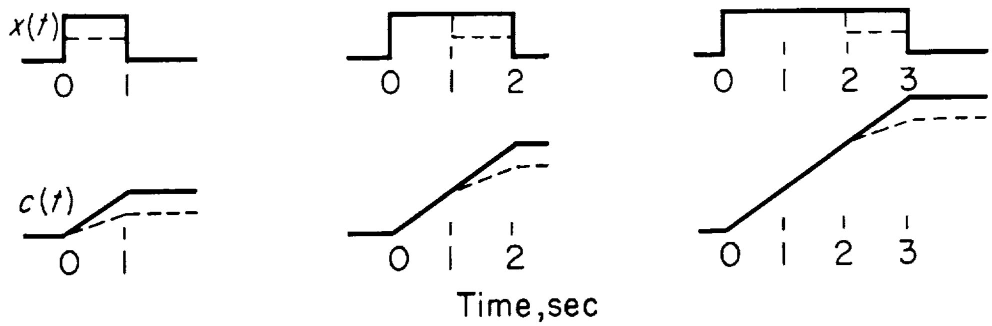
(2) The highest gain permissible at the prescribed sampling rate under the constraints of (1). Thus, for slowly varying inputs, the error will be very small. (Band width of the input should be small compared to the sampling frequency.)

Let us now examine a type 1, first-order plant with a finite settling time design. The block diagram is shown in sketch (aa) with values for $T = 1.0$, $B = 1$, $K = 1$.



Sketch (aa)

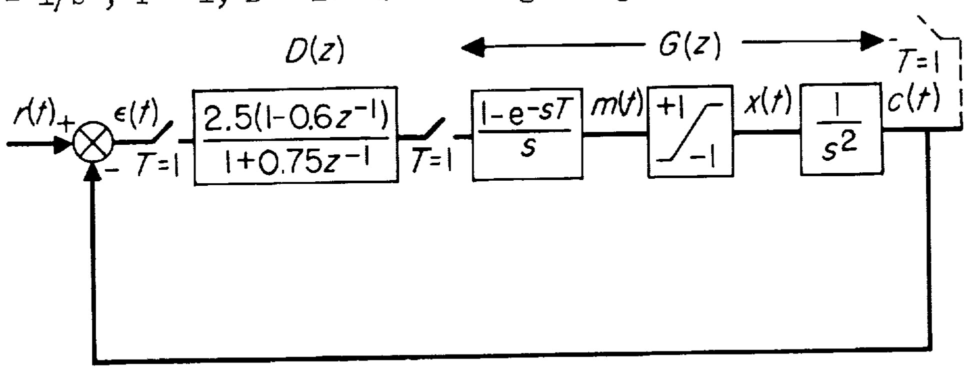
Now consider a step input in the ranges shown in sketch (ab).



Sketch (ab)

Quite obviously, from a study of the step responses, the finite settling time design is optimum with only a gain in the digital controller. A little further thought will show that this system should be optimum for arbitrary inputs, since the output will always move in a direction to reduce the error to zero (with no overshoot) in a minimum time based on the error information available at the present sampling instant. If $T \to 0$ in this design, the gain of the digital controller approaches infinity and the over-all system approaches the continuous system optimum given or referred to in table VI.

Let us examine the design for the simple second-order plant where $G(s) = 1/s^2$, $T = 1$, $B = 1$. This design is given in sketch (ac).



Sketch (ac)

The pulse transfer function

$$\frac{M(z)}{R(z)} = \frac{D(z)}{1 + G(z)D(z)} = 2.5(1 - 0.6z^{-1})(1 - z^{-1})^2 \qquad (70)$$

and for a step of $r(t)$ of magnitude R_o

$$M(z) = R_o(2.5)(1 - 1.6z^{-1} + 0.6z^{-2}) \qquad (71)$$

If $R_o > 1/(2.5)(1.6)$, the value of $m(t)$ exceeds unity (the limit level) and the pulse transfer function given in equation (70) is no longer valid. What happens for a large step of $r(t)$? Certainly the input to the plant is saturated during the first sampling instant (see eq. (71)); therefore, one can compute what $m(t)$ will be at the second sampling instant. This is most conveniently found by subtracting the value of the feedback quantity from the step response of $D(z)$.

$$m(1.0) = \left[\frac{R_o D(z)}{(1 - z^{-1})} - 2.5 \sum_{n=1}^{\infty} \frac{(n)^2 z^{-n}}{2}\right]\Bigg|_{n=1} \tag{72}$$

where $m(1.0)$ means the value of m between the first and second sampling instant.

The first term in the result must be expanded in a power series expression and the value of the coefficient of z^{-1} obtained. The second term is simply the step response of the plant and is valid so long as $m > 1$.

The result in carrying through the calculation is

$$m(1.0) = -0.875\ R_o - 1.25 \tag{73}$$

In other words, for large inputs, the input to the plant will always reverse at the first sampling instant. This is a result of having a pole of $D(z)$ which is lightly damped and is the reason such systems were considered undesirable in chapter II of reference 1.

If the plant input reverses at the first sampling instant, it should be quite obvious from the arguments presented in chapter II of reference 1 that one cannot obtain anywhere close to optimum response with this linear digital controller. Thus, one recognizes that a nonlinear controller of some type must be used. Exactly what means one uses to design this controller is not recognized at present.

In summary, one can state the following about optimum saturated sampled-data systems:

1. They should be finite settling time designs in the linear region.

2. They will never be as good as the continuous system but will approach it as $T \to 0$.

3. Although the first-order case is very simple, extension even to second order probably requires a relatively complex nonlinear digital controller.

4. The determination of the nonlinear digital controller's mathematical operations does not appear to be a simple problem.

The fact that finite settling time designs are very sensitive to parameter variations (see refs. 8 and 9) and to the other difficulties listed probably means that optimum, saturated, sampled-data systems can never exist. Thus probably other designs are required for both nonlinear and linear operation; two such examples were given in chapter IV of reference 1.

APPROXIMATE OPTIMUM SYSTEMS

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This section is devoted to the problem of designing an approximate optimum attitude autopilot for an aircraft. By approximate optimum is meant that for small or slowly varying inputs, if the aircraft has the capability of following a desired input (pitch attitude), the error will be essentially zero. For step inputs that may come about as a result of engaging the system, the response will be near optimum, that is, the error will be reduced to zero in a near minimum time. Disturbance inputs which may come about in actual practice as a result of gusts, will be neglected in this study; however, the resulting design which is a very high gain closed-loop system should have a good response to disturbances.

It will be assumed that the limiting in the system is a result of hydraulic valve bottoming in the control-surface actuator. For simplicity purposes it is further assumed that mechanical limits on the hydraulic valve are equivalent to control-surface rate limits, that is, the actuator is a perfect integrator. The control-surface rate limit will be assumed to be 10° per second. Control-surface position limiting will be neglected in this study.

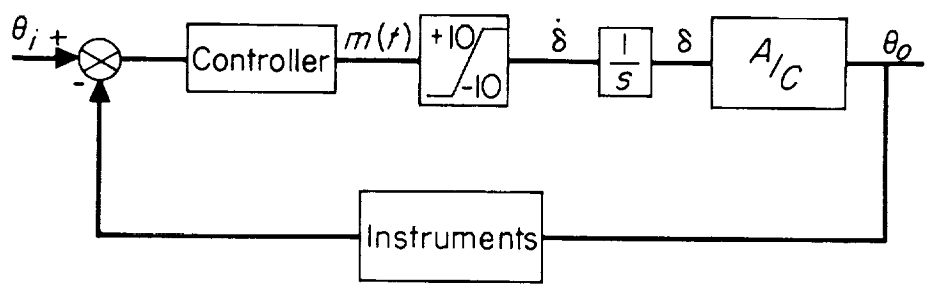
Two transfer functions relating control-surface rate to pitch attitude are considered:

$$\frac{\theta}{\dot{\delta}_e} = \frac{1.5625(s + 1)}{s^2\left[\left(\frac{s}{4}\right)^2 + 2\left(\frac{0.1875}{4}\right) s + 1\right]} \tag{74}$$

$$\frac{\theta}{\dot{\delta}_e} = \frac{3.125(s + 1)}{s^2(0.183\ s + 1)(0.683\ s - 1)} \tag{75}$$

Equation (74) is the approximate linear relationship for the F-86 in normal configuration flying at $M = 0.8$, $h_\rho = 30{,}000$ feet. Equation (75) is the approximate linear relationship for the F-86 with the c.g. shifted aft to give a statically unstable airframe. The flight condition is the same for both examples.

With the assumptions made, the problem reduces to one of designing the controller and instrumentation shown in the block diagram of sketch (ad).



Sketch (ad)

From considerations given in the previous section it is known that the desired instrument transfer function is unity and for very small errors, the linearized controller transfer function should be of the form

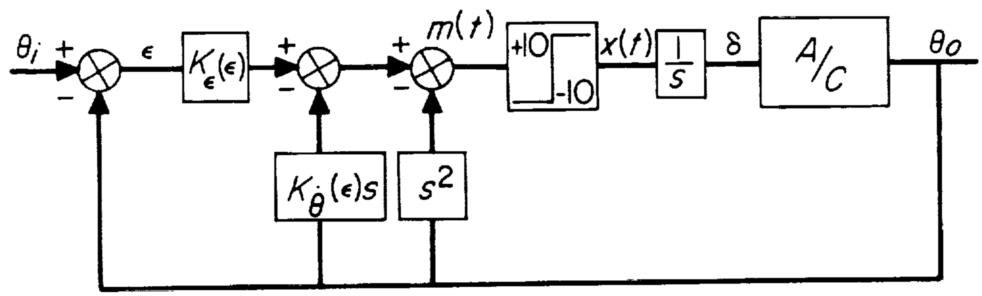
$$m(t) = K(1 + as + bs^2)\epsilon \tag{76}$$

The open loop transfer function system given by sketch (ad) contains three zeros and four poles. Two of the zeros are a result of the controller; the other is from the aircraft transfer function. So long as the zeros are in the left half plane, in theory infinite gain is possible and the closed-loop transfer function θ_o/θ_i will be unity.

The problems associated with differentiation make it impossible to obtain the desired controller operations of equation (76). In general, one must add at least one pole for each differentiation operation. Furthermore, instruments must have dynamics associated with their measurements. The net result is that one cannot in practice obtain an open-loop transfer function with three zeros and only four poles. As a result, the high gain saturating control system will in practice have a limit cycle or chatter. It has been shown (ref. 10) that if this limit cycle is very high in frequency compared with the plant dynamics, one can closely approximate the infinite gain system.

For the unstable dynamics given by equation (75) it was shown in a previous example (for a simpler case) that if one uses error rate, the input is able to drive the system into instability. As a result of this system being higher order, we must neither use error rate nor error acceleration as stabilizing quantities. For our purposes we will use output rate and output acceleration $(\dot{\theta}_o, \ddot{\theta}_o)$ for both aircraft transfer functions (eqs. (74), (75)). The result will be near optimum step response and a finite error to all inputs which are nonconstant.

These assumptions reduce the problem to the design of the system of sketch (ae) for the two plants of equations (74) and (75).



Sketch (ae)

The characteristic equations for the two systems are (under the assumption that the relay may be replaced by an equivalent gain, K):

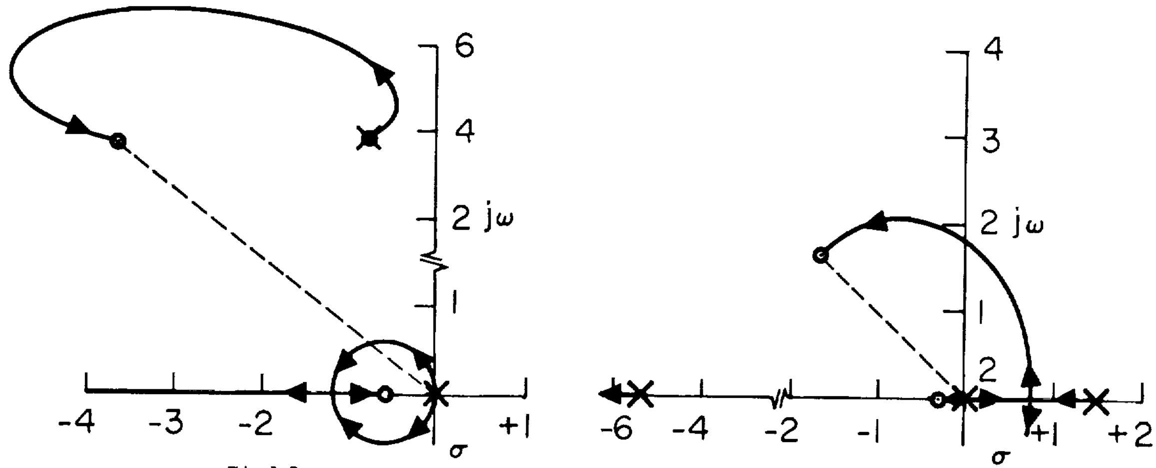
Stable airframe

$$1 + K[s^2 + K_{\dot{\theta}}(\epsilon)s + K_\epsilon(\epsilon)]\overbrace{\left\{\frac{1.5625(s+1)}{s^2\left[\left(\frac{s}{4}\right)^2 + 2\left(\frac{0.1875}{4}\right)s + 1\right]}\right\}}^{G(s)} = 0 \tag{77}$$

Unstable airframe

$$1 + K[s^2 + K_{\dot{\theta}}(\epsilon)s + K_\epsilon(\epsilon)]\overbrace{\left[\frac{3.125(s+1)}{s^2(0.183\ s + 1)(0.683\ s - 1)}\right]}^{G(s)} = 0 \tag{78}$$

The root locus as a function of equivalent limiter gain could be as shown in sketch (af) (for one value of $K_{\dot{\theta}}(\epsilon)$ and $K_\epsilon(\epsilon)$).



Stable

Unstable

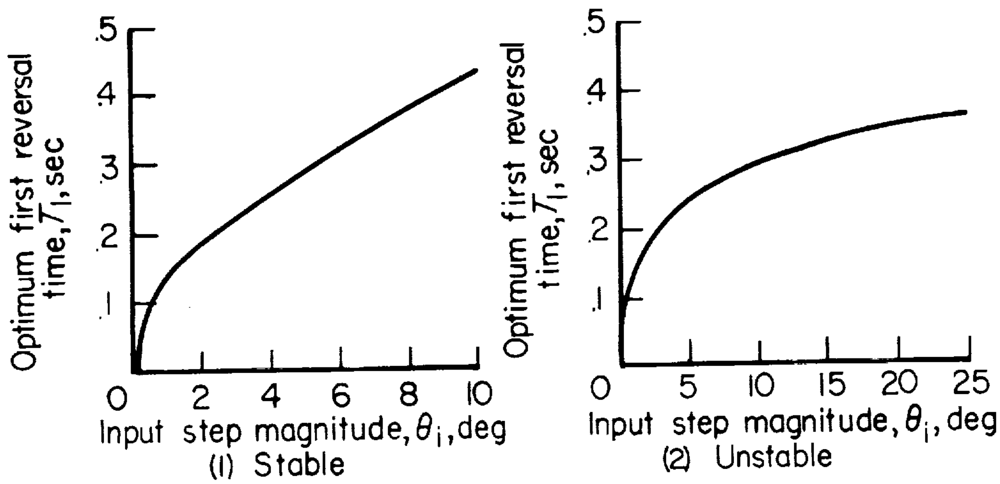
Sketch (af)

As can be seen from sketch (af), the stable system will be lightly damped and the unstable system will diverge for low values of equivalent limiter gain (or large step inputs). To have a good response for large inputs and a very fast system for small inputs the following two conditions will be imposed on the gains $K_\epsilon(\epsilon)$ and $K_{\dot{\theta}}(\epsilon)$.

1. They shall be such that the first reversal time for step inputs is the optimum.

2. They shall only be allowed to vary in a manner such that the damping ratio of the zeros determined by these quantities is a constant value of 0.7.

These two conditions are identical to those used for the third example of reference 1. To impose condition (1) it is necessary to have the optimum first reversal time as a function of the input step magnitude for the two example G(s) transfer functions. This information is available in normalized form in table IV for the stable airframe and in table V for the unstable airframe. Since tabulated data are not directly obtainable for the particular dynamics given by equations (74) and (75), one must interpolate. Nonlinear interpolation was used for these examples to increase the accuracy. The resultant optimum first reversal time curves are plotted in sketch (ag).



Sketch (ag)

To insure that the first reversal time of the system be optimum, it is necessary that $m(t)$, the input to the ideal relay, reverse sign at the optimum first reversal time given in sketch (ag) above. The equation for $m(t)$, the input to the ideal relay, derived from sketch (ae) is

$$m(t) = K_\epsilon(\epsilon)\epsilon + K_{\dot\theta}(\epsilon)\dot\theta_o + \ddot\theta_o \tag{79}$$

For step inputs and for a time of T_1, the first reversal time,

$$m(T_1) = 0 = K_\epsilon(\epsilon)[\theta_i(T_1) - \theta_o(T_1)] + K_{\dot\theta}(\epsilon)\dot\theta_o(T_1) + \ddot\theta_o(T_1) \tag{80}$$

The quantity $\theta_i(T_1)$ is determined from sketch (ae) for the particular plant being studied. The quantities $\theta_o(T_1)$, $\dot\theta_o(T_1)$, $\ddot\theta_o(T_1)$ are the responses of the output, output velocity, and output acceleration, respectively, for a step input $x(t) = Bu(t)$ evaluated at $t = T_1$. Thus, to satisfy the first condition one simply finds an equation of form

$$K_\epsilon(\epsilon)[\epsilon(T_1)] + K_{\dot\theta}(\epsilon)\dot\theta_o(T_1) + \ddot\theta_o(T_1) = 0 \tag{81}$$

where $\epsilon(T_1)$, $\dot\theta_o(T_1)$, and $\ddot\theta_o(T_1)$ are constant for any particular value of T_1.

To satisfy the second condition one evaluates the damping ratios of the zeros determined by the quantities $K_\epsilon(\epsilon)$ and $K_{\dot\theta}(\epsilon)$; namely,

$$\zeta_o = \frac{1}{2}\frac{K_{\dot\theta}(\epsilon)}{\sqrt{K_\epsilon(\epsilon)}} \tag{82}$$

Setting $\zeta_o = 0.7$ in equation (82) one can solve for $K_{\dot\theta}(\epsilon)$ and substitute into equation (81) to obtain a solution for $K_\epsilon(\epsilon)$ for each T_1. Alternatively one can plot $K_{\dot\theta}(\epsilon)$ versus $K_\epsilon(\epsilon)$ from equation (82). Equation (81) is a straight line (for each T_1) on this same plot. The intersection of the two curves is the desired value of $K_\epsilon(\epsilon)$ and $K_{\dot\theta}(\epsilon)$ for the particular choice of T_1.

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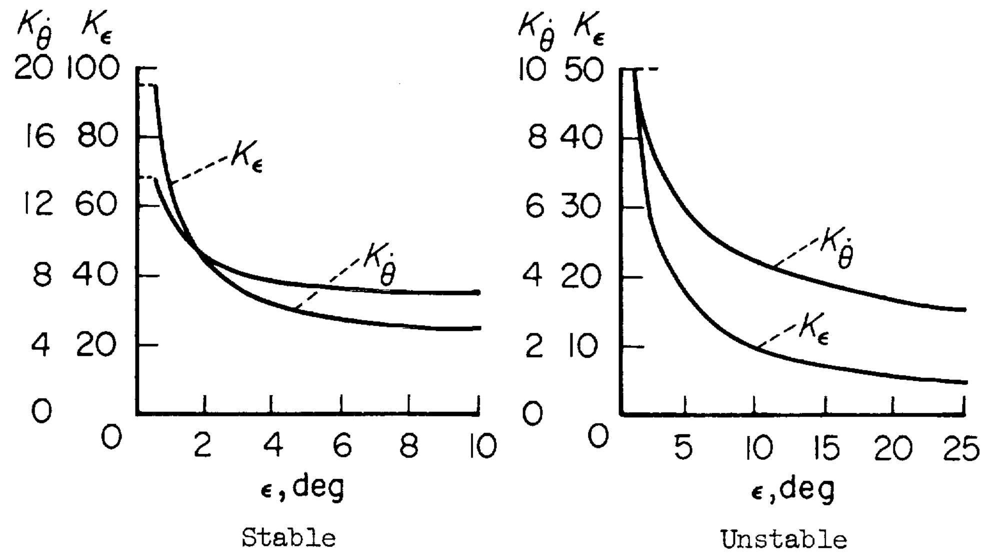
The functions of error were computed using this second alternative. The results are shown in sketch (ah).



Sketch (ah)

As can be seen, both $K_\epsilon(\epsilon)$ and $K_{\dot\theta}(\epsilon)$ approach infinity as $\epsilon \to 0$. This means the zeros (and two of the closed-loop poles for the infinite gain system) go to infinity along the dotted line of sketch (af). For our purposes the gains were arbitrarily made constant for errors less than $1/2^\circ$ for the stable case and less than 1° for the unstable case, as indicated by the dotted lines of sketch (ah). Thus, in the chatter mode of operation (see ref. 10), the system's responses with the stable airframe will be closely approximated by a second-order system with a natural frequency of $\sqrt{K_\epsilon(\epsilon)} = \sqrt{95} = 9.75$ radians per second, and a damping ratio of 0.7.

For the unstable airframe the system's response (in the chatter mode of operation) will be closely approximated by a second-order system of natural frequency $\sqrt{K_\epsilon(\epsilon)} = 6.86$ radians per second and damping ratio of 0.7. In the chatter mode of operation both these systems are very high gain representing very "tight" attitude autopilots. The system will therefore closely follow any normal inputs and should have good response to gusts. In addition, the response of the system to step inputs over the range of inputs considered (10° max in these cases) should be near optimum.

The two systems were simulated on an analog computer and the results of the simulation are presented in figures 7 and 8. Figure 7(a) shows the step responses of the system with stable aerodynamics. Figure 7(b) shows the control-surface deflection belonging to the step responses.

As can be seen from the control-surface deflection, the end of the response has a long tailing off exponential. The motion is a result of the fact that the system is an essentially infinite gain and one closed-loop pole of the system has a time constant of 1 second. As a result of the 1 second time constant zero in the aircraft transfer function, however, this motion does not occur on the output. Aerodynamically speaking, one could say that the pitch attitude is held constant while angle of attack and flight-path angle are slowly changing to take on new trim conditions.

The step responses are near optimum as can be seen by the arrows on the response representing the minimum response time for the particular input.

Figure 8(a) shows the step responses of the attitude control system for the unstable airframe. In this case also the response time is near the minimum. Figure 8(b) shows the control-surface motion for the various steps of figure 8(a). A comparison of figure 7(b) with 8(b) shows a significant difference in the character of the response. The difference is the first reversal time. As was mentioned earlier and shown in reference 1, the first reversal time for stable systems continues to increase with the input step magnitude. As the input goes to infinity, $T_m - T_1$ approaches a constant. For unstable systems, however, T_1 approaches a constant as $T_m \to \infty$. Since we have a near optimum system for both these examples, this fact is quite readily recognized from figures 7(b) and 8(b).

Figure 9 shows the maximum error as a function of frequency for the two systems. The input for both cases was 10 sin ωt. As was hypothesized earlier, there is practically no error for low-frequency inputs. The error does increase with frequency, however, as a result of the use of output derivative feedbacks instead of error derivatives (for the lower frequencies) and a combination of this and limiting at the higher frequencies.

CONCLUSIONS

It has been shown that the switch-time method for synthesis of nonlinear controllers to compensate for the effects of saturation on the input to the plant has very general application. It appears that only two restrictions must be imposed on the nonlinear controller design in order that one obtain a system which has a near optimum step response and well-damped response to other inputs. These restrictions are:

(1) The first reversal time of the bounded variable shall be the optimum first reversal time of the optimum bang-bang control system.

(2) The zeros (in the open loop transfer function) introduced by the controller should change (as a function of error) in such a manner that as the limiter equivalent gain is decreased, the poles of the closed-loop system remain in well-damped regions of the s plane.

To meet restriction (1) it is necessary to have data on the optimum first reversal time as a function of input step magnitude. This report has presented such data for several additional plant transfer functions over those considered in reference 1. In addition, an approximate method for determining such data has been presented. The approximate method has been shown to be reasonably accurate for plants of at least fourth order.

To meet restriction (2) it has been found that if only one zero is introduced by the controller, then as the error increases, this zero should move toward the origin on the negative real axis. For controllers with two zeros, it has been found that if the zeros move toward the origin along a line of constant damping ratio (as the error increases) good characteristics are obtained. Systems with three or more zeros introduced by the controller have not been studied.

It has been shown that if the plant has a zero in the left half plane, the system's response can be made faster than if the zero is neglected. This solution requires some of the plant variables to change as the error and error derivatives remain constant; however, examples have shown that it is a practical solution.

The effects of an unstable pole are to force the controller to keep certain output derivatives from exeeding certain limits; otherwise, control over the plant is lost and the system response diverges. It is interesting to note that if the plant has stable poles, some of the output derivatives are bounded because the plant input is bounded. Thus there are not significant differences between stable and unstable plants in this respect.

With regard to optimum continuous systems it has been shown that with certain restrictions they are derivable from the optimum relay solution. Optimum sampled-data systems, however, need considerably more research before one can derive the optimum controller except for extremely simple plants.

One can generally conclude that saturation in single variable control systems is one type of nonlinearity that lends itself to analytical treatment such that its effects in any particular system can be well understood. Furthermore, it can be concluded that if the effects are undesirable, such as to make the system unstable or oscillatory for large inputs, the switch time method can be used to compute compensating nonlinear functions in the controller which remove these undesirable characteristics.

Ames Research Center
National Aeronautics and Space Administration
Moffett Field, Calif., April 8, 1960

REFERENCES

1. Schmidt, Stanley Francis: The Analysis and Design of Continuous and Sampled-Data Feedback Control Systems With a Saturation Type Nonlinearity. NASA TN D-20, 1959.

2. Truxal, John G.: Automatic Feedback Control System Synthesis. McGraw-Hill Book Co., Inc., 1955, p. 573.

3. Bellman, R., Glicksberg, I., and Gross, O.: On the "Bang-Bang" Control Problem. Quart. Jour. Appl. Math., vol. XIV, no. 1, Apr. 1956, pp. 11-18.

4. Schmidt, Stanley F., and Triplett, William C.: Use of Nonlinearities to Compensate for the Effects of a Rate-Limited Servo on the Response of an Automatically Controlled Aircraft. NACA TN 3387, 1955.

5. Newton, George C., Gould, Leonard A., and Kaiser, James F.: Analytical Design of Linear Feedback Controls. John Wiley & Sons Inc., New York, 1957.

6. Hopkin, A. M., and Wang, P. K. C.: A Relay-Type Feedback Control System for Random Inputs. AIEE Paper 59-218, 1959.

7. Ragazzini, John R., and Franklin, Gene F.: Sampled-Data Control Systems. McGraw-Hill Book Co., Inc., 1958.

8. Schmidt, Stanley F.: Application of Continuous System Design Concepts to the Design of Sampled-Data Systems. AIEE Paper 58-1083, 1958.

9. Schmidt, Stanley F., and Harper, Eleanor V.: Sampled-Data Techniques Applied to a Digital Controller for an Altitude Autopilot. NASA MEMO 4-14-59A, 1959.

10. McLean, John D., and Schmidt, Stanley F.: On the Design of a High-Gain Saturating Control System for use as an Adaptive Autopilot. NASA TN D-305, 1960.

11. Kalman, R. E.: Analysis and Design Principles of Second and Higher Order Saturating Servomechanism. AIEE Paper 55-551 (also available in Trans. AIEE, vol. 74, pt. 2, no. 21, Nov. 1955, pp. 294-308).

12. Yin, Mih: On the Optimum Response of Third-Order Contactor Control Systems. PhD Thesis, Stanford University, Stanford, California, June 1960.

13. Preston, John L.: Nonlinear Control of a Saturating Third-Order Servomechanism. MIT, Servomechanisms Lab., DIC 6897, Tech. Memo. 14, 1954.

14. Doll, H. G., and Stout, T. M.: Design and Analog Computer Analysis of an Optimum Third-Order Nonlinear Servomechanism. ASME Paper 56-IRD-10, 1956.

TABLE I.- TRANSFER FUNCTIONS OF PLANTS CONSIDERED

Case	G(s)	Type
1	$\frac{K}{s}$	1 - first order
2	$\frac{K(\tau_1 s + 1)}{s(\tau_2 s + 1)}$	1 - second order
3	$\frac{K}{s[(s^2/\omega_n^2) + (2\zeta s/\omega_n) + 1]}$	1 - third order
4	$\frac{K}{s^2}$	2 - second order
5	$\frac{K(\tau_1 s + 1)}{s^2(\tau_2 s + 1)}$	2 - third order
6	$\frac{K}{s^2[(s^2/\omega_n^2) + (2\zeta s/\omega_n) + 1]}$	2 - fourth order
7	$\frac{K}{s^3}$	3 - third order
8	$\frac{K}{s(\tau s - 1)}$	1 - second order, unstable
9	$\frac{K(\tau_1 s + 1)}{s[(s^2/\omega_n^2) + (2\zeta s/\omega_n) + 1]}$	1 - third order (with zero)
10	$\frac{K(\tau_1 s + 1)}{s(\tau_2 s + 1)(\tau_3 s - 1)}$	1 - third order (with zero) unstable
11	$\frac{K(\tau_1 s + 1)}{s^2[(s^2/\omega_n^2) + (2\zeta s/\omega_n) + 1]}$	2 - fourth order (with zero)
12	$\frac{K(\tau_1 s + 1)}{s^2(\tau_2 s + 1)(\tau_3 s - 1)}$	2 - fourth order (with zero) unstable

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TABLE II.- NORMALIZED OPTIMUM RESPONSES FOR A TYPE 1 THIRD-ORDER PLANT WITH ZERO

$\zeta = 0.10;\ \omega_n\tau_1 = 0$

$\omega_n T_1$	$\omega_n T_2$	$\omega_n T_m$	b/B	$R_o\omega_n/BK$
0.5	1.2852	1.7423		0.1719
1.0	2.0402	2.8951		.8148
1.5	2.5065	3.7169		1.7038
2.0	2.8570	4.3717		2.6577
2.5	3.1743	4.9161		3.5674
3.0	3.5069	5.3616		4.3479
3.5	3.8828	5.7119		4.9463
4.0	4.3080	5.9868		5.3708
5.0	5.2790	6.4610		5.9029
6.0	6.4358	7.1970		6.3254
7.0	7.7642	8.6216		7.0932
8.0	8.8011	10.0313		8.4292
9.0	9.6275	11.1206		9.8656
10.0	10.4707	11.9576		11.0161
11.0	11.4190	12.6723		11.8343
12.0	12.4878	13.4908		12.5151
13.0	13.6423	14.6078		13.3232
14.0	14.7141	15.8501		14.4219
15.0	15.6500	16.9637		15.6636
16.0	16.5515	17.9120		16.8091
17.0	17.4994	18.7624		17.7637
18.0	18.5203	19.6399		18.5994
19.0	19.5952	20.6550		19.4646
20.0	20.6516	21.7745		20.4713

$\zeta = 0.10;\ \omega_n\tau_1 = 0.5$

$\omega_n T_1$	$\omega_n T_2$	$\omega_n T_m$	b/B	$R_o\omega_n/BK$
.1		.2161	0.4401 -1	.5925 -2
.2		.4537	.1621	.2732 -1
.3		.6958	.3289	.6860 -1
.4		.9298	.5233	.1319
.5		1.1489	.7321	.2171
.6		1.3511	.9470	.3224
.7	1.5335	1.6125		.4455
.8	1.6929	1.8698		.5840
.9	1.8337	2.1033		.7359
1.0	1.9594	2.3178		.8990
1.5	2.4397	3.2012		1.8217
2.0	2.7921	3.8955		2.8112
2.5	3.1063	4.4687		3.7560
3.0	3.4356	4.9298		4.5585
3.5	3.8124	5.2764		5.1516
4.0	4.2413	5.5312		5.5487
5.0	5.2116	5.9412		6.0179
6.0	6.3475	6.5963		6.4012
7.0	7.6836	8.0448		7.1776
8.0	8.7346	9.5183		8.5491
9.0	9.5627	10.6417		10.0163
10.0	10.4060	11.4779		11.1660
11.0	11.3529	12.1625		11.9567
12.0	12.4150	12.9427		12.6128
13.0	13.5678	14.0530		13.4173
14.0	14.6456	15.3235		14.5323
15.0	15.5847	16.4620		15.7926
16.0	16.4865	17.4164		16.9434
17.0	17.4334	18.2539		17.8872
18.0	18.4514	19.1109		18.7081
19.0	19.5245	20.1165		19.5676
20.0	20.5828	21.2459		20.5804
21.0	21.5720	22.3528		21.7088
22.0	22.5202	23.3598		22.8194
23.0	23.4794	24.2835		23.8247
24.0	24.4774	25.1955		24.7407
25.0	25.5108	26.1725		25.6508
26.0	26.5476	27.2261		26.6309
27.0	27.5538	28.2938		27.6863
28.0	28.5305	29.3165		28.7555
29.0	29.5039	30.2861		29.7782
30.0	30.4956	31.2358		30.7446

$\zeta = 0.10;\ \omega_n\tau_1 = 1.0$

$\omega_n T_1$	$\omega_n T_2$	$\omega_n T_m$	b/B	$R_o\omega_n/BK$
.1		.2067	.1746 -1	.1075 -1
.2		.4211	.6658 -1	.4551 -1
.3		.6344	.1409	.1065
.4		.8399	.2337	.1939
.5		1.0335	.3392	.3058
.6		1.2135	.4529	.4394
.7		1.3801	.5715	.5914
.8		1.5340	.6926	.7586
.9		1.6766	.8145	.9379
1.0		1.8093	.9358	1.1265
1.5	2.3196	2.7934		2.1542
2.0	2.6722	3.5907		3.2463
2.5	2.9728	4.2482		4.3026
3.0	3.2879	4.7633		5.1876
3.5	3.6685	5.0964		5.7594
4.0	4.1131	5.2852		6.0591
5.0	5.0905	5.5231		6.3420
6.0		6.1895	.7867	6.5971
7.0		7.5338	.9398	7.4059
8.0	8.6151	9.1176		8.8875
9.0	9.4434	10.3296		10.4429
10.0	10.2868	11.1636		11.5901
11.0	11.2339	11.7699		12.3021
12.0	12.2828	12.4510		12.8853
13.0	13.4320	13.5429		13.6790
14.0	14.5226	14.8881		14.8430
15.0	15.4667	16.0901		16.1567
16.0	16.3687	17.0602		17.3228
17.0	17.3146	17.8648		18.2355
18.0	18.3275	18.6692		19.0142
19.0	19.3969	19.6503		19.8565
20.0	20.4591	20.8055		20.8874
21.0	21.4524	21.9513		22.0464
22.0	22.4017	22.9765		23.1730
23.0	23.3603	23.8892		24.1686
24.0	24.3559	24.7737		25.0619
25.0	25.3870	25.7315		25.9574
26.0	26.4246	26.7910		26.9419
27.0	27.4330	27.8792		28.0131
28.0	28.4110	28.9166		29.0946
29.0	29.3844	29.8849		30.1162
30.0	30.3749	30.8213		31.0715

$\zeta = 0.10;\ \omega_n\tau_1 = 2.0$

$\omega_n T_1$	$\omega_n T_2$	$\omega_n T_m$	b/B	$R_o\omega_n/BK$
.1		.2019	0.1120 -1	.2048 -1
.2		.4034	.4311 -1	.8283 -1
.3		.5987	.9247 -1	.1862
.4		.7840	.1557	.3273
.5		.9573	.2294	.5014
.6		1.1182	.3106	.7030
.7		1.2672	.3970	.9269
.8		1.4054	.4868	1.1682
.9		1.5340	.5783	1.4226
1.0		1.6543	.6705	1.6867
1.5	2.1611	2.3814		3.0592
2.0	2.4953	3.4878		4.4973

$\zeta = 0.10;\ \omega_n\tau_1 = 4.0$

$\omega_n T_1$	$\omega_n T_2$	$\omega_n T_m$	b/B	$R_o\omega_n/BK$
.1		.1995	.9868 -2	.3997 -1
.2		.3944	.3808 -1	.1580
.3		.5801	.8198 -1	.3478
.4		.7543	.1386	.6001
.5		.9160	.2051	.9044
.6		1.0654	.2789	1.2503
.7		1.2036	.3580	1.6283
.8		1.3318	.4405	2.0303
.9		1.4513	.5251	2.4492
1.0		1.5635	.6107	2.8792
1.5	2.0460	2.1422		5.0502

$\zeta = 0.10;\ \omega_n\tau_1 = 8.0$

$\omega_n T_1$	$\omega_n T_2$	$\omega_n T_m$	b/B	$R_o\omega_n/BK$
.1		.1983	.9657 -2	.7896 -1
.2		.3898	.3729 -1	.3085
.3		.5708	.8035 -1	.6720
.4		.7392	.1360	1.1487
.5		.8949	.2015	1.7168
.6		1.0383	.2743	2.3559
.7		1.1708	.3524	3.0483
.8		1.2938	.4341	3.7788
.9		1.4085	.5179	4.5349
1.0		1.5163	.6028	5.3060
1.5	1.9831	2.0946		9.1284

$\zeta = 0.10;\ \omega_n\tau_1 = 16.0$

$\omega_n T_1$	$\omega_n T_2$	$\omega_n T_m$	b/B	$R_o\omega_n/BK$
.1		.1977	.9665 -2	.1569
.2		.3876	.3733 -1	.6097
.3		.5661	.8045 -1	1.3211
.4		.7317	.1362	2.2474
.5		.8843	.2018	3.3447
.6		1.0247	.2748	4.5728
.7		1.1544	.3532	5.8971
.8		1.2746	.4352	7.2885
.9		1.3870	.5194	8.7231
1.0		1.4925	.6046	10.1811

$\zeta = 0.25;\ \omega_n\tau_1 = 0$

$\omega_n T_1$	$\omega_n T_2$	$\omega_n T_m$	b/B	$R_o\omega_n/BK$
.5	1.2567	1.6609		.1475
1.0	2.0301	2.7264		.6662
1.5	2.5573	3.4842		1.3695
2.0	2.9834	4.0876		2.1208
2.5	3.3754	4.6002		2.8495
3.0	3.7672	5.0517		3.5174
3.5	4.1769	5.4612		4.1074
4.0	4.6129	5.8462		4.6203
5.0	5.5733	6.6234		5.4768
6.0	6.6414	7.5465		6.2636
7.0	7.7454	8.6484		7.1577
8.0	8.7842	9.7692		8.2007
9.0	9.7561	10.8091		9.2970
10.0	10.7126	11.7792		10.3540
11.0	11.6898	12.7270		11.3473
12.0	12.6954	13.6971		12.3062
13.0	13.7153	14.7038		13.2732
14.0	14.7294	15.7280		14.2691
15.0	15.7295	16.7447		15.2858
16.0	16.7212	17.7450		16.3026
17.0	17.7139	18.7353		17.3074
18.0	18.7124	19.7264		18.3015
19.0	19.7155	20.7244		19.2933
20.0	20.7192	21.7281		20.2897

$\zeta = 0.25;\ \omega_n\tau_1 = 0.5$

$\omega_n T_1$	$\omega_n T_2$	$\omega_n T_m$	b/B	$R_o\omega_n/BK$
.1		.2129	0.3721 -1	.5741 -2
.2		.4413	.1339	.2564 -1
.3		.6707	.2665	.6253 -1
.4		.8913	.4173	.1174
.5		1.0983	.5755	.1894
.6		1.2905	.7353	.2772
.7		1.4682	.8933	.3785
.8	1.6324	1.6560		.4912
.9	1.7812	1.8761		.6136
1.0	1.9166	2.0773		.7441
1.5	2.4625	2.8971		1.4721
2.0	2.8950	3.5335		2.2434
2.5	3.2889	4.0648		2.9870
3.0	3.6811	4.5250		3.6628
3.5	4.0908	4.9345		4.2528
4.0	4.5266	5.3121		4.7589
4.5	4.9908	5.6792		5.1976
5.0	5.4834	6.0600		5.5931
6.0	6.5455	6.9548		6.3638
7.0	7.6493	8.0562		7.2576
8.0	8.6921	9.1939		8.3097
10.0	10.6233	11.2187		10.4721
12.0	12.6040	13.1249		12.4170
14.0	14.6378	15.1553		14.3797
16.0	16.6305	17.1769		16.4159
18.0	18.6214	19.1565		18.4137
20.0	20.6280	21.1573		20.4013
22.0	22.6285	23.1645		22.4074
24.0	24.6259	25.1611		24.4093
26.0	26.6268	27.1599		26.4062
28.0	28.6274	29.1617		28.4069
30.0	30.6268	31.1614		30.4077

$\zeta = 0.25;\ \omega_n\tau_1 = 1.0$

$\omega_n T_1$	$\omega_n T_2$	$\omega_n T_m$	b/B	$R_o\omega_n/BK$
.1		.2037	.1415 -1	.1043 -1
.2		.4097	.5262 -1	.4290 -1
.3		.6113	.1091	.9781 -1
.4		.8039	.1780	.1740
.5		.9853	.2548	.2695
.6		1.1547	.3362	.3815
.7		1.3127	.4201	.5074
.8		1.4601	.5049	.6448
.9		1.5981	.5893	.7912
1.0		1.7278	.6727	.9449
1.5	2.2855	2.3408		1.7697
2.0	2.7299	3.0687		2.6090
2.5	3.1254	3.6515		3.4008
3.0	3.5168	4.1355		4.1019
3.5	3.9265	4.5449		4.6919
4.0	4.3630	4.9021		5.1762
4.5	4.8268	5.2338		5.5803
5.0	5.3160	5.5700		5.9379
6.0	6.3660	6.3848		6.6528
7.0	7.4696	7.4850		7.5459
8.0	8.5205	8.6707		8.6297
10.0	10.4566	10.7366		10.8233
12.0	12.4336	12.6106		12.7434
14.0	14.4672	14.6394		14.7050
16.0	16.4616	16.6740		16.7507
18.0	18.4519	18.6485		18.7447
20.0	20.4582	20.6468		20.7304
22.0	22.4591	22.6570		22.7388
24.0	24.4564	24.6532		24.7404
26.0	26.4572	26.6511		26.7367
28.0	28.4578	28.6534		28.7377
30.0	30.4573	30.6531		30.7386

$\zeta = 0.25;\ \omega_n\tau_1 = 2.0$

$\omega_n T_1$	$\omega_n T_2$	$\omega_n T_m$	b/B	$R_o\omega_n/BK$
.1		.1990	.9464 -2	.1989 -1
.2		.3928	.3553 -1	.7827 -1
.3		.5774	.7461 -1	.1718
.4		.7510	.1234	.2958
.5		.9130	.1792	.4455
.6		1.0638	.2398	.6159
.7		1.2044	.3036	.8027

Note: A group of digits followed by $^{-n}$ indicates that the decimal point should be n places to the left of the first digit.

TABLE II.- NORMALIZED OPTIMUM RESPONSES FOR A TYPE 1 THIRD-ORDER PLANT WITH ZERO - Continued

$\omega_n T_1$	$\omega_n T_2$	$\omega_n T_m$	b/B	$R_o\omega_n/BK$
$\zeta = 0.25;\ \omega_n\tau_1 = 2.0$				
0.8		1.3359	0.3691	1.0023
.9		1.4594	.4354	1.2115
1.0		1.5761	.5018	1.4276
1.5		2.0868	.8157	2.5446
2.0	2.5260	2.6653		3.6133
2.5	2.9136	3.3940		4.5667
3.0	3.2958	3.9519		5.3602
3.5	3.7056	4.3609		5.9498
4.0	4.1502	4.6548		6.3544
4.5	4.6216	4.8841		6.6410
5.0		5.1115	.9924	6.8732
6.0		6.1705	.7848	7.3992
7.0		7.2745	.7820	8.2896
8.0		8.3166	.8986	9.4807
10.0	10.2526	10.2848		11.7797
12.0		12.2291	.9226	13.6161
14.0		14.2627	.9183	15.5738
16.0		16.2568	.9545	17.6521
18.0		18.2471	.9402	19.6334
20.0		20.2535	.9330	21.6126
22.0		22.2543	.9414	23.6285
24.0		24.2516	.9404	25.6292
26.0		26.2524	.9377	27.6230
28.0		28.2531	.9393	29.6255
30.0		30.2525	.9396	31.6267
$\zeta = 0.25;\ \omega_n\tau_1 = 4.0$				
.1		.1967	.8880 -2	.3882 -1
.2		.3842	.3341 -1	.1495
.3		.5600	.7041 -1	.3217
.4		.7233	.1169	.5444
.5		.8745	.1705	.8074
.6		1.0147	.2290	1.1015
.7		1.1450	.2910	1.4189
.8		1.2670	.3551	1.7533
.9		1.3817	.4203	2.0994
1.0		1.4903	.4858	2.4528
1.5		1.9708	.7986	4.2237
2.0	2.3914	2.6112		5.8283
2.5	2.7307	3.7787		7.3172
$\zeta = 0.25;\ \omega_n\tau_1 = 8.0$				
.1		.1955	.9030 -2	.7672 -1
.2		.3799	.3400 -1	.2921
.3		.5512	.7170 -1	.6224
.4		.7093	.1192	1.0442
.5		.8551	.1739	1.5365
.6		.9898	.2339	2.0817
.7		1.1149	.2975	2.6651
.8		1.2320	.3634	3.2749
.9		1.3422	.4305	3.9014
1.0		1.4467	.4979	4.5369
1.5		1.9116	.8210	7.6565
2.0	2.3079	3.0552		10.4393
$\zeta = 0.25;\ \omega_n\tau_1 = 16.0$				
.1		.1949	.9215 -2	.1525
.2		.3777	.3470 -1	.5775
.3		.5468	.7320 -1	1.2244
.4		.7023	.1217	2.0450
.5		.8454	.1777	2.9974
.6		.9774	.2390	4.0467
.7		1.0999	.3040	5.1645
.8		1.2145	.3714	6.3280
.9		1.3225	.4401	7.5189
1.0		1.4249	.5092	8.7221
1.5		1.8821	.8402	14.5604
$\zeta = 0.50;\ \omega_n\tau_1 = 0$				
.5	1.2188	1.5561		.1186
1.0	2.0141	2.5362		.5080
1.5	2.6105	3.2559		1.0350
2.0	3.1183	3.8491		1.6126
2.5	3.5879	4.3745		2.1988
3.0	4.0446	4.8627		2.7734
3.5	4.5019	5.3316		3.3279
4.0	4.9666	5.7935		3.8604
5.0	5.9277	6.7274		4.8719
6.0	6.9255	7.6985		5.8476
7.0	7.9391	8.7015		6.8234
8.0	8.9507	9.7149		7.8135
9.0	9.9548	10.7241		8.8146
10.0	10.9537	11.7264		9.8191
11.0	11.9513	12.7249		10.8222
12.0	12.9499	13.7228		11.8230
13.0	13.9496	14.7217		12.8225
14.0	14.9500	15.7217		13.8218
15.0	15.9503	16.7220		14.8214
16.0	16.9505	17.7223		15.8214
17.0	17.9505	18.7224		16.8215
18.0	18.9504	19.7224		17.8216
19.0	19.9504	20.7223		18.8216
20.0	20.9503	21.7223		19.8216
$\zeta = 0.50;\ \omega_n\tau_1 = 0.5$				
.1		.2079	0.2668 -1	.5459 -2
.2		.4231	.9272 -1	.2325 -1
.3		.6354	.1797	.5447 -1
.4		.8387	.2754	.9896 -1
.5		1.0307	.3729	.1558
.6		1.2108	.4686	.2235
.7		1.3796	.5609	.3008
.8		1.5383	.6491	.3862
.9		1.6879	.7328	.4785
1.0		1.8295	.8121	.5765
1.5	2.4475	2.5179		1.1229
2.0	2.9663	3.1456		1.7130
2.5	3.4412	3.6896		2.3073
3.0	3.9003	4.1871		2.8866
4.0	4.8228	5.1205		3.9748
5.0	5.7820	6.0465		4.9824
6.0	6.7776	7.0093		5.9542
8.0	8.8020	9.0229		7.9188
10.0	10.8057	11.0371		9.9256
12.0	12.8020	13.0336		11.9295
14.0	14.8020	15.0321		13.9282
16.0	16.8025	17.0327		15.9278
18.0	18.8024	19.0329		17.9280
20.0	20.8024	21.0327		19.9280
22.0	22.8024	23.0327		21.9280
24.0	24.8024	25.0328		23.9280
26.0	26.8024	27.0328		25.9280
$\zeta = 0.50;\ \omega_n\tau_1 = 1.0$				
.1		.1991	.9011 -2	.9945 -2
.2		.3931	.3232 -1	.3917 -1
.3		.5790	.6513 -1	.8611 -1
.4		.7552	.1039	.1487
.5		.9213	.1460	.2247
.6		1.0778	.1898	.3121
.7		1.2254	.2342	.4088
.8		1.3652	.2785	.5133
.9		1.4979	.3220	.6241
1.0		1.6245	.3645	.7400
1.5		2.1906	.5548	1.3642
2.0		2.6896	.7015	2.0119
2.5		3.1580	.8024	2.6444
3.0		3.6155	.8606	3.2452
4.0		4.5378	.8772	4.3395
5.0		5.4980	.8264	5.3284
6.0		6.4956	.7775	6.2819
8.0		8.5209	.7615	8.2406
10.0		10.5238	.7770	10.2532
12.0		12.5200	.7773	12.2572
14.0		14.5201	.7751	14.2550
16.0		16.5206	.7753	16.2548
18.0		18.5205	.7756	18.2551
20.0		20.5205	.7755	20.2551
22.0		22.5205	.7755	22.2550
$\zeta = 0.50;\ \omega_n\tau_1 = 2.0$				
.1		.1946	.6781 -2	.1898 -1
.2		.3773	.2453 -1	.7175 -1
.3		.5477	.5000 -1	.1523
.4		.7065	.8078 -1	.2551
.5		.8546	.1151	.3756
.6		.9933	.1516	.5100
.7		1.1239	.1895	.6551
.8		1.2476	.2280	.8084
.9		1.3653	.2665	.9677
1.0		1.4781	.3048	1.1316
1.5		1.9898	.4830	1.9762
2.0		2.4542	.6263	2.7985
2.5		2.9018	.7275	3.5533
3.0		3.3482	.7866	4.2250
4.0		4.2675	.8036	5.3397
5.0		5.2372	.7518	6.2665
6.0		6.2443	.7024	7.1604
8.0		8.2729	.6862	9.0996
10.0		10.2726	.7018	11.1311
12.0		12.2688	.7021	13.1354
14.0		14.2693	.7000	15.1306
16.0		16.2697	.7002	17.1306
18.0		18.2696	.7004	19.1313
20.0		20.2696	.7004	21.1312
$\zeta = 0.50;\ \omega_n\tau_1 = 4.0$				
0.1		0.1923	0.7352 -2	0.3707 -1
.2		.3693	.2665 -1	.1373
.3		.5318	.5450 -1	.2861
.4		.6815	.8834 -1	.4719
.5		.8200	.1263	.6852
.6		.9492	.1670	.9188
.7		1.0705	.2094	1.1672
.8		1.1854	.2528	1.4258
.9		1.2948	.2965	1.6912
1.0		1.3997	.3401	1.9605
1.5		1.8806	.5451	3.2999
2.0		2.3251	.7122	4.5237
2.5		2.7607	.8310	5.5632
3.0		3.2007	.9006	6.4018
3.5		3.6522	.9271	7.0560
4.0		4.1183	.9207	7.5645
4.5		4.5993	.8941	7.9772
5.0		5.0934	.8596	8.3450
5.5		5.5971	.8268	8.7100
6.0		6.1060	.8014	9.0995
7.0		7.1259	.7788	9.9891
8.0		8.1365	.7825	10.9933
9.0		9.1375	.7935	12.0364
10.0		10.1344	.8008	13.0687
12.0		12.1306	.8011	15.0738
14.0		14.1313	.7986	17.0630
16.0		16.1317	.7988	19.0635
18.0		18.1316	.7991	21.0650
20.0		20.1316	.7991	23.0647
22.0		22.1316	.7990	25.0646
24.0		24.1316	.7990	27.0646
$\zeta = 0.50;\ \omega_n\tau_1 = 8.0$				
.1		.1912	.8061 -2	.7327 -1
.2		.3653	.2924 -1	.2686
.3		.5239	.5982 -1	.5547
.4		.6690	.9706 -1	.9074
.5		.8028	.1389	1.3083
.6		.9273	.1838	1.7433
.7		1.0440	.2307	2.2016
.8		1.1545	.2787	2.6751
.9		1.2599	.3271	3.1572
1.0		1.3610	.3755	3.6428
1.5		1.8269	.6036	6.0016
2.0		2.2620	.7900	8.0579
2.5		2.6920	.9227	9.6898
3.0	3.1291	3.1341		10.8759
3.5	3.5775	3.8195		11.6645
4.0	4.0447	4.2290		12.1397
4.5		4.5279	.9933	12.4188
5.0		5.0235	.9547	12.6144
5.5		5.5285	.9180	12.8153
6.0		6.0387	.8896	13.0783
6.5		6.5499	.8720	13.4261
7.0		7.0596	.8643	13.8551
8.0		8.0700	.8685	14.8778
9.0		9.0705	.8808	15.9759
10.0		10.0671	.8889	17.0445
12.0		12.0632	.8893	19.0513
14.0		14.0641	.8865	21.0278
16.0		16.0645	.8868	23.0297
18.0		18.0643	.8871	25.0326
20.0		20.0643	.8870	27.0320
22.0		22.0643	.8870	29.0317
$\zeta = 0.50;\ \omega_n\tau_1 = 16.0$				
.1		.1907	.8521 -2	.1457
.2		.3633	.3091 -1	.5312
.3		.5199	.6326 -1	1.0922
.4		.6628	.1026	1.7796
.5		.7943	.1469	2.5565
.6		.9164	.1945	3.3955
.7		1.0310	.2442	4.2757
.8		1.1393	.2950	5.1810
.9		1.2427	.3463	6.0989
1.0		1.3420	.3976	7.0196
1.5		1.8008	.6395	11.4320
2.0		2.2314	.8374	15.1676
2.5		2.6587	.9784	17.9954
$\zeta = 0.75;\ \omega_n\tau_1 = 0$				
.5	1.1893	1.4772		.9866 -1
1.0	2.0013	2.4113		.4086
1.5	2.6471	3.1267		.8325
2.0	3.2129	3.7364		1.3106
2.5	3.7382	4.2898		1.8134
3.0	4.2431	4.8121		2.3259
3.5	4.7385	5.3176		2.8405

TABLE II.- NORMALIZED OPTIMUM RESPONSES FOR A TYPE 1 THIRD-ORDER PLANT WITH ZERO - Continued

ω_nT_1	ω_nT_2	ω_nT_m	b/B	$R_o\omega_n/BK$
$\zeta = 0.75;\ \omega_n\tau_1 = 0$				
4.0	5.2305	5.8146		3.3536
5.0	6.2153	6.8013		4.3707
6.0	7.2066	7.7907		5.3774
7.0	8.2035	8.7856		6.3785
8.0	9.2032	9.7843		7.3779
9.0	10.2036	10.7844		8.3771
10.0	11.2040	11.7848		9.3767
11.0	12.2042	12.7850		10.3766
12.0	13.2043	13.7851		11.3766
13.0	14.2043	14.7852		12.3766
$\zeta = 0.75;\ \omega_n\tau_1 = 0.5$				
.1		.2033	0.1704 -1	.5204 -2
.2		.4075	.5746 -1	.2128 -1
.3		.6063	.1090	.4823 -1
.4		.7967	.1641	.8541 -1
.5		.9777	.2191	.1319
.6		1.1494	.2719	.1865
.7		1.3125	.3217	.2483
.8		1.4678	.3683	.3164
.9		1.6161	.4117	.3897
1.0		1.7582	.4519	.4678
1.5		2.3991	.6131	.9075
2.0		2.9658	.7226	1.3955
3.0		3.9996	.8415	2.4211
4.0		4.9888	.8819	3.4522
6.0		6.9649	.8817	5.4760
8.0		8.9611	.8738	7.4758
10.0		10.9619	.8729	9.4746
12.0		12.9622	.8732	11.4744
14.0		14.9622	.8733	13.4745
16.0		16.9621	.8733	15.4745
18.0		18.9621	.8733	17.4745
20.0		20.9621	.8733	19.4745
22.0		22.9621	.8733	21.4745
24.0		24.9622	.8733	23.4745
26.0		26.9622	.8733	25.4745
$\zeta = 0.75;\ \omega_n\tau_1 = 1.0$				
.1		.1948	.4315 -2	.9501 -2
.2		.3789	.1499 -1	.3606 -1
.3		.5526	.2950 -1	.7694 -1
.4		.7165	.4621 -1	.1298
.5		.8716	.6405 -1	.1925
.6		1.0189	.8235 -1	.2635
.7		1.1594	.1007	.3413
.8		1.2940	.1187	.4247
.9		1.4234	.1363	.5129
1.0		1.5483	.1533	.6050
1.5		2.1237	.2281	1.1044
2.0		2.6490	.2848	1.6358
3.0		3.6408	.3515	2.7107
4.0		4.6163	.3755	3.7591
6.0		6.5925	.3753	5.7828
8.0		8.5914	.3706	7.7792
10.0		10.5924	.3701	9.7776
12.0		12.5926	.3703	11.7776
14.0		14.5926	.3703	13.7777
16.0		16.5926	.3703	15.7777
$\zeta = 0.75;\ \omega_n\tau_1 = 2.0$				
.1		.1905	.4329 -2	.1815 -1
.2		.3641	.1516 -1	.6626 -1
.3		.5235	.3015 -1	.1368
.4		.6713	.4780 -1	.2243
.5		.8096	.6709 -1	.3246
.6		.9400	.8736 -1	.4348
.7		1.0639	.1081	.5524
.8		1.1824	.1290	.6757
.9		1.2965	.1499	.8033
1.0		1.4069	.1704	.9340
1.5		1.9218	.2651	1.6084
2.0		2.4048	.3416	2.2783
2.5		2.8777	.3981	2.9184
3.0		3.3505	.4362	3.5220
4.0		4.3104	.4713	4.6322
5.0		5.2917	.4759	5.6601
6.0		6.2866	.4711	6.6556
8.0		8.2885	.4641	8.6397
10.0		10.2900	.4634	10.6368
12.0		12.2900	.4636	12.6373
14.0		14.2900	.4637	14.6375
16.0		16.2899	.4637	16.6375
18.0		18.2900	.4637	18.6374
20.0		20.2900	.4637	20.6374
$\zeta = 0.75;\ \omega_n\tau_1 = 4.0$				
.1		.1884	.5957 -2	.3547 -1
.2		.3566	.2090 -1	.1270
.3		.5089	.4169 -1	.2579
.4		0.6486	0.6629 -1	0.4166
.5		.7783	.9334 -1	.5951
.6		.9000	.1219	.7877
.7		1.0154	.1514	.9902
.8		1.1257	.1812	1.1992
.9		1.2319	.2111	1.4125
1.0		1.3349	.2407	1.6280
1.5		1.8189	.3789	2.6968
2.0		2.2804	.4922	3.6883
2.5		2.7390	.5767	4.5677
3.0		3.2029	.6341	5.3336
3.5		3.6748	.6690	6.0014
4.0		4.1550	.6871	6.5934
5.0		5.1354	.6940	7.6408
6.0		6.1313	.6868	8.6160
8.0		8.1347	.6762	10.5703
10.0		10.1363	.6751	12.5641
12.0		12.1363	.6755	14.5657
14.0		14.1362	.6756	16.5661
16.0		16.1362	.6756	18.5661
18.0		18.1362	.6756	20.5661
20.0		20.1362	.6756	22.5661
22.0		22.1362	.6756	24.5661
24.0		24.1362	.6756	26.5661
26.0		26.1363	.6756	28.5660
28.0		28.1363	.6756	30.5660
$\zeta = 0.75;\ \omega_n\tau_1 = 8.0$				
.1		.1873	.7177 -2	.7014 -1
.2		.3529	.2519 -1	.2487
.3		.5016	.5028 -1	.5006
.4		.6373	.8001 -1	.8028
.5		.7629	.1128	1.1393
.6		.8804	.1474	1.4989
.7		.9918	.1832	1.8736
.8		1.0982	.2194	2.2574
.9		1.2007	.2558	2.6456
1.0		1.3002	.2919	3.0348
1.5		1.7700	.4606	4.9150
2.0		2.2220	.5994	6.5729
2.5		2.6745	.7031	7.9503
3.0		3.1346	.7737	9.0550
3.5		3.6044	.8166	9.9288
4.0		4.0835	.8389	10.6276
5.0		5.0634	.8474	11.7158
6.0		6.0598	.8385	12.6485
8.0		8.0639	.8255	14.5403
10.0		10.0655	.8241	16.5272
12.0		12.0655	.8246	18.5312
14.0		14.0654	.8247	20.5322
16.0		16.0654	.8247	22.5322
$\zeta = 0.75;\ \omega_n\tau_1 = 16.0$				
.1		.1867	.7888 -2	.1395
.2		.3510	.2769 -1	.4921
.3		.4980	.5528 -1	.9865
.4		.6318	.8799 -1	1.5760
.5		.7552	.1240	2.2291
.6		.8708	.1622	2.9239
.7		.9802	.2015	3.6444
.8		1.0847	.2415	4.3791
.9		1.1855	.2815	5.1190
1.0		1.2833	.3213	5.8573
1.5		1.7466	.5074	9.3714
2.0		2.1942	.6605	12.3731
2.5		2.6440	.7750	14.7556
3.0		3.1023	.8529	16.5445
3.5		3.5711	.9003	17.8344
4.0		4.0498	.9249	18.7488
4.5		4.5366	.9341	19.4095
5.0		5.0295	.9343	19.9195
5.5		5.5266	.9301	20.3552
6.0		6.0261	.9245	20.7664
6.5		6.5269	.9192	21.1811
7.0		7.0281	.9150	21.6123
7.5		7.5294	.9120	22.0632
8.0		8.0304	.9101	22.5318
9.0		9.0317	.9086	23.5062
10.0		10.0321	.9086	24.5050
11.0		11.0321	.9089	25.5100
12.0		12.0321	.9091	26.5139
13.0		13.0320	.9092	27.5157
14.0		14.0320	.9092	28.5160
16.0		16.0320	.9092	30.5158
18.0		18.0320	.9092	32.5158
$\zeta = 1.0013;\ \omega_n\tau_1 = 0$				
1.0488	2.0633	2.4008		.3718
2.0976	3.4017	3.8002		1.1920
3.1464	4.5584	4.9784		2.1545
4.1952	5.6501	6.0789		3.1691
5.2440	6.7162	7.1487		4.2042
6.2929	7.7720	8.2060		5.2476
7.3417	8.8236	9.2582		6.2943
8.3905	9.8735	10.3083		7.3423
9.4393	10.9227	11.3576		8.3908
10.4881	11.9717	12.4067		9.4394
11.5369	13.0206	13.4556		10.4881
12.5857	14.0694	14.5044		11.5370
13.6345	15.1182	15.5532		12.5858
14.6833	16.1670	16.6020		13.6346
15.7322	17.2158	17.6508		14.6834
16.7810	18.2647	18.6997		15.7322
17.8298	19.3135	19.7485		16.7810
$\zeta = 1.02;\ \omega_n\tau_1 = 0.5$				
.1		.1988	0.7502 -2	.4956 -2
.2		.3928	.2458 -1	.1949 -1
.3		.5800	.4569 -1	.4289 -1
.4		.7596	.6782 -1	.7430 -1
.5		.9319	.8942 -1	.1128
.6		1.0972	.1098	.1577
.7		1.2562	.1286	.2081
.8		1.4094	.1458	.2635
.9		1.5575	.1615	.3233
1.0		1.7010	.1758	.3869
1.5		2.3651	.2298	.7498
2.0		2.9682	.2638	1.1637
3.0		4.0779	.3002	2.0722
4.0		5.1254	.3163	3.0327
6.0		7.1558	.3270	5.0077
8.0		9.1618	.3292	7.0027
10.0		11.1630	.3296	9.0018
12.0		13.1632	.3297	11.0016
14.0		15.1633	.3297	13.0016
16.0		17.1633	.3297	15.0015
18.0		19.1633	.3297	17.0015
$\zeta = 1.02;\ \omega_n\tau_1 = 1.0$				
.1		.1906	-.3302 -3	.9066 -2
.2		.3657	-.1113 -2	.3322 -1
.3		.5288	-.2144 -2	.6901 -1
.4		.6827	-.3306 -2	.1140
.5		.8290	-.4527 -2	.1665
.6		.9691	-.5765 -2	.2252
.7		1.1041	-.6992 -2	.2889
.8		1.2348	-.8191 -2	.3570
.9		1.3619	-.9350 -2	.4288
1.0		1.4858	-.1046 -1	.5037
1.5		2.0715	-.1527 -1	.9132
2.0		2.6215	-.1887 -1	1.3597
3.0		3.6702	-.2335 -1	2.3064
4.0		4.6893	-.2557 -1	3.2851
6.0		6.7004	-.2713 -1	5.2724
8.0		8.7024	-.2746 -1	7.2701
10.0		10.7028	-.2753 -1	9.2697
12.0		12.7029	-.2754 -1	11.2696
14.0		14.7029	-.2755 -1	13.2695
16.0		16.7029	-.2755 -1	15.2695
$\zeta = 1.02;\ \omega_n\tau_1 = 2.0$				
.1		.1865	.1905 -2	.1734 -1
.2		.3517	.6466 -2	.6123 -1
.3		.5018	.1258 -1	.1233
.4		.6408	.1961 -1	.1984
.5		.7714	.2718 -1	.2830
.6		.8954	.3503 -1	.3747
.7		1.0143	.4300 -1	.4718
.8		1.1291	.5098 -1	.5729
.9		1.2406	.5887 -1	.6772
1.0		1.3495	.6663 -1	.7838
1.5		1.8686	.1022	1.3359
2.0		2.3669	.1313	1.8957
3.0		3.3492	.1707	2.9922
4.0		4.3343	.1917	4.0492
6.0		6.3212	.2071	6.0931
8.0		8.3180	.2104	8.1029
10.0		10.3173	.2111	10.1049
12.0		12.3172	.2113	12.1053
14.0		14.3172	.2113	14.1054
16.0		16.3172	.2113	16.1054
$\zeta = 1.02;\ \omega_n\tau_1 = 4.0$				
.1		.1844	.4579 -2	.3391 -1
.2		.3447	.1557 -1	.1176
.3		.4884	.3037 -1	.2331
.4		.6201	.4748 -1	.3698
.5		.7429	.6599 -1	.5210
.6		.8591	.8531 -1	.6821
.7		.9702	.1050	.8499

TABLE II.- NORMALIZED OPTIMUM RESPONSES FOR A TYPE 1 THIRD-ORDER PLANT WITH ZERO - Continued

$\omega_n T_1$	$\omega_n T_2$	$\omega_n T_m$	b/B	$R_0\omega_n/BK$
ζ = 1.02; $\omega_n\tau_1$ = 4.0				
0.8		1.0774	0.1249	1.0220
.9		1.1816	.1446	1.1968
1.0		1.2834	.1641	1.3730
1.5		1.7718	.2547	2.2469
2.0		2.2470	.3299	3.0727
3.0		3.2007	.4337	4.5340
4.0		4.1715	.4898	5.7878
6.0		6.1484	.5312	7.9763
8.0		8.1432	.5401	10.0174
10.0		10.1421	.5420	12.0258
12.0		12.1419	.5423	14.0274
14.0		14.1418	.5424	16.0277
16.0		16.1418	.5424	18.0278
18.0		18.1418	.5424	20.0278
ζ = 1.02; $\omega_n\tau_1$ = 8.0				
.1		.1834	.0635 -2	.6706 -1
.2		.3413	.2145 -1	.2304
.3		.4818	.4186 -1	.4532
.4		.6099	.6549 -1	.7140
.5		.7291	.9109 -1	.9996
.6		.8416	.1178	1.3011
.7		.9491	.1452	1.6123
.8		1.0528	.1727	1.9288
.9		1.1536	.2001	2.2474
1.0		1.2522	.2272	2.5657
1.5		1.7273	.3535	4.1010
2.0		2.1931	.4588	5.4777
2.5		2.6614	.5417	6.6725
3.0		3.1355	.6044	7.6997
3.5		3.6156	.6504	8.5875
4.0		4.1010	.6834	9.3662
5.0		5.0830	.7228	10.6998
6.0		6.0742	.7416	11.8587
8.0		8.0682	.7542	13.9655
10.0		10.0669	.7568	15.9872
12.0		12.0667	.7573	17.9914
14.0		14.0666	.7574	19.9923
16.0		16.0666	.7574	21.9925
18.0		18.0666	.7574	23.9925
20.0		20.0666	.7574	25.9925
22.0		22.0666	.7574	27.9925
24.0		24.0666	.7574	29.9925
26.0		26.0666	.7574	31.9923
28.0		28.0666	.7574	33.9923
ζ = 1.02; $\omega_n\tau_1$ = 16.0				
.1		.1829	.7264 -2	.1334
.2		.3396	.2472 -1	.4560
.3		.4785	.4825 -1	.8935
.4		.6049	.7550 -1	1.4031
.5		.7222	.1050	1.9580
.6		.8330	.1359	2.5410
.7		.9387	.1674	3.1400
.8		1.0408	.1992	3.7464
.9		1.1400	.2309	4.3540
1.0		1.2372	.2622	4.9580
1.5		1.7063	.4082	7.8243
2.0		2.1678	.5300	10.3114
2.5		2.6332	.6258	12.3804
3.0		3.1053	.6984	14.0687
3.5		3.5841	.7516	15.4417
4.0		4.0686	.7898	16.5686
5.0		5.0495	.8355	18.3185
6.0		6.0402	.8572	19.6754
7.0		7.0359	.8673	20.8403
8.0		8.0339	.8718	21.9152
9.0		9.0330	.8739	22.9489
10.0		10.0326	.8748	23.9638
11.0		11.0324	.8752	24.9705
12.0		12.0323	.8754	25.9735
13.0		13.0323	.8754	26.9746
14.0		14.0323	.8755	27.9755
15.0		15.0323	.8755	28.9755
ζ = 1.50; $\omega_n\tau_1$ = 0				
.5	1.1321	1.3279		.6379 -1
1.0	1.9812	2.2168		.2544
1.5	2.7239	2.9731		.5252
2.0	3.4041	3.6592		.8509
2.5	4.0428	4.3008		1.2152
3.0	4.6507	4.9103		1.6088
3.5	5.2371	5.4979		2.0237
4.0	5.8058	6.0672		2.4556
4.5	6.3611	6.6229		2.9008
5.0	6.9034	7.1656		3.3587
5.5	7.4398	7.7022		3.8225
6.0	7.9718	8.2342		4.2907
6.5	8.4967	8.7594		4.7660
7.0	9.0163	9.2791		5.2464
7.5	9.5329	9.7958		5.7300
8.0	10.0458	10.3089		6.2172
8.5	10.5565	10.8194		6.7064
9.0	11.0636	11.3266		7.1994
9.5	11.5708	11.8338		7.6922
10.0	12.0769	12.3399		8.1862
ζ = 1.50; $\omega_n\tau_1$ = 0.5				
.1		.1916	-0.7579 -2	.4570 -2
.2		.3711	-.2378 -1	.1697 -1
.3		.5427	-.4298 -1	.3581 -1
.4		.7086	-.6251 -1	.6019 -1
.5		.8700	-.8106 -1	.8944 -1
.6		1.0279	-.9806 -1	.1231
.7		1.1827	-.1133	.1607
.8		1.3347	-.1268	.2019
.9		1.4842	-.1386	.2465
1.0		1.6314	-.1490	.2942
1.5		2.3365	-.1837	.5717
2.0		2.9988	-.2016	.9004
3.0		4.2305	-.2172	1.6609
4.0		5.3795	-.2233	2.5088
6.0		7.5418	-.2277	4.3443
8.0		9.6144	-.2292	6.2710
10.0		11.6475	-.2297	8.2376
12.0		13.6629	-.2300	10.2221
14.0		15.6700	-.2301	12.2150
16.0		17.6733	-.2302	14.2117
18.0		19.6748	-.2302	16.2101
20.0		21.6755	-.2302	18.2094
22.0		23.6759	-.2302	20.2090
24.0		25.6760	-.2302	22.2089
26.0		27.6761	-.2302	24.2088
28.0		29.6761	-.2302	26.2088
ζ = 1.50; $\omega_n\tau_1$ = 1.0				
.1		.1839	-.7667 -2	.8388 -2
.2		.3462	-.2468 -1	.2915 -1
.3		.4956	-.4612 -1	.5833 -1
.4		.6367	-.6958 -1	.9374 -1
.5		.7722	-.9375 -1	.1341
.6		.9037	-.1179	.1784
.7		1.0322	-.1415	.2262
.8		1.1586	-.1644	.2770
.9		1.2832	-.1863	.3305
1.0		1.4065	-.2071	.3864
1.5		2.0090	-.2951	.6959
2.0		2.5959	-.3590	1.0451
2.5		3.1707	-.4047	1.4246
3.0		3.7348	-.4375	1.8276
4.0		4.8350	-.4791	2.6859
5.0		5.9052	-.5025	3.5923
6.0		6.9539	-.5163	4.5298
8.0		9.0102	-.5302	6.4595
10.0		11.0366	-.5361	8.4273
12.0		13.0489	-.5387	10.4124
14.0		15.0546	-.5399	12.4055
16.0		17.0573	-.5404	14.4023
18.0		19.0586	-.5407	16.4008
20.0		21.0591	-.5408	18.4001
22.0		23.0594	-.5409	20.3997
24.0		25.0595	-.5409	22.3996
26.0		27.0596	-.5409	24.3995
28.0		29.0596	-.5409	26.3995
ζ = 1.50; $\omega_n\tau_1$ = 2.0				
.1		.1801	-.1922 -2	.1607 -1
.2		.3335	-.6230 -2	.5399 -1
.3		.4715	-.1174 -1	.1050
.4		.5996	-.1789 -1	.1646
.5		.7210	-.2436 -1	.2303
.6		.8378	-.3097 -1	.3003
.7		.9511	-.3761 -1	.3737
.8		1.0619	-.4420 -1	.4497
.9		1.1709	-.5068 -1	.5277
1.0		1.2785	-.5703 -1	.6075
1.5		1.8040	-.8613 -1	1.0238
2.0		2.3199	-.1105	1.4590
2.5		2.8318	-.1307	1.9068
3.0		3.3414	-.1472	2.3641
4.0		4.3560	-.1719	3.3002
5.0		5.3661	-.1886	4.2567
6.0		6.3731	-.1998	5.2272
8.0		8.3813	-.2127	7.1934
10.0		10.3851	-.2186	9.1776
12.0		12.3869	-.2214	11.1703
14.0		14.3877	-.2227	13.1669
16.0		16.3881	-.2233	15.1653
18.0		18.3883	-.2236	17.1646
20.0		20.3884	-.2237	19.1643
22.0		22.3884	-0.2238	21.1641
24.0		24.3885	-.2238	23.1640
26.0		26.3885	-.2238	25.1640
ζ = 1.50; $\omega_n\tau_1$ = 4.0				
.1		.1782	.2404 -2	.3145 -1
.2		.3273	.7805 -2	.1039
.3		.4598	.1474 -1	.1992
.4		.5817	.2251 -1	.3083
.5		.6966	.3073 -1	.4263
.6		.8066	.3917 -1	.5500
.7		.9133	.4768 -1	.6774
.8		1.0175	.5618 -1	.8072
.9		1.1199	.6458 -1	.9384
1.0		1.2210	.7284 -1	1.0704
1.5		1.7167	.1113	1.7284
2.0		2.2072	.1442	2.3696
3.0		3.1901	.1949	3.5896
4.0		4.1782	.2300	4.7417
6.0		6.1648	.2705	6.9173
8.0		8.1587	.2895	8.9994
10.0		10.1559	.2984	11.0378
12.0		12.1545	.3026	13.0557
14.0		14.1539	.3044	15.0638
16.0		16.1536	.3054	17.0679
18.0		18.1535	.3058	19.0697
20.0		20.1535	.3060	21.0705
22.0		22.1534	.3061	23.0709
24.0		24.1534	.3061	25.0710
ζ = 1.50; $\omega_n\tau_1$ = 8.0				
.1		.1772	.4930 -2	.6223 -1
.2		.3242	.1601 -1	.2038
.3		.4540	.3025 -1	.3880
.4		.5730	.4622 -1	.5968
.5		.6848	.6313 -1	.8202
.6		.7918	.8052 -1	1.0523
.7		.8955	.9808 -1	1.2892
.8		.9967	.1156	1.5282
.9		1.0963	.1330	1.7675
1.0		1.1947	.1501	2.0059
1.5		1.6784	.2298	3.1597
2.0		2.1597	.2983	4.2268
3.0		3.1295	.4042	6.1042
4.0		4.1095	.4776	7.7115
6.0		6.0874	.5628	10.4150
8.0		8.0774	.6028	12.7446
10.0		10.0729	.6215	14.8987
12.0		12.0708	.6302	16.9704
14.0		14.0698	.6342	19.0040
16.0		16.0693	.6361	21.0195
18.0		18.0691	.6370	23.0266
20.0		20.0690	.6374	25.0300
22.0		22.0690	.6376	27.0316
24.0		24.0690	.6377	29.0325
26.0		26.0689	.6377	31.0329
28.0		28.0689	.6377	33.0329
ζ = 1.50; $\omega_n\tau_1$ = 16.0				
.1		.1767	.6283 -2	.1238
.2		.3227	.2041 -1	.4038
.3		.4511	.3856 -1	.7658
.4		.5687	.5892 -1	1.1740
.5		.6791	.8050 -1	1.6088
.6		.7846	.1027	2.0583
.7		.8869	.1251	2.5147
.8		.9867	.1475	2.9729
.9		1.0850	.1696	3.4292
1.0		1.1822	.1915	3.8814
1.5		1.6606	.2933	6.0321
2.0		2.1379	.3809	7.9570
2.5		2.6184	.4548	9.6578
3.0		3.1025	.5165	11.1608
3.5		3.5896	.5678	12.4949
4.0		4.0792	.6104	13.6875
5.0		5.0640	.6751	15.7379
6.0		6.0538	.7195	17.4581
7.0		7.0470	.7499	18.9512
8.0		8.0425	.7707	20.2888
10.0		10.0373	.7946	22.6766
12.0		12.0349	.8058	24.8580
14.0		14.0338	.8110	26.9416
16.0		16.0332	.8134	28.9815
18.0		18.0330	.8146	31.0001
20.0		20.0329	.8151	33.0084
22.0		22.0328	.8153	35.0121
24.0		24.0328	.8154	37.0140
26.0		26.0328	.8154	39.0140
28.0		28.0328	.8154	41.0140
30.0		30.0328	.8155	43.0158

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TABLE II.- NORMALIZED OPTIMUM RESPONSES FOR A TYPE 1 THIRD-ORDER PLANT WITH ZERO - Concluded

ω_nT_1	ω_nT_2	ω_nT_m	b/B	$R_o\omega_n/BK$
ζ = 2.00; $\omega_n\tau_1$ = 0				
0.5	1.1032	1.2615		0.5509 -1
1.0	1.9773	2.1560		.2014
1.5	2.7624	2.9456		.4209
2.0	3.4921	3.6769		.6927
2.5	4.1809	4.3661		1.0042
3.0	4.8382	5.0238		1.3473
3.5	5.4708	5.6562		1.7147
4.0	6.0842	6.2698		2.1014
4.5	6.6787	6.8643		2.5068
5.0	7.2567	7.4423		2.9288
5.5	7.8263	8.0119		3.3593
6.0	8.3854	8.5710		3.8003
6.5	8.9373	9.1230		4.2485
7.0	9.4822	9.6679		4.7036
7.5	10.0213	10.2071		5.1645
8.0	10.5540	10.7397		5.6316
8.5	11.0834	11.2691		6.1023
9.0	11.6086	11.7943		6.5770
9.5	12.1305	12.3161		7.0551
10.0	12.6493	12.8350		7.5364
ζ = 2.00; $\omega_n\tau_1$ = 0.5				
.1		.1852	-0.2119 -1	.4229 -2
.2		.3530	-.6405	.1495 -1
.3		.5129	-.1132	.3052 -1
.4		.6689	-.1620	.5013 -1
.5		.8228	-.2075	.7331 -1
.6		.9761	-.2482	.9975 -1
.7		1.1289	-.2837	.1292
.8		1.2814	-.3140	.1616
.9		1.4335	-.3396	.1967
1.0		1.5852	-.3608	.2344
1.5		2.3308	-.4224	.4580
2.0		3.0464	-.4454	.7309
3.0		4.3854	-.4588	1.3852
4.0		5.6285	-.4620	2.1405
6.0		7.9345	-.4634	3.8338
8.0		10.1029	-.4637	5.6653
10.0		12.1980	-.4638	7.5701
12.0		14.2525	-.4639	9.5155
14.0		16.2841	-.4639	11.4839
16.0		18.3025	-.4639	13.4656
18.0		20.3132	-.4639	15.4549
20.0		22.3194	-.4639	17.4486
22.0		24.3230	-.4639	19.4450
24.0		26.3252	-.4639	21.4428
26.0		28.3264	-.4639	23.4416
28.0		30.3272	-.4639	25.4409
30.0		32.3276	-.4639	27.4404
ζ = 2.00; $\omega_n\tau_1$ = 1.0				
.1		.1779	-.1428 -1	.7786 -2
.2		.3299	-.4420 -1	.2587 -1
.3		.4692	-.8055 -1	.5022 -1
.4		.6015	-.1195	.7901 -1
.5		.7298	-.1590	.1113
.6		.8556	-.1980	.1464
.7		.9800	-.2360	.1840
.8		1.1036	-.2725	.2239
.9		1.2268	-.3072	.2659
1.0		1.3499	-.3402	.3099
1.5		1.9671	-.4764	.5565
2.0		2.5887	-.5694	.8420
3.0		3.8279	-.6654	1.5067
4.0		5.0336	-.7011	2.2653
6.0		7.3185	-0.7214	3.9602
8.0		9.4817	-.7262	5.7921
10.0		11.5750	-.7280	7.6970
12.0		13.6288	-.7287	9.6425
14.0		15.6600	-.7291	11.6109
16.0		17.6781	-.7293	13.5926
18.0		19.6887	-.7294	15.5819
20.0		21.6949	-.7294	17.5757
22.0		23.6985	-.7295	19.5720
24.0		25.7006	-.7295	21.5699
26.0		27.7019	-.7295	23.5686
28.0		29.7025	-.7295	25.5679
30.0		31.7030	-.7295	27.5675
ζ = 2.00; $\omega_n\tau_1$ = 2.0				
.1		.1743	-.5370 -2	.1494 -1
.2		.3185	-.1620 -1	.4810 -1
.3		.4477	-.3070 -1	.9094 -1
.4		.5684	-.4593 -1	.1398
.5		.6840	-.6171 -1	.1926
.6		.7962	-.7767 -1	.2484
.7		.9063	-.9356 -1	.3065
.8		1.0150	-.1093	.3665
.9		1.1227	-.1247	.4280
1.0		1.2296	-.1397	.4909
1.5		1.7593	-.2093	.8222
2.0		2.2854	-.2691	1.1765
3.0		3.3318	-.3640	1.9401
4.0		4.3717	-.4334	2.7614
6.0		6.4330	-.5218	4.5234
8.0		8.4735	-.5702	6.3860
10.0		10.4992	-.5972	8.3064
12.0		12.5149	-.6125	10.2600
14.0		14.5244	-.6213	12.2330
16.0		16.5300	-.6264	14.2172
18.0		18.5333	-.6294	16.2080
20.0		20.5353	-.6311	18.2026
22.0		22.5364	-.6321	20.1994
24.0		24.5371	-.6327	22.1976
26.0		26.5375	-.6330	24.1965
28.0		28.5377	-.6332	26.1959
30.0		30.5379	-.6333	28.1955
ζ = 2.00; $\omega_n\tau_1$ = 4.0				
.1		.1725	.4478 -3	.2926 -1
.2		.3128	.1396 -2	.9275 -1
.3		.4373	.2567 -2	.1730
.4		.5527	.3850 -2	.2627
.5		.6627	.5183 -2	.3581
.6		.7692	.6537 -2	.4570
.7		.8735	.7892 -2	.5581
.8		.9762	.9236 -2	.6607
.9		1.0780	.1056 -1	.7642
1.0		1.1791	.1187 -1	.8684
1.5		1.6801	.1797 -1	1.3917
2.0		2.1790	.2335 -1	1.9144
3.0		3.1766	.3219 -1	2.9521
4.0		4.1748	.3897 -1	3.9811
6.0		6.1723	.4812 -1	6.0202
8.0		8.1708	.5348 -1	8.0431
10.0		10.1699	.5663 -1	10.0566
12.0		12.1694	.5846 -1	12.0644
14.0		14.1692	.5954 -1	14.0690
16.0		16.1690	.6017 -1	16.0717
18.0		18.1689	.6054 -1	18.0733
20.0		20.1688	0.6072 -1	20.0740
22.0		22.1688	.6085 -1	22.0746
24.0		24.1688	.6094 -1	24.0750
26.0		26.1688	.6094 -1	26.0750
28.0		28.1688	.6094 -1	28.0750
30.0		30.1687	.6103 -1	30.0754
ζ = 2.00; $\omega_n\tau_1$ = 8.0				
.1		.1716	.3695 -2	.5792 -1
.2		.3101	.1152 -1	.1821
.3		.4322	.2121 -1	.3375
.4		.5451	.3181 -1	.5094
.5		.6525	.4285 -1	.6903
.6		.7565	.5407 -1	.8761
.7		.8582	.6531 -1	1.0643
.8		.9585	.7647 -1	1.2533
.9		1.0578	.8749 -1	1.4421
1.0		1.1565	.9833 -1	1.6301
1.5		1.6468	.1492	2.5467
2.0		2.1366	.1942	3.4170
3.0		3.1200	.2685	5.0278
4.0		4.1079	.3256	6.4970
6.0		6.0923	.4031	9.1328
8.0		8.0836	.4487	11.5059
10.0		10.0785	.4754	13.7249
12.0		12.0756	.4911	15.8530
14.0		14.0740	.5002	17.9280
16.0		16.0730	.5056	19.9718
18.0		18.0724	.5088	21.9976
20.0		20.0721	.5106	24.0125
22.0		22.0719	.5116	26.0213
24.0		24.0718	.5123	28.0268
26.0		26.0717	.5127	30.0296
28.0		28.0717	.5128	32.0307
30.0		30.0717	.5129	34.0318
ζ = 2.00; $\omega_n\tau_1$ = 16.0				
.1		.1712	.5402 -2	.1153
.2		.3087	.1685 -1	.3610
.3		.4297	.3102 -1	.6666
.4		.5414	.4652 -1	1.0030
.5		.6476	.6268 -1	1.3553
.6		.7503	.7910 -1	1.7153
.7		.8509	.9556 -1	2.0781
.8		.9500	.1119	2.4404
.9		1.0482	.1280	2.8005
1.0		1.1459	.1439	3.1568
1.5		1.6315	.2184	4.8635
2.0		2.1176	.2844	6.4336
2.5		2.6056	.3425	7.8743
3.0		3.0955	.3935	9.1998
4.0		4.0797	.4774	11.5582
5.0		5.0681	.5418	13.6009
6.0		6.0596	.5913	15.4007
8.0		8.0484	.6582	18.4833
10.0		10.0421	.6975	21.1187
12.0		12.0384	.7206	23.4905
14.0		14.0363	.7340	25.7077
16.0		16.0351	.7419	27.8357
18.0		18.0344	.7466	29.9107
20.0		20.0339	.7492	31.9540
22.0		22.0337	.7508	33.9795
24.0		24.0336	.7518	35.9948
26.0		26.0335	.7523	38.0038
28.0		28.0334	.7526	40.0089
30.0		30.0334	.7528	42.0114

TABLE III.- NORMALIZED OPTIMUM RESPONSE FOR A TYPE 1 THIRD-ORDER UNSTABLE PLANT WITH ZERO

$\tau_2/\tau_3 = 0.10;\ \tau_1/\tau_2 = 0$

T_1/τ_2	T_2/τ_2	T_m/τ_2	b/B	$R_0/BK\tau_2$
0.2528 -1	0.7528 -1	0.1		0.0281 -4
.5112 -1	.1511	.2		.2529 -4
.7751 -1	.2275	.3		.8445 -4
.1044	.3045	.4		.1996 -3
.1319	.3821	.5		.3905 -3
.1599	.4602	.6		.6730 -3
.1884	.5390	.7		.1067 -2
.2175	.6183	.8		.1590 -2
.2470	.6981	.9		.2261 -2
.2770	.7785	1.0		.3095 -2
.4340	1.1891	1.5		.1032 -1
.6012	1.6132	2.0		.2406 -1
.7767	2.0497	2.5		.4603 -1
.9586	2.4974	3.0		.7766 -1
1.1449	2.9549	3.5		.1200
1.3338	3.4208	4.0		.1740
1.5238	3.8938	4.5		.2400
1.7134	4.3727	5.0		.3185
1.9015	4.8563	5.5		.4096
2.0870	5.3437	6.0		.5132
2.2694	5.8341	6.5		.6294
2.4478	6.3268	7.0		.7579
2.6221	6.8214	7.5		.8986
2.7918	7.3174	8.0		1.0513
2.9566	7.8144	8.5		1.2156
3.1166	8.3123	9.0		1.3913
3.2716	8.8107	9.5		1.5782
3.4216	9.3096	10.0		1.7760
3.7066	10.3082	11.0		2.2032
3.9721	11.3075	12.0		2.6708
4.4474	13.3070	14.0		3.7193
4.8542	15.3069	16.0		4.9054
5.2001	17.3069	18.0		6.2135
5.4925	19.3069	20.0		7.6287
5.7385	21.3069	22.0		9.1368
5.9444	23.3069	24.0		10.7248
6.1163	25.3069	26.0		12.3812
6.2592	27.3069	28.0		14.0953
6.3778	29.3069	30.0		15.8582
6.4759	31.3069	32.0		17.6619
6.5569	33.3069	34.0		19.4998
6.6238	35.3069	36.0		21.3661
6.6789	37.3069	38.0		23.2560
6.7242	39.3069	40.0		25.1653

$\tau_2/\tau_3 = 0.10;\ \tau_1/\tau_2 = 0.5$

T_1/τ_2	T_2/τ_2	T_m/τ_2	b/B	$R_0/BK\tau_2$
.4875 -1		.1	0.4760 -2	.1251 -3
.9542 -1		.2	.1733 -1	.5028 -3
.1405		.3	.3565 -1	.1139 -2
.1844		.4	.5813 -1	.2043 -2
.2275		.5	.8361 -1	.3226 -2
.2698		.6	.1112	.4698 -2
.3117		.7	.1401	.6474 -2
.3532		.8	.1699	.8563 -2
.3945		.9	.2002	.1098 -1
.4355		1.0	.2306	.1374 -1
.6391		1.5	.3776	.3306 -1
.8420		2.0	.5071	.6252 -1
1.0446		2.5	.6155	.1031
1.2464		3.0	.7035	.1555
1.4465		3.5	.7736	.2202
1.6442		4.0	.8287	.2973
1.8387		4.5	.8714	.3869
2.0295		5.0	.9042	.4890
2.2160		5.5	.9291	.6035
2.3979		6.0	.9479	.7302
2.5750		6.5	.9620	.8690
2.7471		7.0	.9725	1.0195
2.9141		7.5	.9802	1.1817
3.0760		8.0	.9858	1.3551
3.2327		8.5	.9899	1.5397
3.3842		9.0	.9928	1.7352
3.5306		9.5	.9950	1.9412
3.6720		10.0	.9965	2.1577
3.9401		11.0	.9983	2.6207
4.1891		12.0	.9992	3.1222
4.6336		14.0	.9998	4.2330
5.0128		16.0	1.0000	5.4744
5.3344		18.0	1.0000	6.8313
5.6056		20.0	1.0000	8.2888
5.8333		22.0	1.0000	9.8335
6.0236		24.0	1.0000	11.4528
6.1822		26.0	1.0000	13.1357
6.3139		28.0	1.0000	14.8722
6.4230		30.0	1.0000	16.6539
6.5133		32.0	1.0000	18.4734
6.5878		34.0	1.0000	20.3243
6.6492		36.0	1.0000	22.2015
6.6998		38.0	1.0000	24.1004
6.7414		40.0	1.0000	26.0172

$\tau_2/\tau_3 = 0.10;\ \tau_1/\tau_2 = 2.0$

T_1/τ_2	T_2/τ_2	T_m/τ_2	b/B	$R_0/BK\tau_2$
0.5048 -1		0.1	-0.7314 -3	0.5002 -3
.1019		.2	-.2850 -2	.1997 -2
.1540		.3	-.6247 -2	.4485 -2
.2068		.4	-.1081 -1	.7956 -2
.2602		.5	-.1644 -1	.1240 -1
.3141		.6	-.2304 -1	.1779 -1
.3684		.7	-.3050 -1	.2413 -1
.4230		.8	-.3873 -1	.3139 -1
.4779		.9	-.4764 -1	.3955 -1
.5329		1.0	-.5715 -1	.4859 -1
.8079		1.5	-.1110	.1063
1.0783		2.0	-.1698	.1830
1.3396		2.5	-.2277	.2761
1.5892		3.0	-.2809	.3835
1.8260		3.5	-.3276	.5033
2.0498		4.0	-.3669	.6342
2.2613		4.5	-.3990	.7753
2.4614		5.0	-.4244	.9260
2.6511		5.5	-.4441	1.0861
2.8315		6.0	-.4592	1.2553
3.0035		6.5	-.4704	1.4338
3.1680		7.0	-.4788	1.6216
3.3255		7.5	-.4849	1.8189
3.4766		8.0	-.4893	2.0255
3.6217		8.5	-.4925	2.2416
3.7612		9.0	-.4948	2.4672
3.8953		9.5	-.4964	2.7021
4.0244		10.0	-.4975	2.9462
4.2679		11.0	-.4988	3.4618
4.4932		12.0	-.4995	4.0125
4.8935		14.0	-.4999	5.2128
5.2334		16.0	-.5000	6.5331
5.5206		18.0	-.5000	7.9588
5.7620		20.0	-.5000	9.4759
5.9641		22.0	-.5000	11.0718
6.1327		24.0	-.5000	12.7347
6.2728		26.0	-.5000	14.4544
6.3890		28.0	-.5000	16.2219
6.4852		30.0	-.5000	18.0296
6.5646		32.0	-.5000	19.8707
6.6301		34.0	-.5000	21.7397
6.6841		36.0	-.5000	23.6318
6.7285		38.0	-.5000	25.5430
6.7650		40.0	-.5000	27.4700

$\tau_2/\tau_3 = 0.10;\ \tau_1/\tau_2 = 3.0$

T_1/τ_2	T_2/τ_2	T_m/τ_2	b/B	$R_0/BK\tau_2$
.5069 -1		.1	-.7099 -3	.7493 -3
.1027		.2	-.2790 -2	.2995 -2
.1559		.3	-.6164 -2	.6725 -2
.2102		.4	-.1075 -1	.1192 -1
.2654		.5	-.1648 -1	.1857 -1
.3216		.6	-.2325 -1	.2664 -1
.3785		.7	-.3101 -1	.3610 -1
.4360		.8	-.3966 -1	.4692 -1
.4942		.9	-.4914 -1	.5907 -1
.5528		1.0	-.5937 -1	.7251 -1
.8501		1.5	-.1193	.1578
1.1477		2.0	-.1883	.2697
1.4385		2.5	-.2602	.4036
1.7174		3.0	-.3301	.5553
1.9815		3.5	-.3947	.7212
2.2289		4.0	-.4520	.8981
2.4594		4.5	-.5008	1.0836
2.6735		5.0	-.5410	1.2759
2.8725		5.5	-.5730	1.4742
3.0579		6.0	-.5980	1.6781
3.2315		6.5	-.6169	1.8877
3.3949		7.0	-.6310	2.1034
3.5492		7.5	-.6414	2.3256
3.6958		8.0	-.6489	2.5550
3.8354		8.5	-.6542	2.7918
3.9688		9.0	-.6580	3.0365
4.0964		9.5	-.6607	3.2893
4.2188		10.0	-.6626	3.5502
4.4483		11.0	-.6648	4.0968
4.6608		12.0	-.6658	4.6758
5.0363		14.0	-.6665	5.9269
5.3543		16.0	-.6666	7.2913
5.6223		18.0	-.6667	8.7553
5.8473		20.0	-.6667	10.3054
6.0353		22.0	-.6667	11.9294
6.1919		24.0	-.6667	13.6162
6.3220		26.0	-.6667	15.3561
6.4297		28.0	-.6667	17.1405
6.5188		30.0	-.6667	18.9623
6.5924		32.0	-.6667	20.8152
6.6530		34.0	-.6667	22.6940
6.7029		36.0	-.6667	24.5942
6.7440		38.0	-.6667	26.5121
6.7777		40.0	-.6667	28.4446

$\tau_2/\tau_3 = 0.10;\ \tau_1/\tau_2 = 5.0$

T_1/τ_2	T_2/τ_2	T_m/τ_2	b/B	$R_0/BK\tau_2$
0.5086 -1		0.1	-0.5938 -3	0.1249 -2
.1034		.2	-.2349 -2	.4991 -2
.1575		.3	-.5222 -2	.1121 -1
.2130		.4	-.9167 -2	.1987 -1
.2699		.5	-.1413 -1	.3093 -1
.3280		.6	-.2007 -1	.4436 -1
.3873		.7	-.2693 -1	.6009 -1
.4476		.8	-.3466 -1	.7806 -1
.5089		.9	-.4319 -1	.9823 -1
.5710		1.0	-.5249 -1	.1205 -1
.8907		1.5	-.1085	.2614
1.2181		2.0	-.1762	.4450
1.5443		2.5	-.2504	.6632
1.8626		3.0	-.3268	.9088
2.1676		3.5	-.4020	1.1750
2.4553		4.0	-.4732	1.4556
2.7231		4.5	-.5382	1.7449
2.9695		5.0	-.5954	2.0378
3.1942		5.5	-.6438	2.3305
3.3982		6.0	-.6834	2.6204
3.5833		6.5	-.7146	2.9066
3.7518		7.0	-.7386	3.1893
3.9063		7.5	-.7564	3.4696
4.0491		8.0	-.7695	3.7493
4.1821		8.5	-.7788	4.0301
4.3069		9.0	-.7854	4.3135
4.4247		9.5	-.7901	4.6008
4.5365		10.0	-.7933	4.8932
4.7446		11.0	-.7969	5.4955
4.9347		12.0	-.7986	6.1239
5.2690		14.0	-.7997	7.4606
5.5507		16.0	-.8000	8.8983
5.7873		18.0	-.8000	10.4254
5.9852		20.0	-.8000	12.0295
6.1502		22.0	-.8000	13.6995
6.2874		24.0	-.8000	15.4252
6.4011		26.0	-.8000	17.1978
6.4952		28.0	-.8000	19.0096
6.5729		30.0	-.8000	20.8542
6.6369		32.0	-.8000	22.7261
6.6897		34.0	-.8000	24.6206
6.7331		36.0	-.8000	26.5338
6.7688		38.0	-.8000	28.4625
6.7981		40.0	-.8000	30.4039

$\tau_2/\tau_3 = 0.25;\ \tau_1/\tau_2 = 0$

T_1/τ_2	T_2/τ_2	T_m/τ_2	b/B	$R_0/BK\tau_2$
.2523 -1	0.7523 -1	.1		.0787 -4
.5092 -1	.1510	.2		.6261 -4
.7705 -1	.2272	.3		.2110 -3
.1036	.3039	.4		.4994 -3
.1306	.3811	.5		.9743 -3
.1580	.4588	.6		.1682 -2
.1857	.5371	.7		.2667 -2
.2139	.6158	.8		.3975 -2
.2423	.6951	.9		.5651 -2
.2711	.7750	1.0		.7737 -2
.4194	1.1823	1.5		.2580 -1
.5729	1.6030	2.0		.6015 -1
.7291	2.0366	2.5		.1150
.8852	2.4821	3.0		.1938
1.0388	2.9383	3.5		.2989
1.1879	3.4038	4.0		.4319
1.3306	3.8773	4.5		.5933
1.4657	4.3572	5.0		.7829
1.5923	4.8423	5.5		.9999
1.7099	5.3315	6.0		1.2432
1.8182	5.8237	6.5		1.5110
1.9175	6.3183	7.0		1.8016
2.0078	6.8145	7.5		2.1134
2.0898	7.3119	8.0		2.4443
2.1638	7.8102	8.5		2.7928
2.2305	8.3090	9.0		3.1571
2.2903	8.8083	9.5		3.5359
2.3440	9.3078	10.0		3.9275
2.4348	10.3072	11.0		4.7448
2.5071	11.3070	12.0		5.5998
2.6095	13.3069	14.0		7.3948
2.6729	15.3069	16.0		9.2680
2.7118	17.3069	18.0		11.1901
2.7356	19.3069	20.0		13.1425
2.7501	21.3069	22.0		15.1135
2.7589	23.3069	24.0		17.0958
2.7643	25.3069	26.0		19.0851
2.7676	27.3069	28.0		21.0786
2.7695	29.3069	30.0		23.0746
2.7707	31.3069	32.0		25.0722
2.7715	33.3069	34.0		27.0708
2.7719	35.3069	36.0		29.0699
2.7722	37.3069	38.0		31.0693
2.7723	39.3069	40.0		33.0690
2.7724	41.3069	42.0		35.0688
2.7725	43.3069	44.0		37.0687

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TABLE III.- NORMALIZED OPTIMUM RESPONSE FOR A TYPE 1 THIRD-ORDER UNSTABLE PLANT WITH ZERO - Continued

τ_2/τ_3 = 0.25; τ_1/τ_2 = 0.5

T_1/τ_2	T_2/τ_2	T_m/τ_2	b/B	$R_o/BK\tau_2$
0.4857^{-1}		0.1	0.5099^{-2}	0.3128^{-3}
.9473^{-1}		.2	.1857^{-1}	.1258^{-2}
.1390		.3	.3818^{-1}	.2849^{-2}
.1819		.4	.6226^{-1}	.5111^{-2}
.2236		.5	.8954^{-1}	.8069^{-2}
.2644		.6	.1191	.1175^{-1}
.3044		.7	.1500	.1620^{-1}
.3438		.8	.1820	.2143^{-1}
.3827		.9	.2144	.2749^{-1}
.4211		1.0	.2469	.3440^{-1}
.6077		1.5	.4037	.8280^{-1}
.7865		2.0	.5410	.1565
.9577		2.5	.6543	.2575
1.1203		3.0	.7445	.3871
1.2737		3.5	.8143	.5454
1.4172		4.0	.8672	.7320
1.5505		4.5	.9064	.9459
1.6734		5.0	.9349	1.1858
1.7862		5.5	.9553	1.4500
1.8891		6.0	.9697	1.7370
1.9826		6.5	.9797	2.0449
2.0673		7.0	.9865	2.3721
2.1437		7.5	.9911	2.7170
2.2125		8.0	.9942	3.0778
2.2743		8.5	.9962	3.4532
2.3297		9.0	.9976	3.8418
2.3792		9.5	.9985	4.2423
2.4235		10.0	.9990	4.6535
2.4981		11.0	.9996	5.5040
2.5572		12.0	.9998	6.3856
2.6406		14.0	1.0000	8.2188
2.6920		16.0	1.0000	10.1160
2.7235		18.0	1.0000	12.0530
2.7428		20.0	1.0000	14.0145
2.7545		22.0	1.0000	15.9911
2.7616		24.0	1.0000	17.9768
2.7659		26.0	1.0000	19.9682
2.7685		28.0	1.0000	21.9629
2.7701		30.0	1.0000	23.9597
2.7711		32.0	1.0000	25.9578
2.7717		34.0	1.0000	27.9566
2.7720		36.0	1.0000	29.9559
2.7723		38.0	1.0000	31.9555
2.7724		40.0	1.0000	33.9552
2.7725		42.0	1.0000	35.9551

τ_2/τ_3 = 0.25; τ_1/τ_2 = 2.0

T_1/τ_2	T_2/τ_2	T_m/τ_2	b/B	$R_o/BK\tau_2$
.5029^{-1}		.1	-.9140^{-3}	.1249^{-2}
.1011		.2	-.3562^{-2}	.4992^{-2}
.1522		.3	-.7805^{-2}	.1121^{-1}
.2036		.4	-.1351^{-1}	.1988^{-1}
.2550		.5	-.2053^{-1}	.3097^{-1}
.3065		.6	-.2874^{-1}	.4443^{-1}
.3579		.7	-.3801^{-1}	.6022^{-1}
.4091		.8	-.4823^{-1}	.7828^{-1}
.4600		.9	-.5926^{-1}	.9857^{-1}
.5105		1.0	-.7101^{-1}	.1210
.7551		1.5	-.1368	.2634
.9816		2.0	-.2065	.4498
1.1864		2.5	-.2722	.6717
1.3685		3.0	-.3293	.9215
1.5290		3.5	-.3758	1.1935
1.6700		4.0	-.4119	1.4837
1.7940		4.5	-.4388	1.7895
1.9033		5.0	-.4582	2.1098
2.0000		5.5	-.4718	2.4438
2.0857		6.0	-.4813	2.7911
2.1620		6.5	-.4877	3.1513
2.2300		7.0	-.4920	3.5239
2.2906		7.5	-.4948	3.9084
2.3446		8.0	-.4967	4.3041
2.3928		8.5	-.4979	4.7102
2.4357		9.0	-.4987	5.1260
2.4739		9.5	-.4991	5.5506
2.5079		10.0	-.4995	5.9832
2.5650		11.0	-.4998	6.8697
2.6100		12.0	-.4999	7.7799
2.6732		14.0	-.5000	9.6536
2.7120		16.0	-.5000	11.5760
2.7357		18.0	-.5000	13.5285
2.7502		20.0	-.5000	15.4996
2.7590		22.0	-.5000	17.4820
2.7643		24.0	-.5000	19.4713
2.7676		26.0	-.5000	21.4648
2.7696		28.0	-.5000	23.4609
2.7707		30.0	-.5000	25.4585
2.7715		32.0	-.5000	27.4570
2.7719		34.0	-.5000	29.4562
2.7722		36.0	-.5000	31.4556
2.7723		38.0	-.5000	33.4553
2.7724		40.0	-.5000	35.4551
2.7725		42.0	-.5000	37.4550

τ_2/τ_3 = 0.25; τ_1/τ_2 = 3.0

T_1/τ_2	T_2/τ_2	T_m/τ_2	b/B	$R_o/BK\tau_2$
0.5050^{-1}		0.1	-0.9556^{-3}	0.1873^{-2}
.1019		.2	-.3755^{-2}	.7486^{-2}
.1540		.3	-.8293^{-2}	.1681^{-1}
.2068		.4	-.1446^{-1}	.2979^{-1}
.2600		.5	-.2215^{-1}	.4638^{-1}
.3136		.6	-.3124^{-1}	.6651^{-1}
.3674		.7	-.4164^{-1}	.9008^{-1}
.4213		.8	-.5321^{-1}	.1170
.4752		.9	-.6586^{-1}	.1472
.5289		1.0	-.7947^{-1}	.1806
.7923		1.5	-.1584	.3907
1.0393		2.0	-.2467	.6617
1.2628		2.5	-.3344	.9776
1.4596		3.0	-.4141	1.3232
1.6293		3.5	-.4814	1.6857
1.7741		4.0	-.5349	2.0566
1.8973		4.5	-.5753	2.4313
2.0026		5.0	-.6046	2.8086
2.0933		5.5	-.6252	3.1890
2.1720		6.0	-.6393	3.5739
2.2409		6.5	-.6488	3.9646
2.3015		7.0	-.6551	4.3624
2.3551		7.5	-.6593	4.7677
2.4025		8.0	-.6619	5.1809
2.4446		8.5	-.6637	5.6019
2.4819		9.0	-.6648	6.0305
2.5151		9.5	-.6655	6.4662
2.5446		10.0	-.6659	6.9086
2.5940		11.0	-.6664	7.8112
2.6328		12.0	-.6666	8.7341
2.6872		14.0	-.6667	10.6255
2.7206		16.0	-.6667	12.5588
2.7410		18.0	-.6667	14.5180
2.7534		20.0	-.6667	16.4932
2.7609		22.0	-.6667	18.4781
2.7655		24.0	-.6667	20.4690
2.7683		26.0	-.6667	22.4634
2.7700		28.0	-.6667	24.4600
2.7710		30.0	-.6667	26.4580
2.7716		32.0	-.6667	28.4567
2.7720		34.0	-.6667	30.4560
2.7722		36.0	-.6667	32.4555
2.7724		38.0	-.6667	34.4552
2.7725		40.0	-.6667	36.4551

τ_2/τ_3 = 0.25; τ_1/τ_2 = 5.0

T_1/τ_2	T_2/τ_2	T_m/τ_2	b/B	$R_o/BK\tau_2$
.5066^{-1}		.1	-.8905^{-3}	.3123^{-2}
.1026		.2	-.3522^{-2}	.1248^{-1}
.1556		.3	-.7829^{-2}	.2801^{-1}
.2095		.4	-.1374^{-1}	.4963^{-1}
.2643		.5	-.2118^{-1}	.7725^{-1}
.3198		.6	-.3006^{-1}	.1107
.3758		.7	-.4031^{-1}	.1499
.4322		.8	-.5183^{-1}	.1947
.4889		.9	-.6454^{-1}	.2449
.5458		1.0	-.7836^{-1}	.3002
.8285		1.5	-.1610	.6478
1.0991		2.0	-.2583	1.0934
1.3473		2.5	-.3604	1.6076
1.5661		3.0	-.4584	2.1600
1.7524		3.5	-.5455	2.7226
1.9067		4.0	-.6174	3.2737
2.0325		4.5	-.6732	3.8009
2.1349		5.0	-.7142	4.3012
2.2189		5.5	-.7431	4.7778
2.2888		6.0	-.7629	5.2367
2.3479		6.5	-.7760	5.6844
2.3986		7.0	-.7847	6.1262
2.4425		7.5	-.7903	6.5663
2.4809		8.0	-.7939	7.0075
2.5146		8.5	-.7961	7.4514
2.5444		9.0	-.7976	7.8990
2.5708		9.5	-.7985	8.3510
2.5941		10.0	-.7991	8.8072
2.6329		11.0	-.7996	9.7323
2.6634		12.0	-.7999	10.6724
2.7060		14.0	-.8000	12.5878
2.7321		16.0	-.8000	14.5358
2.7480		18.0	-.8000	16.5040
2.7576		20.0	-.8000	18.4847
2.7635		22.0	-.8000	20.4730
2.7671		24.0	-.8000	22.4658
2.7692		26.0	-.8000	24.4615
2.7706		28.0	-.8000	26.4589
2.7714		30.0	-.8000	28.4573
2.7718		32.0	-.8000	30.4563
2.7721		34.0	-.8000	32.4557
2.7723		36.0	-.8000	34.4554
2.7724		38.0	-.8000	36.4552
2.7725		40.0	-.8000	38.4550

τ_2/τ_3 = 0.50; τ_1/τ_2 = 0

T_1/τ_2	T_2/τ_2	T_m/τ_2	b/B	$R_o/BK\tau_2$
0.2515^{-1}	0.7516^{-1}	0.1		0.1562^{-4}
.5059^{-1}	.1507	.2		.1249^{-3}
.7630^{-1}	.2265	.3		.4215^{-3}
.1022	.3027	.4		.9983^{-3}
.1284	.3794	.5		.1948^{-2}
.1547	.4564	.6		.3362^{-2}
.1812	.5339	.7		.5332^{-2}
.2079	.6118	.8		.7946^{-2}
.2346	.6902	.9		.1129^{-1}
.2614	.7691	1.0		.1546^{-1}
.3955	1.1712	1.5		.5148^{-1}
.5270	1.5869	2.0		.1197
.6525	2.0165	2.5		.2278
.7690	2.4596	3.0		.3812
.8742	2.9153	3.5		.5822
.9668	3.3820	4.0		.8303
1.0464	3.8577	4.5		1.1226
1.1134	4.3406	5.0		1.4544
1.1687	4.8288	5.5		1.8202
1.2139	5.3210	6.0		2.2142
1.2502	5.8158	6.5		2.6312
1.2793	6.3125	7.0		3.0664
1.3023	6.8104	7.5		3.5161
1.3205	7.3090	8.0		3.9770
1.3349	7.8082	8.5		4.4467
1.3461	8.3077	9.0		4.9231
1.3549	8.8074	9.5		5.4049
1.3618	9.3072	10.0		5.8907
1.3714	10.3070	11.0		6.8711
1.3773	11.3070	12.0		7.8593
1.3808	12.3070	13.0		8.8521
1.3830	13.3069	14.0		9.8478
1.3843	14.3069	15.0		10.8452
1.3851	15.3069	16.0		11.8436
1.3856	16.3069	17.0		12.8426
1.3858	17.3069	18.0		13.8420
1.3860	18.3069	19.0		14.8417
1.3861	19.3069	20.0		15.8414
1.3862	20.3069	21.0		16.8413

τ_2/τ_3 = 0.50; τ_1/τ_2 = 0.5

T_1/τ_2	T_2/τ_2	T_m/τ_2	b/B	$R_o/BK\tau_2$
.4827^{-1}		.1	0.5666^{-2}	.6259^{-3}
.9359^{-1}		.2	.2063^{-1}	.2515^{-2}
.1366		.3	.4240^{-1}	.5699^{-2}
.1776		.4	.6911^{-1}	.1022^{-1}
.2171		.5	.9935^{-1}	.1614^{-1}
.2552		.6	.1320	.2351^{-1}
.2922		.7	.1663	.3240^{-1}
.3282		.8	.2016	.4287^{-1}
.3632		.9	.2374	.5498^{-1}
.3973		1.0	.2732	.6879^{-1}
.5563		1.5	.4448	.1651
.6971		2.0	.5919	.3099
.8202		2.5	.7092	.5049
.9260		3.0	.7984	.7487
1.0154		3.5	.8635	1.0375
1.0895		4.0	.9094	1.3662
1.1503		4.5	.9409	1.7289
1.1995		5.0	.9620	2.1199
1.2390		5.5	.9759	2.5340
1.2705		6.0	.9848	2.9665
1.2955		6.5	.9905	3.4138
1.3152		7.0	.9941	3.8726
1.3307		7.5	.9964	4.3405
1.3428		8.0	.9978	4.8155
1.3524		8.5	.9986	5.2960
1.3598		9.0	.9992	5.7808
1.3656		9.5	.9995	6.2690
1.3702		10.0	.9997	6.7598
1.3765		11.0	.9999	7.7470
1.3804		12.0	1.0000	8.7393
1.3827		13.0	1.0000	9.7346
1.3841		14.0	1.0000	10.7318
1.3850		15.0	1.0000	11.7301
1.3855		16.0	1.0000	12.7290
1.3858		17.0	1.0000	13.7284
1.3860		18.0	1.0000	14.7280
1.3861		19.0	1.0000	15.7278
1.3862		20.0	1.0000	16.7276

τ_2/τ_3 = 0.50; τ_1/τ_2 = 2.0

T_1/τ_2	T_2/τ_2	T_m/τ_2	b/B	$R_o/BK\tau_2$
.4996^{-1}		.1	-.1219^{-2}	.2499^{-2}
.9976^{-1}		.2	-.4747^{-2}	.9980^{-2}
.1492		.3	-.1039^{-1}	.2240^{-1}
.1981		.4	-.1797^{-1}	.3969^{-1}
.2464		.5	-.2727^{-1}	.6176^{-1}
.2939		.6	-.3811^{-1}	.8849^{-1}
.3404		.7	-.5031^{-1}	.1197
.3860		.8	-.6368^{-1}	.1554
.4304		.9	-.7803^{-1}	.1952
.4736		1.0	-.9319^{-1}	.2391
.6697		1.5	-.1756	.5119

TABLE III.- NORMALIZED OPTIMUM RESPONSE FOR A TYPE 1 THIRD-ORDER UNSTABLE PLANT WITH ZERO - Continued

T_1/τ_2	T_2/τ_2	T_m/τ_2	b/B	$R_o/BK\tau_2$
τ_2/τ_3 = 0.50; τ_1/τ_2 = 2.0				
0.8302		2.0	-0.2572	0.8541
.9567		2.5	-.3274	1.2413
1.0544		3.0	-.3820	1.6552
1.1293		3.5	-.4217	2.0848
1.1867		4.0	-.4492	2.5249
1.2310		4.5	-.4675	2.9731
1.2652		5.0	-.4795	3.4285
1.2918		5.5	-.4872	3.8906
1.3126		6.0	-.4920	4.3589
1.3288		6.5	-.4951	4.8326
1.3414		7.0	-.4970	5.3111
1.3513		7.5	-.4981	5.7937
1.3590		8.0	-.4989	6.2797
1.3650		8.5	-.4993	6.7686
1.3697		9.0	-.4996	7.2598
1.3734		9.5	-.4997	7.7528
1.3762		10.0	-.4998	8.2473
1.3802		11.0	-.4999	9.2395
1.3826		12.0	-.5000	10.2348
1.3840		13.0	-.5000	11.2319
1.3849		14.0	-.5000	12.2301
1.3855		15.0	-.5000	13.2291
1.3858		16.0	-.5000	14.2284
1.3860		17.0	-.5000	15.2280
1.3861		18.0	-.5000	16.2278
1.3862		19.0	-.5000	17.2276
τ_2/τ_3 = 0.50; τ_1/τ_2 = 3.0				
.5017^{-1}		.1	-.1365^{-2}	.3748^{-2}
.1006		.2	-.5362^{-2}	.1497^{-1}
.1510		.3	-.1183^{-1}	.3358^{-1}
.2012		.4	-.2061^{-1}	.5947^{-1}
.2510		.5	-.3153^{-1}	.9249^{-1}
.3004		.6	-.4440^{-1}	.1324
.3490		.7	-.5904^{-1}	.1791
.3968		.8	-.7527^{-1}	.2321
.4437		.9	-.9290^{-1}	.2913
.4894		1.0	-.1117	.3564
.6981		1.5	-.2175	.7562
.8682		2.0	-.3270	1.2445
.9991		2.5	-.4241	1.7742
1.0963		3.0	-.5012	2.3109
1.1674		3.5	-.5574	2.8374
1.2196		4.0	-.5962	3.3494
1.2583		4.5	-.6219	3.8491
1.2875		5.0	-.6386	4.3408
1.3097		5.5	-.6492	4.8282
1.3268		6.0	-.6559	5.3140
1.3400		6.5	-.6600	5.8000
1.3502		7.0	-.6626	6.2873
1.3582		7.5	-.6642	6.7761
1.3644		8.0	-.6651	7.2666
1.3692		8.5	-.6657	7.7587
1.3730		9.0	-.6661	8.2523
1.3759		9.5	-.6663	8.7471
1.3782		10.0	-.6665	9.2429
1.3814		11.0	-.6666	10.2370
1.3833		12.0	-.6666	11.2333
1.3845		13.0	-.6667	12.2310
1.3852		14.0	-.6667	13.2296
1.3856		15.0	-.6667	14.2287
1.3859		16.0	-.6667	15.2282
1.3861		17.0	-.6667	16.2279
1.3861		18.0	-.6667	17.2277
1.3862		19.0	-.6667	18.2276
τ_2/τ_3 = 0.50; τ_1/τ_2 = 5.0				
.5034^{-1}		.1	-.1385^{-2}	.6245^{-2}
.1012		.2	-.5476^{-2}	.2494^{-1}
.1524		.3	-.1216^{-1}	.5595^{-1}
.2038		.4	-.2133^{-1}	.9908^{-1}
.2551		.5	-.3283^{-1}	.1540
.3061		.6	-.4652^{-1}	.2205
.3566		.7	-.6226^{-1}	.2980
.4066		.8	-.7987^{-1}	.3862
.4557		.9	-.9918^{-1}	.4844
.5039		1.0	-.1200	.5922
.7260		1.5	-.2406	1.2512
.9078		2.0	-.3715	2.0417
1.0454		2.5	-.4921	2.8694
1.1434		3.0	-.5899	3.6626
1.2110		3.5	-.6620	4.3879
1.2574		4.0	-.7116	5.0431
1.2899		4.5	-.7443	5.6417
1.3132		5.0	-.7653	6.2001
1.3303		5.5	-.7785	6.7320
1.3431		6.0	-.7868	7.2476
1.3529		6.5	-.7919	7.7537
1.3604		7.0	-.7950	8.2545
1.3661		7.5	-0.7970	8.7526
1.3706		8.0	-.7982	9.2496
1.3741		8.5	-.7989	9.7462
1.3768		9.0	-.7993	10.2430
1.3789		9.5	-.7996	10.7401
1.3805		10.0	-.7997	11.2377
1.3828		11.0	-.7999	12.2339
1.3842		12.0	-.8000	13.2315
1.3850		13.0	-.8000	14.2299
1.3855		14.0	-.8000	15.2289
1.3858		15.0	-.8000	16.2283
1.3860		16.0	-.8000	17.2280
1.3861		17.0	-.8000	18.2278
1.3862		18.0	-.8000	19.2276
τ_2/τ_3 = 1.0; τ_1/τ_2 = 0				
.2499^{-1}	0.7501^{-1}	.1		.3121^{-4}
.4994^{-1}	.1501	.2		.2497^{-3}
.7479^{-1}	.2252	.3		.8424^{-3}
.9950^{-1}	.3005	.4		.1995^{-2}
.1240	.3760	.5		.3891^{-2}
.1483	.4517	.6		.6711^{-2}
.1723	.5277	.7		.1064^{-1}
.1960	.6040	.8		.1584^{-1}
.2194	.6806	.9		.2248^{-1}
.2423	.7577	1.0		.3075^{-1}
.3496	1.1504	1.5		.1015
.4418	1.5582	2.0		.2327
.5166	1.9834	2.5		.4337
.5736	2.4264	3.0		.7054
.6148	2.8852	3.5		1.0408
.6431	3.3569	4.0		1.4278
.6617	3.8383	4.5		1.8532
.6736	4.3264	5.0		2.3054
.6812	4.8188	5.5		2.7754
.6858	5.3142	6.0		3.2568
.6887	5.8113	6.5		3.7453
.6904	6.3096	7.0		4.2383
.6915	6.8085	7.5		4.7340
.6921	7.3079	8.0		5.2314
.6925	7.8075	8.5		5.7298
.6928	8.3072	9.0		6.2289
.6929	8.8071	9.5		6.7283
.6930	9.3070	10.0		7.2280
.6931	9.8069	10.5		7.7277
τ_2/τ_3 = 1.0; τ_1/τ_2 = 0.5				
.4767^{-1}		.1	.6797^{-2}	.1252^{-2}
.9131^{-1}		.2	.2472^{-1}	.5028^{-2}
.1316		.3	.5076^{-1}	.1139^{-2}
.1691		.4	.8262^{-1}	.2041^{-1}
.2043		.5	.1186	.3220^{-1}
.2373		.6	.1573	.4685^{-1}
.2683		.7	.1977	.6448^{-1}
.2976		.8	.2391	.8516^{-1}
.3253		.9	.2808	.1090
.3514		1.0	.3223	.1360
.4609		1.5	.5156	.3205
.5399		2.0	.6709	.5847
.5946		2.5	.7846	.9185
.6310		3.0	.8626	1.3067
.6545		3.5	.9140	1.7340
.6693		4.0	.9468	2.1879
.6786		4.5	.9673	2.6592
.6842		5.0	.9800	3.1415
.6877		5.5	.9878	3.6306
.6899		6.0	.9926	4.1240
.6911		6.5	.9955	4.6200
.6919		7.0	.9973	5.1175
.6924		7.5	.9983	5.6160
.6927		8.0	.9990	6.1151
.6929		8.5	.9994	6.6146
.6930		9.0	.9996	7.1142
.6930		9.5	.9998	7.6140
τ_2/τ_3 = 1.0; τ_1/τ_2 = 2.0				
.4933^{-1}		.1	-.1827^{-2}	.4995^{-2}
.9714^{-1}		.2	-.7108^{-2}	.1993^{-1}
.1432		.3	-.1553^{-1}	.4464^{-1}
.1873		.4	-.2675^{-1}	.7887^{-1}
.2293		.5	-.4042^{-1}	.1223
.2690		.6	-.5618^{-1}	.1743
.3064		.7	-.7367^{-1}	.2346
.3414		.8	-.9252^{-1}	.3023
.3739		.9	-.1124	.3770
.4040		1.0	-.1329	.4578
.5208		1.5	-.2359	.9302
.5922		2.0	-.3230	1.4615
.6340		2.5	-.3863	2.0046

T_1/τ_2	T_2/τ_2	T_m/τ_2	b/B	$R_m/BK\tau_2$
0.6582		3.0	-0.4287	2.5411
.6723		3.5	-.4559	3.0672
.6807		4.0	-.4730	3.5846
.6856		4.5	-.4835	4.0957
.6886		5.0	-.4900	4.6027
.6904		5.5	-.4939	5.1070
.6915		6.0	-.4963	5.6096
.6921		6.5	-.4977	6.1112
.6925		7.0	-.4986	6.6122
.6928		7.5	-.4992	7.1128
.6929		8.0	-.4995	7.6131
.6930		8.5	-.4997	8.1134
.6931		9.0	-.4998	8.6135
τ_2/τ_3 = 1.0; τ_1/τ_2 = 3.0				
.4953^{-1}		.1	-.2183^{-2}	.7493^{-2}
.9790^{-1}		.2	-.8564^{-2}	.2988^{-1}
.1448		.3	-.1885^{-1}	.6691^{-1}
.1900		.4	-.3273^{-1}	.1181
.2333		.5	-.4984^{-1}	.1829
.2744		.6	-.6977^{-1}	.2606
.3132		.7	-.9213^{-1}	.3501
.3495		.8	-.1165	.4504
.3833		.9	-.1424	.5604
.4146		1.0	-.1694	.6788
.5347		1.5	-.3076	1.3533
.6051		2.0	-.4264	2.0691
.6439		2.5	-.5130	2.7513
.6651		3.0	-.5708	3.3822
.6768		3.5	-.6076	3.9690
.6835		4.0	-.6305	4.5244
.6874		4.5	-.6446	5.0590
.6897		5.0	-.6533	5.5803
.6911		5.5	-.6585	6.0934
.6919		6.0	-.6617	6.6014
.6924		6.5	-.6637	7.1062
.6927		7.0	-.6648	7.6092
.6929		7.5	-.6656	8.1109
.6930		8.0	-.6660	8.6120
.6930		8.5	-.6663	9.1127
.6931		9.0	-.6664	9.6131
τ_2/τ_3 = 1.0; τ_1/τ_2 = 5.0				
.4969^{-1}		.1	-.2374^{-2}	.1249^{-1}
.9853^{-1}		.2	-.9371^{-2}	.4980^{-1}
.1462		.3	-.2077^{-1}	.1115
.1923		.4	-.3628^{-1}	.1967
.2376		.5	-.5559^{-1}	.3045
.2791		.6	-.7831^{-1}	.4334
.3191		.7	-.1040	.5818
.3568		.8	-.1323	.7478
.3919		.9	-.1626	.9291
.4244		1.0	-.1945	1.1235
.5482		1.5	-.3609	2.2080
.6179		2.0	-.5066	3.2973
.6538		2.5	-.6131	4.2581
.6720		3.0	-.6839	5.0753
.6814		3.5	-.7287	5.7806
.6864		4.0	-.7564	6.4094
.6892		4.5	-.7735	6.9890
.6908		5.0	-.7839	7.5378
.6918		5.5	-.7902	8.0675
.6923		6.0	-.7941	8.5857
.6926		6.5	-.7964	9.0967
.6928		7.0	-.7978	9.6034
.6930		7.5	-.7987	10.1074
.6930		8.0	-.7992	10.6099
.6931		8.5	-.7995	11.1114
τ_2/τ_3 = 2.0; τ_1/τ_2 = 0				
.2467^{-1}	0.7470^{-1}	.1		.6236^{-4}
.4863^{-1}	.1489	.2		.4991^{-3}
.7179^{-1}	.2226	.3		.1681^{-2}
.9408^{-1}	.2961	.4		.3973^{-2}
.1154	.3693	.5		.7731^{-2}
.1358	.4424	.6		.1330^{-1}
.1551	.5156	.7		.2099^{-1}
.1734	.5889	.8		.3113^{-1}
.1905	.6625	.9		.4400^{-1}
.2066	.7365	1.0		.5985^{-1}
.2702	1.1155	1.5		.1906
.3090	1.5166	2.0		.4151
.3297	1.9433	2.5		.7272
.3395	2.3931	3.0		1.1071
.3438	2.8604	3.5		1.5332
.3455	3.3397	4.0		1.9885
.3462	3.8269	4.5		2.4616
.3464	4.3191	5.0		2.9454
.3465	4.8143	5.5		3.4356

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TABLE III.- NORMALIZED OPTIMUM RESPONSE FOR A TYPE 1 THIRD-ORDER UNSTABLE PLANT WITH ZERO - Concluded

T_1/τ_2	T_2/τ_2	T_m/τ_2	b/B	$R_o/BK\tau_2$
τ_2/τ_3 = 2.0; τ_1/τ_2 = 0.5				
0.4649 -1		0.1	0.9051 -2	0.2501 -2
.8678 -1		.2	.3281 -1	.1002 -1
.1219		.3	.6706 -1	.2262 -1
.1527		.4	.1085	.4037 -1
.1796		.5	.1547	.6334 -1
.2033		.6	.2038	.9156 -1
.2240		.7	.2541	.1250
.2420		.8	.3046	.1637
.2577		.9	.3544	.2074
.2712		1.0	.4029	.2561
.3151		1.5	.6116	.5640
.3341		2.0	.7573	.9531
.3418		2.5	.8510	1.3909
.3448		3.0	.9092	1.8558
.3459		3.5	.9448	2.3358
.3463		4.0	.9665	2.8241
.3465		4.5	.9797	3.3172
τ_2/τ_3 = 2.0; τ_1/τ_2 = 2.0				
.4805 -1		.1	-.3040 -2	.9977 -2
.9196 -1		.2	-.1177 -1	.3963 -1
.1315		.3	-.2551 -1	.8811 -1
.1664		.4	-.4345 -1	.1540
.1970		.5	-.6470 -1	.2355
.2232		.6	-.8836 -1	.3304
.2454		.7	-.1136	.4363
.2641		.8	-.1395	.5509
.2796		.9	-.1656	.6720
.2923		1.0	-.1912	.7977
.3280		1.5	-.3018	1.4475
.3402		2.0	-.3775	2.0745
.3444		2.5	-.4252	2.6617
.3458		3.0	-.4545	3.2175
.3463		3.5	-.4724	3.7522
.3465		4.0	-.4833	4.2736
.3465		4.5	-.4898	4.7866
τ_2/τ_3 = 2.0; τ_1/τ_2 = 3.0				
.4824 -1		.1	-.3815 -2	.1496 -1
.9263 -1		.2	-.1489 -1	.5941 -1
.1328		.3	-.3251 -1	.1320
.1685		.4	-.5577 -1	.2303
.1997		.5	-.8359 -1	.3514
.2265		.6	-.1148	.4916
.2491		.7	-.1483	.6468
.2680		.8	-.1830	.8131
.2835		.9	-.2179	.9868
0.2962		1.0	-0.2523	1.1647
.3305		1.5	-.4014	2.0432
.3414		2.0	-.5030	2.8263
.3448		2.5	-.5669	3.5109
.3460		3.0	-.6060	4.1261
.3464		3.5	-.6299	4.6969
.3465		4.0	-.6443	5.2400
τ_2/τ_3 = 2.0; τ_1/τ_2 = 5.0				
.4839 -1		.1	-.4345 -2	.2494 -1
.9319 -1		.2	-.1707 -1	.9897 -1
.1339		.3	-.3751 -1	.2197
.1702		.4	-.6471 -1	.3831
.2020		.5	-.9753 -1	.5836
.2293		.6	-.1346	.8146
.2523		.7	-.1747	1.0688
.2714		.8	-.2164	1.3389
.2870		.9	-.2585	1.6183
.2996		1.0	-.3001	1.9010
.3327		1.5	-.4806	3.2373
.3425		2.0	-.6033	4.3316
.3453		2.5	-.6802	5.2103
.3462		3.0	-.7272	5.9438
.3464		3.5	-.7558	6.5863
.3465		4.0	-.7732	7.1730
τ_2/τ_3 = 5.0; τ_1/τ_2 = 0				
.2372 -1	0.7379 -1	.1		.1558 -3
.4476 -1	.1454	.2		.1238 -2
.6309 -1	.2151	.3		.4128 -2
.7873 -1	.2835	.4		.9625 -2
.9182 -1	.3510	.5		.1842 -1
.1026	.4181	.6		.3105 -1
.1113	.4853	.7		.4794 -1
.1182	.5529	.8		.6937 -1
.1235	.6213	.9		.9547 -1
.1276	.6908	1.0		.1263
.1368	1.0604	1.5		.3472
.1384	1.4686	2.0		.6603
.1386	1.9080	2.5		1.0389
τ_2/τ_3 = 5.0; τ_1/τ_2 = 0.5				
.4297 -1		.1	.1571 -1	.6208 -2
.7385 -1		.2	.5577 -1	.2442 -1
.9548 -1		.3	.1107	.5367 -1
.1103		.4	.1733	.9271 -1
.1203		.5	.2384	.1402
0.1269		0.6	0.3027	0.1949
.1312		.7	.3642	.2556
.1339		.8	.4219	.3212
.1357		.9	.4753	.3910
.1369		1.0	.5243	.4643
.1385		1.5	.7106	.8678
.1386		2.0	.8244	1.3106
τ_2/τ_3 = 5.0; τ_1/τ_2 = 2.0				
.4427 -1		.1	-.6625 -2	.2471 -1
.7720 -1		.2	-.2493 -1	.9545 -1
.1001		.3	-.5162 -1	.2031
.1151		.4	-.8311 -1	.3361
.1245		.5	-.1163	.4838
.1302		.6	-.1493	.6381
.1336		.7	-.1807	.7941
.1357		.8	-.2100	.9487
.1369		.9	-.2371	1.1003
.1376		1.0	-.2618	1.2484
.1386		1.5	-.3553	1.9334
τ_2/τ_3 = 5.0; τ_1/τ_2 = 3.0				
.4443 -1		.1	-.8635 -2	.3705 -1
.7763 -1		.2	-.3272 -1	.1429
.1007		.3	-.6815 -1	.3031
.1157		.4	-.1101	.4990
.1251		.5	-.1546	.7136
.1307		.6	-.1986	.9345
.1340		.7	-.2407	1.1540
.1359		.8	-.2799	1.3677
.1371		.9	-.3160	1.5738
.1377		1.0	-.3490	1.7715
.1386		1.5	-.4737	2.6440
τ_2/τ_3 = 5.0; τ_1/τ_2 = 5.0				
.4455 -1		.1	-.1017 -1	.6174 -1
.7799 -1		.2	-.3876 -1	.2378
.1012		.3	-.8110 -1	.5031
.1163		.4	-.1315	.8249
.1256		.5	-.1849	1.1736
.1311		.6	-.2380	1.5276
.1343		.7	-.2885	1.8740
.1362		.8	-.3357	2.2060
.1372		.9	-.3791	2.5209
.1378		1.0	-.4187	2.8179
.1386		1.5	-.5684	4.0650

TABLE IV.- NORMALIZED OPTIMUM RESPONSES FOR A TYPE 2 FOURTH-ORDER PLANT WITH ZERO

ω_nT_1	ω_nT_2	ω_nT_3	ω_nT_m	b/B	$R_o\omega_n^2/BK$
ζ = 0.10; $\omega_n\tau_1$ = 0					
0.5	1.5061	2.4715	2.9308		0.2050
1.0	2.4965	3.8584	4.7240		1.5036
1.5	3.1995	4.6451	5.8914		3.7642
2.0	3.7771	5.1630	6.7719		6.5393
2.5	4.3012	5.5559	7.5092		9.5765
3.0	4.8091	5.8930	8.1675		12.7548
3.5	5.3273	6.2178	8.7810		16.0227
4.0	5.8847	6.5713	9.3733		19.3772
4.5	6.5280	7.0118	9.9677		22.8769
5.0	7.3584	7.6569	10.5972		26.6670
5.5	8.6154	8.7735	11.3163		31.0297
6.0	10.8398	10.9577	12.2357		36.2864
6.5	13.0501	13.5887	14.0775		42.4894
7.0	14.1977	15.0782	15.7614		49.4334
7.5	15.1214	16.1535	17.0642		57.0636
8.0	15.9189	16.9835	18.1293		65.3088
8.5	16.6414	17.6578	19.0328		74.0104
9.0	17.3310	18.2503	19.8386		83.0742
9.5	18.0204	18.8143	20.5881		92.4397
10.0	18.7402	19.3962	21.3118		102.0811
10.5	19.5319	20.0508	22.0377		112.0403
11.0	20.4650	20.8632	22.7966		122.3828
11.5	21.6020	21.9130	23.6220		133.2165
12.0	23.0171	23.3084	24.5827		144.6568
12.5	24.5443	24.9449	25.8014		156.7788
13.0	25.8343	26.4370	27.2056		169.5661
13.5	26.9012	27.6595	28.5166		182.9750
14.0	27.8381	28.6714	29.6669		196.9730
14.5	28.7013	29.5481	30.6934		211.4666
15.0	29.5228	30.3358	31.6259		226.4358
ζ = 0.10; $\omega_n\tau_1$ = 0.5					
.4802 -1	.1475		.2	-0.1890 -2	.1253 -3
.9299 -1	.2899		.4	-.1253 -1	.1008 -2
.1360	.4272		.6	-.3536 -1	.3443 -2
.1780	.5603		.8	-.7084 -1	.8299 -2
.2196	.6901		1.0	-.1181	.1657 -1
.3252	1.0049		1.5	-.2810	.6016 -1
.4383	1.3138		2.0	-.4979	.1564
.7018	1.9204	2.9686	3.0		.6328
.9802	2.4272	3.6970	4.0		1.5125
ζ = 0.10; $\omega_n\tau_1$ = 1.0					
.4911 -1	.1489		.2	-.4078 -3	.2503 -3
.9684 -1	.2954		.4	-.2964 -2	.2008 -2
.1437	.4392		.6	-.9123 -2	.6812 -2
.1903	.5804		.8	-.1979 -1	.1627 -1
.2370	.7192		1.0	-.3550 -1	.3211 -1
.3568	1.0575		1.5	-.9853 -1	.1124
.4862	1.3877		2.0	-.1971	.2804
.8023	2.0467		3.0	-.5111	1.0897
1.2517	2.7405	3.9887	4.0		3.0332
ζ = 0.10; $\omega_n\tau_1$ = 2.0					
.4970 -1	.1496		.2	-.1369 -3	.5004 -3
.9906 -1	.2980		.4	-.1044 -2	.4012 -2
.1485	.4451		.6	-.3367 -2	.1359 -1
.1984	.5907		.8	-.7639 -2	.3238 -1
.2491	.7348		1.0	-.1431 -1	.6369 -1
.3820	1.0881		1.5	-.4390 -1	.2204
.5291	1.4332		2.0	-.9584 -1	.5409
.8981	2.1124		3.0	-.2857	2.0128
1.4432	2.8184		4.0	-.6248	5.3583
ζ = 0.10; $\omega_n\tau_1$ = 4.0					
0.5000 -1	0.1499		0.2	-0.6176 -4	0.1001 -2
.1003	.2993		.4	-.4829 -3	.8021 -2
.1511	.4479		.6	-.1595 -2	.2716 -1
.2030	.5956		.8	-.3705 -2	.6467 -1
.2563	.7421		1.0	-.7103 -2	.1271
.3983	1.1022		1.5	-.2302 -1	.4377
.5586	1.4529		2.0	-.5282 -1	1.0666
.7457	1.7945		2.5	-.1006	2.1543
.9699	2.1292		3.0	-.1704	3.8633
ζ = 0.10; $\omega_n\tau_1$ = 8.0					
.5016 -1	.1500		.2	-.3059 -4	.2001 -2
.1009	.2999		.4	-.2422 -3	.1604 -1
.1525	.4493		.6	-.8097 -3	.5430 -1
.2055	.5979		.8	-.1904 -2	.1293
.2602	.7455		1.0	-.3693 -2	.2539
.4075	1.1083		1.5	-.1231 -1	.8727
.5759	1.4600		2.0	-.2897 -1	2.1178
.7741	1.7986		2.5	-.5637 -1	4.2458
1.0122	2.1242		3.0	-.9702 -1	7.5252
ζ = 0.10; $\omega_n\tau_1$ = 16.0					
.5023 -1	.1501		.2	-.1540 -4	.4003 -2
.1012	.3002		.4	-.1227 -3	.3208 -1
.1532	.4499		.6	-.4128 -3	.1806
.2068	.5990		.8	-.9764 -3	.2585
.2623	.7471		1.0	-.1905 -2	.5075
.4125	1.1109		1.5	-.6442 -2	1.7427
.5852	1.4624		2.0	-.1536 -1	4.2186
.7894	1.7979		2.5	-.3018 -1	8.4184
1.0344	2.1160		3.0	-.5229 -1	14.8052
ζ = 0.25; $\omega_n\tau_1$ = 0					
.5	1.4949	2.3961	2.8024		.1702
1.0	2.5448	3.8009	4.5122		1.2094
1.5	3.3615	4.6917	5.6602		3.0567
2.0	4.0807	5.3559	6.5502		5.4192
2.5	4.7676	5.9242	7.3131		8.1249
3.0	5.4624	6.4699	8.0149		11.0983
3.5	6.2011	7.0496	8.6971		14.3261
4.0	7.0135	7.7072	9.3872		17.8490
4.5	7.9460	8.5061	10.1201		21.7439
5.0	9.0425	9.5100	10.9349		26.1118
5.5	10.3104	10.7539	11.8871		31.0588
6.0	11.6378	12.1510	13.0263		36.6344
6.5	12.8284	13.4541	14.2514		42.8564
7.0	13.9272	14.6581	15.4617		49.6402
7.5	14.9338	15.7335	16.5994		56.9950
8.0	15.8895	16.7188	17.6587		64.8703
8.5	16.8127	17.6405	18.6553		73.2419
9.0	17.7023	18.4919	19.5792		82.0943
9.5	18.6491	19.3934	20.4885		91.3236
10.0	19.6565	20.3853	21.4574		101.0258
ζ = 0.25; $\omega_n\tau_1$ = 0.5					
.4836 -1	.1479		.2	-.1644 -2	.1254 -3
.9425 -1	.2915		.4	-.1090 -1	.1008 -2
.1387	.4310		.6	-.3076 -1	.3440 -2
.1824	.5670		.8	-.6159 -1	.8285 -2
.2261	.7005		1.0	-.1026	.1652 -1
.3385	1.0277		1.5	-.2430	.5964 -1

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TABLE IV.- NORMALIZED OPTIMUM RESPONSES FOR A TYPE 2 FOURTH-ORDER PLANT WITH ZERO - Continued

$\omega_n T_1$	$\omega_n T_2$	$\omega_n T_3$	$\omega_n T_m$	b/B	$R_o \omega_n^2/BK$
ζ = 0.25; $\omega_n \tau_1$ = 0.5					
0.4602	1.3535		2.0	-0.4269	0.1538
.5972	1.6853		2.5	-.6477	.3301
.9124	2.3380	3.4256	3.5		1.0240
1.2745	2.9615	4.1870	4.5		2.2400
1.7534	3.6757	4.9223	5.5		4.3324
2.3530	4.4955	5.6425	6.5		7.4602
3.0456	5.4343	6.3887	7.5		11.5655
ζ = 0.25; $\omega_n \tau_1$ = 1.0					
.4947 -1	.1493		.2	-.3399 -3	.2503 -3
.9821 -1	.2970		.4	-.2471 -2	.2008 -2
.1467	.4429		.6	-.7603 -2	.6809 -2
.1954	.5872		.8	-.1649 -1	.1625 -1
.2447	.7299		1.0	-.2957 -1	.3204 -1
.3732	1.0823		1.5	-.8190 -1	.1118
.5141	1.4326		2.0	-.1631	.2771
.8598	2.1526		3.0	-.4144	1.0519
1.3414	2.9452		4.0	-.7925	2.8370
ζ = 0.25; $\omega_n \tau_1$ = 2.0					
.5006 -1	.1499		.2	-.1190 -3	.5005 -3
.1005	.2996		.4	-.9078 -3	.4011 -2
.1517	.4487		.6	-.2927 -2	.1358 -1
.2039	.5973		.8	-.6639 -2	.3235 -1
.2577	.7452		1.0	-.1243 -1	.6358 -1
.4010	1.1130		1.5	-.3806 -1	.2193
.5626	1.4798		2.0	-.8281 -1	.5357
.9723	2.2292		3.0	-.2431	1.9547
1.5593	3.0458		4.0	-.5136	5.0079
2.3682	3.9920		5.0	-.8762	10.0944
ζ = 0.25; $\omega_n \tau_1$ = 4.0					
.5037 -1	.1503		.2	-.5720 -4	.1001 -2
.1017	.3008		.4	-.4470 -3	.8019 -2
.1544	.4515		.6	-.1476 -2	.2714 -1
.2089	.6020		.8	-.3428 -2	.6460 -1
.2654	.7523		1.0	-.6567 -2	.1268
.4190	1.1265		1.5	-.2123 -1	.4356
.5961	1.4990		2.0	-.4855 -1	1.0568
.8058	1.8719		2.5	-.9193 -1	2.1200
1.3646	2.6385		3.5	-.2380	6.1157
2.1689	3.4909		4.5	-.4640	13.1341
ζ = 0.25; $\omega_n \tau_1$ = 8.0					
.5053 -1	.1504		.2	-.2944 -4	.2002 -2
.1024	.3014		.4	-.2329 -3	.1604 -1
.1559	.4528		.6	-.7785 -3	.5426 -1
.2115	.6042		.8	-.1830 -2	.1291
.2697	.7555		1.0	-.3546 -2	.2534
.4292	1.1321		1.5	-.1179 -1	.8684
.6158	1.5052		2.0	-.2765 -1	2.0982
.8392	1.8752		2.5	-.5348 -1	4.1800
1.4386	2.6195		3.5	-.1423	11.7570
ζ = 0.25; $\omega_n \tau_1$ = 16.0					
.5061 -1	.1505		.2	-.1511 -4	.4003 -2
.1027	.3017		.4	-.1203 -3	.3207 -1
.1566	.4534		.6	-.4047 -3	.1085
.2129	.6052		.8	-.9567 -3	.2581
.2719	.7569		1.0	-.1865 -2	.5064
.4347	1.1344		1.5	-.6290 -2	1.7339
.6265	1.5070		2.0	-.1493 -1	4.1791
.8572	1.8738		2.5	-.2918 -1	8.2885
1.1373	2.2359		3.0	-.5017 -1	14.4569
ζ = 0.50; $\omega_n \tau_1$ = 0					
0.5	1.4860	2.3159	2.6598		0.1319
1.0	2.6095	3.7520	4.2847		.9122
1.5	3.5710	4.8085	5.4752		2.3689
2.0	4.4574	5.6844	6.4538		4.3825
2.5	5.3274	6.4958	7.3366		6.8496
3.0	6.2052	7.2951	8.1798		9.7408
3.5	7.1083	8.1201	9.0234		13.0538
4.0	8.1246	9.0985	9.9477		16.7050
4.5	9.1587	10.1119	10.9063		20.8561
5.0	10.1779	11.1221	11.8889		25.5598
5.5	11.1779	12.1221	12.8889		30.8107
ζ = 0.50; $\omega_n \tau_1$ = 0.5					
.4893 -1	.1486		.2	-0.1233 -2	.1253 -3
.9634 -1	.2943		.4	-.8171 -2	.1007 -2
.1430	.4373		.6	-.2305 -1	.3432 -2
.1897	.5782		.8	-.4608 -1	.8244 -2
.2368	.7176		1.0	-.7658 -1	.1638 -1
.3594	1.0646		1.5	-.1792	.5836 -1
.4931	1.4158		2.0	-.3090	.1477
.6424	1.7782		2.5	-.4568	.3094
1.0016	2.5552		3.5	-.7860	.9725
1.4287	3.3730	4.4443	4.5		2.2092
1.9007	4.2134	5.3127	5.5		4.0405
2.4417	5.1499	6.2082	6.5		6.6428
3.0215	6.1639	7.1424	7.5		9.9752
ζ = 0.5; $\omega_n \tau_1$ = 1.0					
.5006 -1	.1499		.2	-.2265 -3	.2502 -3
.1005	.2997		.4	-.1646 -2	.2006 -2
.1516	.4491		.6	-.5061 -2	.6793 -2
.2038	.5983		.8	-.1096 -1	.1618 -1
.2573	.7475		1.0	-.1962 -1	.3182 -1
.3992	1.1222		1.5	-.5389 -1	.1098
.5561	1.5032		2.0	-.1058	.2678
.9323	2.3044		3.0	-.2559	.9645
1.4110	3.1853		4.0	-.4514	2.4123
1.9888	4.1582		5.0	-.6612	4.8025
2.6256	5.2043		6.0	-.8426	8.0890
3.2678	6.2877		7.0	-.9604	12.0388
4.4440	8.4505		9.0	-.9871	21.0322
5.4623	10.5277		11.0	-.8691	30.9252
6.4124	12.5266		13.0	-.7717	42.0995
7.3684	14.4982		15.0	-.7406	55.2518
8.3518	16.4769		17.0	-.7498	70.7319
9.3546	18.4706		19.0	-.7680	88.5147
10.3630	20.4734		21.0	-.7792	108.4113
ζ = 0.5; $\omega_n \tau_1$ = 2.0					
.5067 -1	.1506		.2	-.8924 -4	.5003 -3
.1029	.3022		.4	-.6803 -3	.4006 -2
.1569	.4547		.6	-.2191 -2	.1355 -1
.2131	.6081		.8	-.4961 -2	.3220 -1
.2717	.7625		1.0	-.9268 -2	.6312 -1
.4312	1.1530		1.5	-.2814 -1	.2156
.6133	1.5529		2.0	-.6042 -1	.5186
.8232	1.9658		2.5	-.1074	1.0286
1.3429	2.8475		3.5	-.2454	2.8793
1.9958	3.8187		4.5	-.4272	5.0272
2.7330	4.8672		5.5	-.6158	10.3320
4.1811	7.0561		7.5	-.8750	20.8069
5.3879	9.2198		9.5	-.9181	31.9569
6.3529	11.2929		11.5	-.8101	42.9159
7.2309	13.2791		13.5	-.7018	54.6799
8.1589	15.2434		15.5	-.6655	68.9510
9.1465	17.2230		17.5	-.6735	86.1143
10.1617	19.2198		19.5	-.6919	105.7668

TABLE IV.- NORMALIZED OPTIMUM RESPONSES FOR A TYPE 2 FOURTH-ORDER PLANT WITH ZERO - Continued

$\omega_n T_1$	$\omega_n T_2$	$\omega_n T_3$	$\omega_n T_m$	b/B	$R_o \omega_n^2/BK$
$\zeta = 0.5;\ \omega_n \tau_1 = 4.0$					
0.5099 -1	0.1509		0.2	-0.4956 -4	0.1001 -2
.1042	.3034		.4	-.3870 -3	.8009 -2
.1599	.4573		.6	-.1276 -2	.2707 -1
.2185	.6126		.8	-.2958 -2	.6429 -1
.2803	.7690		1.0	-.5653 -2	.1259
.4519	1.1657		1.5	-.1811 -1	.4281
.6532	1.5715		2.0	-.4085 -1	1.0233
.8908	1.9890		2.5	-.7590 -1	2.0110
1.4959	2.8738		3.5	-.1860	5.4771
2.2707	3.8416		4.5	-.3395	11.0210
4.0268	5.9609		6.5	-.6580	25.8268
5.6545	8.1275		8.5	-.8885	42.3835
$\zeta = 0.5;\ \omega_n \tau_1 = 8.0$					
.5115 -1	.1510		.2	-.2750 -4	.2001 -2
.1048	.3040		.4	-.2174 -3	.1602 -1
.1614	.4585		.6	-.7256 -3	.5412 -1
.2214	.6146		.8	-.1702 -2	.1285
.2851	.7719		1.0	-.3290 -2	.2514
.4638	1.1705		1.5	-.1083 -1	.8532
.6768	1.5766		2.0	-.2505 -1	2.0313
.9320	1.9918		2.5	-.4755 -1	3.9678
1.5923	2.8614		3.5	-.1202	10.5876
3.4139	4.8213		5.5	-.3356	32.7151
5.3321	6.9395		7.5	-.5356	58.0204
7.1241	9.0394		9.5	-.7087	85.0474
8.8832	11.1305		11.5	-.8757	115.4568
$\zeta = 0.5;\ \omega_n \tau_1 = 16.0$					
.5123 -1	.1511		.2	-.1462 -4	.4002 -2
.1052	.3042		.4	-.1163 -3	.3203 -1
.1622	.4591		.6	-.3906 -3	.1082
.2229	.6155		.8	-.9215 -3	.2569
.2875	.7732		1.0	-.1792 -2	.5025
.4701	1.1723		1.5	-.5982 -2	1.7031
.6896	1.5776		2.0	-.1400 -1	4.0453
.9544	1.9896		2.5	-.2685 -1	7.8704
1.2716	2.4106		3.0	-.4512 -1	13.4113
3.0381	4.2541		5.0	-.1605	50.2963
5.0827	6.3265		7.0	-.2820	93.4738
7.0018	8.3965		9.0	-.3882	136.2389
$\zeta = 0.75;\ \omega_n \tau_1 = 0$					
.5	1.4863	2.2708	2.5690		.1024
1.0	2.6800	3.7856	4.2111		.7183
1.5	3.6975	4.8829	5.3709		2.0022
2.0	4.6743	5.8647	6.3807		3.8115
2.5	5.6071	6.8049	7.3957		6.2306
3.0	6.6048	7.8037	8.3973		8.9842
3.5	7.6033	8.8026	9.3983		12.2387
4.0	8.6001	9.7999	10.4000		16.0025
4.5	9.6000	10.8000	11.4000		20.2513
5.0	10.6000	11.8000	12.4000		25.0000
5.5	11.6000	12.8000	13.4000		30.2500
$\zeta = 0.75;\ \omega_n \tau_1 = 0.5$					
.4949 -1	.1493		.2	-.8219 -3	.1251 -3
.9842 -1	.2971		.4	-.5442 -2	.1005 -2
.1473	.4435		.6	-.1533 -1	.3417 -2
.1967	.5891		.8	-.3055 -1	.8179 -2
.2470	.7343		1.0	-.5058 -1	.1617 -1
.3783	1.0992		1.5	-.1165	.5665 -1
.5208	1.4717		2.0	-.1964	.1402
.8479	2.2559		3.0	-.3682	.5144
1.2366	3.1041		4.0	-.5302	1.2990
1.6814	4.0148		5.0	-.6664	2.6244
2.1676	4.9753		6.0	-.7695	4.5555
2.6777	5.9680		7.0	-.8388	7.0923
3.1972	6.9775		8.0	-.8788	10.1983
4.2337	9.0083		10.0	-.9018	17.9589
5.2528	11.0295		12.0	-.8931	27.6400
6.2586	13.0382		14.0	-.8816	39.2163
7.2587	15.0400		16.0	-.8751	52.7308
8.2576	17.0394		18.0	-.8728	68.2257
9.2566	19.0385		20.0	-.8725	85.7226
10.2562	21.0380		22.0	-.8728	105.2271

$\omega_n T_1$	$\omega_n T_2$	$\omega_n T_3$	$\omega_n T_m$	b/B	$R_o \omega_n^2/BK$
$\zeta = 0.75;\ \omega_n \tau_1 = 1.0$					
0.5066 -1	0.1506		0.2	-0.1133 -3	0.2501 -3
.1027	.3023		.4	-.8220 -3	.2002 -2
.1565	.4552		.6	-.2522 -2	.6765 -2
.2120	.6093		.8	-.5448 -2	.1606 -1
.2695	.7646		1.0	-.9714 -2	.3144 -1
.4228	1.1596		1.5	-.2633 -1	.1068
.5915	1.5661		2.0	-.5067 -1	.2549
.7768	1.9863		2.5	-.8110 -1	.5007
1.1982	2.8720		3.5	-.1525	1.3738
1.6785	3.8156		4.5	-.2258	2.8581
2.1976	4.8030		5.5	-.2891	4.9938
2.7339	5.8158		6.5	-.3361	7.7454
3.2715	6.8386		7.5	-.3660	11.0486
4.3244	8.8808		9.5	-.3870	19.1082
5.3451	10.9032		11.5	-.3837	28.9816
6.3484	12.9102		13.5	-.3765	40.7045
7.3464	14.9104		15.5	-.3719	54.3631
8.3441	16.9090		17.5	-.3701	70.0142
9.3429	18.9080		19.5	-.3698	87.6783
10.3425	20.9075		21.5	-.3700	107.3555
$\zeta = 0.75;\ \omega_n \tau_1 = 2.0$					
.5128 -1	.1512		.2	-.5949 -4	.5001 -3
.1053	.3048		.4	-.4527 -3	.3999 -2
.1621	.4607		.6	-.1455 -2	.1349 -1
.2220	.6188		.8	-.3284 -2	.3196 -1
.2853	.7791		1.0	-.6111 -2	.6236 -1
.4587	1.1904		1.5	-.1830 -1	.2097
.6559	1.6174		2.0	-.3849 -1	.4937
1.1243	2.5223		3.0	-.1020	1.6177
1.6790	3.4920		4.0	-.1870	3.6186
2.7857	4.5088		5.0	-.2770	6.4820
2.9062	5.5494		6.0	-.3568	10.0589
4.0957	7.6376		8.0	-.4580	18.7198
5.1814	9.6930		10.0	-.4883	28.8283
6.1980	11.7140		12.0	-.4839	40.3542
7.1900	13.7168		14.0	-.4732	53.5975
8.1805	15.7142		16.0	-.4663	68.8016
9.1751	17.7116		18.0	-.4635	86.0589
10.1733	19.7103		20.0	-.4630	105.3716
$\zeta = 0.75;\ \omega_n \tau_1 = 4.0$					
.5161 -1	.1515		.2	-.4191 -4	.9999 -3
.1066	.3059		.4	-.3269 -3	.7993 -2
.1653	.4631		.6	-.1075 -2	.2695 -1
.2279	.6229		.8	-.2484 -2	.6378 -1
.2947	.7853		1.0	-.4727 -2	.1243
.4821	1.2023		1.5	-.1492 -1	.4163
.7013	1.6354		2.0	-.3296 -1	.9737
1.2418	2.5519		3.0	-.9496 -1	3.1289
1.9039	3.5313		4.0	-.1863	6.8073
2.6406	4.5560		5.0	-.2923	11.8074
4.1436	6.6586		7.0	-.4925	24.3369
5.5287	8.7604		9.0	-.6341	38.6541
6.7389	10.8326		11.0	-.7032	53.6221
7.7906	12.8669		13.0	-.7119	68.7701
8.7663	14.8735		15.0	-.6964	84.7986
9.7357	16.8701		17.0	-.6828	102.6709
$\zeta = 0.75;\ \omega_n \tau_1 = 8.0$					
.5177 -1	.1517		.2	-.2556 -4	.2000 -2
.1073	.3065		.4	-.2017 -3	.1598 -1
.1669	.4642		.6	-.6717 -3	.5387 -1
.2310	.6247		.8	-.1570 -2	.1274
.2999	.7878		1.0	-.3021 -2	.2483
.4955	1.2063		1.5	-.9798 -2	.8293
.7285	1.6397		2.0	-.2218 -1	1.9328
1.3173	2.5509		3.0	-.6662 -1	6.1338
2.0575	3.5178		4.0	-.1349	13.0935
3.7730	5.5592		6.0	-.3034	32.6550
5.5142	7.6431		8.0	-.4678	55.8863
7.1848	9.7265		10.0	-.6146	81.4821
8.7623	11.8051		12.0	-.7393	108.8689

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TABLE IV.- NORMALIZED OPTIMUM RESPONSES FOR A TYPE 2 FOURTH-ORDER PLANT WITH ZERO - Continued

$\omega_n T_1$	$\omega_n T_2$	$\omega_n T_3$	$\omega_n T_m$	b/B	$R_o \omega_n^2/BK$
$\zeta = 0.75;\ \omega_n \tau_1 = 16.0$					
0.5185 -1	0.1517		0.2	-0.1413 -4	0.3999 -2
.1077	.3068		.4	-.1122 -3	.3196 -1
.1678	.4648		.6	-.3759 -3	.1077
.2326	.6256		.8	-.8837 -3	.2548
.3026	.7890		1.0	-.1710 -2	.4961
.5027	1.2077		1.5	-.5623 -2	1.6552
.7433	1.6402		2.0	-.1289 -1	3.8491
1.0285	2.0857		2.5	-.2411 -1	7.3027
1.3595	2.5439		3.0	-.3945 -1	12.1162
3.0397	4.4883		5.0	-.1314	42.5620
4.9321	6.5275		7.0	-.2381	81.3945
6.8108	8.5801		9.0	-.3413	123.2668
$\zeta = 1.01;\ \omega_n \tau_1 = 0$					
.50	1.4669	2.1794	2.4250		.9230 -1
1.0	2.7049	3.7414	4.0729		.6130
1.5	3.8405	5.0533	5.4256		1.6816
2.0	4.9245	6.2440	6.6391		3.3028
2.5	5.9769	7.3633	7.7727		5.4634
3.0	7.0020	8.4186	8.8332		8.1672
3.5	8.0289	9.4802	9.9025		11.3698
4.0	9.0365	10.4962	10.9195		15.1056
4.5	10.0448	11.5146	11.9399		19.3418
5.0	11.0507	12.5290	12.9564		24.0757
5.5	12.0585	13.5461	13.9754		29.3154
6.0	13.0587	14.5463	14.9753		35.0660
6.5	14.0588	15.5463	15.9752		41.3152
7.0	15.0587	16.5464	16.9753		48.0634
7.5	16.0589	17.5469	17.9760		55.3118
8.0	17.0658	18.5632	18.9949		63.0523
$\zeta = 1.02;\ \omega_n \tau_1 = 0.5$					
.5011 -1	.1500		.2	-.3775 -3	.1252 -3
.1006	.3000		.4	-.2499 -2	.1003 -2
.1519	.4501		.6	-.7016 -2	.3396 -2
.2040	.6006		.8	-.1393 -1	.8085 -2
.2573	.7516		1.0	-.2293 -1	.1588 -1
.3965	1.1335		1.5	-.5179 -1	.5446 -1
.5452	1.5239		2.0	-.8497 -1	.1313
.8734	2.3360		3.0	-.1495	.4558
1.2414	3.1910		4.0	-.2020	1.0954
1.6439	4.0836		5.0	-.2410	2.1365
2.0738	5.0066		6.0	-.2687	3.6391
2.5246	5.9526		7.0	-.2881	5.6399
2.9905	6.9152		8.0	-.3015	8.1575
3.4672	7.8896		9.0	-.3106	11.1992
3.9514	8.8722		10.0	-.3168	14.7658
4.4407	9.8605		11.0	-.3210	18.8549
4.9336	10.8526		12.0	-.3239	23.4631
5.4288	11.8473		13.0	-.3258	28.5865
5.9256	12.8438		14.0	-.3271	34.2219
6.4234	13.8414		15.0	-.3279	40.3663
6.9220	14.8398		16.0	-.3285	47.0176
7.4210	15.8388		17.0	-.3289	54.1738
7.9204	16.8381		18.0	-.3292	61.8337
8.4200	17.8376		19.0	-.3293	69.9963
8.9197	18.8373		20.0	-.3295	78.6608
9.9194	20.8370		22.0	-.3296	97.4934
10.9192	22.8368		24.0	-.3296	118.3289
11.9192	24.8368		26.0	-.3297	141.1659
12.9191	26.8367		28.0	-.3297	166.0035
13.9191	28.8367		30.0	-.3297	192.8414
$\zeta = 1.02;\ \omega_n \tau_1 = 1.0$					
.5130 -1	.1513		.2	.9033 -5	.2500 -3
.1052	.3052		.4	.6559 -4	.1996 -2
.1616	.4617		.6	.2007 -3	.6721 -2
.2205	.6207		.8	.4315 -3	.1588 -1
.2818	.7822		1.0	.7651 -3	.3088 -1
.4453	1.1963		1.5	.2035 -2	.1028
.6221	1.6240		2.0	.3819 -2	.2390
1.0100	2.5141		3.0	.8216 -2	.7679
1.4335	3.4399		4.0	.1271 -1	1.7143
1.8823	4.3906		5.0	.1662 -1	3.1348
2.3484	5.3582		6.0	.1972 -1	5.0578
2.8260	6.3371		7.0	.2205 -1	7.4955
3.3114	7.3232		8.0	.2374 -1	10.4510
3.8017	8.3142		9.0	.2494 -1	13.9235
4.2954	9.3083		10.0	0.2577 -1	17.9106
4.7913	10.3045		11.0	.2634 -1	22.4094
5.2886	11.3019		12.0	.2673 -1	27.4173
5.7868	12.3003		13.0	.2700 -1	32.9322
6.2856	13.2992		14.0	.2718 -1	38.9521
6.7848	14.2985		15.0	.2730 -1	45.4759
7.2843	15.2980		16.0	.2738 -1	52.5024
7.7840	16.2977		17.0	.2744 -1	60.0308
8.2838	17.2975		18.0	.2747 -1	68.0607
8.7836	18.2974		19.0	.2750 -1	76.5916
9.2835	19.2973		20.0	.2751 -1	85.6232
10.2834	21.2972		22.0	.2753 -1	105.1879
11.2834	23.2971		24.0	.2754 -1	126.7535
12.2833	25.2971		26.0	.2754 -1	150.3197
13.2833	27.2971		28.0	.2754 -1	175.8861
14.2833	29.2971		30.0	.2755 -1	203.4526
$\zeta = 1.02;\ \omega_n \tau_1 = 2.0$					
.5194 -1	.1519		.2	-.2734 -4	.4996 -3
.1078	.3076		.4	-.2077 -3	.3987 -2
.1676	.4669		.6	-.6650 -3	.1340 -1
.2314	.6299		.8	-.1494 -2	.3158 -1
.2991	.7963		1.0	-.2764 -2	.6123 -1
.4849	1.2267		1.5	-.8107 -2	.2016
.6925	1.6759		2.0	-.1662 -1	.4620
1.1605	2.6191		3.0	-.4142 -1	1.4270
1.6742	3.6024		4.0	-.7176 -1	3.0356
2.2087	4.6065		5.0	-.1022	5.2722
2.7481	5.6189		6.0	-.1292	8.0870
3.2842	6.6329		7.0	-.1513	11.4330
3.8140	7.6455		8.0	-.1685	15.2770
4.3370	8.6558		9.0	-.1813	19.6004
4.8541	9.6636		10.0	-.1905	24.3953
5.3665	10.6694		11.0	-.1971	29.6608
5.8752	11.6736		12.0	-.2016	35.3994
6.3812	12.6765		13.0	-.2047	41.6154
6.8854	13.6785		14.0	-.2069	48.3132
7.3882	14.6799		15.0	-.2083	55.4968
7.8902	15.6809		16.0	-.2093	63.1697
8.3915	16.6815		17.0	-.2100	71.3345
8.8923	17.6819		18.0	-.2104	79.9934
9.3929	18.6822		19.0	-.2107	89.1480
9.8933	19.6824		20.0	-.2109	98.7996
10.8938	21.6826		22.0	-.2111	119.5967
11.8940	23.6827		24.0	-.2112	142.3893
12.8940	25.6828		26.0	-.2113	167.1796
13.8941	27.6828		28.0	-.2113	193.9688
14.8941	29.6828		30.0	-.2113	222.7575
$\zeta = 1.02;\ \omega_n \tau_1 = 4.0$					
.5227 -1	.1522		.2	-.3364 -4	.9988 -3
.1092	.3087		.4	-.2619 -3	.7969 -2
.1710	.4693		.6	-.8584 -3	.2676 -1
.2377	.6337		.8	-.1973 -2	.6301 -1
.3094	.8020		1.0	-.3731 -2	.1220
.5109	1.2378		1.5	-.1153 -1	.3999
.7428	1.6932		2.0	-.2481 -1	.9106
1.2855	2.6507		3.0	-.6738 -1	2.7605
1.9016	3.6502		4.0	-.1257	5.7302
2.5534	4.6713		5.0	-.1910	9.6788
3.2130	5.7008		6.0	-.2561	14.4199
3.8636	6.7314		7.0	-.3161	19.7931
4.4964	7.7595		8.0	-.3685	25.6781
5.1077	8.7835		9.0	-.4121	31.9904
5.6971	9.8031		10.0	-.4470	38.6765
6.2664	10.8184		11.0	-.4740	45.7100
6.8183	11.8299		12.0	-.4942	53.0865
7.3563	12.8384		13.0	-.5089	60.8167
7.8833	13.8446		14.0	-.5194	68.9205
8.4023	14.8489		15.0	-.5267	77.4207
8.9154	15.8519		16.0	-.5318	86.3393
9.4244	16.8539		17.0	-.5353	95.6959
9.9305	17.8553		18.0	-.5376	105.5061
10.4346	18.8563		19.0	-.5392	115.7824
10.9374	19.8569		20.0	-.5403	126.5343
11.9405	21.8576		22.0	-.5415	149.4905
12.9419	23.8579		24.0	-.5420	174.4106
13.9425	25.8581		26.0	-.5422	201.3130
14.9428	27.8582		28.0	-.5423	230.2069
15.9429	29.8582		30.0	-.5424	261.0966

TABLE IV.- NORMALIZED OPTIMUM RESPONSES FOR A TYPE 2 FOURTH-ORDER PLANT WITH ZERO - Continued

$\omega_n T_1$	$\omega_n T_2$	$\omega_n T_3$	$\omega_n T_m$	b/B	$R_o \omega_n^2/BK$
$\zeta = 1.02;\ \omega_n \tau_1 = 8.0$					
0.5244^{-1}	0.1523		0.2	-0.2345^{-4}	0.1998^{-2}
.1099	.3092		.4	-.1847^{-3}	.1593^{-1}
.1728	.4703		.6	-.6127^{-3}	.5349^{-1}
.2411	.6354		.8	-.1425^{-2}	.1259
.3151	.8042		1.0	-.2724^{-2}	.2436
.5259	1.2413		1.5	-.8648^{-2}	.7964
.7733	1.6970		2.0	-.1907^{-1}	1.8074
1.3684	2.6519		3.0	-.5414^{-1}	5.4243
2.0663	3.6460		4.0	-.1051	11.1110
2.8248	4.6615		5.0	-.1658	18.4949
3.6105	5.6870		6.0	-.2309	27.1654
4.4024	6.7161		7.0	-.2966	36.8144
5.1880	7.7457		8.0	-.3606	47.2332
5.9597	8.7743		9.0	-.4214	58.2750
6.7128	9.8010		10.0	-.4780	69.8233
7.4436	10.8254		11.0	-.5296	81.7747
8.1494	11.8471		12.0	-.5756	94.0330
8.8282	12.8660		13.0	-.6156	106.5100
9.4792	13.8819		14.0	-.6493	119.1329
10.1027	14.8949		15.0	-.6769	131.8521
10.7005	15.9053		16.0	-.6988	144.6486
11.2755	16.9133		17.0	-.7156	157.5345
11.8314	17.9192		18.0	-.7280	170.5485
12.3718	18.9235		19.0	-.7371	183.7453
12.9005	19.9266		20.0	-.7435	197.1851
13.9343	21.9303		22.0	-.7510	225.0144
14.9501	23.9320		24.0	-.7545	254.3928
15.9572	25.9328		26.0	-.7561	285.5386
16.9604	27.9331		28.0	-.7568	318.5694
17.9619	29.9333		30.0	-.7571	353.5449
$\zeta = 1.02;\ \omega_n \tau_1 = 16.0$					
.5253^{-1}	.1524		.2	-.1359^{-4}	.3995^{-2}
.1103	.3095		.4	-.1077^{-3}	.3186^{-1}
.1737	.4708		.6	-.3594^{-3}	.1070
.2429	.6361		.8	-.8406^{-3}	.2516
.3181	.8052		1.0	-.1616^{-2}	.4868
.5339	1.2423		1.5	-.5200^{-2}	1.5894
.7900	1.6971		2.0	-.1161^{-1}	3.5997
1.4159	2.6465		3.0	-.3367^{-1}	10.7362
2.1635	3.6313		4.0	-.6653^{-1}	21.7946
2.9887	4.6356		5.0	-.1066	35.8876
3.8557	5.6498		6.0	-.1507	52.1135
4.7415	6.6683		7.0	-.1967	69.8518
5.6337	7.6886		8.0	-.2431	88.7412
6.5257	8.7094		9.0	-.2895	108.5830
7.4139	9.7302		10.0	-.3355	129.2639
8.2962	10.7506		11.0	-.3807	150.7110
9.1711	11.7705		12.0	-.4251	172.8677
10.0372	12.7899		13.0	-.4684	195.6816
10.8933	13.8086		14.0	-.5106	219.0971
11.7380	14.8266		15.0	-.5514	243.0519
12.5695	15.8439		16.0	-.5907	267.4746
13.3860	16.8603		17.0	-.6282	292.2822
14.1856	17.8758		18.0	-.6637	317.3785
14.9658	18.8902		19.0	-.6969	342.6547
15.7242	19.9035		20.0	-.7276	367.9898
17.1663	21.9262		22.0	-.7800	418.3243
18.4972	23.9431		24.0	-.8193	467.4505
19.7169	25.9545		26.0	-.8453	514.9228
20.8452	27.9611		28.0	-.8605	561.0995
21.9122	29.9646		30.0	-.8684	606.9106
$\zeta = 1.5;\ \omega_n \tau_1 = 0$					
.5	1.4788	2.1565	2.3554		.6055^{-1}
1.0	2.7742	3.7852	4.0222		.4570
1.5	3.9937	5.2372	5.4870		1.3250
2.0	5.1628	6.5809	6.8362		2.7136
2.5	6.2944	7.8469	8.1050		4.6418
3.0	7.3991	9.0578	9.3176		7.1112
3.5	8.4824	10.2255	10.4863		10.1212
4.0	9.5492	11.3598	11.6212		13.6696
4.5	10.6033	12.4683	12.7301		17.7484
5.0	11.6466	13.5552	13.8175		22.3612
5.5	12.6819	14.6263	14.8887		27.4890
6.0	13.7120	15.6865	15.9491		33.1393
6.5	14.7354	16.7335	16.9963		39.3103
7.0	15.7552	17.7730	18.0358		45.9928
7.5	16.7713	18.8055	19.0684		53.1826
8.0	17.7839	19.8306	20.0936		60.8881
8.5	18.7953	20.8536	21.1166		69.0974
9.0	19.8032	21.8695	22.1325		77.8152
9.5	20.8115	22.8861	23.1492		87.0380
10.0	21.8176	23.8982	24.1613		96.7672
$\zeta = 1.5;\ \omega_n \tau_1 = 0.5$					
.5119^{-1}	.1513		.2	0.4098^{-3}	.1252^{-3}
.1045	.3052		.4	.2700^{-2}	.9964^{-3}
.1596	.4615		.6	.7527^{-2}	.3342^{-2}
.2161	.6198		.8	.1479^{-1}	.7865^{-2}
.2739	.7799		1.0	.2403^{-1}	.1523^{-1}
.4227	1.1857		1.5	.5217^{-1}	.5014^{-1}
.5765	1.5970		2.0	.8180^{-1}	.1158
.8978	2.4306		3.0	.1316	.3718
1.2369	3.2782		4.0	.1651	.8452
1.5942	4.1408		5.0	.1863	1.5923
1.9691	5.0190		6.0	.1999	2.6639
2.3604	5.9126		7.0	.2087	4.1043
2.7667	6.8204		8.0	.2145	5.9508
3.1866	7.7412		9.0	.2186	8.2340
3.6183	8.6737		10.0	.2214	10.9782
4.0605	9.6164		11.0	.2235	14.2023
4.5118	10.5680		12.0	.2250	17.9206
4.9707	11.5273		13.0	.2261	22.1434
5.4364	12.4931		14.0	.2269	26.8780
5.9076	13.4645		15.0	.2276	32.1290
6.3836	14.4406		16.0	.2281	37.8991
6.8636	15.4207		17.0	.2285	44.1898
7.3469	16.4041		18.0	.2288	51.0015
7.8331	17.3904		19.0	.2291	58.3337
8.3216	18.3789		20.0	.2293	66.1855
9.3042	20.3616		22.0	.2296	83.4430
10.2922	22.3497		24.0	.2298	102.7627
11.2840	24.3415		26.0	.2299	124.1329
12.2784	26.3359		28.0	.2300	147.5430
13.2746	28.3321		30.0	.2301	172.9842
$\zeta = 1.5;\ \omega_n \tau_1 = 1.0$					
.5243^{-1}	.1525		.2	.2258^{-3}	.2496^{-3}
.1094	.3102		.4	.1629^{-2}	.1982^{-2}
.1703	.4728		.6	.4943^{-2}	.6613^{-2}
.2346	.6398		.8	.1051^{-1}	.1544^{-1}
.3016	.8108		1.0	.1839^{-1}	.2960^{-1}
.4774	1.2509		1.5	.4696^{-1}	.9442^{-1}
.6606	1.7027		2.0	.8428^{-1}	.2100
1.0373	2.6208		3.0	.1670	.6245
1.4234	3.5452		4.0	.2437	1.3243
1.8193	4.4729		5.0	.3073	2.3534
2.2257	5.4047		6.0	.3580	3.7512
2.6429	6.3417		7.0	.3975	5.5519
3.0705	7.2846		8.0	.4281	7.7842
3.5079	8.2337		9.0	.4518	10.4718
3.9540	9.1891		10.0	.4701	13.6335
4.4080	10.1502		11.0	.4844	17.2838
4.8690	11.1168		12.0	.4956	21.4337
5.3361	12.0883		13.0	.5044	26.0910
5.8083	13.0640		14.0	.5114	31.2612
6.2851	14.0435		15.0	.5170	36.9476
6.7656	15.0263		16.0	.5214	43.1522
7.2493	16.0118		17.0	.5250	49.8758
7.7358	16.9997		18.0	.5279	57.1184
8.2245	17.9896		19.0	.5303	64.8793
8.7151	18.9812		20.0	.5322	73.1575
9.7009	20.9684		22.0	.5350	91.2614
10.6911	22.9596		24.0	.5369	111.4194
11.6844	24.9535		26.0	.5382	133.6210
12.6798	26.9494		28.0	.5391	157.8570
13.6767	28.9465		30.0	.5396	184.1196
$\zeta = 1.5;\ \omega_n \tau_1 = 2.0$					
.5310^{-1}	.1531		.2	.2965^{-4}	.4987^{-3}
.1122	.3125		.4	.2241^{-3}	.3957^{-2}
.1770	.4777		.6	.7114^{-3}	.1318^{-1}
.2468	.6484		.8	.1580^{-2}	.3068^{-1}
.3212	.8240		1.0	.2880^{-2}	.5861^{-1}
.5221	1.2802		1.5	.8093^{-2}	.1846

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TABLE IV.- NORMALIZED OPTIMUM RESPONSES FOR A TYPE 2 FOURTH-ORDER PLANT WITH ZERO - Continued

ω_nT_1	ω_nT_2	ω_nT_s	ω_nT_m	b/B	$R_0\omega_n^2/BK$
ζ = 1.5; $\omega_n\tau_1$ = 2.0					
0.7375	1.7533		2.0	0.1582 $^{-1}$	0.4037
1.1884	2.7245		3.0	.3616 $^{-1}$	1.1492
1 6489	3.7080		4.0	.5908 $^{-1}$	2.3214
2.1134	4.6951		5.0	.8163 $^{-1}$	3.9341
2.5812	5.6836		6.0	.1025	6.0019
3.0523	6.6734		7.0	.1210	8.5386
3.5270	7.6642		8.0	.1372	11.5562
4.0050	8.6562		9.0	.1512	15.0638
4.4862	9.6493		10.0	.1631	19.0684
4.9701	10.6433		11.0	.1731	23.5750
5.4565	11.6381		12.0	.1816	28.5870
5.9450	12.6338		13.0	.1887	34.1066
6.4354	13.6301		14.0	.1947	40.1350
6.9273	14.6270		15.0	.1996	46.6728
7.4206	15.6244		16.0	.2038	53.7202
7.9150	16.6222		17.0	.2072	61.2769
8.4103	17.6204		18.0	.2101	69.3424
8.9065	18.6189		19.0	.2124	77.9161
9.4032	19.6176		20.0	.2144	86.9975
10.3984	21.6157		22.0	.2174	106.6803
11.3950	23.6144		24.0	.2194	128.3854
12.3927	25.6135		26.0	.2208	152.1080
13.3911	27.6129		28.0	.2217	177.8442
14.3901	29.6125		30.0	.2224	205.5907
ζ = 1.5; $\omega_n\tau_1$ = 4.0					
.5345 $^{-1}$	.1534		.2	-.1900 $^{-4}$	.9972 $^{-3}$
.1138	.3135		.4	-.1470 $^{-3}$	.7907 $^{-2}$
.1807	.4798		.6	-.4775 $^{-3}$	.2631 $^{-1}$
.2540	.6518		.8	-.1084 $^{-2}$	.6118 $^{-1}$
.3330	.8290		1.0	-.2020 $^{-2}$	.1167
.5519	1.2900		1.5	-.5977 $^{-2}$	.3659
.7938	1.7693		2.0	-.1226 $^{-1}$	.7945
1.3179	2.7567		3.0	-.3058 $^{-1}$	2.2191
1.8679	3.7600		4.0	-.5391 $^{-1}$	4.3741
2.4272	4.7682		5.0	-.7949 $^{-1}$	7.2064
2.9884	5.7776		6.0	-.1054	10.6667
3.5477	6.7868		7.0	-.1305	14.7142
4 1029	7.7953		8.0	-.1538	19.3159
4.6531	8.8029		9.0	-.1751	24.4458
5.1980	9.8095		10.0	-.1942	30.0840
5.7374	10.8153		11.0	-.2110	36.2158
6.2717	11.8203		12.0	-.2257	42.8306
6.8013	12.8246		13.0	-.2384	49.9215
7.3267	13.8282		14.0	-.2492	57.4843
7.8482	14.8313		15.0	-.2585	65.5168
8.3665	15.8338		16.0	-.2663	74.0185
8.8819	16.8360		17.0	-.2729	82.9901
9.3948	17.8378		18.0	-.2785	92.4329
9.9056	18.8393		19.0	-.2832	102 3487
10.4147	19.8405		20.0	-.2871	112.7397
11.4285	21.8424		22.0	-.2930	134.9561
12.4380	23.8437		24.0	-.2971	159.0998
13.4446	25.8446		26.0	-.3000	185.1863
14.4491	27.8453		28.0	-.3019	213.2292
15.4522	29.8457		20.0	-.3033	243.2389
ζ = 1.5; $\omega_n\tau_1$ = 8.0					
.5362 $^{-1}$	.1535		.2	-.1971 $^{-4}$	.1994 $^{-2}$
.1146	.3139		.4	-.1544 $^{-3}$	.1581 $^{-1}$
.1827	.4807		.6	-.5073 $^{-3}$	.5258 $^{-1}$
.2579	.6532		.8	-.1165 $^{-2}$	.1222
.3396	.8308		1.0	-.2195 $^{-2}$	.2330
.5693	1.2927		1.5	-.6667 $^{-2}$	.7285
.8286	1.7725		2.0	-.1403 $^{-1}$	1.5773
1.4077	2.7608		3.0	-.3674 $^{-1}$	4.3735
2.0370	3.7657		4.0	-.6784 $^{-1}$	8.5432
2.6949	4.7767		5.0	-.1045	13.9274
3.3683	5.7899		6.0	-.1446	20.3725
4.0485	6.8037		7.0	-.1862	27.7439
4.7293	7.8173		8.0	-.2280	35.9273
5.4061	8.8303		9.0	-.2690	44.8263
6.0758	9.8424		10.0	-.2084	54.3601
6.7361	10.8535		11.0	-.3456	64.4612

ω_nT_1	ω_nT_2	ω_nT_s	ω_nT_m	b/B	$R_0\omega_n^2/BK$
ζ = 1.5; $\omega_n\tau_1$ = 8.0					
7.3854	11.8636		12.0	-0.3804	75.0732
8.0230	12.8727		13.0	-.4126	86.1493
8.6483	13.8808		14.0	-.4419	97.6521
9.2613	14.8880		15.0	-.4683	109.5515
9.8622	15.8943		16.0	-.4920	121.8253
10.4516	16.8998		17.0	-.5130	134.4581
11.0302	17.9046		18.0	-.5314	147.4404
11.5987	18.9087		19.0	-.5475	160.7682
12.1580	19.9123		20.0	-.5614	174.4423
ζ = 1.5; $\omega_n\tau_1$ = 16.0					
.5371 $^{-1}$	.1536		.2	-.1265 $^{-4}$	.3990 $^{-2}$
.1150	.3142		.4	-.9957 $^{-4}$	.3161 $^{-1}$
.1837	.4811		.6	-.3291 $^{-3}$	.1051
.2599	.6538		.8	-.7601 $^{-3}$	.2442
.3430	.8315		1.0	-.1440 $^{-2}$	.4654
.5788	1.2933		1.5	-.4433 $^{-2}$	1.4538
.8480	1.7725		2.0	-.9446 $^{-2}$	3.1428
1.4608	2.7579		3.0	-.2536 $^{-1}$	8.6825
2.1428	3.7591		4.0	-.4795 $^{-1}$	16.8844
2.8714	4.7664		5.0	-.7562 $^{-1}$	27.3811
3.6319	5.7763		6.0	-.1069	39.8194
4.4137	6.7875		7.0	-.1408	53.8951
5.2087	7.7991		8.0	-.1762	69.3584
6.0110	8.8107		9.0	-.2125	86.0090
6.8159	9.8222		10.0	-.2492	103.6886
7.6199	10.8334		11.0	-.2858	122.2709
8.4205	11.8442		12.0	-.3220	141.6543
9.2156	12.8545		13.0	-.3576	161.7549
10.0035	13.8643		14.0	-.3924	182.5011
10.7829	14.8736		15.0	-.4262	203.8297
11.5527	15.8824		16.0	-.4588	225.6834
12.3118	16.8907		17.0	-.4901	248.0084
13.0594	17.8984		18.0	-.5201	270.7539
13.7948	18.9056		19.0	-.5487	293.8707
14.5174	19.9123		20.0	-.5756	317.3113
ζ = 2.0; $\omega_n\tau_1$ = 0					
.5	1.4819	2.1243	2.2851		.0443
1.0	2.8138	3.8071	3.9862		.3690
1.5	4.0798	5.3431	5.5264		1.1165
2.0	5.3099	6.8051	6.9899		2.3150
2.5	6.5017	8.1884	8.3737		4.0314
3.0	7.6579	9.5009	9.6864		6.2888
3.5	8.7910	10.7675	10.9530		9.0772
4.0	9.9023	11.9901	12.1757		12.4183
4.5	10.9965	13.1782	13.3638		16.3031
5.0	12.0706	14.3269	14.5126		20.7466
5.5	13.1471	15.4800	15.6657		25.6714
6.0	14.2075	16.6005	16.7863		31.1648
6.5	15.2583	17.7021	17.8879		37.1886
7.0	16.3028	18.7911	18.9769		43.7366
7.5	17.3408	19.8673	20.0529		50.8034
8.0	18.3685	20.9228	21.1085		58.4224
8.5	19.4042	21.9941	22.1797		66.5052
9.0	20.4227	23.0311	23.2168		75.1658
9.5	21.4511	24.0878	24.2735		84.2782
10.0	22.4654	25.1160	25.3016		93.9667
ζ = 2.0; $\omega_n\tau_1$ = 0.5					
.5231 $^{-1}$	.1526		.2	.1225 $^{-2}$	.1260 $^{-3}$
.1084	.3104		.4	.8028 $^{-2}$	.9865 $^{-3}$
.1671	.4726		.6	.2214 $^{-1}$	.3270 $^{-2}$
.2274	.6381		.8	.4292 $^{-1}$	.7580 $^{-2}$
.2886	.8057		1.0	.6868 $^{-1}$	.1444 $^{-1}$
.4430	1.2288		1.5	.1430	.4559 $^{-1}$
.5979	1.6518		2.0	.2157	.1014
.9098	2.4912		3.0	.3255	.3086
1.2290	3.3262		4.0	.3890	.6780
1.5591	4.1648		5.0	.4230	1.2518
1.9019	5.0121		6.0	.4408	2.0714
2.2579	5.8705		7.0	.4504	3.1758
2.6269	6.7408		8.0	.4557	4.6013

TABLE IV.- NORMALIZED OPTIMUM RESPONSES FOR A TYPE 2 FOURTH-ORDER PLANT WITH ZERO - Concluded

$\omega_n T_1$	$\omega_n T_2$	$\omega_n T_3$	$\omega_n T_m$	b/B	$R_o \omega_n^2/BK$
ζ = 2.0; $\omega_n \tau_1$ = 0.5					
3.0082	7.6229		9.0	0.4587	6.3803
3.4012	8.5163		10.0	.4604	8.5422
3.8050	9.4204		11.0	.4615	11.1123
4.2188	10.3344		12.0	.4622	14.1120
4.6418	11.2574		13.0	.4627	17.5608
5.0731	12.1888		14.0	.4630	21.4749
5.5120	13.1278		15.0	.4632	25.8665
5.9577	14.0735		16.0	.4634	30.7474
6.4096	15.0255		17.0	.4635	36.1269
6.8670	15.9829		18.0	.4636	42.0089
7.3293	16.9453		19.0	.4637	48.4018
7.7960	17.9119		20.0	.4637	55.3059
ζ = 2.0; $\omega_n \tau_1$ = 1.0					
.5360 -1	.1538		.2	.4500 -5	.2498 -5
.1137	.3153		.4	.3226 -2	.1961 -2
.1788	.4837		.6	.9676 -2	.6464 -2
.2477	.6578		.8	.2028 -1	.1486 -1
.3189	.8363		1.0	.3488 -1	.2800 -1
.5017	1.2943		1.5	.8521 -1	.8544 -1
.6854	1.7589		2.0	.1470	.1829
1.0496	2.6878		3.0	.2764	.5172
1.4129	3.6089		4.0	.3920	1.0680
1.7795	4.5224		5.0	.4858	1.8706
2.1516	5.4303		6.0	.5574	2.9567
2.5305	6.3353		7.0	.6095	4.3557
2.9171	7.2402		8.0	.6461	6.0954
3.3121	8.1477		9.0	.6712	8.2028
3.7157	9.0598		10.0	.6882	10.7035
4.1280	9.9779		11.0	.6998	13.6210
4.5488	10.9026		12.0	.7077	16.9761
4.9776	11.8341		13.0	.7132	20.7865
5.4138	12.7723		14.0	.7171	25.0674
5.8570	13.7169		15.0	.7199	29.8313
6.3064	14.6674		16.0	.7219	35.0880
6.7615	15.6232		17.0	.7235	40.8455
7.2218	16.5841		18.0	.7246	47.1102
7.6867	17.5494		19.0	.7255	53.8872
8.1557	18.5188		20.0	.7262	61.1792
ζ = 2.0; $\omega_n \tau_1$ = 2.0					
.5430 -1	.1544		.2	.8857 -4	.4980 -3
.1167	.3174		.4	.6652 -3	.3914 -2
.1861	.4882		.6	.2086 -2	.1287 -1
.2612	.6658		.8	.4561 -2	.2950 -1
.3406	.8488		1.0	.8172 -2	.5535 -1
.5499	1.3219		1.5	.2191 -1	.1665
.7656	1.8066		2.0	.4106 -1	.3494
1.1975	2.7861		3.0	.8860 -1	.9422
1.6261	3.7668		4.0	.1408	1.8557
2.0542	4.7469		5.0	.1926	3.1159
2.4846	5.7265		6.0	.2419	4.7496
2.9189	6.7062		7.0	.2874	6.7814
3.3577	7.6864		8.0	.3287	9.2329
3.8014	8.6672		9.0	.3658	12.1224
4.2500	9.6489		10.0	.3989	15.4658
4.7034	10.6315		11.0	.4281	19.2762
5.1613	11.6153		12.0	.4540	23.5645
5.6234	12.6001		13.0	.4766	28.3398
6.0895	13.5860		14.0	.4965	33.6092
6.5592	14.5731		15.0	.5139	39.3787
7.0321	15.5613		16.0	.5292	45.6528
7.5080	16.5505		17.0	.5424	52.4351
7.9867	17.5407		18.0	.5540	59.7281
8.4677	18.5319		19.0	.5641	67.5339
8.9510	19.5239		20.0	.5730	75.8536
ζ = 2.0; $\omega_n \tau_1$ = 4.0					
0.5466 -1	0.1547		0.2	-0.3791 -5	0.9943 -3
.1184	.3183		.4	-.2909 -4	.7820 -2
.1903	.4901		.6	-.9330 -4	.2569 -1
.2692	.6688		.8	-.2085 -3	.5880 -1
.3538	.8531		1.0	-.3816 -3	.1102
.5827	1.3306		1.5	-.1077 -2	.3295
.8256	1.8213		2.0	-.2118 -2	.6865
1.3267	2.8167		3.0	-.4985 -2	1.8134
1.8339	3.8169		4.0	-.8526 -2	3.4781
2.3427	4.8178		5.0	-.1242 -1	5.6704
2.8518	5.8190		6.0	-.1643 -1	8.3846
3.3609	6.8201		7.0	-.2042 -1	11.6161
3.8697	7.8212		8.0	-.2427 -1	15.3610
4.3780	8.8222		9.0	-.2793 -1	19.6160
4.8858	9.8231		10.0	-.3135 -1	24.3781
5.3930	10.8240		11.0	-.3451 -1	29.6449
5.8996	11.8248		12.0	-.3740 -1	35.4144
6.4056	12.8255		13.0	-.4004 -1	41.6849
6.9110	13.8262		14.0	-.4242 -1	48.4551
7.4159	14.8268		15.0	-.4456 -1	55.7241
7.9202	15.8273		16.0	-.4648 -1	63.4910
8.4241	16.8278		17.0	-.4819 -1	71.7553
8.9276	17.8282		18.0	-.4971 -1	80.5167
9.4307	18.8286		19.0	-.5107 -1	89.7748
9.9334	19.8289		20.0	-.5227 -1	99.5295
ζ = 2.0; $\omega_n \tau_1$ = 8.0					
.5485 -1	.1548		.2	-.1582 -4	.1988 -2
.1192	.3187		.4	-.1229 -3	.1563 -1
.1925	.4909		.6	-.3988 -3	.5134 -1
.2735	.6699		.8	-.9013 -3	.1174
.3612	.8545		1.0	-.1668 -2	.2198
.6021	1.3327		1.5	-.4834 -2	.6561
.8634	1.8243		2.0	-.9758 -2	1.3634
1.4190	2.8223		3.0	-.2416 -1	3.5783
1.9997	3.8263		4.0	-.4335 -1	6.8033
2.5960	4.8320		5.0	-.6598 -1	10.9719
3.2023	5.8384		6.0	-.9097 -1	16.0212
3.8146	6.8450		7.0	-.1174	21.8912
4.4297	7.8516		8.0	-.1445	28.5253
5.0451	8.8581		9.0	-.1718	35.8715
5.6587	9 8643		10.0	- 1986	43.8822
6.2690	10.8703		11.0	-.2247	52.5152
6.8749	11.8759		12.0	-.2498	61.7330
7.4754	12.8811		13.0	-.2736	71.5032
8.0701	13.8859		14.0	-.2961	81.7978
8.6586	14.8903		15.0	-.3171	92.5928
9.2408	15.8943		16.0	-.3366	103.8684
9.8167	16.8980		17.0	-.3547	115.6083
10.3863	17.9014		18.0	-.3712	127.7995
10.9499	18.9044		19.0	-.3864	140.4320
11.5079	19.9071		20.0	-.4001	153.4983
ζ = 2.0; $\omega_n \tau_1$ = 16.0					
.5494 -1	.1549		.2	-.1164 -4	.3976 -2
.1197	.3189		.4	-.9095 -4	.3126 -1
.1936	.4912		.6	-.2968 -3	.1026
.2758	.6704		.8	-.6745 -3	.2347
.3651	.8551		1.0	-.1255 -2	.4392
.6126	1.3331		1.5	-.3687 -2	1.3094
.8847	1.8243		2.0	-.7544 -2	2.7182
1.4747	2.8212		3.0	-.1919 -1	7.1173
2.1068	3.8242		4.0	-.3533 -1	13.4920
2.7703	4.8290		5.0	-.5515 -1	21.6771
3.4580	5.8347		6.0	-.7791 -1	31.5114
4.1645	6.8410		7.0	-.1029	42.8396
4.8849	7.8475		8.0	-.1297	55.5164
5.6150	8.8541		9.0	-.1576	69.4099
6.3513	9.8608		10.0	-.1863	84.4019

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TABLE V.- NORMALIZED OPTIMUM RESPONSES FOR A TYPE 2 FOURTH-ORDER UNSTABLE PLANT WITH ZERO

T_1/τ_2	T_2/τ_2	T_3/τ_2	T_m/τ_2	b/B	$R_o/BK\tau_2^2$
$\tau_2/\tau_3 = 0.10;\ \tau_1/\tau_2 = 0$					
0.2936 -1	0.1002	0.1708	0.2		0.3448 -5
.6008 -1	.2030	.3430	.4		.5733 -5
.9120 -1	.3068	.5156	.6		.3102 -4
.1230	.4120	.6890	.8		.1052 -3
.1555	.5186	.8632	1.0		.2600 -3
.2395	.7917	1.3022	1.5		.1305 -2
.3272	1.0734	1.7462	2.0		.4089 -2
.4182	1.3633	2.1951	2.5		.9880 -2
.5119	1.6608	2.6488	3.0		.2025 -1
.6081	1.9653	3.1072	3.5		.3699 -1
.7062	2.2760	3.5698	4.0		.6212 -1
.8058	2.5924	4.0366	4.5		.9781 -1
.9065	2.9136	4.5070	5.0		.1464
1.0080	3.2390	4.9810	5.5		.2101
1.1099	3.5679	5.4580	6.0		.2916
1.2120	3.8998	5.9378	6.5		.3931
1.3139	4.2341	6.4201	7.0		.5173
1.4156	4.5703	6.9047	7.5		.6665
1.5167	4.9080	7.3913	8.0		.8432
1.6172	5.2467	7.8796	8.5		1.0498
1.7168	5.5862	8.3694	9.0		1.2888
1.8155	5.9262	8.8606	9.5		1.5625
1.9133	6.2663	9.3530	10.0		1.8732
2.1054	6.9461	10.3407	11.0		2.6149
2.2927	7.6243	11.3316	12.0		3.5318
2.4749	8.2997	12.3248	13.0		4.6414
2.6518	8.9716	13.3198	14.0		5.9605
2.8232	9.6394	14.3162	15.0		7.5054
2.9893	10.3028	15.3135	16.0		9.2918
3.1499	10.9616	16.3116	17.0		11.3350
3.3052	11.6154	17.3102	18.0		13.6498
3.4552	12.2644	18.3092	19.0		16.2503
3.5999	12.9084	19.3085	20.0		19.1503
$\tau_2/\tau_3 = 0.10;\ \tau_1/\tau_2 = 0.5$					
.4875 -1	.1485		.2	-0.8623 -3	.1252 -4
.9544 -1	.2940		.4	-.5701 -2	.1003 -3
.1406	.4366		.6	-.1601 -1	.3408 -3
.1847	.5767		.8	-.3182 -1	.8144 -3
.2280	.7149		1.0	-.5245 -1	.1607 -2
.3342	1.0544		1.5	-.1194	.5600 -2
.4393	1.3898		2.0	-.1982	.1379 -1
.5443	1.7242		2.5	-.2801	.2804 -1
.6495	2.0595		3.0	-.3600	.5041 -1
.7549	2.3962		3.5	-.4351	.8313 -1
.8606	2.7346		4.0	-.5041	.1285
.9663	3.0747		4.5	-.5664	.1890
1.0718	3.4162		5.0	-.6222	.2669
1.1769	3.7590		5.5	-.6717	.3649
1.2815	4.1027		6.0	-.7154	.4852
1.3854	4.4470		6.5	-.7538	.6304
1.4885	4.7917		7.0	-.7874	.8028
1.5907	5.1365		7.5	-.8167	1.0049
1.6918	5.4812		8.0	-.8422	1.2390
1.7917	5.8257		8.5	-.8643	1.5075
1.8906	6.1697		9.0	-.8835	1.8126
1.9881	6.5131		9.5	-.9001	2.1566
2.0844	6.8558		10.0	-.9145	2.5418
2.2731	7.5387		11.0	-.9375	3.4443
2.4563	8.2176		12.0	-.9546	4.5376
2.6340	8.8922		13.0	-.9672	5.8380
2.8062	9.5621		14.0	-.9764	7.3620
2.9729	10.2271		15.0	-.9831	9.1254
3.1341	10.8871		16.0	-.9880	11.1432
3.2898	11.5420		17.0	-.9915	13.4305
3.4403	12.1918		18.0	-.9940	16.0014
3.5855	12.8366		19.0	-.9958	18.8696
3.7256	13.4763		20.0	-.9971	22.0484
3.8603	14.1108		21.0	-.9980	25.5591
3.9906	14.7409		22.0	-.9986	29.3956
$\tau_2/\tau_3 = 0.10;\ \tau_1/\tau_2 = 2.0$					
.5048 -1	.1505		.2	.3567 -4	.4999 -4
.1018	.3021		.4	.2714 -3	.3996 -3
.1540	.4549		.6	.8709 -3	.1347 -2
.2068	.6087		.8	.1963 -2	.3186 -2
.2601	.7637		1.0	.3643 -2	.6208 -2
.3951	1.1559		1.5	.1082 -1	.2077 -1
.5311	1.5536		2.0	.2255 -1	.4866 -1
.6669	1.9556		2.5	.3869 -1	.9370 -1
.8014	2.3601		3.0	.5872 -1	.1593
.9340	2.7657		3.5	.8188 -1	.2484
1.0638	3.1711		4.0	.1074	.3638
1.1908	3.5752		4.5	.1344	.5079
1.3148	3.9770		5.0	.1622	.6829
1.4357	4.3759		5.5	.1902	.8907
1.5534	4.7713		6.0	.2178	1.1333
1.6681	5.1628		6.5	.2446	1.4124
1.7798	5.5502		7.0	0.2703	1.7297
1.8886	5.9333		7.5	.2947	2.0868
1.9946	6.3120		8.0	.3174	2.4852
2.0979	6.6864		8.5	.3385	2.9266
2.1987	7.0565		9.0	.3578	3.4125
2.2970	7.4224		9.5	.3755	3.9444
2.3929	7.7843		10.0	.3914	4.5239
2.5783	8.4967		11.0	.4184	5.8321
2.7556	9.1952		12.0	.4396	7.3504
2.9255	9.8813		13.0	.4558	9.0926
3.0887	10.5567		14.0	.4680	11.0733
3.2456	11.2227		15.0	.4771	13.3069
3.3966	11.8804		16.0	.4838	15.8086
3.5422	12.5307		17.0	.4885	18.5924
3.6824	13.1744		18.0	.4920	21.6745
3.8177	13.8121		19.0	.4944	25.0663
3.9480	14.4441		20.0	.4961	28.7834
$\tau_2/\tau_3 = 0.10;\ \tau_1/\tau_2 = 3.0$					
.5069 -1	.1507		.2	.2328 -4	.7493 -4
.1027	.3030		.4	.1800 -3	.5992 -3
.1559	.4567		.6	.5870 -3	.2019 -2
.2101	.6121		.8	.1344 -2	.4775 -2
.2653	.7691		1.0	.2533 -2	.9299 -2
.4064	1.1682		1.5	.7811 -2	.3107 -1
.5507	1.5760		2.0	.1687 -1	.7262 -1
.6962	1.9911		2.5	.2995 -1	.1394
.8414	2.4119		3.0	.4696 -1	.2362
.9853	2.8366		3.5	.6756 -1	.3668
1.1268	3.2637		4.0	.9129 -1	.5348
1.2654	3.6917		4.5	.1176	.7428
1.4006	4.1194		5.0	.1459	.9934
1.5323	4.5458		5.5	.1757	1.2886
1.6602	4.9699		6.0	.2064	1.6304
1.7845	5.3911		6.5	.2377	2.0203
1.9052	5.8088		7.0	.2691	2.4600
2.0222	6.2226		7.5	.3002	2.9508
2.1358	6.6319		8.0	.3308	3.4940
2.2460	7.0367		8.5	.3605	4.0906
2.3529	7.4365		9.0	.3891	4.7415
2.4566	7.8313		9.5	.4165	5.4478
2.5573	8.2208		10.0	.4424	6.2101
2.7498	8.9840		11.0	.4894	7.9055
2.9315	9.7259		12.0	.5296	9.8329
3.1032	10.4475		13.0	.5629	11.9980
3.2658	11.1501		14.0	.5896	14.4076
3.4202	11.8358		15.0	.6103	17.0714
3.5674	12.5066		16.0	.6261	20.0008
3.7080	13.1648		17.0	.6379	23.2090
3.8427	13.8123		18.0	.6464	26.7113
3.9719	14.4508		19.0	.6526	30.5212
$\tau_2/\tau_3 = 0.10;\ \tau_1/\tau_2 = 5.0$					
.5086 -1	.1509		.2	.1176 -4	.1250 -3
.1034	.3036		.4	.9212 -4	.9985 -3
.1574	.4582		.6	.3043 -3	.3364 -2
.2129	.6147		.8	.7055 -3	.7954 -2
.2697	.7731		1.0	.1347 -2	.1549 -1
.4166	1.1773		1.5	.4283 -2	.5169 -1
.5688	1.5926		2.0	.9531 -2	.1207
.7243	2.0178		2.5	.1742 -1	.2313
.8811	2.4513		3.0	.2809 -1	.3909
1.0377	2.8915		3.5	.4152 -1	.6056
1.1929	3.3368		4.0	.5760 -1	.8802
1.3455	3.7857		4.5	.7610 -1	1.2184
1.4949	4.2368		5.0	.9678 -1	1.6232
1.6406	4.6890		5.5	.1194	2.0969
1.7823	5.1414		6.0	.1436	2.6417
1.9200	5.5930		6.5	.1692	3.2592
2.0535	6.0434		7.0	.1959	3.9511
2.1830	6.4919		7.5	.2236	4.7188
2.3084	6.9381		8.0	.2519	5.5638
2.4299	7.3816		8.5	.2807	6.4873
2.5476	7.8220		9.0	.3098	7.4904
2.6616	8.2592		9.5	.3390	8.5743
2.7721	8.6927		10.0	.3682	9.7398
2.9828	9.5478		11.0	.4260	12.3175
3.1804	10.3852		12.0	.4819	15.2257
3.3657	11.2029		13.0	.5349	18.4614
3.5392	11.9987		14.0	.5838	22.0170
3.7016	12.7711		15.0	.6278	25.8807
3.8534	13.5188		16.0	.6662	30.0388
$\tau_2/\tau_3 = 0.25;\ \tau_1/\tau_2 = 0$					
.2962 -1	.1007	0.1710	.2		.8701 -5
.5980 -1	.2025	.3427	.4		.1668 -4
.9056 -1	.3056	.5151	.6		.8416 -4
.1219	.4100	.6881	.8		.2652 -3

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TABLE V.- NORMALIZED OPTIMUM RESPONSES FOR A TYPE 2 FOURTH-ORDER UNSTABLE PLANT WITH ZERO - Continued

$\tau_2/\tau_3 = 0.25;\ \tau_1/\tau_2 = 0$

T_1/τ_2	T_2/τ_2	T_3/τ_2	T_m/τ_2	b/B	$R_o/BK\tau_2^2$
0.1537	0.5155	0.8619	1.0		0.6484 -3
.2352	.7846	1.2995	1.5		.3261 -2
.3191	1.0608	1.7418	2.0		.1022 -1
.4047	1.3436	2.1889	2.5		.2469 -1
.4917	1.6324	2.6408	3.0		.5057 -1
.5792	1.9266	3.0973	3.5		.9234 -1
.6669	2.2253	3.5584	4.0		.1550
.7542	2.5280	4.0238	4.5		.2438
.8406	2.8337	4.4931	5.0		.3645
.9259	3.1420	4.9661	5.5		.5226
1.0095	3.4520	5.4426	6.0		.7240
1.0912	3.7633	5.9221	6.5		.9744
1.1708	4.0751	6.4043	7.0		1.2793
1.2481	4.3872	6.8890	7.5		1.6443
1.3230	4.6989	7.3759	8.0		2.0746
1.3953	5.0100	7.8647	8.5		2.5752
1.4650	5.3202	8.3552	9.0		3.1509
1.5320	5.6291	8.8471	9.5		3.8062
1.5963	5.9366	9.3403	10.0		4.5455
1.7170	6.5467	10.3297	11.0		6.2917
1.8271	7.1495	11.3223	12.0		8.4184
1.9273	7.7445	12.3172	13.0		10.9509
2.0179	8.3317	13.3137	14.0		13.9106
2.0998	8.9111	14.3114	15.0		17.3160
2.1734	9.4832	15.3098	16.0		21.1825
2.2395	10.0483	16.3088	17.0		25.5232
2.2988	10.6069	17.3081	18.0		30.3487

$\tau_2/\tau_3 = 0.25;\ \tau_1/\tau_2 = 0.5$

T_1/τ_2	T_2/τ_2	T_3/τ_2	T_m/τ_2	b/B	$R_o/BK\tau_2^2$
.4857 -1	.1483		.2	-0.9237 -3	.3128 -4
.9475 -1	.2932		.4	-.6106 -2	.2510 -3
.1391	.4348		.6	-.1715 -1	.8519 -3
.1821	.5736		.8	-.3405 -1	.2035 -2
.2240	.7100		1.0	-.5611 -1	.4016 -2
.3257	1.0438		1.5	-.1276	.1400 -1
.4245	1.3716		2.0	-.2116	.3447 -1
.5213	1.6966		2.5	-.2988	.7008 -1
.6165	2.0206		3.0	-.3836	.1260
.7100	2.3442		3.5	-.4630	.2076
.8017	2.6678		4.0	-.5356	.3206
.8913	2.9911		4.5	-.6008	.4708
.9787	3.3140		5.0	-.6586	.6640
1.0636	3.6362		5.5	-.7094	.9058
1.1459	3.9575		6.0	-.7537	1.2018
1.2256	4.2776		6.5	-.7920	1.5573
1.3024	4.5962		7.0	-.8251	1.9776
1.3765	4.9131		7.5	-.8533	2.4675
1.4476	5.2283		8.0	-.8774	3.0318
1.5159	5.5415		8.5	-.8979	3.6748
1.5814	5.8526		9.0	-.9151	4.4009
1.6440	6.1616		9.5	-.9297	5.2141
1.7039	6.4684		10.0	-.9419	6.1178
1.8156	7.0754		11.0	-.9607	8.2111
1.9170	7.6735		12.0	-.9736	10.7061
2.0087	8.2631		13.0	-.9825	13.6247
2.0914	8.8443		14.0	-.9885	16.9857
2.1659	9.4178		15.0	-.9925	20.8049
2.2328	9.9840		16.0	-.9951	25.0956
2.2929	10.5437		17.0	-.9969	29.8660

$\tau_2/\tau_3 = 0.25;\ \tau_1/\tau_2 = 2.0$

T_1/τ_2	T_2/τ_2	T_3/τ_2	T_m/τ_2	b/B	$R_o/BK\tau_2^2$
.5029 -1	.1503		.2	.4459 -4	.1250 -3
.1011	.3014		.4	.3392 -3	.9989 -3
.1522	.4533		.6	.1088 -2	.3367 -2
.2035	.6060		.8	.2452 -2	.7964 -2
.2549	.7595		1.0	.4551 -2	.1551 -1
.3829	1.1464		1.5	.1350 -1	.5188 -1
.5089	1.5370		2.0	.2809 -1	.1214
.6314	1.9295		2.5	.4807 -1	.2334
.7495	2.3222		3.0	.7270 -1	.3961
.8626	2.7136		3.5	.1010	.6162
.9703	3.1021		4.0	.1318	.8999
1.0724	3.4864		4.5	.1640	1.2519
1.1690	3.8657		5.0	.1967	1.6766
1.2602	4.2391		5.5	.2290	2.1773
1.3461	4.6062		6.0	.2602	2.7571
1.4270	4.9667		6.5	.2897	3.4184
1.5031	5.3203		7.0	.3172	4.1634
1.5748	5.6672		7.5	.3424	4.9941
1.6422	6.0074		8.0	.3652	5.9123
1.7057	6.3412		8.5	.3856	6.9198
1.7655	6.6689		9.0	.4035	8.0185
1.8218	6.9909		9.5	.4191	9.2102
1.8750	7.3075		10.0	.4326	10.4971
1.9724	7.9263		11.0	.4539	13.3647
2.0594	8.5284		12.0	.4690	16.6389
2.1372	9.1166		13.0	.4794	20.3381
2.2067	9.6933		14.0	.4865	24.4790
2.2690	10.2602		15.0	.4913	29.0785

$\tau_2/\tau_3 = 0.25;\ \tau_1/\tau_2 = 3.0$

T_1/τ_2	T_2/τ_2	T_3/τ_2	T_m/τ_2	b/B	$R_o/BK\tau_2^2$
0.5050 -1	0.1505		0.2	0.3134 -4	0.1875 -3
.1019	.3022		.4	.2423 -3	.1498 -2
.1540	.4552		.6	.7901 -3	.5048 -2
.2067	.6094		.8	.1808 -2	.1194 -1
.2599	.7650		1.0	.3409 -2	.2324 -1
.3936	1.1593		1.5	.1050 -1	.7761 -1
.5268	1.5608		2.0	.2265 -1	.1813
.6576	1.9678		2.5	.4013 -1	.3475
.7843	2.3785		3.0	.6275 -1	.5876
.9061	2.7910		3.5	.8997 -1	.9106
1.0220	3.2036		4.0	.1211	1.3238
1.1318	3.6145		4.5	.1552	1.8326
1.2352	4.0224		5.0	.1915	2.4412
1.3323	4.4260		5.5	.2291	3.1526
1.4233	4.8243		6.0	.2673	3.9689
1.5084	5.2164		6.5	.3053	4.8912
1.5879	5.6018		7.0	.3426	5.9199
1.6619	5.9798		7.5	.3786	7.0547
1.7310	6.3502		8.0	.4128	8.2947
1.7952	6.7127		8.5	.4450	9.6388
1.8551	7.0671		9.0	.4747	11.0854
1.9109	7.4135		9.5	.5018	12.6330
1.9628	7.7521		10.0	.5262	14.2801
2.0563	8.4070		11.0	.5671	17.8693
2.1380	9.0349		12.0	.5980	21.8503
2.2095	9.6401		13.0	.6204	26.2292
2.2725	10.2266		14.0	.6360	31.0194

$\tau_2/\tau_3 = 0.25;\ \tau_1/\tau_2 = 5.0$

T_1/τ_2	T_2/τ_2	T_3/τ_2	T_m/τ_2	b/B	$R_o/BK\tau_2^2$
.5066 -1	.1507		.2	.1764 -4	.3124 -3
.1026	.3029		.4	.1382 -3	.2496 -2
.1555	.4567		.6	.4564 -3	.8410 -2
.2095	.6121		.8	.1058 -2	.1989 -1
.2641	.7692		1.0	.2020 -2	.3871 -1
.4031	1.1692		1.5	.6420 -2	.1292
.5434	1.5791		2.0	.1427 -1	.3013
.6826	1.9977		2.5	.2605 -1	.5769
.8186	2.4234		3.0	.4191 -1	.9736
.9500	2.8545		3.5	.6177 -1	1.5050
1.0756	3.2890		4.0	.8538 -1	2.1813
1.1946	3.7255		4.5	.1123	3.0095
1.3066	4.1622		5.0	.1422	3.9941
1.4116	4.5978		5.5	.1745	5.1378
1.5096	5.0313		6.0	.2087	6.4419
1.6008	5.4616		6.5	.2443	7.9067
1.6855	5.8879		7.0	.2810	9.5315
1.7640	6.3095		7.5	.3182	11.3147
1.8367	6.7258		8.0	.3556	13.2540
1.9039	7.1362		8.5	.3929	15.3462
1.9660	7.5402		9.0	.4297	17.5871
2.0234	7.9373		9.5	.4656	19.9715
2.0762	8.3269		10.0	.5003	22.4935
2.1698	9.0825		11.0	.5651	27.9217

$\tau_2/\tau_3 = 0.5;\ \tau_1/\tau_2 = 0$

T_1/τ_2	T_2/τ_2	T_3/τ_2	T_m/τ_2	b/B	$R_o/BK\tau_2^2$
.2949 -1	.1004	0.1709	.2		.2192 -5
.5935 -1	.2017	.3423	.4		.3336 -4
.8953 -1	.3037	.5142	.6		.1683 -3
.1200	.4066	.6866	.8		.5316 -3
.1507	.5103	.8597	1.0		.1295 -2
.2280	.7729	1.2950	1.5		.6512 -2
.3055	1.0401	1.7346	2.0		.2039 -1
.3825	1.3114	2.1789	2.5		.4919 -1
.4582	1.5862	2.6281	3.0		.1005
.5318	1.8639	3.0821	3.5		.1831
.6029	2.1440	3.5411	4.0		.3063
.6708	2.4256	4.0048	4.5		.4799
.7352	2.7083	4.4731	5.0		.7140
.7959	2.9914	4.9455	5.5		1.0181
.8525	3.2744	5.4218	6.0		1.4017
.9051	3.5568	5.9017	6.5		1.8733
.9536	3.8383	6.3846	7.0		2.4407
.9982	4.1185	6.8703	7.5		3.1108
1.0388	4.3973	7.3584	8.0		3.8898
1.0758	4.6744	7.8486	8.5		4.7828
1.1093	4.9498	8.3405	9.0		5.7941
1.1395	5.2234	8.8339	9.5		6.9271
1.1667	5.4952	9.3285	10.0		8.1847
1.2130	6.0336	10.3206	11.0		11.0820
1.2500	6.5655	11.3155	12.0		14.4978
1.2793	7.0916	12.3122	13.0		18.4381
1.3025	7.6127	13.3102	14.0		22.9050
1.3207	8.1297	14.3089	15.0		27.8983

$\tau_2/\tau_3 = 0.50;\ \tau_1/\tau_2 = 0.5$

T_1/τ_2	T_2/τ_2	T_3/τ_2	T_m/τ_2	b/B	$R_o/BK\tau_2^2$
.4827 -1	.1480		.2	-.1026 -2	.6256 -4
.9360 -1	.2919		.4	-.6780 -2	.5018 -3
.1366	.4319		.6	-.1902 -1	.1703 -2
.1778	.5683		.8	-.3774 -1	.4066 -2

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TABLE V.- NORMALIZED OPTIMUM RESPONSES FOR A TYPE 2 FOURTH-ORDER UNSTABLE PLANT WITH ZERO - Continued

T_1/τ_2	T_2/τ_2	T_3/τ_2	T_m/τ_2	b/B	$R_o/BK\tau_2^2$
τ_2/τ_3 = 0.50; τ_1/τ_2 = 0.5					
0.2174	0.7019		1.0	-0.6213 -1	0.8019 -2
.3115	1.0263		1.5	-.1409	.2793 -1
.3999	1.3416		2.0	-.2330	.6868 -1
.4835	1.6514		2.5	-.3281	.1393
.5625	1.9576		3.0	-.4198	.2498
.6370	2.2608		3.5	-.5048	.4101
.7068	2.5614		4.0	-.5814	.6304
.7719	2.8596		4.5	-.6492	.9207
.8322	3.1551		5.0	-.7081	1.2902
.8878	3.4481		5.5	-.7588	1.7474
.9389	3.7384		6.0	-.8018	2.2999
.9855	4.0260		6.5	-.8381	2.9546
1.0279	4.3108		7.0	-.8683	3.7174
1.0663	4.5930		7.5	-.8934	4.5932
1.1010	4.8725		8.0	-.9140	5.5862
1.1323	5.1496		8.5	-.9309	6.6999
1.1604	5.4242		9.0	-.9447	7.9372
1.1856	5.6966		9.5	-.9558	9.3001
1.2081	5.9669		10.0	-.9648	10.7906
1.2461	6.5017		11.0	-.9778	14.1585
1.2764	7.0298		12.0	-.9862	18.0473
1.3002	7.5523		13.0	-.9914	22.4608
1.3190	8.0703		14.0	-.9947	27.3986
τ_2/τ_3 = 0.50; τ_1/τ_2 = 2.0					
.4997 -1	.1500		.2	.5944 -4	.2499 -3
.9975 -1	.3002		.4	.4521 -3	.1997 -2
.1492	.4506		.6	.1450 -2	.6729 -2
.1981	.6013		.8	.3265 -2	.1591 -1
.2463	.7523		1.0	.6054 -2	.3098 -1
.3629	1.1308		1.5	.1790 -1	.1033
.4725	1.5096		2.0	.3706 -1	.2409
.5739	1.8869		2.5	.6299 -1	.4609
.6665	2.2610		3.0	.9447 -1	.7768
.7502	2.6301		3.5	.1299	1.1991
.8251	2.9926		4.0	.1675	1.7350
.8917	3.3475		4.5	.2057	2.3890
.9507	3.6939		5.0	.2432	3.1637
1.0027	4.0315		5.5	.2788	4.0595
1.0484	5.3603		6.0	.3118	5.0763
1.0886	4.6803		6.5	.3416	6.2131
1.1239	4.9920		7.0	.3681	7.4688
1.1549	5.2960		7.5	.3911	8.8424
1.1821	5.5929		8.0	.4108	10.3334
1.2060	5.8835		8.5	.4275	11.9415
1.2271	6.1685		9.0	.4414	13.6669
1.2456	6.4485		9.5	.4529	15.5104
1.2619	6.7243		10.0	.4623	17.4727
1.2891	7.2653		11.0	.4762	21.7577
1.3103	7.7954		12.0	.4851	26.5308
1.3270	8.3177		13.0	.4907	31.7991
τ_2/τ_3 = 0.50; τ_1/τ_2 = 3.0					
.5017 -1	.1502		.2	.4477 -4	.3749 -3
.1006	.3011		.4	.3461 -3	.2996 -2
.1509	.4526		.6	.1128 -2	.1009 -1
.2011	.6050		.8	.2580 -2	.2385 -1
.2509	.7582		1.0	.4860 -2	.4642 -1
.3724	1.1448		1.5	.1493 -1	.1547
.4878	1.5359		2.0	.3207 -1	.3599
.5952	1.9299		2.5	.5647 -1	.6865
.6933	2.3247		3.0	.8759 -1	1.1531
.7817	2.7183		3.5	.1244	1.7724
.8604	3.1085		4.0	.1655	2.5517
.9297	3.4937		4.5	.2094	3.4940
.9903	3.8723		5.0	.2547	4.5982
1.0430	4.2430		5.5	.3000	5.8606
1.0886	4.6051		6.0	.3444	7.2755
1.1280	4.9579		6.5	.3866	8.8354
1.1619	5.3010		7.0	.4261	10.5324
1.1910	5.6345		7.5	.4623	12.3585
1.2162	5.9584		8.0	.4949	14.3061
1.2378	6.2732		8.5	.5236	16.3688
1.2564	6.5794		9.0	.5487	18.5415
1.2726	6.8778		9.5	.5701	20.8205
1.2865	7.1689		10.0	.5883	23.2038
1.3092	7.7331		11.0	.6159	28.2815
τ_2/τ_3 = 0.50; τ_1/τ_2 = 5.0					
.5034 -1	.1504		.2	.2744 -4	.6248 -3
.1012	.3018		.4	.2149 -3	.4992 -2
.1524	.4542		.6	.7096 -3	.1681 -1
.2037	.6078		.8	.1644 -2	.3974 -1
.2549	.7627		1.0	.3137 -2	.7733 -1
.3810	1.1558		1.5	.9948 -2	.2575
.5019	1.5570		2.0	.2203 -1	.5988
.6153	1.9653		2.5	.3998 -1	1.1406
.7194	2.3790		3.0	.6386 -1	1.9121
.8130	2.7962		3.5	.9327 -1	2.9311
0.8960	3.2148		4.0	0.1275	4.2061
.9685	3.6328		4.5	.1657	5.7370
1.0313	4.0486		5.0	.2069	7.5184
1.0852	4.4607		5.5	.2502	9.5402
1.1311	4.8677		6.0	.2946	11.7899
1.1701	5.2688		6.5	.3395	14.2534
1.2031	5.6629		7.0	.3839	16.9152
1.2309	6.0495		7.5	.4275	19.7592
1.2543	6.4279		8.0	.4694	22.7689
1.2740	6.7976		8.5	.5094	25.9273
1.2905	7.1581		9.0	.5470	29.2180
1.3045	7.5091		9.5	.5819	32.6249
τ_2/τ_3 = 1.0; τ_1/τ_2 = 0					
.2927 -1	.1000	0.1707	.2		.0408 -4
.5846 -1	.2000	.3415	.4		.6660 -4
.8747 -1	.3000	.5125	.6		.3364 -3
.1162	.4000	.6838	.8		.1061 -2
.1446	.5000	.8554	1.0		.2582 -2
.2136	.7500	1.2864	1.5		.1294 -1
.2788	1.0000	1.7212	2.0		.4028 -1
.3392	1.2500	2.1608	2.5		.9646 -1
.3943	1.5000	2.6057	3.0		.1953
.4434	1.7500	3.0566	3.5		.3517
.4866	2.0000	3.5134	4.0		.5806
.5239	2.2500	3.9761	4.5		.8962
.5556	2.5000	4.4444	5.0		1.3111
.5822	2.7500	4.9178	5.5		1.8360
.6042	3.0000	5.3958	6.0		2.4793
.6223	3.2500	5.8777	6.5		3.2474
.6369	3.5000	6.3631	7.0		4.1447
.6487	3.7500	6.8513	7.5		5.1739
.6581	4.0000	7.3419	8.0		6.3366
.6656	4.2500	7.8344	8.5		7.6335
.6715	4.5000	8.3285	9.0		9.0644
.6762	4.7500	8.8238	9.5		10.6291
.6799	5.0000	9.3201	10.0		12.3266
.6851	5.5000	10.3149	11.0		16.1173
.6882	6.0000	11.3118	12.0		20.4300
.6902	6.5000	12.3098	13.0		25.2587
.6913	7.0000	13.3087	14.0		30.5987
.6920	7.5000	14.3080	15.0		36.4466
.6925	8.0000	15.3075	16.0		42.7998
.6927	8.5000	16.3073	17.0		49.6566
.6929	9.0000	17.3071	18.0		57.0158
.6930	9.5000	18.3070	19.0		64.8766
.6931	10.0000	19.3069	20.0		73.2384
τ_2/τ_3 = 1.0; τ_1/τ_2 = 0.5					
.4768 -1	.1474		.2	-.1231 -2	.1250 -3
.9131 -1	.2893		.4	-.8118 -2	.1002 -2
.1316 -1	.4259		.6	-.2272 -1	.3396 -2
.1692 -1	.5579		.8	-.4496 -1	.8096 -2
.2044	.6859		1.0	-.7379 -1	.1594 -1
.2836	.9922		1.5	-.1659	.5517 -1
.3524	1.2844		2.0	-.2719	.1346
.4119	1.5671		2.5	-.3789	.2704
.4629	1.8430		3.0	-.4796	.4784
.5061	2.1136		3.5	-.5700	.7738
.5423	2.3801		4.0	-.6489	1.1693
.5723	2.6433		4.5	-.7159	1.6758
.5968	2.9038		5.0	-.7719	2.3013
.6167	3.1622		5.5	-.8180	3.0519
.6327	3.4188		6.0	-.8555	3.9316
.6455	3.6741		6.5	-.8857	4.9428
.6557	3.9282		7.0	-.9099	6.0869
.6638	4.1815		7.5	-.9292	7.3642
.6701	4.4340		8.0	-.9444	8.7747
.6751	4.6860		8.5	-.9565	10.3178
.6791	4.9376		9.0	-.9659	11.9931
.6822	5.1888		9.5	-.9734	13.7995
.6846	5.4398		10.0	-.9792	15.7366
.6880	5.9411		11.0	-.9873	19.9991
.6900	6.4420		12.0	-.9923	24.7754
.6913	6.9425		13.0	-.9953	30.0610
τ_2/τ_3 = 1.0; τ_1/τ_2 = 2.0					
.4933 -1	.1494		.2	.8913 -4	.4997 -3
.9714 -1	.2978		.4	.6773 -3	.3990 -2
.1432	.4454		.6	.2169 -2	.1343 -1
.1873	.5922		.8	.4871 -2	.3168 -1
.2293	.7383		1.0	.9003 -2	.6150 -1
.3242	1.1006		1.5	.2632 -1	.2032
.4042	1.4578		2.0	.5360 -1	.4669
.4696	1.8089		2.5	.8924 -1	.8767
.5217	2.1523		3.0	.1306	1.4456
.5625	2.4872		3.5	.1747	2.1771
.5938	2.8128		4.0	.2189	3.0684
.6177	3.1289		4.5	.2612	4.1122

TABLE V.- NORMALIZED OPTIMUM RESPONSES FOR A TYPE 2 FOURTH-ORDER UNSTABLE PLANT WITH ZERO - Continued

T_1/τ_2	T_2/τ_2	T_3/τ_2	T_m/τ_2	b/B	$R_o/BK\tau_2^2$
$\tau_2/\tau_3 = 1.0;\ \tau_1/\tau_2 = 2.0$					
0.6358	3.4359		5.0	0.3001	5.3000
.6495	3.7342		5.5	.3347	6.6233
.6598	4.0247		6.0	.3648	8.0747
.6677	4.3081		6.5	.3905	9.6487
.6736	4.5855		7.0	.4119	11.3413
.6781	4.8577		7.5	.4296	13.1500
.6816	5.1256		8.0	.4440	15.0736
.6842	5.3899		8.5	.4557	17.1118
.6862	5.6513		9.0	.4651	19.2652
.6878	5.9103		9.5	.4725	21.5341
.6890	6.1674		10.0	.4784	23.9198
.6906	6.6774		11.0	.4868	29.0453
$\tau_2/\tau_3 = 1.0;\ \tau_1/\tau_2 = 3.0$					
.4955^{-1}	.1496		.2	.7160^{-4}	.7494^{-3}
.9790^{-1}	.2987		.4	.5531^{-3}	.5985^{-2}
.1448	.4475		.6	.1800^{-2}	.2013^{-1}
.1900	.5962		.8	.4109^{-2}	.4750^{-1}
.2333	.7449		1.0	.7716^{-2}	.9220^{-1}
.3317	1.1169		1.5	.2345^{-1}	.3042
.4149	1.4893		2.0	.4958^{-1}	.6978
.4827	1.8610		2.5	.8553^{-1}	1.3059
.5361	2.2303		3.0	.1294	2.1432
.5770	2.5952		3.5	.1788	3.2088
.6078	2.9540		4.0	.2308	4.4906
.6305	3.3055		4.5	.2833	5.9705
.6472	3.6485		5.0	.3342	7.6281
.6594	3.9825		5.5	.3821	9.4432
.6682	4.3073		6.0	.4261	11.3977
.6746	4.6230		6.5	.4655	13.4763
.6794	4.9297		7.0	.5003	15.6669
.6828	5.2282		7.5	.5303	17.9603
.6854	5.5190		8.0	.5558	20.3509
.6872	5.8030		8.5	.5772	22.8353
.6887	6.0810		9.0	.5949	25.4125
.6897	6.3539		9.5	.6095	28.0833
.6905	6.6224		10.0	.6213	30.8496
$\tau_2/\tau_3 = 1.0;\ \tau_1/\tau_2 = 5.0$					
.4969^{-1}	.1498		.2	.4702^{-4}	.1249^{-2}
.9853^{-1}	.2995		.4	.3680^{-3}	.9975^{-2}
.1462	.4492		.6	.1214^{-2}	.3355^{-1}
.1923	.5994		.8	.2807^{-2}	.7916^{-1}
.2367	.7501		1.0	.5341^{-2}	.1536
.3384	1.1303		1.5	.1677^{-1}	.5069
.4247	1.5162		2.0	.3658^{-1}	1.1618
.4949	1.9075		2.5	.6505^{-1}	2.1705
.5496	2.3029		3.0	.1013	3.5516
.5908	2.7004		3.5	.1438	5.2954
.6209	3.0978		4.0	.1908	7.3734
.6425	3.4932		4.5	.2403	9.7484
.6578	3.8849		5.0	.2908	12.3821
.6685	4.2714		5.5	.3412	15.2387
.6759	4.6517		6.0	.3903	18.2865
.6810	5.0249		6.5	.4375	21.4981
.6846	5.3903		7.0	.4823	24.8495
.6871	5.7472		7.5	.5241	28.3197
.6888	6.0953		8.0	.5626	31.8906
$\tau_2/\tau_3 = 2.0;\ \tau_1/\tau_2 = 0$					
.2884^{-1}	.9917^{-1}	0.1703	.2		.0823^{-4}
.5668^{-1}	.1967	.3400	.4		.1329^{-3}
.8335^{-1}	.2926	.5093	.6		.6698^{-3}
.1087	.3870	.6783	.8		.2104^{-2}
.1327	.4799	.8473	1.0		.5098^{-2}
.1860	.7069	1.2709	1.5		.2513^{-1}
.2294	.9280	1.6986	2.0		.7656^{-1}
.2635	1.1459	2.1324	2.5		.1785
.2891	1.3628	2.5737	3.0		.3504
.3077	1.5809	3.0232	3.5		.6102
.3208	1.8014	3.4806	4.0		.9725
.3298	2.0251	3.9454	4.5		1.4485
.3358	2.2524	4.4166	5.0		2.0462
.3397	2.4832	4.8935	5.5		2.7705
.3422	2.7173	5.3750	6.0		3.6245
.3439	2.9542	5.8604	6.5		4.6095
.3449	3.1937	6.3488	7.0		5.7263
.3455	3.4352	6.8396	7.5		6.9746
.3459	3.6784	7.3325	8.0		8.3541
.3462	3.9230	7.8269	8.5		9.8643
.3463	4.1688	8.3225	9.0		11.5044
.3464	4.4155	8.8190	9.5		13.2740
.3465	4.6628	9.3163	10.0		15.1723
$\tau_2/\tau_3 = 2.0;\ \tau_1/\tau_2 = 0.5$					
.4648^{-1}	.1461		.2	-.1638^{-2}	.2497^{-3}
.8675^{-1}	.2841		.4	-.1075^{-1}	.1994^{-2}
.1218	.4143		.6	-.2988^{-1}	.6720^{-2}
.1525	.5378		.8	-.5861^{-1}	.1591^{-1}
0.1793	0.6555		1.0	-0.9524^{-1}	0.3106^{-1}
.2326	.9305		1.5	-.2084	.1049
.2703	1.1873		2.0	-.3320	.2483
.2965	1.4340		2.5	-.4500	.4816
.3142	1.6755		3.0	-.5547	.8207
.3259	1.9150		3.5	-.6436	1.2768
.3335	2.1543		4.0	-.7170	1.8571
.3384	2.3943		4.5	-.7765	2.5656
.3415	2.6354		5.0	-.8242	3.4044
.3434	2.8779		5.5	-.8621	4.3742
.3446	3.1216		6.0	-.8921	5.4750
.3454	3.3665		6.5	-.9156	6.7064
.3458	3.6123		7.0	-.9341	8.0679
.3461	3.8590		7.5	-.9486	9.5588
.3463	4.1063		8.0	-.9599	11.1784
.3464	4.3543		8.5	-.9687	12.9263
.3465	4.6026		9.0	-.9756	14.8021
$\tau_2/\tau_3 = 2.0;\ \tau_1/\tau_2 = 2.0$					
.4805^{-1}	.1482		.2	.1484^{-3}	.9983^{-3}
.9198^{-1}	.2931		.4	.1123^{-2}	.7943^{-2}
.1315	.4351		.6	.3573^{-2}	.2657^{-1}
.1666	.5746		.8	.7954^{-2}	.6217^{-1}
.1973	.7118		1.0	.1454^{-1}	.1194
.2562	1.0471		1.5	.4093^{-1}	.3807
.2939	1.3734		2.0	.7950^{-1}	.8362
.3166	1.6923		2.5	.1257	1.4927
.3298	2.0046		3.0	.1748	2.3383
.3372	2.3101		3.5	.2229	3.3544
.3413	2.6091		4.0	.2678	4.5224
.3436	2.9015		4.5	.3079	5.8271
.3449	3.1877		5.0	.3429	7.2572
.3456	3.4682		5.5	.3726	8.8052
.3460	3.7436		6.0	.3975	10.4662
.3462	4.0143		6.5	.4181	12.2376
.3464	4.2813		7.0	.4349	14.1186
.3465	4.5449		7.5	.4484	16.1092
.3465	4.8058		8.0	.4593	18.2104
$\tau_2/\tau_3 = 2.0;\ \tau_1/\tau_2 = 3.0$					
.4824^{-1}	.1484		.2	.1252^{-3}	.1497^{-2}
.9265^{-1}	.2941		.4	.9633^{-3}	.1192^{-1}
.1329	.4376		.6	.3116^{-2}	.3985^{-1}
.1687	.5793		.8	.7052^{-2}	.9326^{-1}
.2002	.7198		1.0	.1310^{-1}	.1791
.2604	1.0679		1.5	.3836^{-1}	.5702
.2984	1.4144		2.0	.7730^{-1}	1.2480
.3207	1.7603		2.5	.1264	2.2151
.3330	2.1049		3.0	.1813	3.4438
.3395	2.4466		3.5	.2381	4.8968
.3429	2.7836		4.0	.2938	6.5391
.3446	3.1145		4.5	.3466	8.3415
.3456	3.4383		5.0	.3952	10.2814
.3460	3.7544		5.5	.4389	12.3416
.3463	4.0626		6.0	.4776	14.5094
.3464	4.3631		6.5	.5111	16.7756
.3465	4.6562		7.0	.5398	19.1344
$\tau_2/\tau_3 = 2.0;\ \tau_1/\tau_2 = 5.0$					
.4840^{-1}	.1486		.2	.8611^{-4}	.2496^{-2}
.9321^{-1}	.2949		.4	.6716^{-3}	.1986^{-1}
.1340	.4395		.6	.2202^{-2}	.6644^{-1}
.1705	.5831		.8	.5051^{-2}	.1555
.2026	.7264		1.0	.9509^{-2}	.2987
.2641	1.0860		1.5	.2878^{-1}	.9506
.3024	1.4520		2.0	.5981^{-1}	2.0766
.3242	1.8258		2.5	.1006	3.6708
.3356	2.2062		3.0	.1482	5.6741
.3413	2.5903		3.5	.1996	8.0141
.3441	2.9753		4.0	.2525	10.6271
.3454	3.3585		4.5	.3053	13.4641
.3460	3.7381		5.0	.3568	16.4884
.3463	4.1125		5.5	.4065	19.6720
.3464	4.4804		6.0	.4536	22.9924
.3465	4.8408		6.5	.4977	26.4307
$\tau_2/\tau_3 = 5.0;\ \tau_1/\tau_2 = 0$					
.2754^{-1}	.9669^{-1}	0.1691	.2		.2063^{-4}
.5135^{-1}	.1870	.3356	.4		.3272^{-3}
.7131^{-1}	.2716	.5003	.6		.1619^{-2}
.8757^{-1}	.3516	.6640	.8		.4965^{-2}
.1005	.4280	.8275	1.0		.1169^{-1}
.1212	.6106	1.2394	1.5		.5295^{-1}
.1312	.7916	1.6604	2.0		.1474
.1357	.9782	2.0926	2.5		.3153
.1375	1.1735	2.5360	3.0		.5730
.1382	1.3777	2.9895	3.5		.9330
.1385	1.5902	3.4517	4.0		1.4049
.1385	1.8100	3.9214	4.5		1.9955

TABLE V.- NORMALIZED OPTIMUM RESPONSES FOR A TYPE 2 FOURTH-ORDER UNSTABLE PLANT WITH ZERO - Concluded

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T_1/τ_2	T_2/τ_2	T_3/τ_2	T_m/τ_2	b/B	$R_o/BK\tau_2^2$
$\tau_2/\tau_3 = 5.0;\ \tau_1/\tau_2 = 0.5$					
0.4295^{-1}	0.1422		0.2	-0.2838^{-2}	0.6188^{-3}
.7370^{-1}	.2692		.4	-.1813^{-1}	.4824^{-2}
.9509^{-1}	.3830		.6	-.4851^{-1}	.1570^{-1}
.1097	.4869		.8	-.9117^{-1}	.3572^{-1}
.1195	.5840		1.0	-.1419	.6679^{-1}
.1320	.8116		1.5	-.2813	.2024
.1364	1.0327		2.0	-.4147	.4338
.1379	1.2553		2.5	-.5303	.7728
.1384	1.4818		3.0	-.6264	1.2273
.1386	1.7124		3.5	-.7044	1.8029
$\tau_2/\tau_3 = 5.0;\ \tau_1/\tau_2 = 2.0$					
.4428^{-1}	.1446		.2	.3236^{-3}	.2475^{-2}
.7729^{-1}	.2797		.4	.2389^{-2}	.1922^{-1}
.1003	.4076		.6	.7313^{-2}	.6192^{-1}
.1155	.5310		.8	.1552^{-1}	.1382
.1251	.6519		1.0	.2689^{-1}	.2519
.1353	.9514		1.5	.6612^{-1}	.7009
.1379	1.2522		2.0	.1143	1.3654
.1385	1.5536		2.5	.1651	2.2166
.1386	1.8532		3.0	.2146	3.2311
$\tau_2/\tau_3 = 5.0;\ \tau_1/\tau_2 = 3.0$					
.4444^{-1}	.1449		.2	.2837^{-3}	.3713^{-2}
.7775^{-1}	.2809		.4	.2130^{-2}	.2884^{-1}
.1011	.4110		.6	.6634^{-2}	.9293^{-1}
.1164	.5378		.8	.1431^{-1}	.2074
.1259	.6637		1.0	.2520^{-1}	.3778
.1358	.9822		1.5	.6426^{-1}	1.0462
.1381	1.3102		2.0	.1147	2.0235
.1385	1.6447		2.5	.1708	3.2571
.1386	1.9812		3.0	.2284	4.7049
$\tau_2/\tau_3 = 5.0;\ \tau_1/\tau_2 = 5.0$					
.4457^{-1}	.1451		.2	.2018^{-3}	.6188^{-2}
.7813^{-1}	.2820		.4	.1537^{-2}	.4809^{-1}
.1017	.4138		.6	.4853^{-2}	.1550
.1171	.5436		.8	.1062^{-1}	.3461
.1266	.6739		1.0	.1894^{-1}	.6300
.1362	1.0106		1.5	.4977^{-1}	1.7398
.1382	1.3664		2.0	.9128^{-1}	3.3468
.1386	1.7368		2.5	.1393	5.3515

TABLE VI.- REFERENCES FOR OPTIMUM SYSTEMS

Case	Plant transfer function	References	Results or comments
1	$\frac{K}{s}$		$x(t) = B \operatorname{sgn} \epsilon(t)$
2	$\frac{K(\tau_1 s + 1)}{s(\tau_2 s + 1)}$	11 for $\tau_1 = 0$; for $\tau_1/\tau_2 \geq 0.5$ optimum derived here	for $\tau_1 = 0$ $x(t) = B \operatorname{sgn} \left\{\epsilon(t) + \frac{\tau_2 \dot{\epsilon}(t)}{KB} \left[\frac{1 - \ln\left(1 + \frac{\lvert\dot{\epsilon}\rvert}{KB}\right)}{\lvert\dot{\epsilon}\rvert KB}\right]\right\}$
3	$\frac{K}{s[(s^2/\omega_n^2) + (2\zeta/\omega_n)s + 1]}$	12	$x(t) = B \operatorname{sgn} f(\epsilon, \dot{\epsilon}, \ddot{\epsilon})$ complex function
4	$\frac{K}{s^2}$	11	$x(t) = B \operatorname{sgn}\left(\epsilon + \frac{\dot{\epsilon}\lvert\dot{\epsilon}\rvert}{2KB}\right)$
5	$\frac{K(\tau_1 s + 1)}{s^2(\tau_2 s + 1)}$	13 for $\tau_1 = 0$	$x(t) = B \operatorname{sgn} f(\epsilon, \dot{\epsilon}, \ddot{\epsilon})$ for $\tau_1 = 0$
6	$\frac{K}{s^2[(s^2/\omega_n^2) + (2\zeta/\omega_n)s + 1]}$		$x(t) = B \operatorname{sgn} f(\epsilon, \dot{\epsilon}, \ddot{\epsilon}, \dddot{\epsilon})$
7	$\frac{K}{s^3}$	14	$x(t) = B \operatorname{sgn} f(\epsilon, \dot{\epsilon}, \ddot{\epsilon})$
8	$\frac{K}{s(\tau s - 1)}$	Approximate optimum derived here	
9	$\frac{K(\tau_1 s + 1)}{s[(s^2/\omega_n^2) + (2\zeta/\omega_n)s + 1]}$	None	
10	$\frac{K(\tau_1 s + 1)}{s(\tau_2 s + 1)(\tau_3 s - 1)}$		
11	$\frac{K(\tau_1 s + 1)}{s^2[(s^2/\omega_n^2) + (2\zeta/\omega_n)s + 1]}$	Approximate optimum derived here	
12	$\frac{K(\tau_1 s + 1)}{s^2(\tau_2 s + 1)(\tau_3 s - 1)}$		

Note: A number of the references use output derivative feedbacks rather than error derivatives. As has been pointed out, error derivatives are necessary if the optimum is to be obtained for arbitrary inputs.

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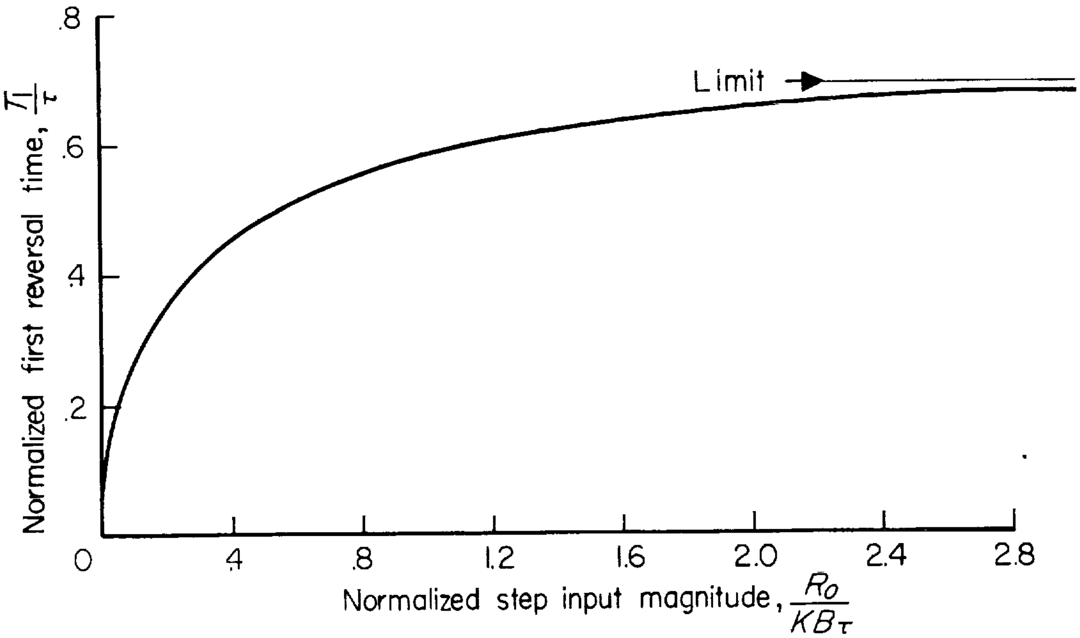
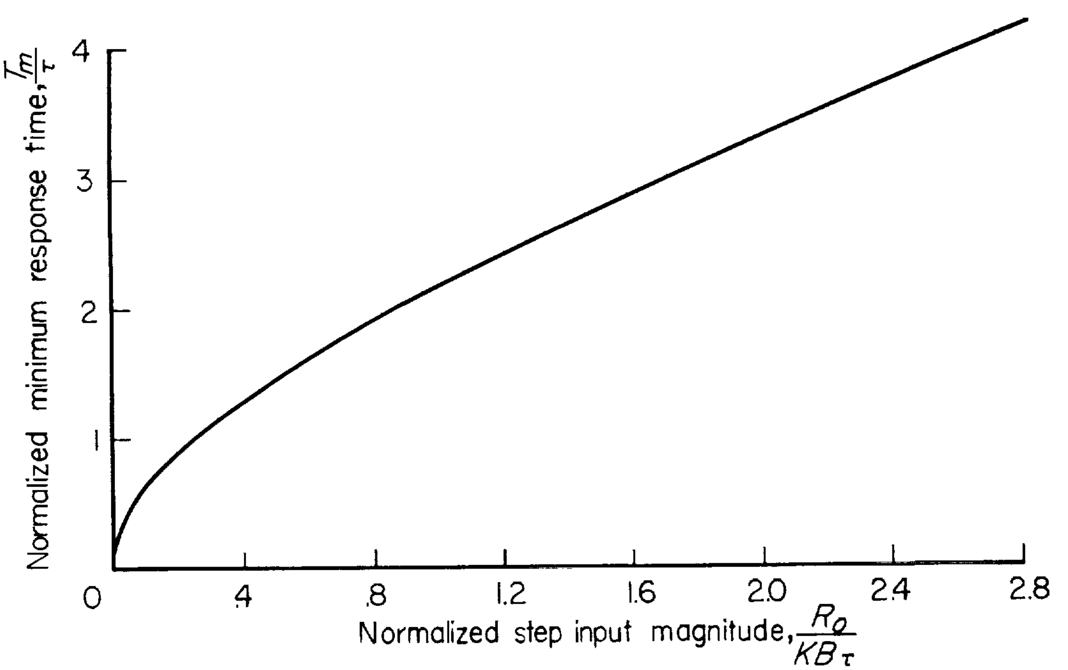


(a) Optimum first reversal time.



(b) Minimum response time.

Figure 1.- Optimum response times for a type 1 second-order unstable plant.

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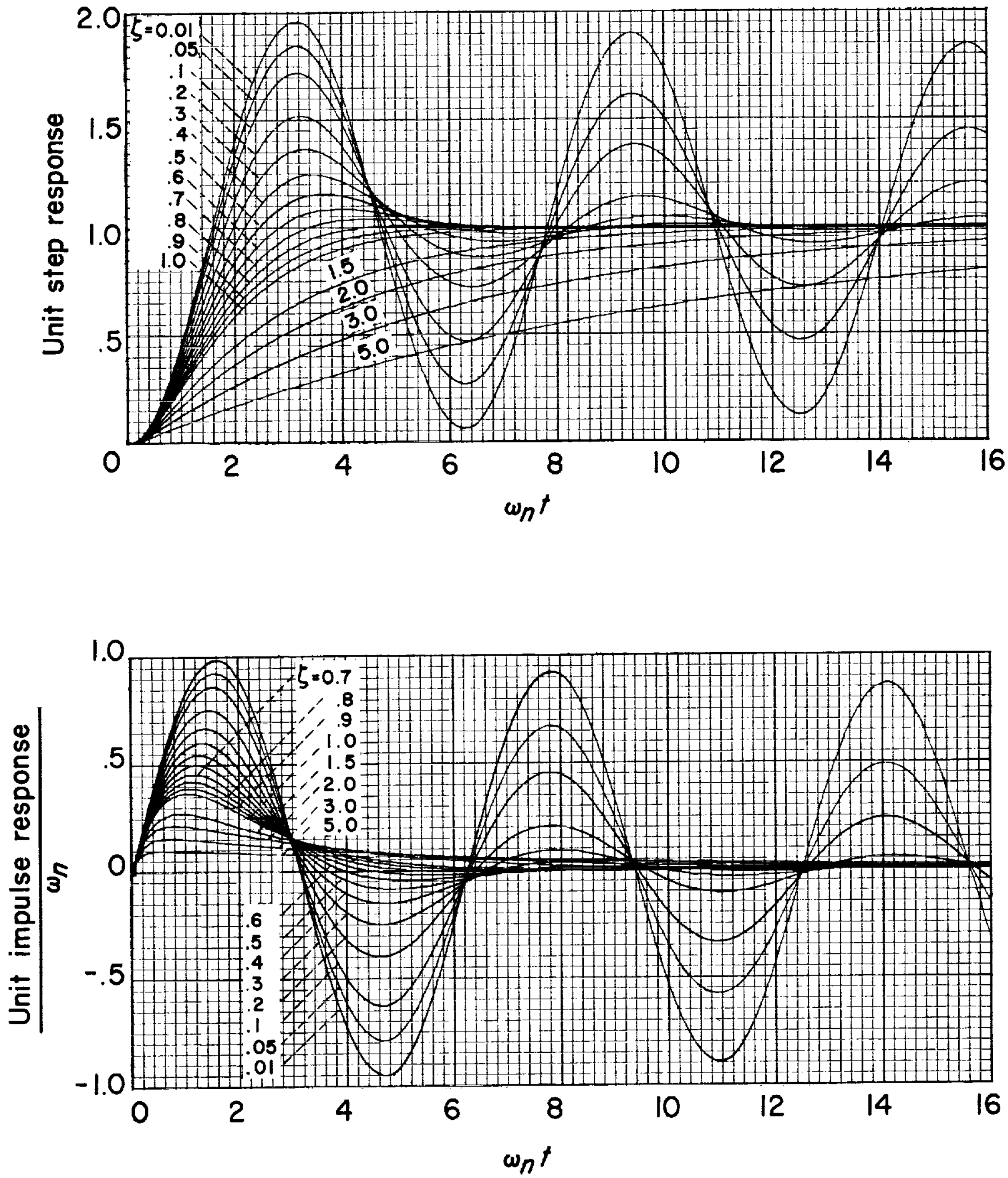

Figure 2.- Responses of a second-order system.

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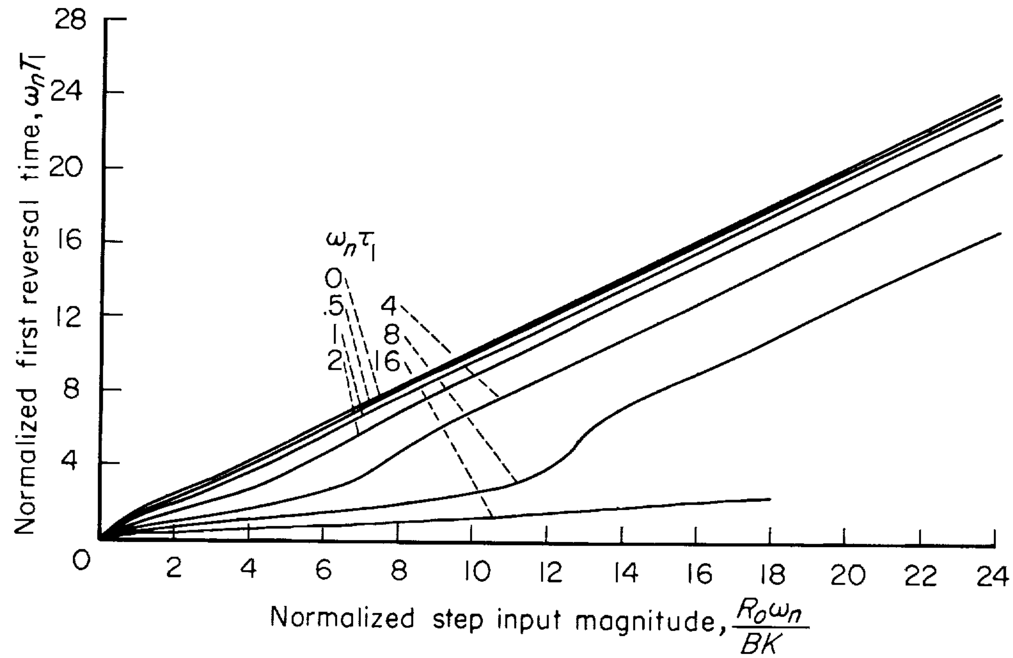
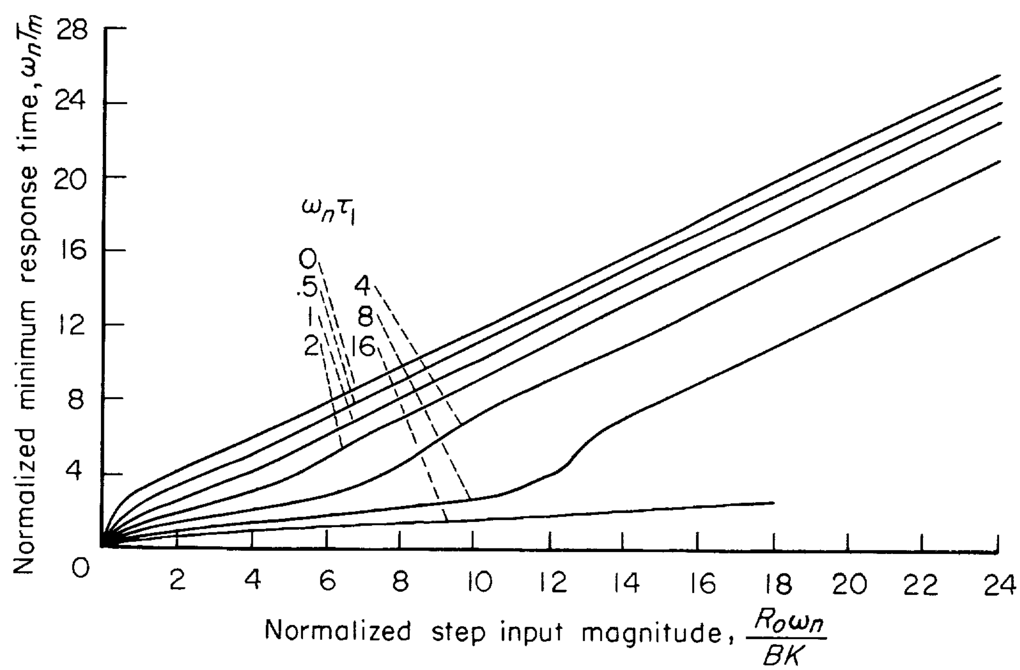
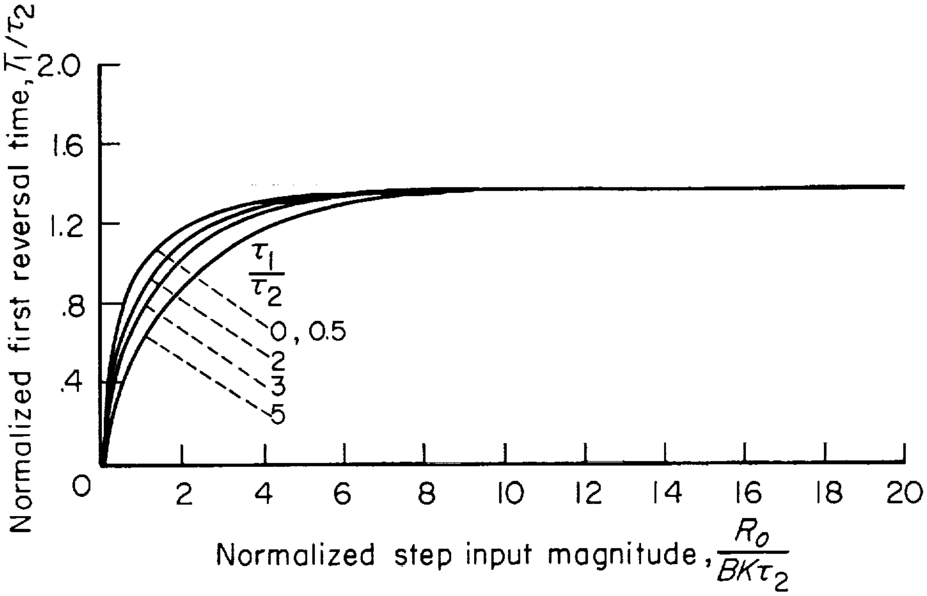


(a) Optimum first reversal time.

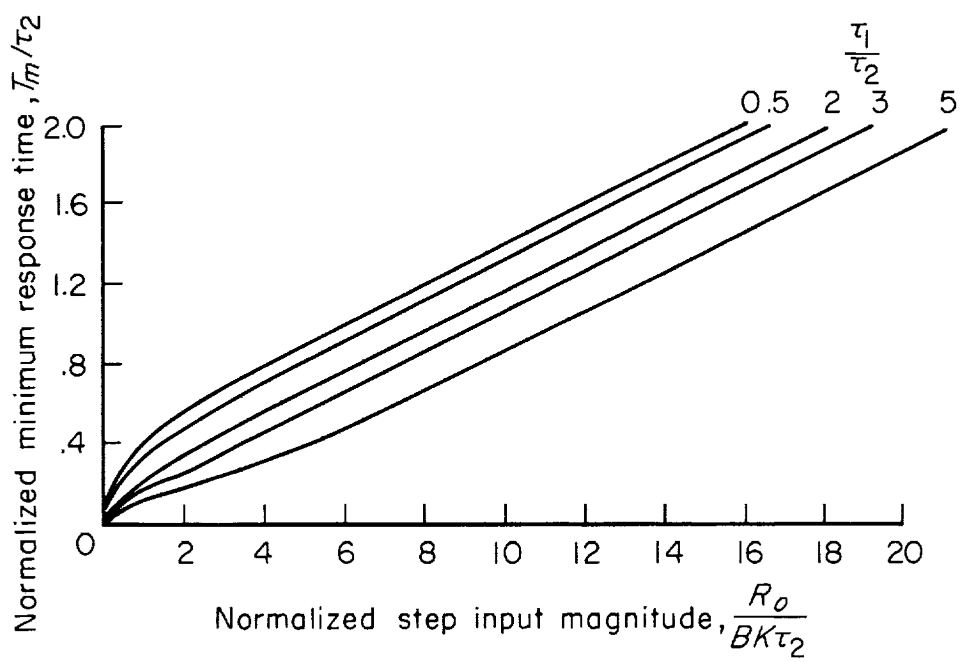


(b) Minimum response time.

Figure 3.- Optimum response times for a type 1 third-order plant with zero (ζ = 0.5).



(a) Optimum first reversal time.



(b) Minimum response time.

Figure 4.- Optimum response times for a type 1 third-order unstable plant with zero (τ_2/τ_3 = 0.5).

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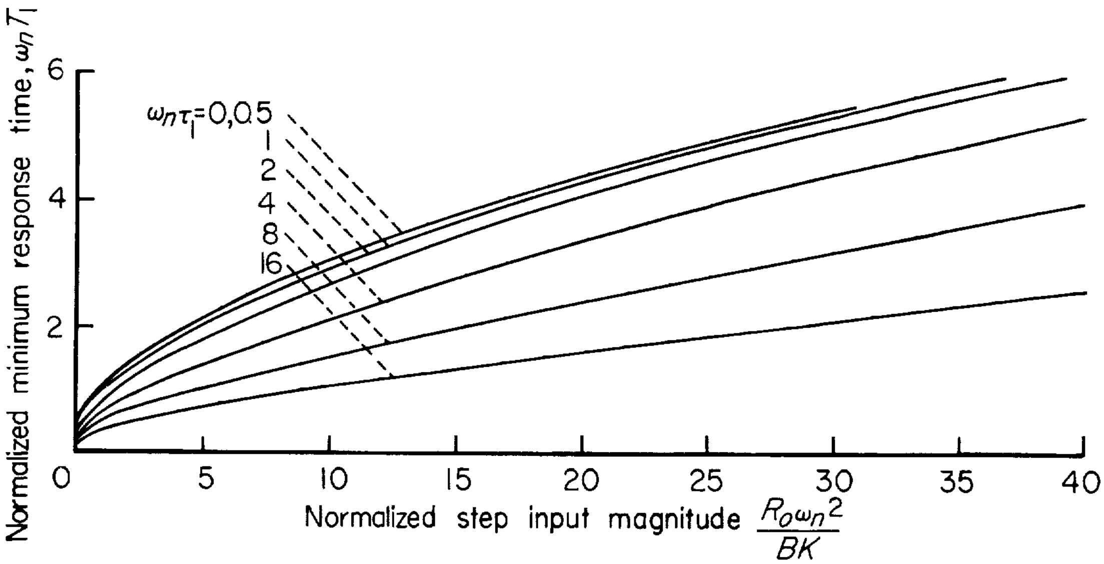
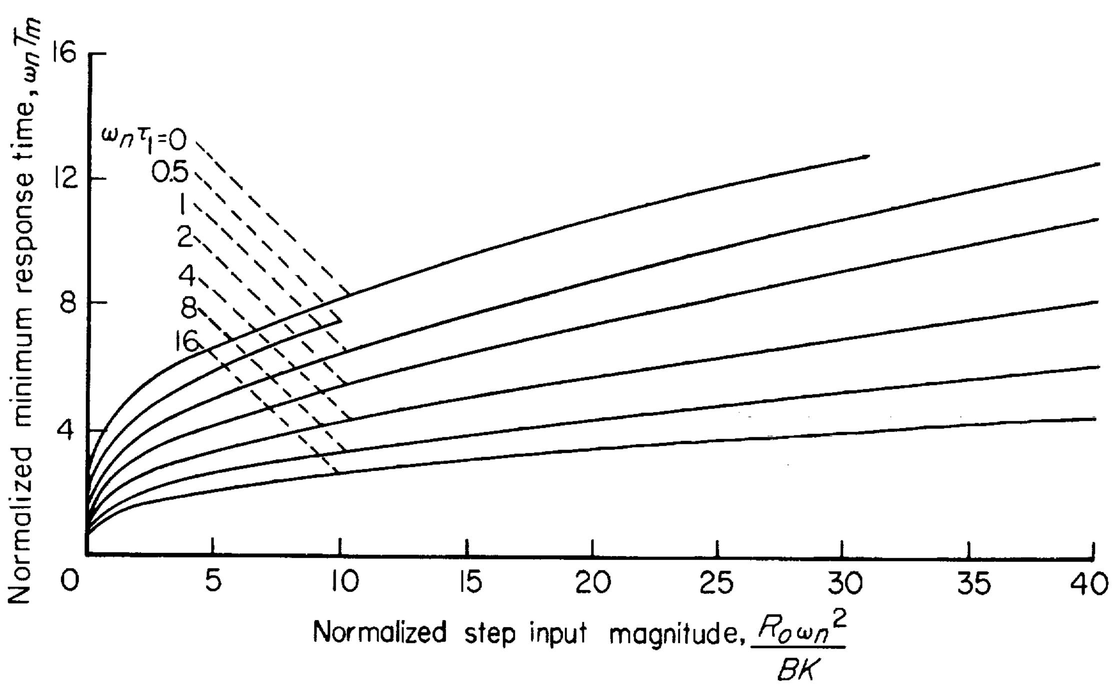


(a) Optimum first reversal time.



(b) Minimum response time.

Figure 5.- Optimum response times for a type 2 fourth-order plant with zero (ζ = 0.5).

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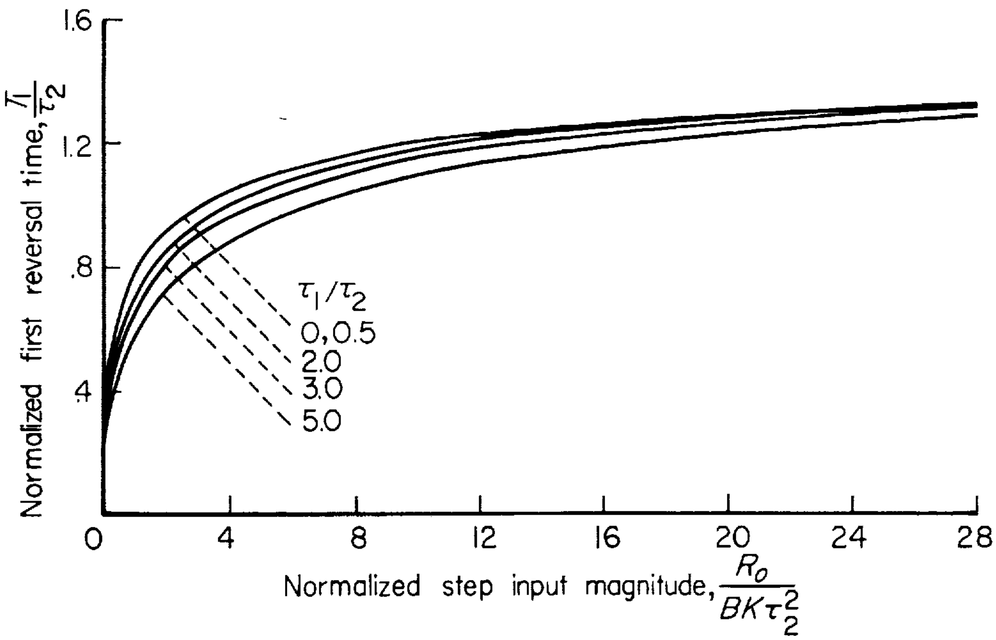
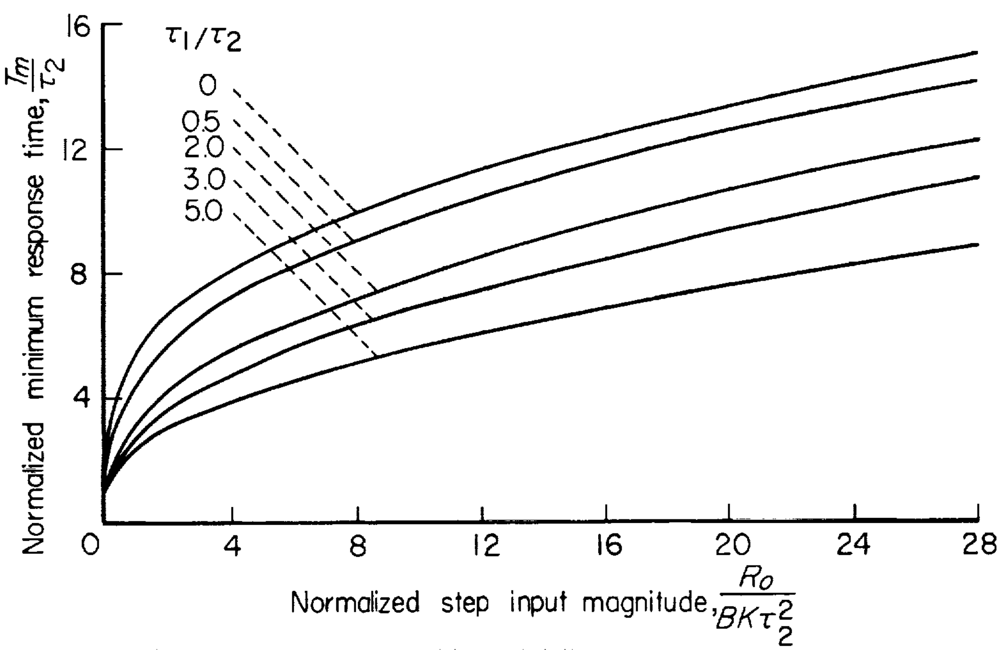


(a) Optimum first reversal time.



(b) Minimum response time.

Figure 6.- Optimum response times for a type 2 fourth-order unstable plant with zero ($\tau_2/\tau_3 = 0.5$).

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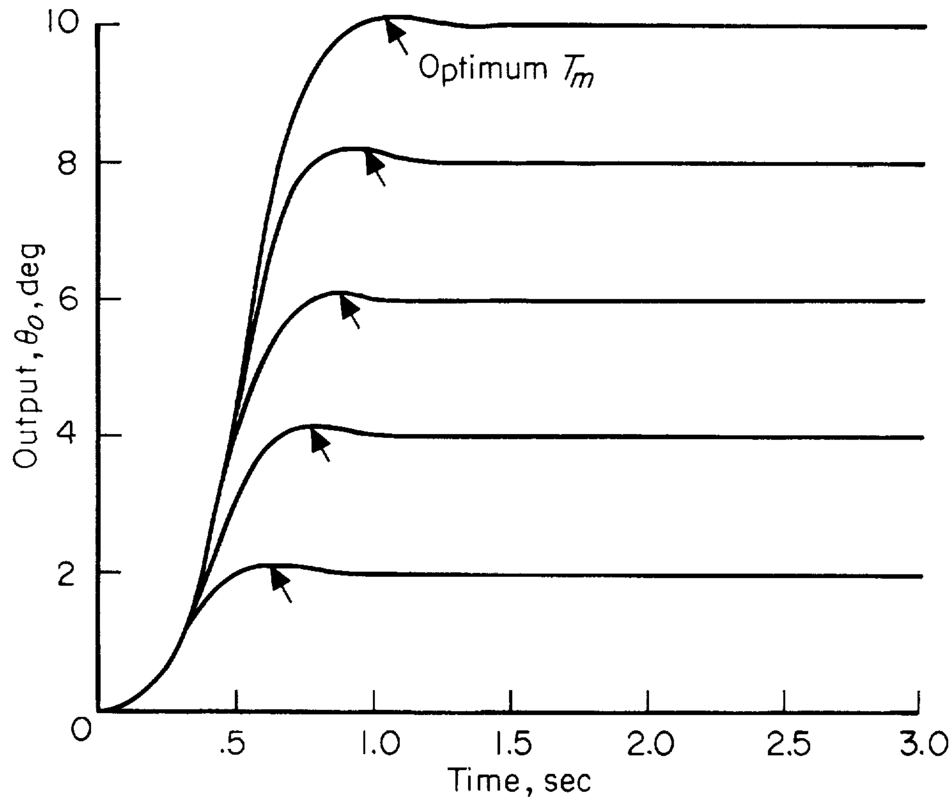

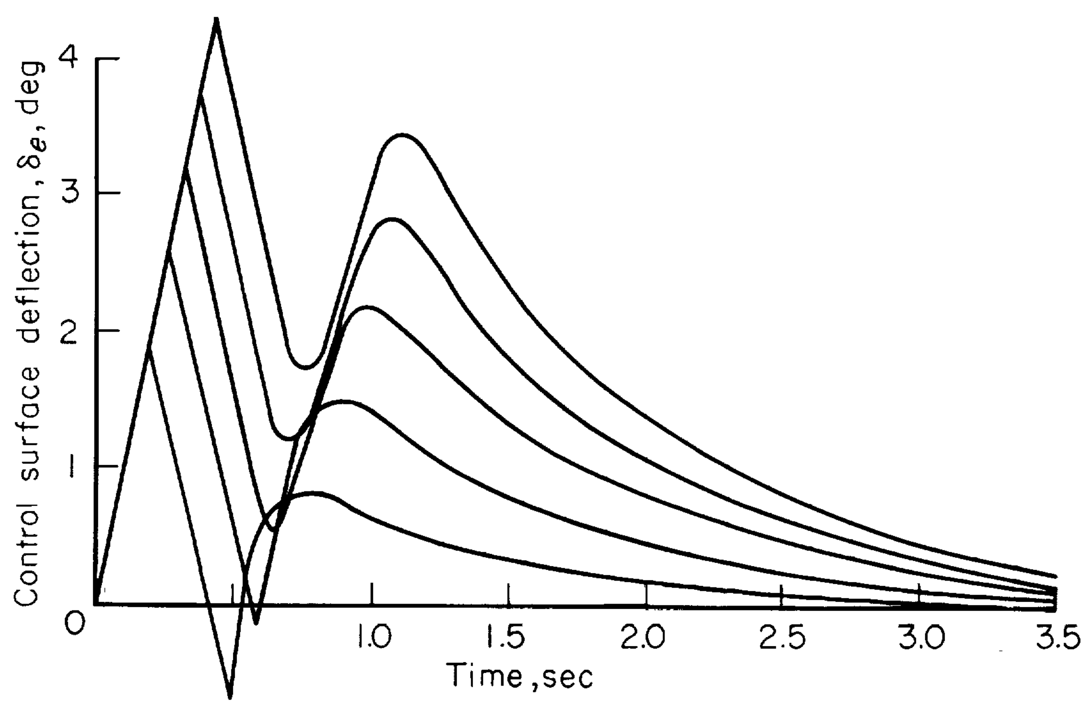

Figure 7.- Responses of attitude control system with stable aerodynamics.

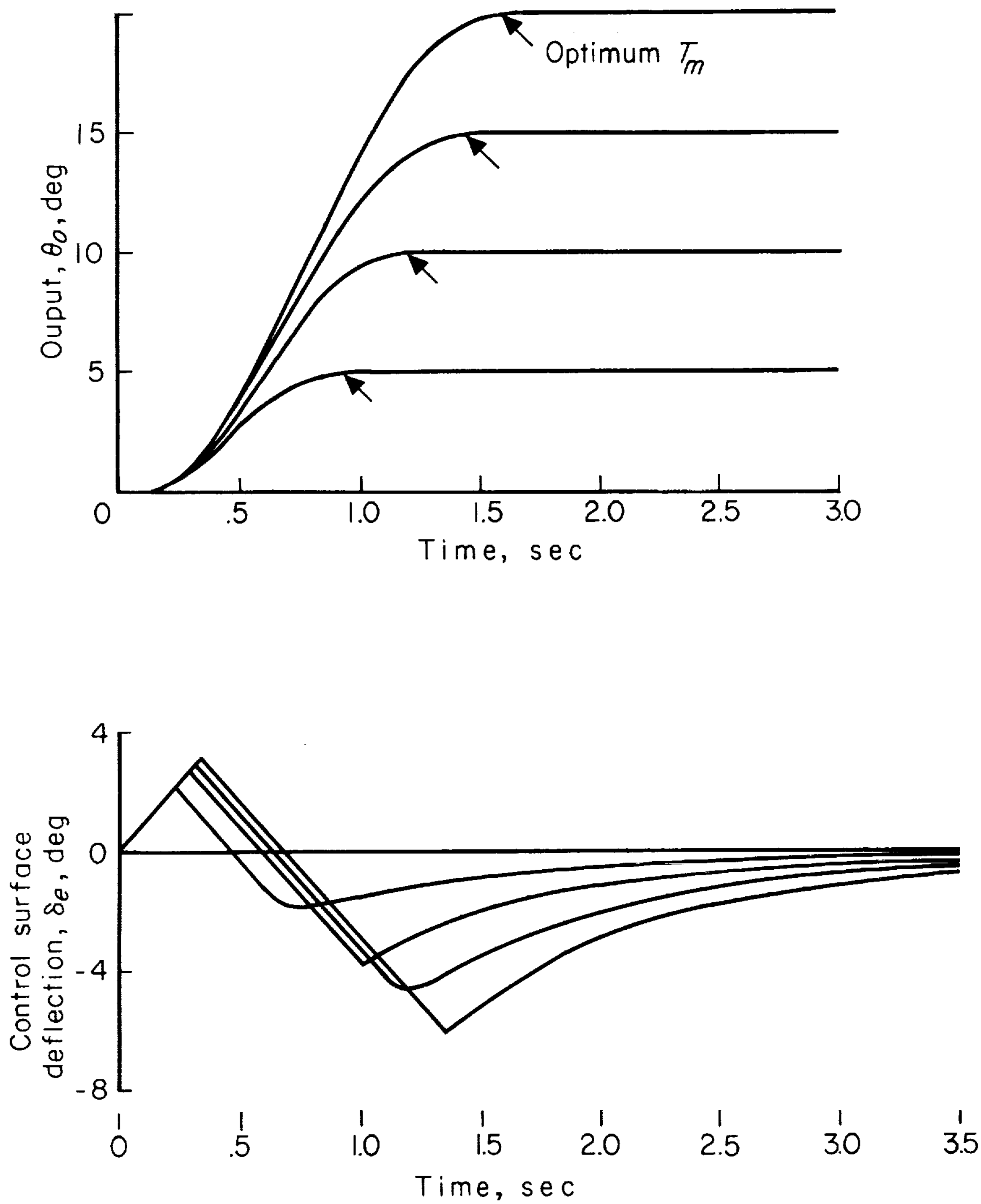

Figure 8.- Responses of attitude control system for unstable aerodynamics.

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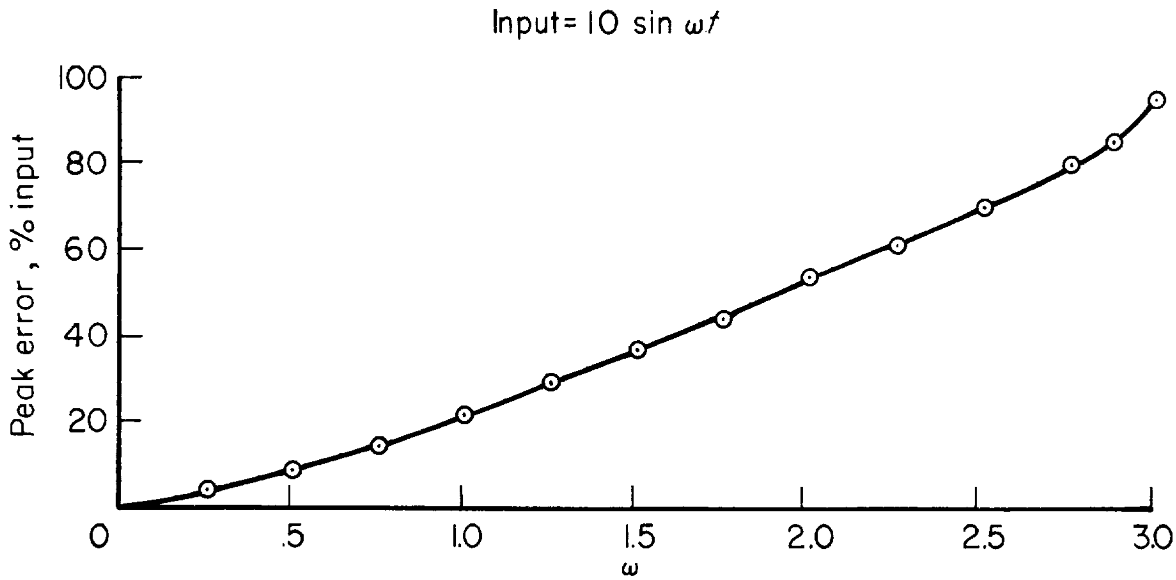
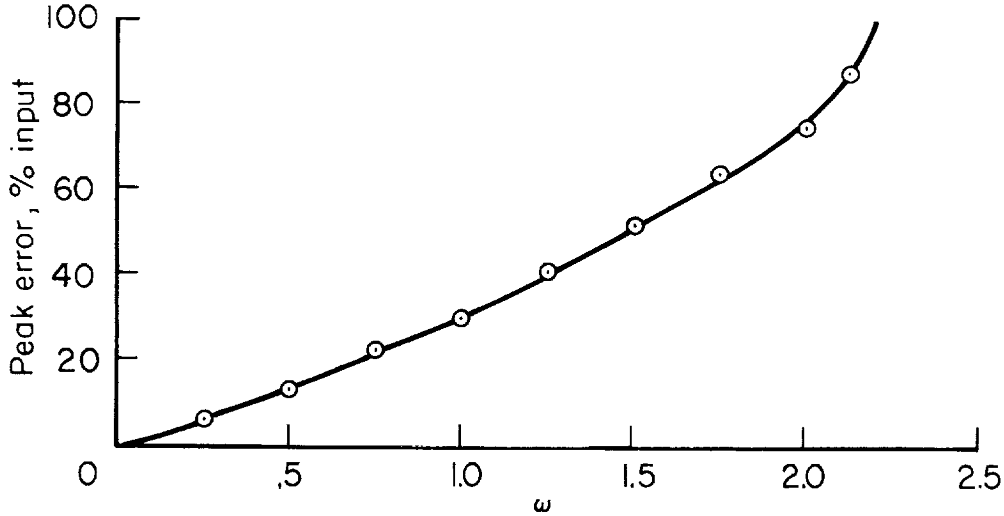


(a) Stable.



(b) Unstable.

Figure 9.- Error magnitude versus frequency for the two example attitude control systems.

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