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ESTIMATE OF SHOCK STANDOFF DISTANCE AHEAD OF A GENERAL STAGNATION POINT

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SUMMARY

The shock standoff distance ahead of a general rounded stagnation point has been estimated under the assumption of a constant-density shock layer. It is found that, with the exception of almost-two-dimensional bodies with very strong shock waves, the present theoretical calculations and the experimental data of Zakkay and Visich for toroids are well represented by the relation

$$\frac{\Delta_{3D}}{R_{s,x}} = \left(\frac{\Delta_{ax\ sym}}{R_s}\right)\left(\frac{2}{K + 1}\right)$$

where Δ is the shock standoff distance, $R_{s,x}$ is the smaller principal shock radius, and K is the ratio of the smaller to the larger of the principal shock radii.

INTRODUCTION

In recent years much attention has been given to the problem of the inviscid flow about blunt shapes, particularly about bodies of revolution and cylinders. The theories proposed range from those that consider the shock layer to be of constant density to exact numerical integrations of the compressible flow equations. (A summary of these techniques and an extensive list of references are given in ref. 1.) The inviscid flow about general blunt shapes (finite bodies with unequal principal curvatures) has, however, received very little attention. Hayes (ref. 2) has derived an expression for the shock standoff distance ahead of a general stagnation point for a constant-density shock layer. This expression includes centrifugal corrections to the pressure and velocity distributions. It was, however, not evaluated in reference 2, perhaps because the centrifugal effects depended on shock shape away from the stagnation point and a representative three-dimensional shock or body shape is difficult to choose.

The present analysis retreats from that of Hayes (ref. 2) in that centrifugal effects are neglected. In this respect it is the three-dimensional analog of an earlier analysis by Hayes (ref. 3). The shock location is a function only of the density ratio across the shock and the ratio of the principal radii of curvature of the shock. It is independent of whether the body is, for example, an ellipsoid or a toroid.

Because of the many assumptions and simplifications that are made in the present analysis, it is not reasonable to expect the theory to yield precise absolute values for the shock standoff distance ahead of a general stagnation point. It will be shown, however, to give reasonably good results for the ratio of this standoff distance to that for an axially symmetric body. Then, this ratio together with exact solutions for axially symmetric bodies such as those of Van Dyke and Gordon (ref. 4) may give reasonably good results for general bodies.

The main analysis is preceded by a very approximate calculation (OVERSIMPLIFIED METHOD) which, in spite of its crudity, yields a result that is in general agreement with the main analysis. The present results are also compared with the experimental data of Zakkay and Visich (ref. 5).

SYMBOLS

C	shock-layer velocity gradient, eq. (1)
C_x, C_z	shock-layer velocity gradients in principal directions
K	ratio of smaller to larger principal radius of shock wave, $R_{s,x}/R_{s,z}$
k	ratio of free-stream density to that behind a normal shock, ρ_∞/ρ_1
M_∞	free-stream Mach number
p	static pressure
R	radius of curvature
R_x, R_z	principal radii of curvature
u,v,w	velocities in x, y, and z directions, respectively
X	ξ/x

x	coordinate in shock layer identified with smaller principal radius of curvature of shock
y	direction normal to body
z	coordinate in shock layer identified with larger principal radius of curvature of shock
α	$\sqrt{2k/(1 - 2k)}$
γ	ratio of specific heats
Δ	shock standoff distance
ζ	dummy variable in shock surface in z direction
ξ	dummy variable in shock surface in x direction
ρ	density

Subscripts:

ax sym	axially symmetric
b	body
s	shock
1	conditions in shock layer
2D	two-dimensional
3D	three-dimensional
∞	free-stream conditions

OVERSIMPLIFIED METHOD

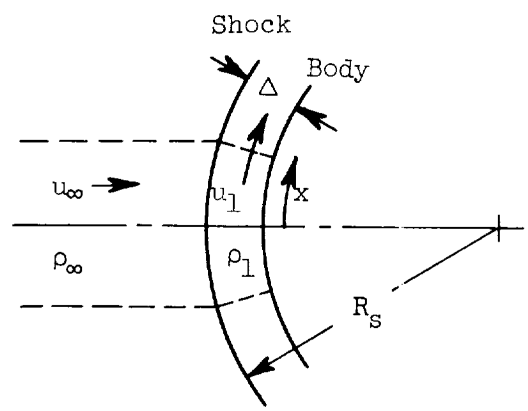
In this approximation the constant-density shock layer is of uniform thickness in the neighborhood of the stagnation point, which is the region under consideration. The free-stream Mach number is very large, the shock layer is very thin $\left(\frac{\Delta}{R_s} \ll 1, \frac{R_b}{R_s} \approx 1\right)$, and the pressure distribution is assumed to be Newtonian and constant along a normal to the body. The velocity in the shock layer is also assumed constant along a normal to the body and equal to the velocity adjacent to the body.

Within these assumptions the shock-layer velocity may be expressed $u_1 \approx Cx$. For $\frac{R_b}{R_s} \approx 1$ and $\left[(\gamma - 1)M_\infty^2\right] \to \infty$, the velocity gradient is $C = \frac{u_\infty}{R_s}\sqrt{\frac{\gamma - 1}{\gamma}}$. But in this limit $k = \frac{\rho_\infty}{\rho_1} \approx \frac{\gamma - 1}{\gamma + 1}$ or $\gamma \approx \frac{1 + k}{1 - k}$, so the shock-layer velocity gradient can be written

$$C = \frac{u_\infty}{R_s}\sqrt{\frac{2k}{1 + k}} \tag{1}$$

Three cases are now considered:

Case I: Axially Symmetric Flow



From continuity considerations (see sketch),

$$\rho_\infty u_\infty \pi x^2 = \rho_1 u_1 2\pi x \, \Delta_{ax\ sym}$$

Using equation (1), the shock-layer thickness becomes

$$\Delta_{ax\ sym} \approx \frac{kR_s}{2}\sqrt{\frac{1 + k}{2k}}$$

Case II: Two-Dimensional Flow

The continuity equation per unit span is

$$\rho_\infty u_\infty x = \rho_1 u_1 \Delta_{2D}$$

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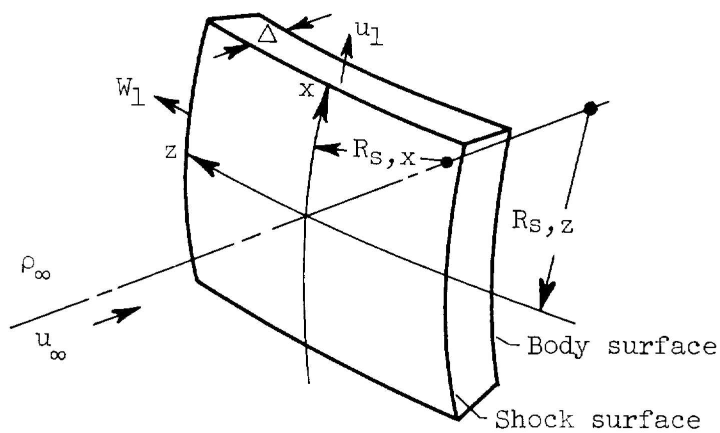
so that the shock-layer thickness becomes

$$\Delta_{2D} \approx kR_s \sqrt{\frac{1 + k}{2k}}$$

which is just twice that for the axially symmetric case.

Case III: Three-Dimensional Flow



Let $R_{s,x}$ be the smaller of the principal shock radii and let $K = R_{s,x}/R_{s,z}$. For this case, the continuity equation in each quadrant can be written

$$\rho_\infty u_\infty xz = \rho_1 u_1 z \, \Delta_{3D} + \rho_1 w_1 x \, \Delta_{3D}$$

or

$$\Delta_{3D} = \frac{\rho_\infty u_\infty}{\rho_1 (C_x + C_z)}$$

$$= \frac{k \sqrt{\frac{1 + k}{2k}}}{\frac{1}{R_{s,x}} + \frac{1}{R_{s,z}}}$$

In terms of $R_{s,x}$ and K,

$$\Delta_{3D} = \frac{kR_{s,x} \sqrt{\frac{1 + k}{2k}}}{1 + K}$$

Note that in all cases the standoff distances are of the form $(\Delta/R_{s,x}) \sim \sqrt{k}$, which is not appropriate to rounded bodies. This incorrect form results from assuming that at a given x and z the velocities in the interior of the shock layer are equal to those at the body surface, so that in effect all the fluid in the shock layer entered through the normal portion of the shock wave as did that fluid at the body surface. Nevertheless, the ratio of the preceding results for a given density ratio k, namely,

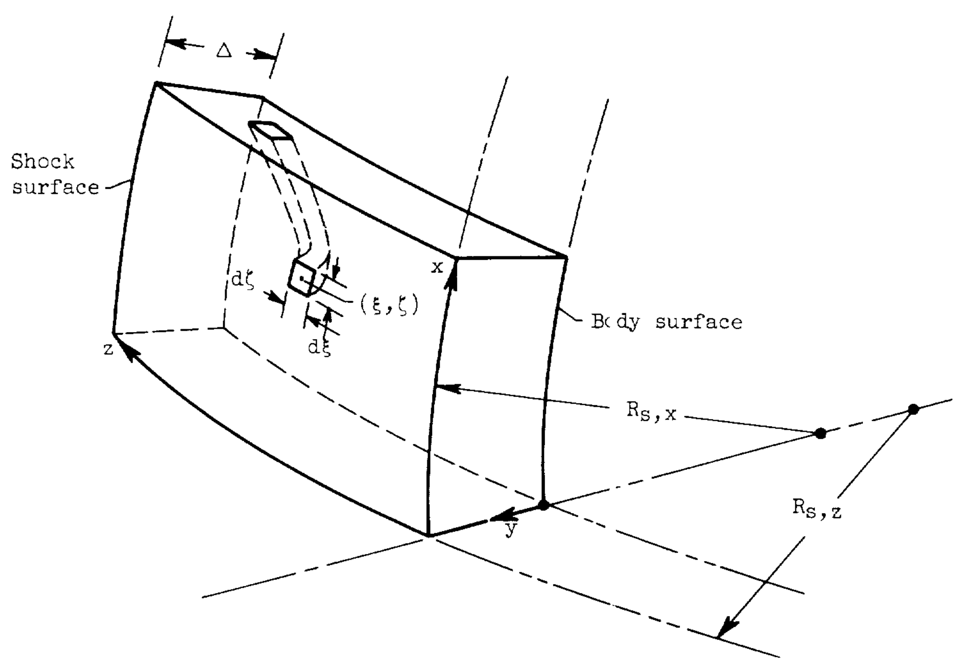
$$\frac{\left(\frac{\Delta}{R_{s,x}}\right)_{3D}}{\left(\frac{\Delta}{R_s}\right)_{ax\ sym}} = \frac{2}{1 + \bar{K}} \tag{2}$$

will turn out to well represent the results of the following analysis.

ANALYSIS

The analysis for the three-dimensional case will now be somewhat improved. In particular, the variation of velocities u_1 and v_1 across the shock layer will be taken into account.

Consider the flow in a quadrant of the constant-density shock layer ahead of a three-dimensional stagnation point. The coordinate system is considered locally Cartesian.



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The pressure distribution is taken to be Newtonian. It is considered a function of x and z only, thus independent of the normal coordinate y. In the neighborhood of the stagnation point it may be written for $M_\infty \gg 1$, $k \ll 1$

$$p_1 \approx \left(1 - \frac{k}{2}\right)\rho_\infty u_\infty^2 \left(1 - \frac{x^2}{R_{s,x}^2} + \ldots\right)\left(1 - \frac{z^2}{R_{s,z}^2} + \ldots\right)$$

$$\approx \left(1 - \frac{k}{2}\right)\rho_\infty u_\infty^2 \left(1 - \frac{x^2}{R_{s,x}^2} - \frac{z^2}{R_{s,z}^2} + \ldots\right) \tag{3}$$

With the aid of Bernoulli's relation, the velocities at a point in the shock layer can be estimated. Consider the streamline that passes through the shock at the point (ξ,ζ). Bernoulli's relation states:

$$\left(p_1 + \rho_1 \frac{u_1^2 + v_1^2 + w_1^2}{2}\right)_{\xi,\zeta} = \left(p_1 + \rho_1 \frac{u_1^2 + v_1^2 + w_1^2}{2}\right)_{x,z} \tag{4}$$

From the oblique shock relations in the limit $M_\infty \gg 1$, $k \ll 1$

$$\frac{\left(u_1^2 + v_1^2 + w_1^2\right)_{\xi,\zeta}}{u_\infty^2} \approx 1 - (1 - k^2)\left(1 - \frac{\xi^2}{R_{s,x}^2} - \frac{\zeta^2}{R_{s,z}^2} + \ldots\right) \tag{5}$$

Upon substituting relations (3) and (5) into equation (4),

$$\frac{\left(u_1^2 + v_1^2 + w_1^2\right)_{x,z}}{u_\infty^2} = \left(\frac{\xi}{R_{s,x}}\right)^2 + \left(\frac{\zeta}{R_{s,z}}\right)^2 + 2k\left(\frac{x^2 - \xi^2}{R_{s,x}^2} + \frac{z^2 - \zeta^2}{R_{s,z}^2}\right) + \mathcal{O}(k^2) \tag{6}$$

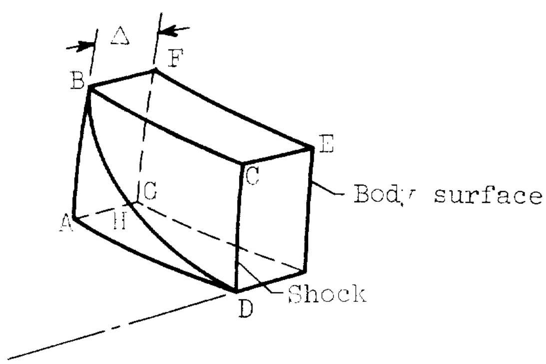
For the portion of the shock layer being considered, the principal shock radii are assumed constant. From equation (3), the pressure gradient in a given principal direction is then independent of the other principal coordinate so that $u_1 = u_1(x,\xi)$ and $w_1 = w_1(z,\zeta)$. Since, in addition, the normal velocity v/u_∞ is of order k (this can be verified a posteriori), the shock-layer velocities in the x and z directions are, respectively,

$$\frac{u_1^2}{u_\infty^2} = \left(\frac{\xi}{R_{s,x}}\right)^2 + 2k\left(\frac{x^2 - \xi^2}{R_{s,x}^2}\right) + \mathcal{O}(k^2) \tag{7}$$

$$\frac{w_1^2}{u_\infty^2} = \left(\frac{\zeta}{R_{s,z}}\right)^2 + 2k\left(\frac{z^2 - \zeta^2}{R_{s,z}^2}\right) + \mathcal{O}(k^2) \tag{8}$$

For the stagnation streamline ($\xi = \zeta = 0$), $\frac{u_1}{u_\infty} \approx \frac{x}{R_{s,x}} \sqrt{2k}$ and $\frac{w_1}{u_\infty} \approx \frac{z}{R_{s,z}} \sqrt{2k}$. These are the velocities obtained upon assuming as in the aforepresented OVERSIMPLIFIED METHOD that all the flow enters the shock layer through the normal portion of the bow shock wave. Currently, however, the shock-layer velocity u_1 at x depends also on the height ξ at which the streamline in question traversed the curved bow shock wave. The velocity w_1 at z depends similarly on ζ.

The shock displacement distance is now to be determined. From continuity, the mass flow into the shock-layer quadrant through face ABCD must equal the sum of the flows leaving through the top face BCEF and the side face ABFG.



But since the normal velocities through the top and side faces depend on ξ and ζ, respectively, the division of outflow between the top and side faces must be determined to properly carry out the mass balance. In other words, the curve BHD must be found such that the inflow through area BCDH exits the top face of the control volume while the inflow through ABHD exits the side face of the control volume. The line BHD is the intersection of the stream surface through the dividing line BF with the shock wave and is found by tracing streamlines back from line BF, which has the coordinates (x,z). Once this is done, the shock displacement distance can be written either as

$$\Delta_{3D} = \int_0^z \frac{\rho_\infty u_\infty \xi_{BHD}(\zeta)d\zeta}{\rho_1 x w_1(\zeta,z)} = \frac{k}{x}\int_0^z \frac{\xi_{BHD}(\zeta)d\zeta}{\frac{w_1}{u_\infty}(\zeta,z)} \tag{9a}$$

or

$$\Delta_{3D} = \int_0^x \frac{\rho_\infty u_\infty \zeta_{BHD}(\xi)d\xi}{\rho_1 z u_1(\xi,x)} = \frac{k}{z}\int_0^x \frac{\zeta_{BHD}(\xi)d\xi}{\frac{u_1}{u_\infty}(\xi,x)} \tag{9b}$$

The equation of a streamline is

$$\frac{dx}{u_1(x,\xi)} = \frac{dz}{w_1(z,\zeta)} \tag{10a}$$

Upon substitution of the velocities from equations (7) and (8) and integrating between the prescribed limits

$$\int_\xi^x \frac{R_{s,x}\,dx}{\sqrt{\xi^2(1 - 2k) + 2kx^2}} = \int_\zeta^z \frac{R_{s,z}\,dz}{\sqrt{\zeta^2(1 - 2k) + 2kz^2}} \tag{10b}$$

the equation of line BHD in the form suitable for equation (9b) is found to be

$$\zeta = \frac{\alpha z}{\sinh\left[(1 - K)\sinh^{-1}\alpha + K \sinh^{-1}\alpha\,\frac{x}{\xi}\right]} \tag{11}$$

where

$$\alpha \equiv \sqrt{\frac{2k}{1 - 2k}} \tag{12}$$

From equations (9b) and (11) the expression for standoff distance is

$$\Delta_{3D} = \frac{kR_{s,x}}{\sqrt{1 - 2k}}\int_0^x \frac{\alpha\,d\xi}{\sinh\left[(1 - K)\sinh^{-1}\alpha + K \sinh^{-1}\alpha\,\frac{x}{\xi}\right]\sqrt{\xi^2 + \alpha^2x^2}} \tag{13}$$

By letting $X = \xi/x$, equation (13) can be written

$$\Delta_{3D} = \frac{kR_{s,x}}{\sqrt{1 - 2k}} \int_0^1 \frac{\alpha \, dX}{\sinh\left[(1 - K)\sinh^{-1}\alpha + K \sinh^{-1} \frac{\alpha}{X}\right]\sqrt{X^2 + \alpha^2}} \tag{14}$$

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Equation (14) is exactly the expression that would be obtained by neglecting centrifugal effects in equation (4.5.7) of Hayes' analysis (ref. 2).

In general, equation (14) must be integrated numerically. However, in the special cases of two-dimensional and axially symmetric flow, closed-form expressions can be obtained. These are:

K = 0, two-dimensional flow:

$$\Delta_{2D} = \frac{kR_{s,x}}{\sqrt{1 - 2k}} \int_0^1 \frac{dX}{\sqrt{X^2 + \alpha^2}} = \frac{kR_{s,x}}{\sqrt{1 - 2k}} \sinh^{-1} \sqrt{\frac{1 - 2k}{2k}} \tag{15}$$

K = 1, axially symmetric flow:

$$\Delta_{ax\ sym} = \frac{kR_{s,x}}{\sqrt{1 - 2k}} \int_0^1 \frac{X \, dX}{\sqrt{X^2 + \alpha^2}} = \frac{kR_{s,x}}{1 + \sqrt{2k}} \tag{16}$$

These two special results were obtained by Hayes (ref. 3).

Equations (15) and (16) are now compared with exact solutions. In figure 1(a), it is seen that equation (16) agrees rather closely with the calculations of Van Dyke and Gordon (ref. 4) for $k \lesssim 0.1$, while at higher density ratios ($k \lesssim 0.3$) it overestimates the exact solutions by no more than 10 percent. Equation (15) for two-dimensional flow is less satisfactory in that it overestimates the results of Van Dyke (ref. 6), Belotserkovskii (ref. 7), and Uchida and Yasuhara (ref. 8) by 20 to 50 percent. This poor agreement is reflected also in the ratio of two-dimensional to axially symmetric standoff distances as shown in figure 1(b).

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For the three-dimensional case $(0 < K < 1)$ with density ratio k greater than zero, equation (14) was numerically integrated on a desk calculator using Simpson's rule. The results are given in table I and are also plotted in figure 2. The results shown for $k = 0$ (table I(b), fig. 2(b)) were obtained by evaluating the integrand of equation (14) for $\alpha \to 0$ and then integrating. The result is

$$\left(\frac{\Delta_{3D}}{kR_{s,x}}\right)_{k\to 0} = \frac{1}{1 - K} \ln \frac{1}{K} \tag{17}$$

The curves of figure 2 are seen to be quite regular, which is perhaps not very surprising. However, the resulting ratios of three-dimensional to axially symmetric standoff distances given in table II and plotted in figure 3 show a more interesting result, namely, that most calculated points show surprisingly good agreement with the crudely derived expression (2) from the OVERSIMPLIFIED METHOD. The exceptions are for bodies approaching the two-dimensional, for example, for $K \lesssim 0.2$ at density ratios $k \lesssim 0.1$.

Before proceeding to a comparison with experiment, it must be realized that the present analysis yields no information regarding the variation of shock-layer thickness about the body and leaves the body shape unspecified. It is therefore not suited to determining the shock standoff distance ahead of a given body. Considering some of the results for a sphere (ref. 2), it is doubtful whether much is gained by considering centrifugal effects and higher order pressure terms in a constant-density approach. It seems rather that, if the constant-density solution is to be improved on, the compressible flow equations should be solved exactly as done in references 4, 6, 7, and 8 for axially symmetric and two-dimensional bodies.

COMPARISON WITH EXPERIMENT

The only pertinent experiments known to the author are those at $M_\infty = 3$ and $M_\infty = 8$ by Zakkay and Visich (ref. 5). The three-dimensional body tested was a toroid. The vital statistics of the experiments and the theoretical comparison are given in the following table (shock radii unfortunately had to be measured from the schlieren photographs presented in ref. 5).

	$M_\infty = 3$	$M_\infty = 8$
Experiment - (ref. 5)		
$R_{b,x}/R_{b,z}$	0.255	0.255
$K = R_{s,x}/R_{s,z}$	.45	.34
$\Delta/R_{b,x}$	.408	.267
$\Delta/R_{s,x}$	.194	.152
Theory		
k	0.259	0.180
$\Delta/R_{s,x}$ (from fig. 2)	.208	.169
(eq. 18)	.187	.157
(eq. 19)	.186	.156

The experimental data for $\Delta/R_{s,x}$ are compared with three theoretical estimates. The first is that value taken directly from figure 2(a) for the pertinent density ratio and shock radius ratio. The second calculation is according to the relation

$$\frac{\Delta_{3D}}{R_{s,x}} = \left(\frac{\Delta_{ax\ sym}}{R_s}\right)_{Van\ Dyke} \left[\frac{\left(\frac{\Delta_{3D}}{R_{s,x}}\right)}{\left(\frac{\Delta_{ax\ sym}}{R_s}\right)}\right] \tag{18}$$

where the ratio of three-dimensional to axially symmetric standoff distances for the proper density ratio k is taken from figure 3. The third calculation uses equation (2) for the aforementioned ratio:

$$\frac{\Delta_{3D}}{R_{s,x}} = \left(\frac{\Delta_{ax\ sym}}{R_s}\right)_{Van\ Dyke} \left(\frac{2}{K + 1}\right) \tag{19}$$

For both equations (18) and (19) the axially symmetric standoff distance is taken from Van Dyke and Gordon (ref. 4), whose data are partially shown in figure 1.

The theoretical estimates agree well with the experimental standoff distances; in fact, the agreement is better than might be expected, considering the author's ability to measure shock radii from photographs. The results using equation (18) or (19) are within 5 percent of the measured values. The theoretical estimate from figure 2 is somewhat higher, which indicates primarily that the present constant-density approximation overestimates the absolute standoff distance for all bodies. This has already been shown for two-dimensional and axially symmetric bodies in figure 1.

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CONCLUDING REMARKS

The shock standoff distance ahead of a general rounded stagnation point has been estimated under the assumption of a constant-density shock layer. It is found that many of the present theoretical calculations as well as the two experimental points of Zakkay and Visich are well represented by the relation

$$\frac{\Delta_{3D}}{R_{s,x}} = \left(\frac{\Delta_{ax\ sym}}{R_s}\right)\left(\frac{2}{K + 1}\right)$$

where K is the ratio of the smaller to the larger principal shock radius. The exceptional cases are bodies approaching the two-dimensional ($K \lesssim 0.2$) with shock layers whose density is much larger than that of the free stream ($k \lesssim 0.10$). In comparing with experiment, the axially symmetric standoff distance was taken from the exact solutions of Van Dyke and Gordon.

Unfortunately, a constant-density theory gives no information regarding the body shape corresponding to a given shock wave and is therefore not suited for obtaining the standoff distance ahead of a given body. In fact, from the experience with the two-dimensional and axially symmetric problems, about the only solutions that adequately relate the flow field to the body are the exact compressible flow solutions. The present results may nevertheless serve as a semiquantitative guide to the phenomenon in the absence of exact solutions.

Lewis Research Center
National Aeronautics and Space Administration
Cleveland, Ohio, June 5, 1961

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2. Hayes, Wallace D.: Constant Density Solutions. Ch. IV of Hypersonic Flow Theory, Academic Press, Inc., 1959, pp. 162-165.

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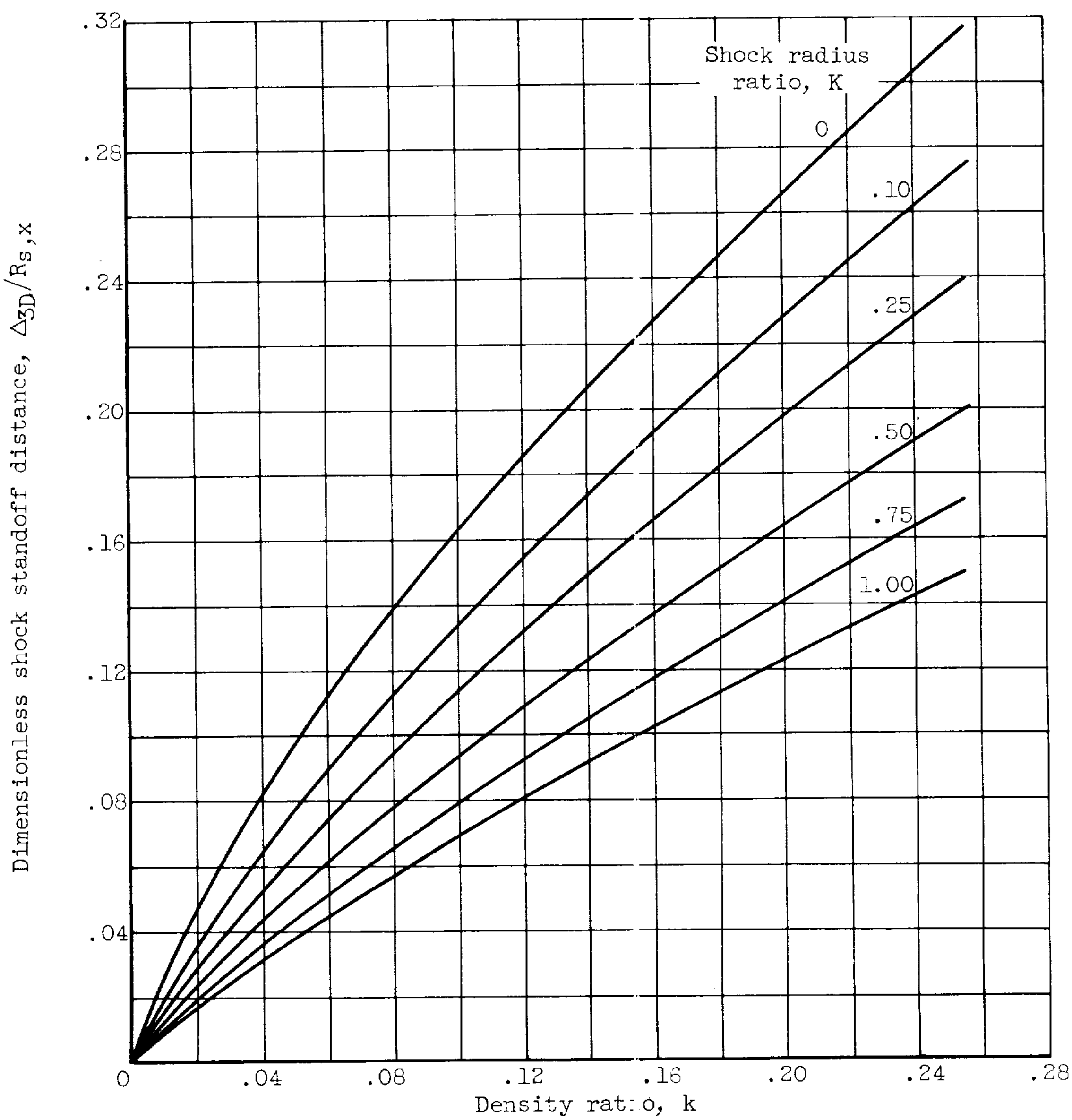
standoff distances.

Figure 1. - Comparison of exact solutions for two-dimensional and axially symmetric shock standoff distances with Hayes' simple theory (ref. 3).

TABLE II. - RATIO OF THREE DIMENSIONAL TO AXIALLY



(a) $\Delta_{3D}/R_{s,x}$.

Figure 2. - Shock standoff distance for three-dimensional bodies.

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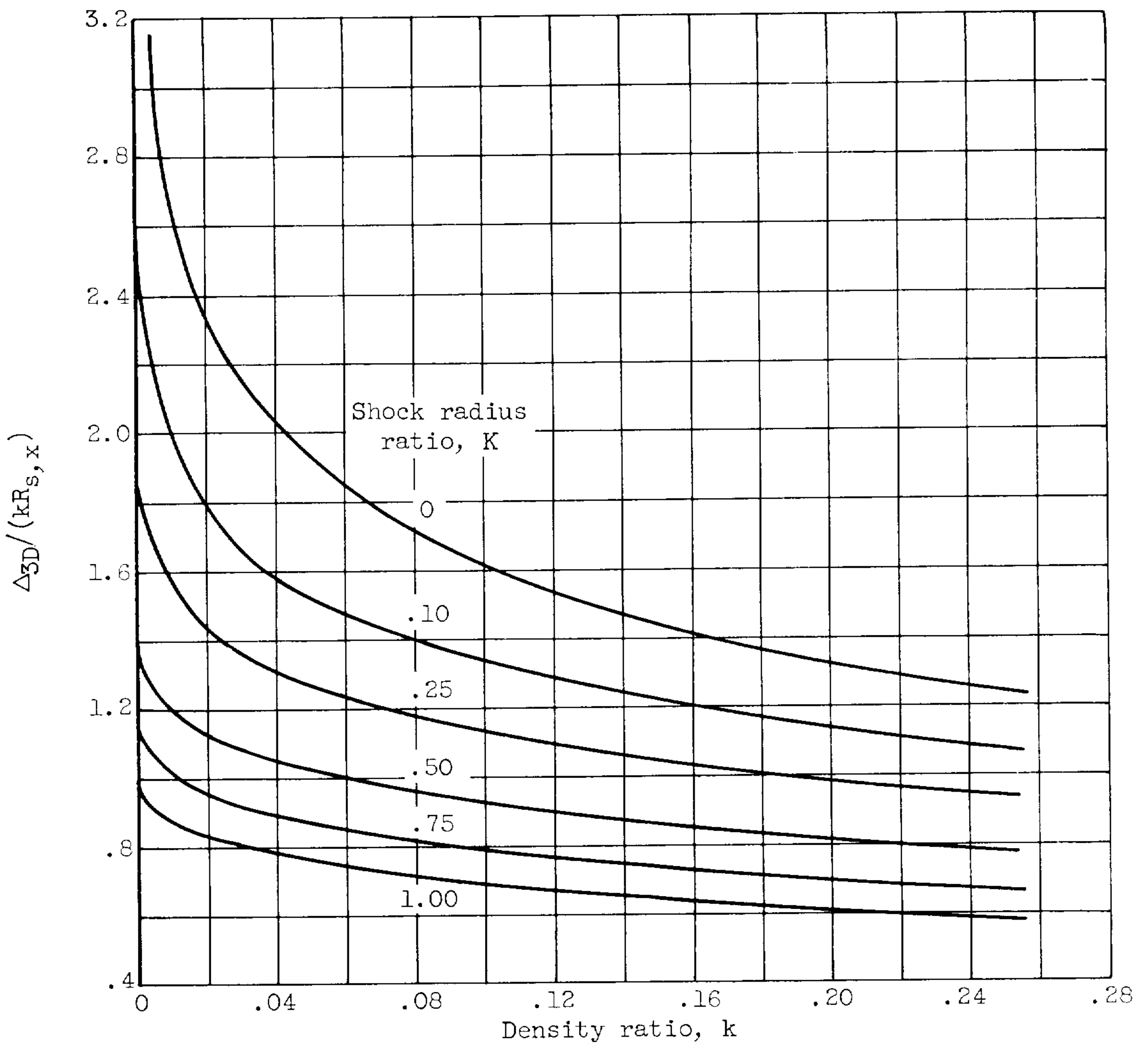

(b) $\Delta_{3D}/(kR_{s,x})$.

Figure 2. - Concluded. Shock standoff distance for three-dimensional bodies.

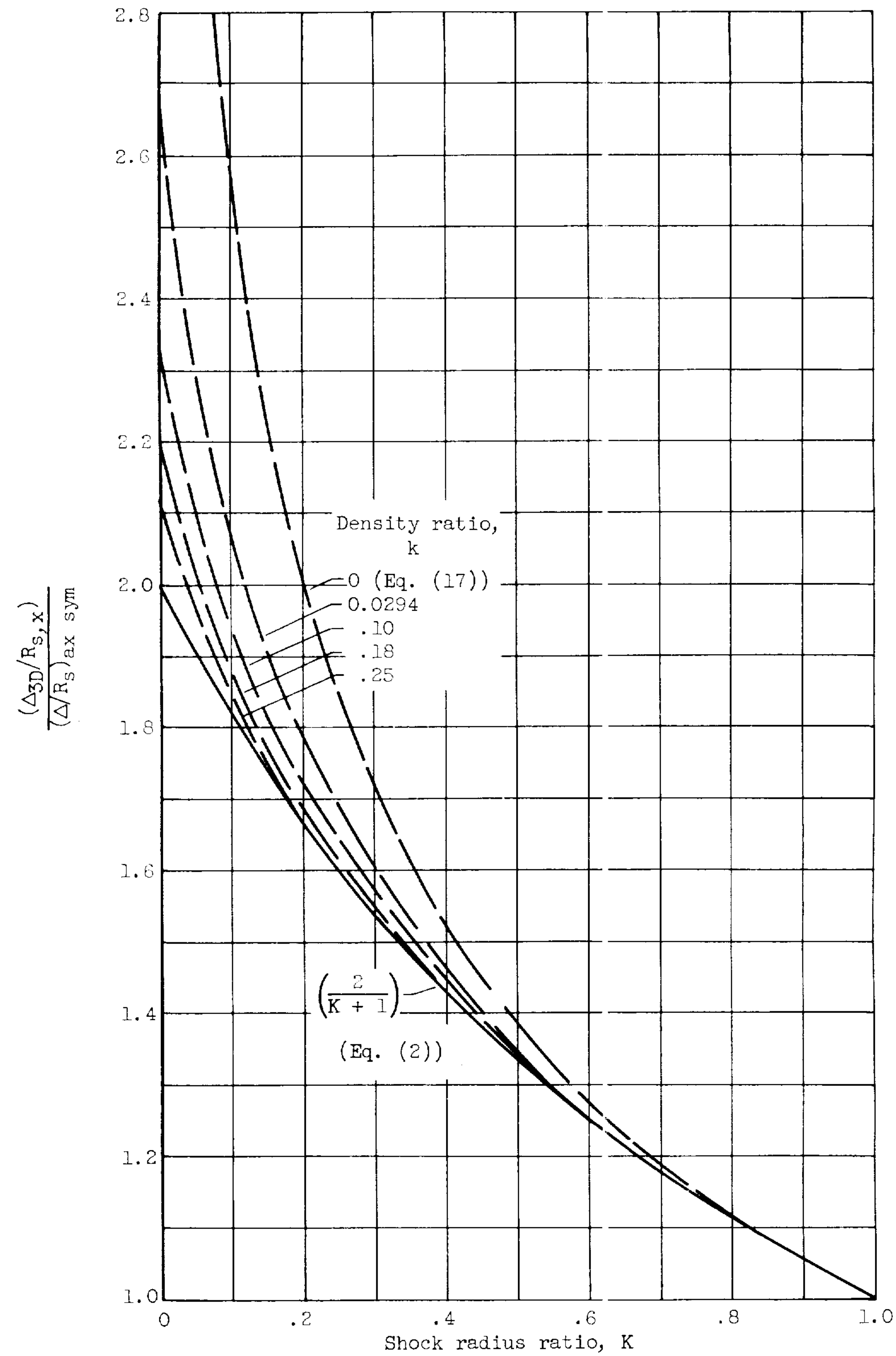

Figure 3. - Ratio of three-dimensional to axially symmetric shock standoff distances.