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TECHNICAL NOTE
D- 233

DETERMINATION OF AZIMUTH ANGLE AT BURNOUT FOR PLACING A

SATELLITE OVER A SELECTED EARTH·POSITION

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SUMMARY

Expressions are presented for relating the satellite position in
the orbital plane with the projected latitude and longitude on a rotating
earth surface. An expression is also presented for determining the azi-
muth angle at a given burnout position on the basis of a selected passage
position on the earth's surface.

Examples are presented of a satellite launched eastward and one
launched westward, each passing over a selected position sometime after
having completed three orbits. Incremental changes from the desired
latitude and longitude due to the earth's oblateness are included in the
iteration for obtaining the azimuth angles of the two examples. The
results for both cases are then compared with those obtained from a com-
puting program using an oblate rotating earth. Changes from the selected
latitude and longitude resulting from incremental changes from the burn-
out azimuth angle and latitude are also analyzed.

INTRODUCTION

In the recovery of an artificial earth satellite it is necessary to
bring the satellite over a preselected point above the earth from which
the reentry is to be initiated. It is the purpose of the present paper
to determine the azimuth angle which must exist at a given burnout point
in order to have the satellite vehicle pass over any particular earth
position after a selected small number of orbital passes. For this pur-
pose equations are developed which give the latitude and longitude of a
point on the surface of a rotating spherical earth directly under the
satellite at any time. An expression is developed which, with the orbit
characteristics and the selected passage position on the earth's surface,
will establish after only a few iterations the required azimuth angle at
the burnout point.

Elliptical orbits about a rotating spherical earth are first assumed.
Although this simplification ignores the effects of the earth's oblateness,

it is convenient to examine the oblateness effects later as incremental changes which can be accounted for in an iteration procedure. Departures from the desired latitude and longitude passage position resulting from incremental changes in the azimuth angle and latitude at burnout are also analyzed.

SYMBOLS

a	semimajor axis of elliptic orbit
E	eccentric anomaly, defined by equation (9)
e	eccentricity of orbit, defined by equation (6)
g_O	gravitational constant at earth surface
i	inclination angle of orbital plane
N	nodal point at which satellite crosses equator
n	number of completed orbital passes, referenced to launch latitude
O	center of earth
P	perigee location in orbital plane
p	semilatus rectum of ellipse, defined by equation (1)
R	radius of earth
r	distance of satellite from earth center
S	position of satellite in orbital plane
T	satellite period, defined by equation (7)
$t(\theta)$	time from perigee, min
V	satellite velocity, defined by equation (3)
V_c	satellite velocity for circular orbit, defined by equation (4)
γ	elevation angle between local horizon and velocity vector, positive upwards

θ   angle in orbital plane measured from perigee point; also, true anomaly

λ   longitude measured east of prime meridian at Greenwich

$\Delta\lambda_{NS} = \lambda_N - \lambda_S$

$\Delta\lambda_{N1} = \lambda_N - \lambda_1$

ϕ geocentric latitude measured north or south of equator

ψ azimuth angle measured clockwise from North

Ω angle defining rotation of orbital plane about earth's axis

ω angle in orbital plane between nodal point and perigee and, also, argument of perigee, deg

ω_E angular velocity of the earth, 0.25068 deg/min (taken as 0.25 deg/min in this report)

Subscripts:

N   nodal point

n   orbit number

S   any satellite position

1   initial or burnout position

2   selected position

2e   equivalent selected position in first orbit

1-2e   increment between initial or burnout position and equivalent position

SATELLITE POSITION RELATIONSHIPS

If drag or accelerating forces acting parallel to the path of the satellite are neglected, the initial conditions of speed V_1, radial distance r_1, and elevation angle γ_1 at final-stage burnout determine the orbit characteristics of a satellite. It is the inclination angle

of the orbital plane as determined by the azimuth angle ψ_1 at the burnout latitude ϕ_1, however, which determines the range of latitude that will be covered by the satellite. It should, therefore, be possible for a given value of V_1, r_1, and γ_1 to determine an initial azimuth angle ψ_1 which will place a satellite vehicle over a selected position on the rotating earth surface after a selected small number of orbits. A method for determining this initial azimuth angle is presented in this report.

The orbit relations obtained from elementary celestial mechanics for a body acted upon by a radial force of attraction proportional to the inverse square of the distance are first reviewed. Reference conditions for relating the position of the satellite in the orbital plane with the projected position on the earth surface are established. These results are then modified so that the latitude and longitude of the satellite trace on a rotating earth surface may be obtained. An expression is derived which will establish the azimuth angle at burnout required to place a satellite over a selected earth position. Expressions for the incremental changes from the desired latitude and longitude due to errors in the azimuth angle and to the earth's oblateness are presented in appendix A.

Elliptical Orbit Relations

From elementary orbital mechanics (ref. 1), the characteristics of an orbit in terms of the initial conditions at burnout r_1, V_1, and γ_1 can be obtained from the following expressions:

The equations for distance of satellite from earth center (see fig. 1) are

$$r = \frac{p}{1 + e \cos \theta} \tag{1a}$$

$$r = \frac{a(1 - e^2)}{1 + e \cos \theta} \tag{1b}$$

$$r = a(1 - e \cos E) \tag{1c}$$

where

$$\frac{p}{r_1} = \left(\frac{V_1}{V_c}\right)^2 \cos^2\gamma_1 \tag{2}$$

Satellite velocity is given by

$$V^2 = g_0 R^2 \left(\frac{2}{r} - \frac{1}{a}\right) \tag{3}$$

Satellite velocity for a circular orbit at radius r_1 is

$$V_c^2 = \frac{g_0 R^2}{r_1} \tag{4}$$

Angle in orbit plane between perigee and burnout point is given by

$$\tan \theta_1 = \tan \gamma_1 \left(\frac{\frac{p}{r_1}}{\frac{p}{r_1} - 1}\right) \tag{5}$$

Eccentricity is

$$e = \frac{1}{\cos \theta_1}\left(\frac{p}{r_1} - 1\right) \tag{6}$$

Satellite period in seconds is

$$T = 2\pi \sqrt{\frac{R}{g_0}} \sqrt{\left(\frac{a}{R}\right)^3} \tag{7}$$

Time from perigee is expressed as

$$t(\theta) = \frac{T}{2\pi}(E - e \sin E) \tag{8}$$

Eccentric anomaly (fig. 1(b)) is given by

$$E = 2 \tan^{-1}\left(\sqrt{\frac{1 - e}{1 + e}} \tan \frac{\theta}{2}\right) \tag{9}$$

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Relations Between Orbit and Earth Systems of Coordinates

In the present paper spherical-triangle relations are used for relating the satellite position in the orbital plane to its trace on the earth surface. By using the inclination angle i of the orbit plane (which is numerically equal to the maximum latitude reached) and a reference or nodal point N, it is possible to obtain the desired position relations between the equatorial and orbital planes. The angular motion of a satellite in an elliptic orbit is usually related to a perigee passage; therefore, in the present paper time is referenced to the perigee point. The orientation of a satellite orbit and the projection of the orbit plane on a spherical earth are illustrated in figure 2.

<u>Latitude position</u>.- The inclination angle i, the angle between the orbit plane and the equatorial plane (fig. 2(a)), is fixed at the instant of burnout by the latitude ϕ_1 and azimuth ψ_1. If the equations for a right spherical triangle are applied to the triangle bounded by the equator, the projected orbit plane, and the meridian or longitude through the position ϕ_1, the pertinent relation for the situation pictured in figure 2(b) is

$$\cos i = \cos \phi_1 \sin \psi_1 \qquad (0° < \psi_1 < 180°) \tag{10a}$$

For a retrograde or westward launching, which occurs when the azimuth angle at burnout exceeds 180°, the relation is

$$\cos i = -\cos \phi_1 \sin \psi_1 \qquad (180° < \psi_1 < 360°) \tag{10b}$$

The position of the satellite in the plane of the orbit at the instant of burnout is specified by the angle θ_1 (eq. (5)) measured relative to perigee. The perigee point in turn is related to one of the two nodes at the intersection of the projected orbital plane and the equatorial plane by the angle ω. The relationship for the situation shown in figure 2(b) is

$$\frac{\sin \phi_1}{\sin i} = \sin(\omega + \theta_1) \tag{11a}$$

or

$$\frac{\tan \phi_1}{|\cos \psi_1|} = \tan(\omega + \theta_1) \tag{11b}$$

For determination of the argument of perigee ω, with θ_1, ϕ_1, and ψ_1 or i given, the inverse angle relations implied by equations (11a) and (11b) are ordinarily most conveniently resolved in favor of placing the angle $\omega + \theta_1$ in the first or fourth quadrants - that is, $90° > (\omega + \theta_1) > -90°$; thus, for reference purposes the node which is closest to the point of burnout is selected.

Subsequent to burnout, the latitude of the satellite at any point in the orbit ϕ_S is given by relations similar to equation (11a), that is

$$\sin \phi_S = \sin i \sin(\omega + \theta_S) \tag{12}$$

For use of equation (12) it is convenient to regard the angle i as positive if the reference node is an ascending node, one where the equator is crossed from south to north. Conversely, if the reference node is a descending node, i should be regarded as a negative angle. The combinations of launch direction and burnout latitude, which lead to positive or negative values of i, together with the range of the azimuth ψ_1 are summarized in table I.

Longitude position.- The longitude of a point on the earth's surface is measured by the arc of the equator intercepted between the prime meridian at Greenwich and the meridian through the point. A convenient reference for determining the longitude of a satellite is the node, the point

of intersection of the equatorial and orbital planes, at the instant of burnout. At a subsequent time, the nodal point will have shifted westward by the product of the earth's angular velocity ω_E and the orbiting time.

As given in equations (8) and (9), time is regarded as a function of the orbital angle θ and is conveniently computed from perigee, $\theta = 0^\circ$. In the present paper, orbiting time is measured from burnout, $\theta = \theta_1$. Time, therefore, at any instant in the first orbit may be given by $t(\theta_S) - t(\theta_1)$ or, in general, by $t(\theta_S) - t(\theta_1) + nT$ where T is the period of the satellite in minutes, and n is the number of completed orbits. (An orbit is completed when the latitude at burnout ϕ_1 is reached again in the same direction as at launch.)

From figure 2(b), the longitude of the reference node $\left(\lambda_N\right)_{ref}$ and the longitude of burnout λ_1 are related by

$$\left(\lambda_N\right)_{ref} = \lambda_1 - \Delta\lambda_{N1} \tag{13}$$

where

$$\tan \Delta\lambda_{N1} = \sin \phi_1 \tan \psi_1 \tag{14}$$

From figure 2(c), at time $t(\theta_S)$

$$\tan \Delta\lambda_{NS} = \cos i \tan(\omega + \theta_S) \tag{15}$$

and the longitude of the node is given by

$$\lambda_N = \left(\lambda_N\right)_{ref} - \omega_E\left[t(\theta_S) - t(\theta_1) + nT\right] \tag{16}$$

Hence, the longitude of the satellite at any time is

$$\lambda_S = \left(\lambda_N\right)_{ref} - \omega_E\left[nT + t(\theta_S) - t(\theta_1)\right] + \Delta\lambda_{NS} \tag{17}$$

EXPRESSION FOR PLACING A SATELLITE OVER A SELECTED

POSITION

The requirement that a satellite with burnout position ϕ_1, λ_1 pass over a selected position ϕ_2, λ_2 after the completion of n orbits is equivalent to the requirement that, during the first orbit, the satellite pass over an equivalent position with latitude ϕ_2 the same as that of the selected position but with longitude λ_{2e} displaced eastward from λ_2 by an amount sufficient to compensate for the rotation of the earth during the n complete orbits, that is, by the polar hour angle $n\omega_E T$. (See fig. 3(a).) The longitude of this equivalent position is thus given by the relation

$$\lambda_{2e} = \lambda_2 + n\omega_E T \tag{18}$$

The azimuth angle ψ_1 which will place the satellite over the position ϕ_2, λ_{2e} at time $t(\theta_{2e})$ will, therefore, n orbits later place it over the selected position ϕ_2, λ_2.

The trace of the satellite position from the point ϕ_1, λ_1 to the point ϕ_2, λ_{2e} is not a great circle, since during the time interval $t(\theta_{2e}) - t(\theta_1)$ when the satellite passes through the orbital angle $\theta_{2e} - \theta_1$, the earth rotates eastward through the polar hour angle $\omega_E\left[t(\theta_{2e}) - t(\theta_1)\right]$. The difference in polar hour angle (which is a difference in longitude) between the meridian λ_1 at $t(\theta_1)$ and the meridian through the point ϕ_2, λ_{2e} at $t(\theta_{2e})$ will be

$$\Delta\lambda_{1-2e} = \lambda_{2e} - \lambda_1 + \omega_E\left[t(\theta_{2e}) - t(\theta_1)\right] \tag{19}$$

Since the latitudes ϕ_1 and ϕ_2 are known, the orbit angle $\theta_{2e} - \theta_1$ (fig. 3(b)), may be obtained from equation (19) and the relation

$$\cos(\theta_{2e} - \theta_1) = \sin \phi_2 \sin \phi_1 + \cos \phi_2 \cos \phi_1 \cos \Delta\lambda_{1-2e} \tag{20}$$

(In the application of equation (20) it may be noted that since the angles $\theta_{2e} - \theta_1$ and $\Delta\lambda_{1-2e}$ are both portions of intersecting great circles, $\theta_{2e} - \theta_1$ will be less than (or greater than) 180° when $\Delta\lambda_{1-2e}$ is less than (or greater than) 180°.)

In the use of equations (19) and (20) an iterative procedure is required, since the time $t(\theta_{2e})$ from perigee to the equivalent position is not known initially. A satisfactory first approximation is to assume that

$$\left[t(\theta_{2e}) - t(\theta_1)\right] \approx \frac{T}{360}(\lambda_{2e} - \lambda_1) \tag{21}$$

This value, substituted into equation (19), yields a first approximation for $\Delta\lambda_{1-2e}$. This approximation substituted into equation (20) yields a first approximation for θ_{2e} from which, by equation (8), a new value of $t(\theta_{2e})$ is obtained. The iteration of these three steps is continued until the value of θ_{2e} converges. For convenience these equations are listed in appendix B.

The effects of the earth's oblateness, discussed in appendix A, may be included as small modifications of the values of ϕ_2 and λ_{2e} obtained in the first iteration. The equations are given in appendix B, and examples are shown in the following section.

When convergence of the value of θ_{2e} has been attained, the azimuth angle at burnout ψ_1 may now be calculated from the sine law (fig. 3(b)) by

$$\sin \psi_1 = \frac{\sin \Delta\lambda_{1-2e} \cos \phi_2}{\sin(\theta_{2e} - \theta_1)} \tag{22}$$

and the inclination angle by equations (10):

$$\cos i = \pm\sin \psi_1 \cos \phi_1$$

The argument of perigee ω may be calculated from the following equation which is equation (11a):

$$\sin(\omega + \theta_1) = \frac{\sin \phi_1}{\sin i}$$

The position equations given in the previous section and listed in appendix B may be used to check final values and to compute the complete orbit path.

In orbit calculations, such as those described in this paper, a number of occasions arise when ambiguities of the antitrigonometric functions must be resolved in order to place angles in the proper quadrants. The following conventions have been used in this paper:

Longitude λ: range, 0° to 360°; positive, counterclockwise, eastward; negative, clockwise, westward

Azimuth ψ_1: range, 0° to 360°; clockwise from north; orbits exhibit normal rotation when $0^\circ < \psi_1 < 180^\circ$; orbits exhibit retrograde motion when $180^\circ < \psi_1 < 360^\circ$

Orbit angle θ: range, 0° to 360°; positive for normal rotation; negative for retrograde motion

Inclination angle i: range, 0° to 90°; measured from equator; positive at ascending node; negative at descending node

Nodal point N: sometimes specified as ascending node; reference node utilized herein is selected on basis of proximity to burnout point

EXAMPLE ORBIT CALCULATIONS

In order to illustrate the application of the developed expressions, a temporary satellite orbit has been selected just outside the earth's atmosphere. Examples are given of a satellite launched eastward and one launched westward, each passing over a selected position after having completed three orbits. The two directions of launch illustrate that the procedure is not limited by the direction of launch. The positions were considered favorable positions from which to initiate a reentry leading to a selected recovery area. Incremental changes from the desired latitude and longitude due to the earth's oblateness, as discussed in

appendix A, are included in the procedure for obtaining the azimuth angles.
The results for both cases are then compared with the results obtained
by the method of reference 2 of a three-degree-of-freedom calculation
for an oblate rotating earth performed on an IBM 704 electronic data
processing machine. The data from this calculation are referred to here-
inafter as three-degree-of-freedom data. Changes from the selected lati-
tude and longitude, resulting from incremental changes from the burnout
condition ψ_1 and ϕ_1 are also analyzed.

The orbit characteristics of a satellite vehicle launched into an
orbit 120 nautical miles above the earth's surface with an initial veloc-
ity $V_1 = 1.010V_c$ and an elevation angle $\gamma = 0.50°$ are presented in
table II. The satellite radial distance r and time t calculated by
equations (1), (8), and (9) are plotted as a function of θ in figure 4.
With the three parameters - speed, radial distance, and elevation angle -
given, only the azimuth angle at burnout ψ_1 remains to be calculated
to establish the orbit completely.

Eastward Launch, Case A

The satellite is to be launched eastward from a burnout position
$\phi_1=28.50°$ N.,$\lambda_1=279.45°$ E. and is to pass over a position
$\phi_2=34.00°$ N.,$\lambda_2=241.00°$ E. after completing three orbits. The steps of
the procedure for determining the azimuth angle at burnout are outlined
and illustrated in appendix B.

The azimuth angle ψ_1 is determined on the basis of an equivalent
position ϕ_2,λ_{2e} which must be passed in the first orbit. After the
first iteration, it is desired to incorporate the oblateness effects as
corrections to the desired position and to ϕ_2,λ_{2e}. For the orbiting
time $3T + t(\theta_{2e}) - t(\theta_1) \approx 281.615$ minutes, there results a change in
latitude of $0.086°$ and a change in longitude of $1.028°$. These values
are subtracted from the original latitude and longitude to give a cor-
rected position $\phi_2=33.914°$ N.,$\lambda_2=239.972°$ E. and hence a corrected
$\lambda_{2e}=308.661°$ E. With these substitutions into equations (19) and (20),
the iteration for θ_{2e} is continued and final values of the desired
quantities are obtained as shown in appendix B.

By using the given initial conditions and ψ_1 calculated by the
method of this paper, an orbit was calculated by a three-degree-of-
freedom method for an oblate rotating earth (ref. 2). A trace of the
resulting orbit is presented in figure 5(a). The method of the present
paper indicated that the selected position would be reached in

281.316 minutes. The three-degree-of-freedom data indicated that at
t = 281.316 minutes the satellite is located at $\phi=33.98°$ N., $\lambda=240.62°$ E.,
which is 18.6 nautical miles west of the desired position, and would reach
the desired position at t = 281.398 minutes. This agreement is con-
sidered good.

With the calculated azimuth and inclination angles, errors in the
desired latitude and longitude due to insertion errors in ψ_1 and ϕ_1
are examined. The results for case A are presented in figure 6(a). The
variations of Δi, $\Delta\phi_2$, and $\Delta\lambda_2$ with $\Delta\psi_1$ and $\Delta\phi_1$ are considered
to be small, the slopes being dependent on equations (A1) and (A3).

Westward Launch, Case B

The same iteration procedure is used for the westward launch as
used for the eastward launch with pertinent sign changes. As shown in
appendix B, first approximations are made and resulting values are used
to compute oblateness effects. These corrections are applied to ϕ_2
and λ_2, and final values are then derived.

As in case A, an orbit was calculated by a three-degree-of-freedom
method by using the given initial conditions and ψ_1 calculated by the
method shown in appendix B.

The trace of the resulting orbit is shown in figure 5(b). The method
of this paper indicated that the desired position would be reached in
335.608 minutes. The three-degree-of-freedom data indicated that at
t = 335.608 minutes the satellite is located at $\phi=20.63°$ N., $\lambda=320.67°$ E.,
a point 52.2 nautical miles west of the desired position.

The variations of Δi, $\Delta\phi_2$, and $\Delta\lambda_2$ with $\Delta\psi_1$ and $\Delta\phi_1$ for
case B as derived in equations (A1) to (A3) are presented in figure 6(b).
The results for both cases indicate that the error in the desired lati-
tude and longitude is a function of the orbital angle $\omega + \theta_2$, the azimuth
angle at burnout ψ_1, and the burnout latitude ϕ_1.

CONCLUDING REMARKS

It is possible by the use of a simple iteration procedure to estab-
lish the azimuth angle at burnout on the basis of a selected latitude and
longitude and a prescribed number of complete orbits for a given satellite
period and orbit eccentricity for either an eastward or westward launch.

Incremental changes from the desired latitude and longitude positions due to the perigee motion and rotation of the orbital plane can conveniently be taken into account in the procedure. The magnitude of the changes for a given orbit is a function of the time in orbit and the inclination angle of the orbital plane. A comparison of the results for both the eastward and westward launches with the results obtained from a three-degree-of-freedom calculation for an oblate rotating earth shows very good agreement. This indicates that there are no apparent limitations as to launch direction or selected passage point. The error in the desired latitude and longitude is a function of the orbital angle, the azimuth angle at burnout, and the burnout latitude.

Langley Research Center,
National Aeronautics and Space Administration,
Langley Field, Va., November 27, 1959.

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APPENDIX A

INCREMENTAL CHANGES FROM DESIRED POSITIONS

Incremental changes in the azimuth angle ψ_1 and the resulting influence on the desired satellite passage point ϕ_2, λ_2 are presented along with incremental changes due to the effect of the earth's oblateness. Changes in the desired burnout conditions V_1, r_1, and γ_1 introduce changes in the orbit elements which are beyond the scope of this report and will therefore not be considered.

The error in the inclination angle Δi obtained by differentiating equation (10a) with respect to ψ_1 and ϕ_1 is

$$\Delta i = -\frac{\cos \phi_1 \cos \psi_1}{\sin i} \Delta \psi_1 + \frac{\sin \psi_1 \sin \phi_1}{\sin i} \Delta \phi_1 \tag{A1}$$

The error in the desired satellite latitude obtained by differentiating equation (10a), making use of equation (A1), and neglecting $\Delta \theta_{2e}$ is

$$\Delta \phi_2 = \pm \frac{\sin(\omega + \theta_{2e})}{\cos \phi_2 \tan i}(\sin \psi_1 \sin \phi_1 \Delta \phi_1 - \cos \phi_1 \cos \psi_1 \Delta \psi_1)$$

$$\pm \frac{\sin i \cos(\omega + \theta_{2e})}{\cos \phi_2} \Delta \omega \tag{A2}$$

Similarly, the error in the desired longitude obtained by substituting equation (15) into equation (17) and differentiating, making use of equation (A1), and neglecting $\Delta \theta_{2e}$ is

$$\Delta \lambda_2 = \frac{\tan(\omega + \theta_{2e})}{1 + \cos^2 i \, \tan^2(\omega + \theta_{2e})}(\cos \phi_1 \cos \psi_1 \Delta \psi_1 - \sin \psi_1 \sin \phi_1 \Delta \phi_1)$$

$$+ \frac{\cos i \, \sec^2(\omega + \theta_{2e})}{1 + \cos^2 i \, \tan^2(\omega + \theta_{2e})} \Delta \omega \tag{A3}$$

The errors in ϕ_2 and λ_2 as presented by equations (A2) and (A3) are thus made up of errors in the insertion conditions ϕ_1 and ψ_1 and errors due to perigee motion $\Delta\omega$. These errors are as follows:

$$\Delta\phi_2 = \left(\Delta\phi_2\right)_{ie} + \left(\Delta\phi_2\right)_{pm}$$

and

$$\Delta\lambda_2 = \left(\Delta\lambda_2\right)_{ie} + \left(\Delta\lambda_2\right)_{pm}$$

where the subscripts ie and pm represent insertion errors and perigee motion (oblateness effects), respectively, and

$$\left(\Delta\phi_2\right)_{pm} = \pm \frac{\sin i \cos\left(\omega + \theta_2\right)}{\cos \phi_2} \Delta\omega \tag{A4}$$

and

$$\left(\Delta\lambda_2\right)_{pm} = \frac{\cos i \sec^2\left(\omega + \theta_{2e}\right)}{1 + \cos^2 i \tan^2\left(\omega + \theta_{2e}\right)} \Delta\omega \tag{A5}$$

The rotation of the perigee point $\Delta\omega$ is defined as

$$\Delta\omega = \frac{d\omega}{dt} \Delta t \tag{A6}$$

where the approximate mean rate of the argument of perigee motion or rotation of the major axis in deg/min (ref. 3) is

$$\frac{d\omega}{dt} = 3.4722 \times 10^{-3}\left(\frac{R}{p}\right)^2\left(\frac{R}{a}\right)^{3/2}\left(5 \cos^2 i - 1\right) \tag{A7}$$

The rotation of the major axis is in the same direction as the satellite if $i < 63.4°$, in the opposite direction if $i > 63.4°$ and is zero when $i = 63.4°$. The other main effect of the earth's oblateness is the rotation of the orbital plane $\Delta\Omega$ defined as

$$\Delta\Omega = \frac{d\Omega}{dt} \Delta t \tag{A8}$$

The orbital plane rotates about the earth's axis in the opposite direction to the earth's rotation and the approximate mean rate of rotation of the orbital plane in deg/min (ref. 3) is

$$\frac{d\Omega}{dt} = -6.9444 \times 10^{-3} \left(\frac{R}{p}\right)^2 \left(\frac{R}{a}\right)^{3/2} \cos i \qquad (A9)$$

The rotation of the orbital plane affects only the incremental longitude so that a $\Delta\Omega$ term must be included in equation (A3) to give the total oblateness effect on the longitude of the desired passage point. The magnitudes of $\Delta\omega$ and $\Delta\Omega$ are obtained by multiplying equations (A7) and (A9) by the orbiting time $nT + t(\theta)_2 - t(\theta)_1$ expressed in minutes.

Changes from the desired satellite passage point due to oblateness effect can be taken into account in the iteration procedure previously discussed. The desired passage point becomes

$$\phi_2 - (\Delta\phi_2)_{total}, \quad \lambda_2 - (\Delta\lambda_2)_{total}$$

These values are substituted into equation (19) before a second iteration is performed.

APPENDIX B

EQUATIONS FOR CALCULATING AZIMUTH ANGLE, OBLATENESS EFFECTS, AND SATELLITE POSITION AND ILLUSTRATIVE EXAMPLES

Azimuth Angle

The calculation of the azimuth angle at burnout involves the use of equations (21), (18), (19), (20), (8), (22), and (10) as follows:

$$\left[t(\theta_{2e}) - t(\theta_1)\right] \approx \frac{T}{360}(\lambda_{2e} - \lambda_1)$$

$$\Delta\lambda_{1-2e} = \left(\lambda_2 + n\omega_E T\right) + \omega_E\left[t(\theta_{2e}) - t(\theta_1)\right] - \lambda_1$$

$$\cos(\theta_{2e} - \theta_1) = \sin\phi_2 \sin\phi_1 + \cos\phi_2 \cos\phi_1 \cos\Delta\lambda_{1-2e}$$

$$t(\theta_{2e}) = \frac{T}{2\pi}(E - e\sin E)$$

$$\sin\psi_1 = \frac{\sin\Delta\lambda_{1-2e}\cos\phi_2}{\sin(\theta_{2e} - \theta_1)}$$

$$\cos i = \pm\sin\psi_1 \cos\phi_1$$

Oblateness Effects

The oblateness effects are calculated through the use of the following equations:

$$\Delta\omega = 3.4722 \times 10^{-3}\left(\frac{R}{p}\right)^2\left(\frac{R}{a}\right)^{3/2}\left(5\cos^2 i - 1\right)\left[nT + t(\theta_{2e}) - t(\theta_1)\right]$$

$$\left(\Delta\phi_2\right)_{pm} = \pm\frac{\sin i \cos(\omega + \theta_{2e})}{\cos\phi_2}\Delta\omega$$

$$\Delta\Omega = -6.9444 \times 10^{-3}\left(\frac{R}{p}\right)^2\left(\frac{R}{a}\right)^{3/2}\cos i\left[nT + t\left(\theta_{2e}\right) - t\left(\theta_1\right)\right]$$

$$\left(\Delta\lambda_2\right)_{pm} = \frac{\cos i \; \sec^2\left(\omega + \theta_{2e}\right)}{1 + \cos^2 i \; \tan^2\left(\omega + \theta_{2e}\right)}\Delta\omega + \Delta\Omega$$

These equations make use of equations (A5) to (A9).

Satellite Position

The calculation of the satellite position involves the use of equations (12), (17), (11a), (14), (13), and (15). For convenience, these equations are repeated as follows:

$$\sin \phi_S = \sin i \; \sin\left(\omega + \theta_S\right)$$

$$\lambda_S = \left(\lambda_N\right)_{ref} + \Delta\lambda_{NS} - \omega_E\left[nT + t\left(\theta_S\right) - t\left(\theta_1\right)\right]$$

$$\sin\left(\omega + \theta_1\right) = \frac{\sin \phi_1}{\sin i}$$

$$\tan \Delta\lambda_{N1} = \sin \phi_1 \; \tan \psi_1$$

$$\left(\lambda_N\right)_{ref} = \lambda_1 - \Delta\lambda_{N1}$$

$$\tan \Delta\lambda_{NS} = \cos i \; \tan\left(\omega + \theta_S\right)$$

Illustrative Examples of Orbit Calculations

In the following sections examples of the orbit calculations for cases A (eastward launch) and B (westward launch) are given to demonstrate the application of the method.

Case A, eastward launch.- For case A, eastward launch, the given launch position is $\phi_1=28.50°$ N.,$\lambda_1=279.45°$ E.; the selected position is $\phi_2=34.00°$ N.,$\lambda_2=241.00°$ E.; the number of orbital passes n is 3; and the equivalent selected longitude λ_{2e} is 309.689° E. The values of the first approximations are

$$\left[t(\theta_{2e}) - t(\theta_1)\right] \approx 7.693$$

$$\Delta\lambda_{1-2e} = 32.162$$

$$\theta_{2e} = 51.89$$

$$t(\theta_{2e}) = 12.702$$

$$\psi_1 = 70.468$$

$$i = 34.081$$

The values found in the calculation of the oblateness effects are

$$\Delta\omega = 1.965$$

$$\Delta\phi_2 = 0.086$$

$$\Delta\Omega = -1.340$$

$$\Delta\lambda_2 = 1.028$$

Corrected $\phi_2 = 33.914°$ N. Corrected $\lambda_2 = 239.972°$ E.

The iteration procedure, as explained in the text of this paper, yields the following values:

Parameter	2d iteration	3d iteration	4th iteration
$t(\theta_{2e}) - t(\theta_1)$	6.860	6.603	6.594
$\Delta\lambda_{1-2e}$	30.926	30.862	30.859
θ_{2e}	50.866	50.809	50.806
$t(\theta_{2e})$	12.445	12.436	12.436

The final values are as follows:

$$\psi_1 = 70.541$$

$$i = 34.043$$

$$\left(\lambda_N\right)_{ref} = 225.971$$

$$\omega = 34.497$$

Case B, westward launch.- For case B, westward launch, the given launch position is $\phi_1 = 34.00^\circ$ N., $\lambda_1 = 241.00^\circ$ E.; the selected position is $\phi_2 = 20.00^\circ$ N., $\lambda_2 = 320.00^\circ$ E.; the number of orbital passes n is 3; and the equivalent selected longitude λ_{2e} is 28.689° E. The values of the first approximations are

$$t\left(\theta_{2e} - \theta_1\right) = 54.012$$

$$\Delta\lambda_{1-2e} = -198.808$$

$$\theta_{2e} = -260.863$$

$$t\left(\theta_{2e}\right) = 66.996$$

$$\psi_1 = 201.203$$

$$i = -72.553$$

The values found in the calculation of the oblateness effects are

$$\Delta\omega = -0.531$$

$$\Delta\phi_2 = 0.503$$

$$\Delta\Omega = -0.578$$

$$\Delta\lambda_2 = -0.759$$

Corrected $\phi_2 = 19.497°$ N. Corrected $\lambda_2 = 320.759°$ E.

The iteration procedure yields the following values:

Parameter	2d iteration	3d iteration	4th iteration
$t(\theta_{2e}) - t(\theta_1)$	61.154	60.848	60.853
$\Delta\lambda_{1-2e}$	-196.263	-196.340	-196.339
θ_{2e}	-259.664	-259.685	-259.684
$t(\theta_{2e})$	66.690	66.695	66.695

The final values are as follows:

$$\psi_1 = 198.721$$

$$i = -74.569$$

$$(\lambda_N)_{ref} = 230.269$$

$$\omega = 59.427$$

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1. Moulton, Forest Ray: An Introduction to Celestial Mechanics. Second rev. ed., The Macmillan Co., c.1914.

2. Nielsen, Jack N., Goodwin, Frederick K., and Mersman, William A.: Three-Dimensional Orbits of Earth Satellites, Including Effects of Earth Oblateness and Atmospheric Rotation. NASA MEMO 12-4-58A, 1958.

3. King-Hele, D. G., and Gilmore, D. M. C.: The Effect of the Earth's Oblateness on the Orbit of a Near Satellite. Tech. Note No. G.W.475, British R.A.E., Oct. 1957.

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TABLE I

MAGNITUDE OF AZIMUTH ANGLE ψ_1 AND SIGN OF INCLINATION ANGLE i

FOR A GIVEN BURNOUT LATITUDE AND LAUNCH DIRECTION

Burnout latitude, ϕ_1	Launch direction	Inclination angle, i	Azimuth angle, ψ_1, deg
N.(+) S.(-)	NE. SE.	+ -	0 to 90 90 to 180
N.(+) S.(-)	NW. SW.	+ -	270 to 360 180 to 270
N.(+) S.(-)	SE. NE.	- +	90 to 180 0 to 90
N.(+) S.(-)	SW. NW.	- +	180 to 270 270 to 360

TABLE II

ORBIT CHARACTERISTICS USED FOR SAMPLE CALCULATIONS

Satellite velocity at burnout, V_1, ft/sec 25761.345
Distance of satellite from center of earth at
    burnout, r_1, ft 21,637,933
Elevation angle, γ_1, deg 0.5
Velocity for circular orbit, V_c, ft/sec 25506.28
p/r_1 . . . . 1.020022269
Semimajor axis, a, ft 22081775.58
Angle between perigee and burnout, θ_1, deg . . . . ±23.969
Eccentricity, e 0.0219118
Period, T, min 91.585
Time from perigee to burnout, $t(\theta_1)$, min 5.842
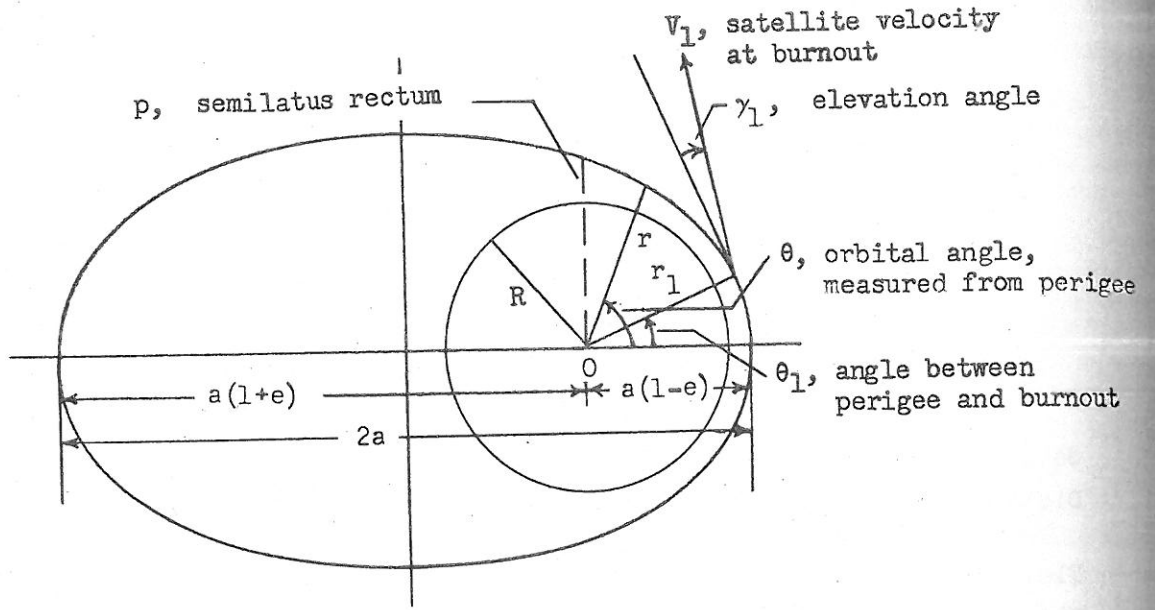
(a) Initial conditions at burnout.
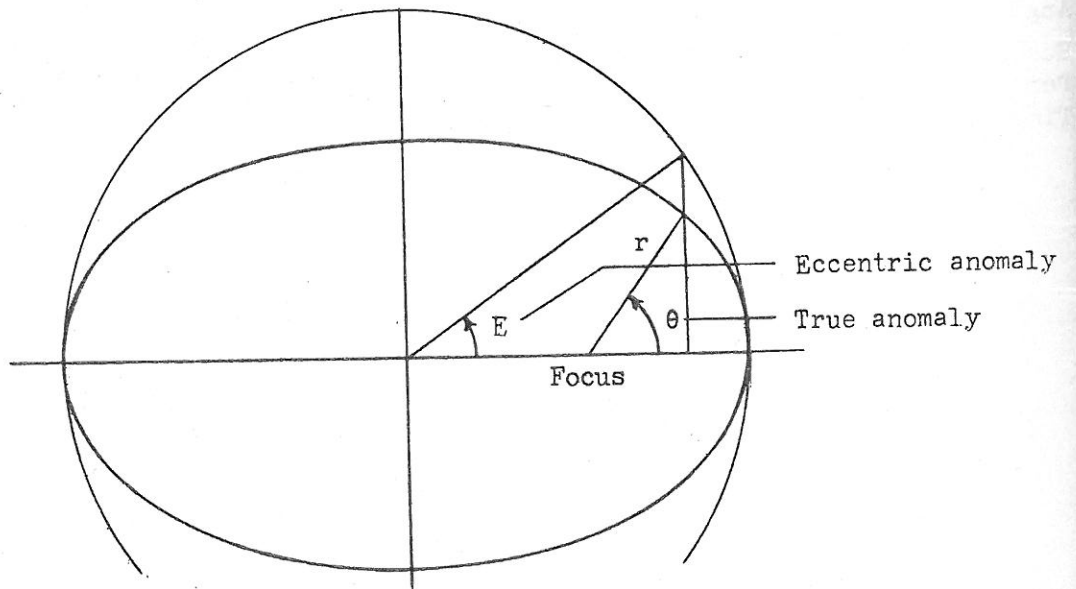
(b) Eccentric and true anomaly.

Figure 1.- Geometry of elliptic orbits. Arrows indicate positive
direction of angles.

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burnout
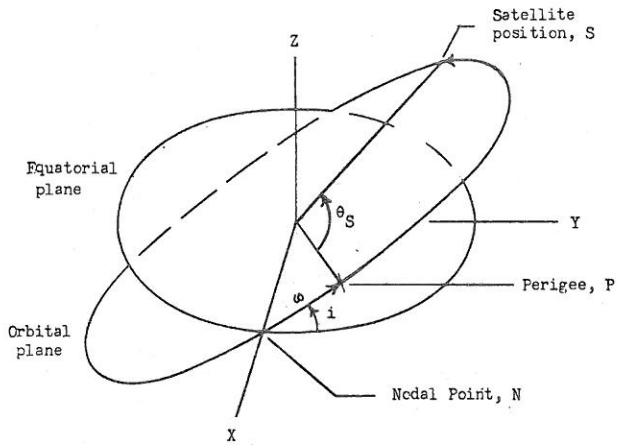
(a) Relationship between orbital plane and equatorial plane.

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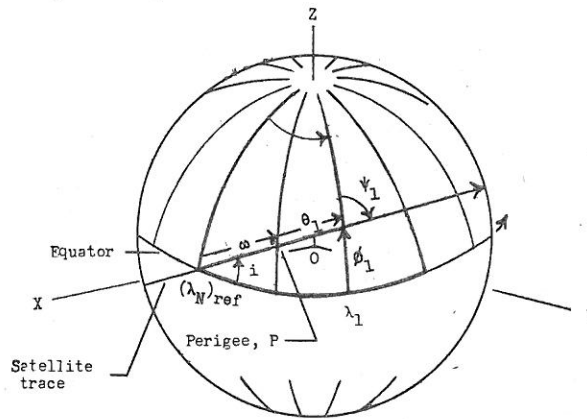
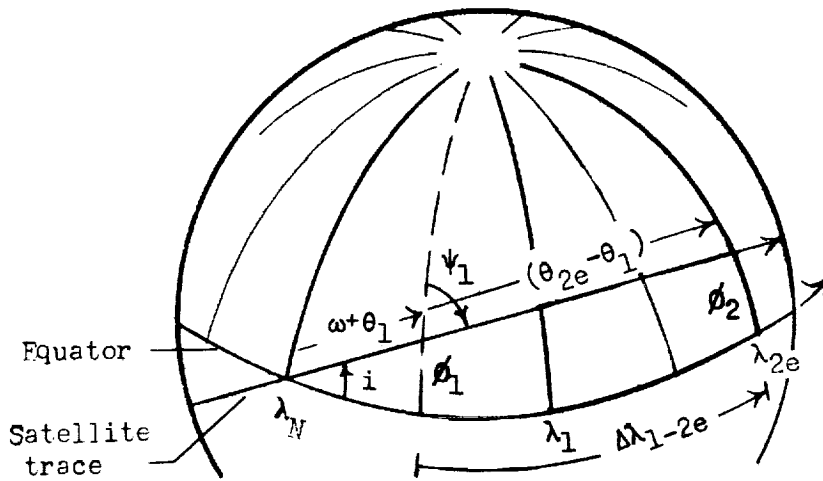(b) Geometry of satellite path over earth. Time = $t(\theta_1)$.

ve
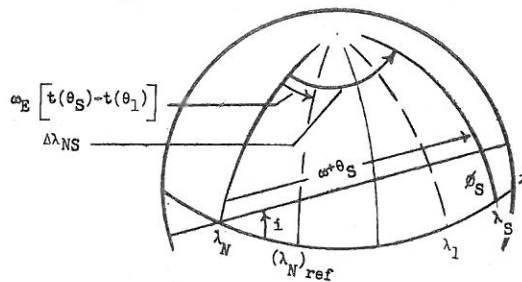
(c) Geometry of satellite path over earth. Time = $t(\theta_S)$.

Figure 2.- Satellite orientation over spherical earth. Arrows indicate positive direction.
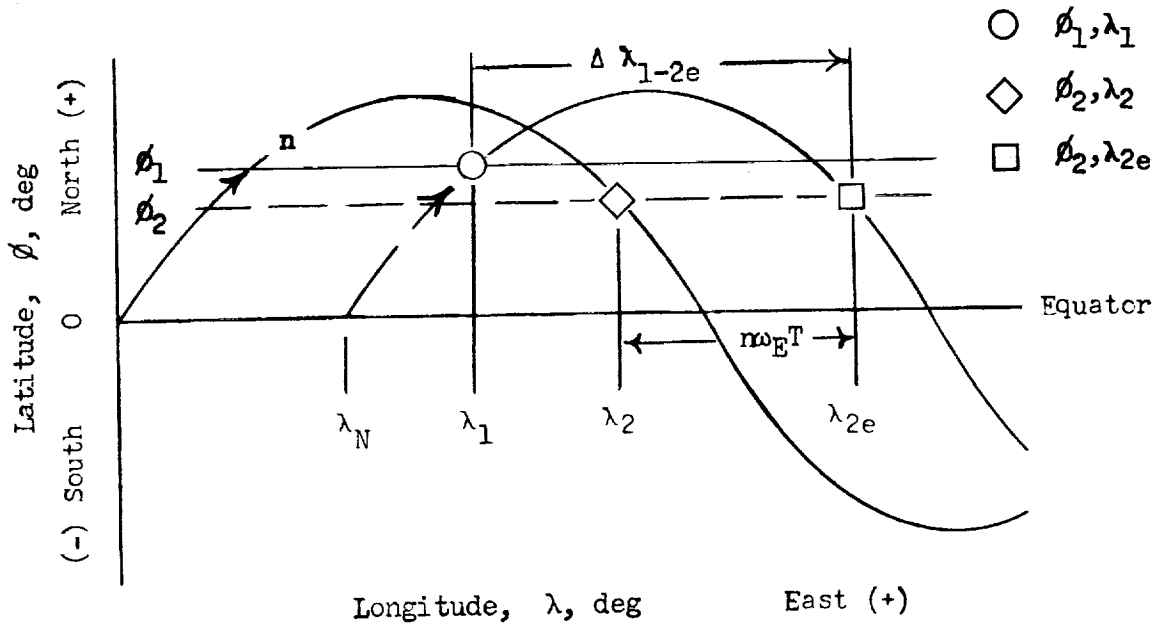
(a) Eastward displacement of λ_{2e} from λ_2.

(b) Relationship between orbit angles and earth positions at
time $t(\theta_{2e})$.

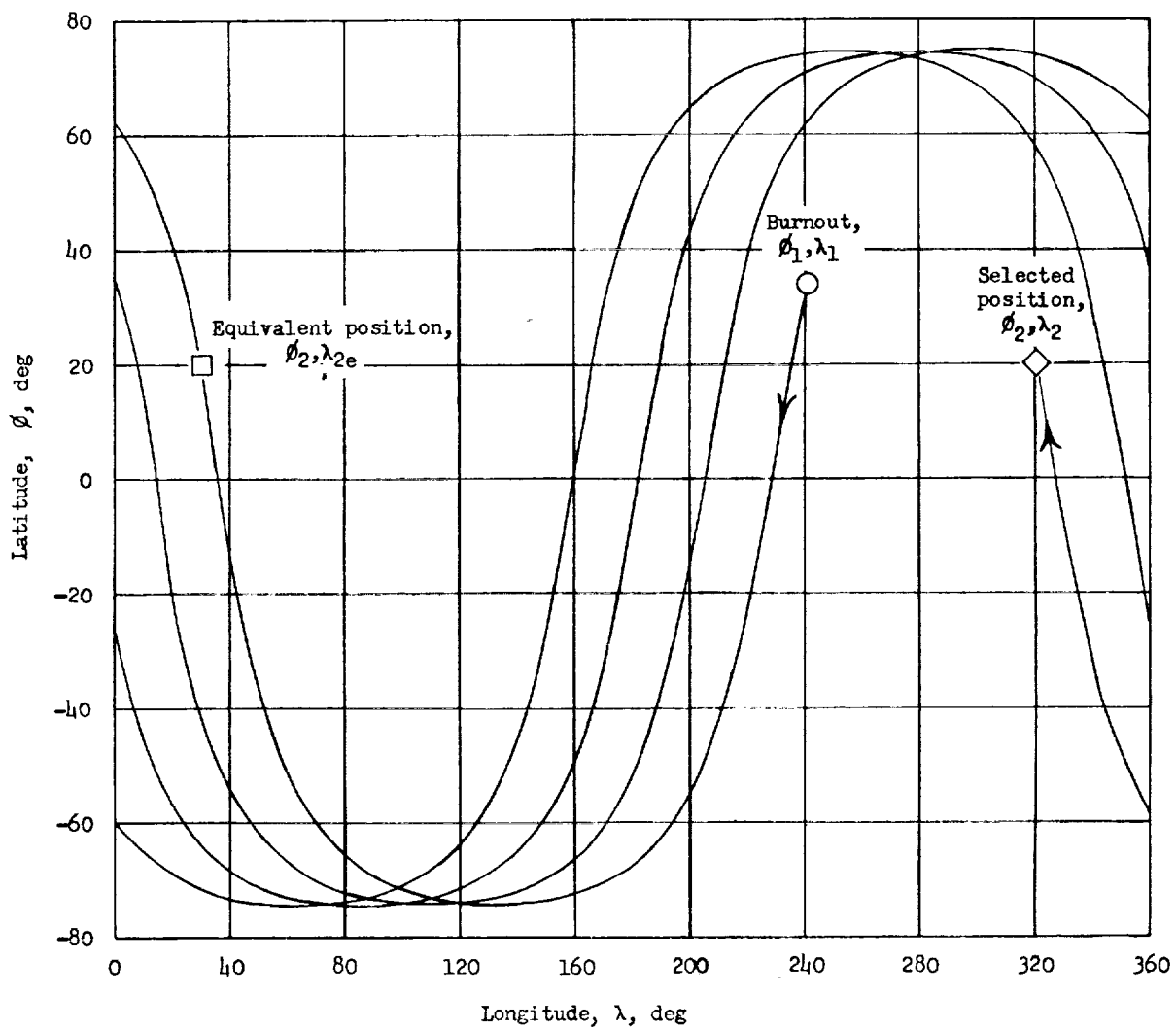Figure 3.- Relation between selected position and equivalent position.
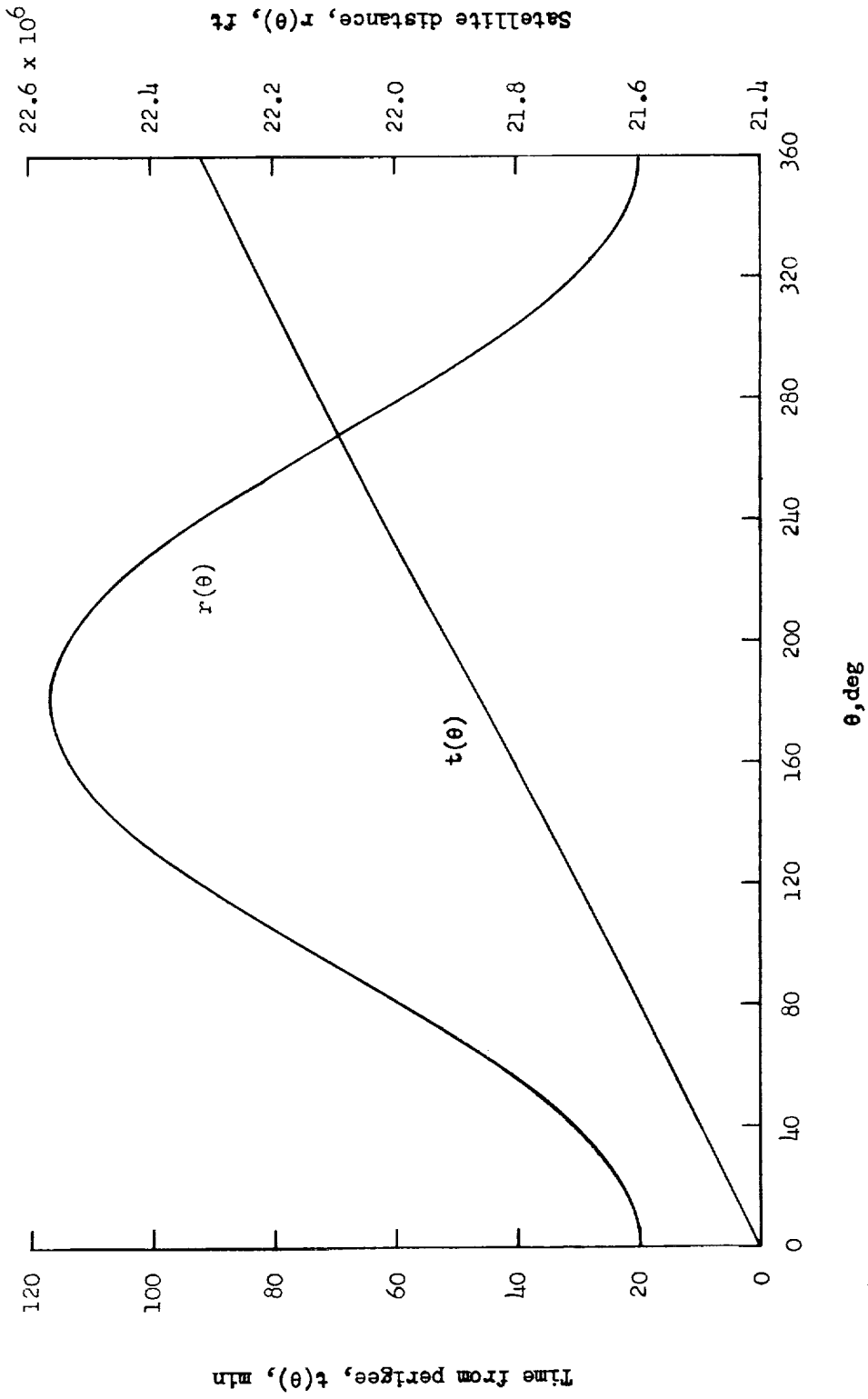
Figure 4.- Satellite time and distance as a function of θ. r_apogee = 22565628 ft;
r_perigee = 21597924 ft; T = 91.585 min.
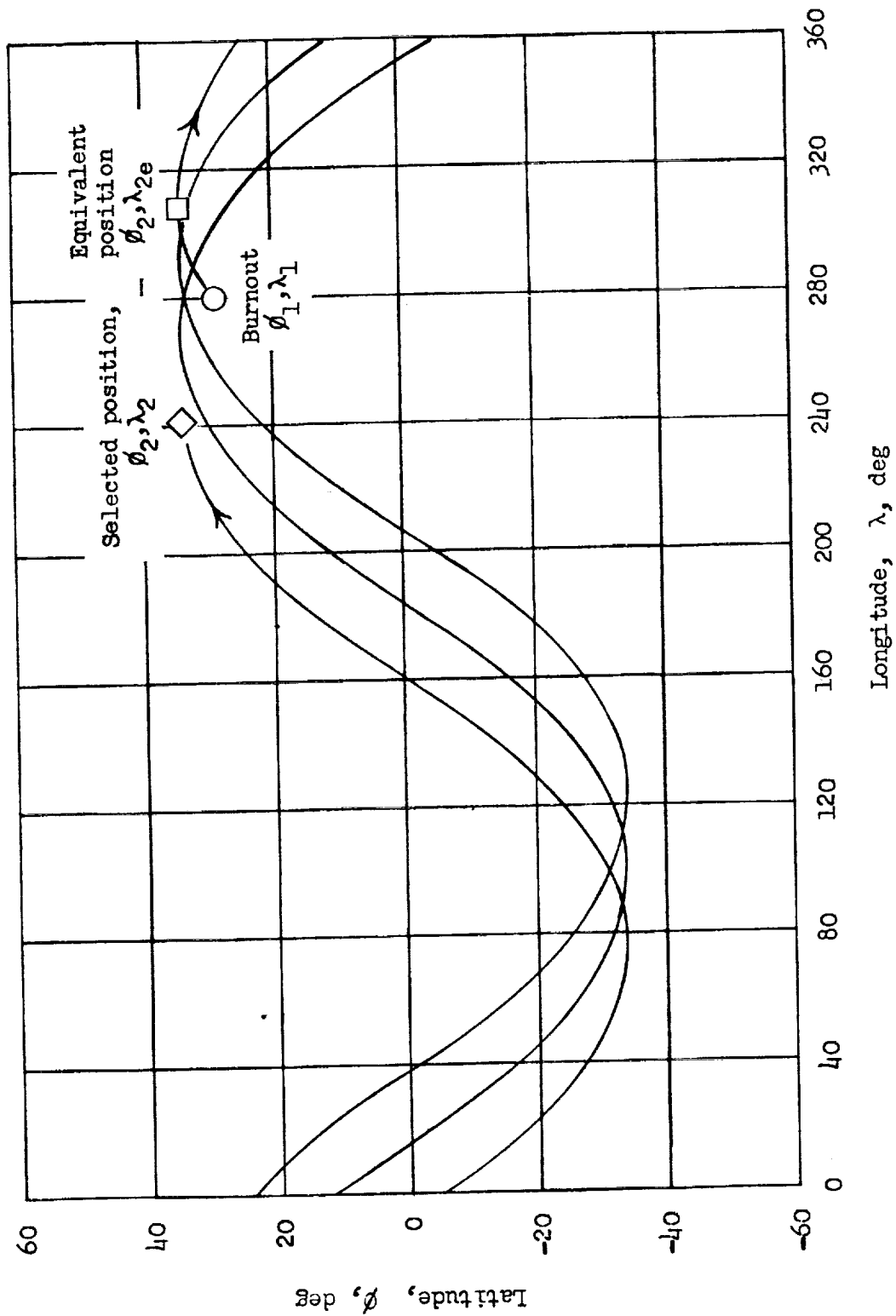
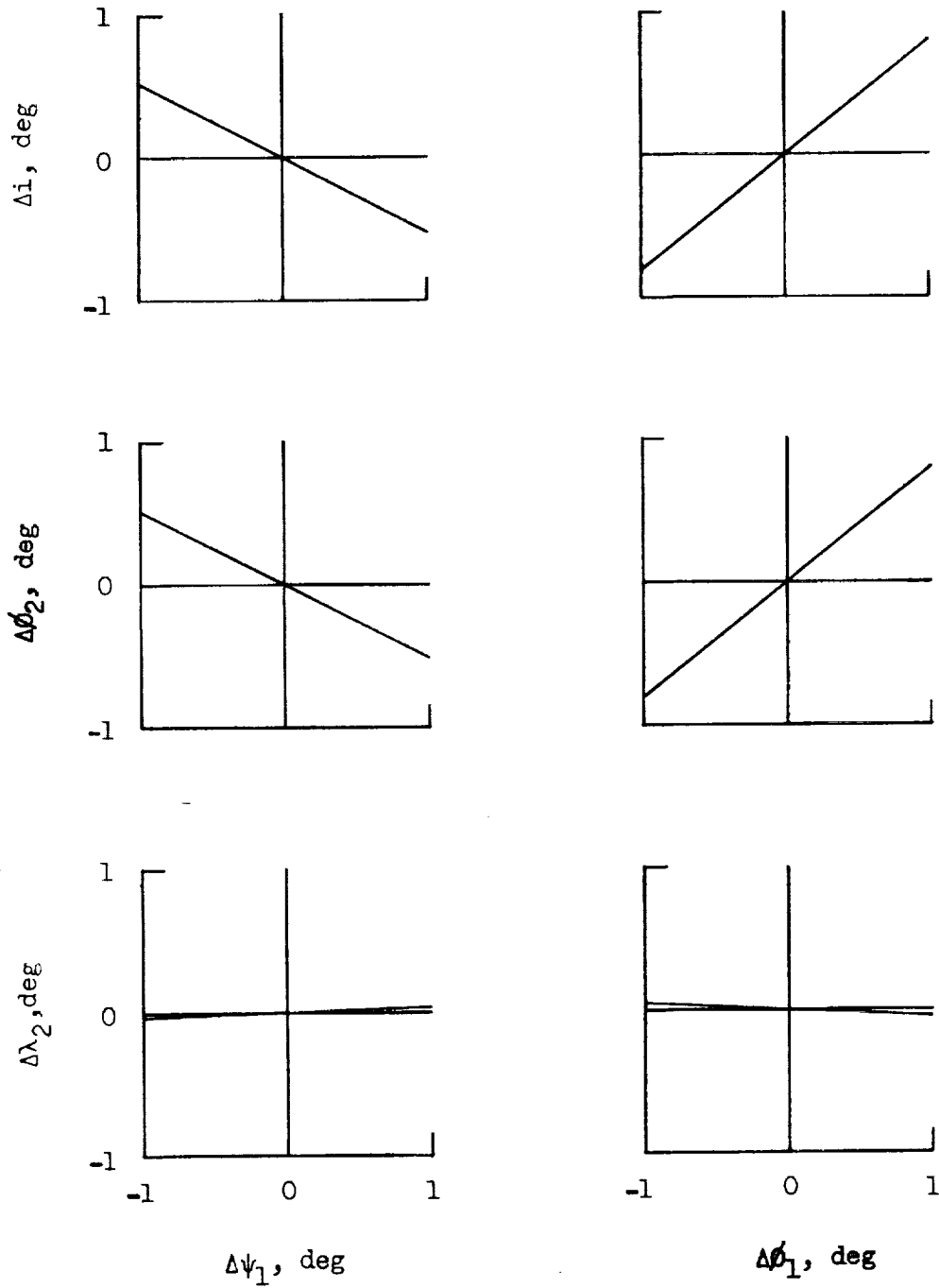(a) Case A, eastward launch.
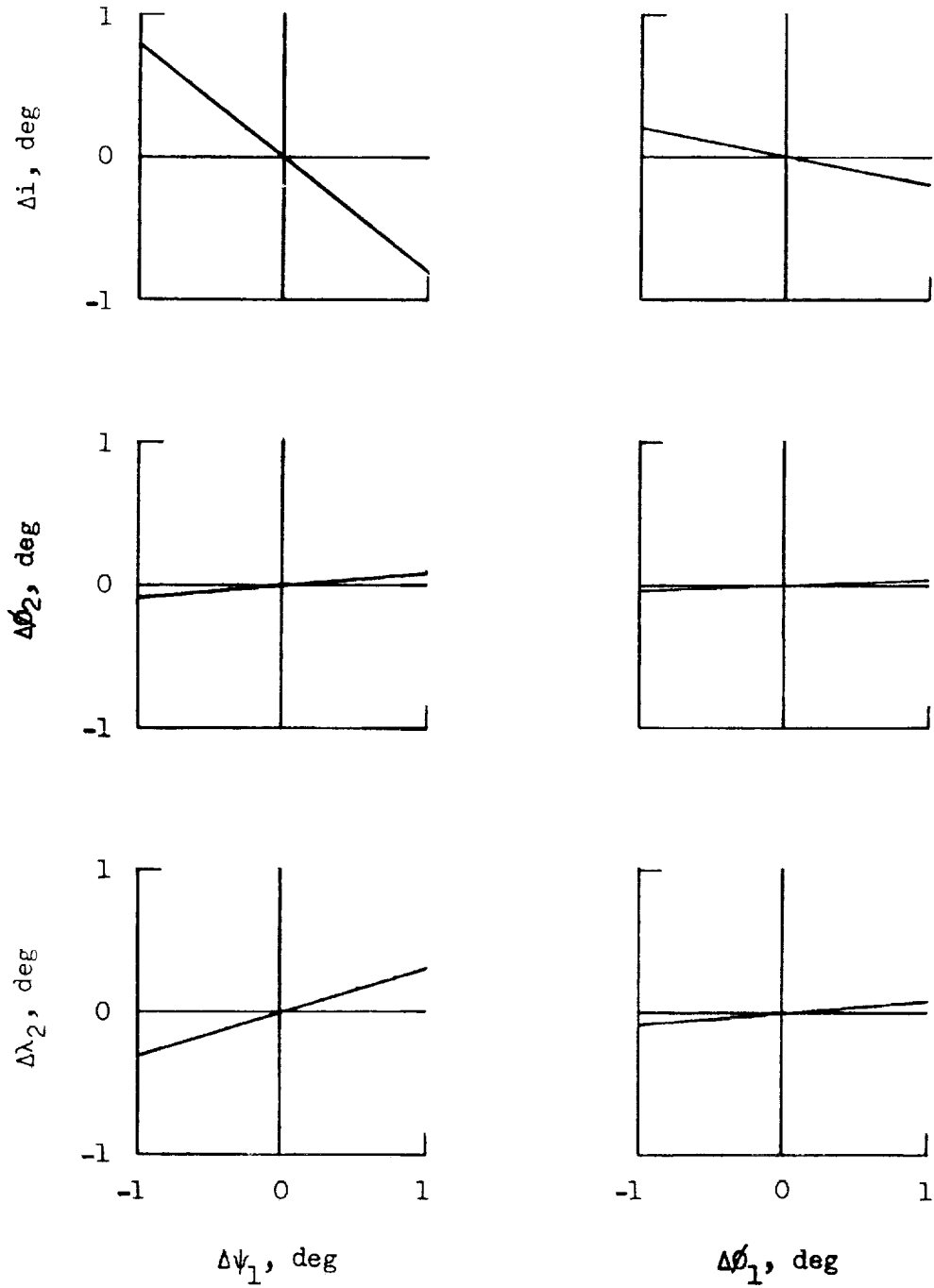
Figure 5.- Satellite position trace on earth surface.

(b) Case B, westward launch.

Figure 5.- Concluded.

(a) Case A, eastward launch.

Figure 6.- Errors in $\Delta\phi_2$ and $\Delta\lambda_2$ due to errors in Δi.

(b) Case B, westward launch.

Figure 6.- Concluded.