NASA TR R-114

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

TECHNICAL REPORT
R-114

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1961

For sale by the Superintendent of Documents, U.S. Government Printing Office, Washington 25, D.C. Yearly subscription, $15: foreign, $19;
single copy price varies according to size- - - - - - - - - Price 35 cents

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SUMMARY

The theory of the steady-flow, linear type of crossed-field, d-c plasma accelerator, idealized for the case of no wall friction and no transfer of heat as such to or from the plasma and specialized to the case of constant static temperature and constant cross-sectional area of flow, is developed from the individual equations of motion of the three components of the plasma. The results are more comprehensive, more nearly complete, and more accurate than previous results on this type of accelerator. The effect of the ion cyclotron angle $\omega\tau$, *which is the product of the ion cyclotron frequency and the ion mean free time between collisions with neutral particles and which is proportional to the axial component of the ion slip velocity, on both Joule heating rate and accelerator length is included in the results and is shown to be small only for values of about* 10^{-3} *radian or less.*

INTRODUCTION

The theory of the steady-flow, linear type of crossed-field, d-c plasma accelerator, idealized for the case of no wall friction and no heat loss from the plasma and specialized to the case of constant static temperature and constant cross-sectional area of flow, is developed herein. The theory of the accelerator for the same case has previously been treated (for example, in refs. 1 and 2) on the basis of the equations for the one-dimensional flow of the center of mass of the plasma. The results so obtained have two faults. The first, which is lack of completeness, can be remedied by additional analysis based on the equations of motion of the electrons and the ions. For example, expressions for the axial component of the applied electric field, an important quantity that could not be obtained from the analyses based on the equation of motion of the center of mass, have been so derived and several are discussed and compared in reference 3. Furthermore, it is desirable to have equations for certain additional information, such as the magnitude of the deviation of the average ion velocity direction from the direction of the average neutral-particle velocity. Apparently, the equations for obtaining this and similar results have not been presented except in reference 2 and there the drag of the Coulomb forces was not taken into account. In the previous analyses of the accelerator, a less accurate form of the generalized Ohm's law was used than is used in the present analysis. Because of the use of the more accurate form, the results of the present analysis are more nearly exact, but, as it turns out, the improvement in accuracy is significant only for sufficiently large values of the cyclotron angle $\omega\tau$ of the ions.

The present paper develops the theory of the accelerator from the individual equations of motion of the three components of the plasma. It thus constitutes a single unified treatment that obtains the basic results that have been obtained previously but obtains them with somewhat greater accuracy, and also the additional results (such as axial component of the electric field and ion velocity vector) that come only from the equations of motion of the charged particles. It is a more comprehensive treatment than previous ones and therefore should serve to clarify certain aspects about which there may have been some uncertainty in the past. It makes rather easy the taking into account of the long-range forces between charged particles, does not require any consideration of the components of the tensor form of the electrical conductivity, and shows the effect of ion slip on both Joule heating rate and accelerator length.

In reference 2, for better insight into the plasma physics that is involved in the accelerator, an analysis from a microscopic, or particle, point of view was presented. A portion of that analysis is repeated herein to emphasize what the actual basic physics of the acceleration process is and thus to provide a background for the analysis from the macroscopic point of view.

SYMBOLS

The rationalized mksq system of units is used herein.

B	magnetic induction, webers/m²
c^2	constant, $\gamma M^2 u^2 = \frac{m_3\beta^2\epsilon_{23}^2}{kTe^2B^2}$, dimensionless
c_p	specific heat at constant pressure, joules/kg-°K
e	charge on positive ion, coulombs
E	electric field strength, volts/m
j	current density, amp/m²
k	Boltzmann's constant, joules/particle-°K
m	mass of particle, kg
M	Mach number, herein identical to $\mathbf{M}_3$
n	number density, particles/m³
p	pressure, newtons/m²
Q	collision cross section, m²
t	time, sec
T	temperature, °K
u	dimensionless variable, $\frac{n_3\epsilon_{23}}{eB}$
v	velocity, m/sec
w	velocity of ion in moving coordinate system, m/sec
x	distance in axial direction, m
y	distance in vertical direction, m
Z	collision rate per unit volume, m⁻³ sec⁻¹
α	degree of ionization, $\frac{n_1}{n_3+n_1}\approx\frac{n_1}{n_3}$
β	rate of flow of neutral particles per unit area, $n_3v_{3,x}$, constant, particles/m²-sec
γ	ratio of specific heats
ϵ	interaction parameter, newtons-m²-sec
ϵ_0	permittivity of vacuum, $(36\pi\times10^9)^{-1}$ farad/m
ζ	dimensionless length, $\frac{2\alpha\beta\epsilon_{23}(1+\lambda)}{kT_3}x$
λ	constant, $\frac{\epsilon_{13}}{\epsilon_{23}}$, dimensionless
Λ	quantity in expression for cross section for long-range encounters, dimensionless
μ	constant, $\alpha\frac{\epsilon_{12}}{\epsilon_{23}}$, dimensionless
ρ	mass density, kg/m³
σ	scalar electrical conductivity, mhos/m
τ	mean free time, sec
ω	cyclotron frequency, sec⁻¹

Subscripts:

c	center of mass
f	final
o	initial
x	component in x-direction
y	component in y-direction
i, j	indices indicating species
1	electron
2	ion
3	neutral particle

Symbols defined as vector quantities by use of bold-face type are represented in italic type to indicate scalar quantities.

PRELIMINARY BACKGROUND MATERIAL

A schematic diagram of a longitudinal section along a continuous-flow, linear type of crossed-field, d-c plasma accelerator is shown in figure 1. The cathodes are shown in the upper wall and the anodes in the lower wall, the imposed magnetic field is directed normal to and out of the plane of the paper, and the electric field is in the plane of the paper and directed upward, perhaps at a slight angle to the vertical for reasons to be discussed subsequently.

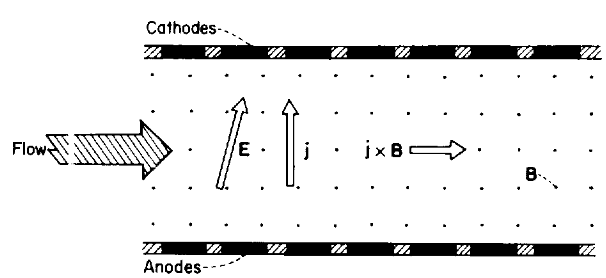

FIGURE 1.—Schematic of plasma accelerator.

Concepts of the basic mechanism of the plasma accelerator, that is, concepts as to the manner in which the charged particles in a three-component slightly ionized plasma accelerate the bulk of the plasma, can be developed from two points of view. From the microscopic point of view, the behavior of an electron and of an ion in electric and magnetic fields that are normal to each other is examined. If there were no collisions, the paths of both ions and electrons would consist of a helical motion with a superposed drift. The two-dimensional projection of these paths in a plane normal to the magnetic field would be a series of cycloids that lie along equipotential lines in a direction normal to both the electric and the magnetic field. Because of the difference in mass, the size of the cycle differs for ions and electrons, and because of the difference in the sign of their electric charges, the sense of rotation is opposite for the two. But the average direction and magnitude of velocity are the same for the two, $\frac{\mathbf{E}\times\mathbf{B}}{\mathbf{B}\cdot\mathbf{B}}$, in the absence of collisions. There are, of course, collisions, and the effects and the results of these are discussed in the next section.

From both the microscopic and the macroscopic point of view, it will be seen that the accelerating force on the plasma derives principally from the ions which, between collisions, are accelerated by the fields and then, by collisions with neutral particles, drive the neutral particles along the channel. From the macroscopic point of view, the driving force is also shown to be the Lorentz force on a current in a magnetic field, $\mathbf{j}\times\mathbf{B}$ per unit volume, where the current will be shown to consist principally of the flow of electrons. The microscopic method is used herein to provide insight into the physics of the process and the macroscopic method is used to provide quantitative results for use in designing accelerators.

QUALITATIVE ANALYSIS FROM PARTICLE POINT OF VIEW

Because of the large difference in the masses of electrons and ions, these two species behave differently in crossed fields in a three-component plasma; that is, they have different mean free times, different cyclotron frequencies, and different velocity directions and magnitudes. The desirable behavior on the part of each is discussed first.

Because of its comparatively small mass, an electron cannot impart much momentum at a collision with a neutral particle. It is therefore not undesirable that the electron make many cycles between collisions with neutral particles and thus have its velocity vector directed essentially at random just before collision.

On the other hand, the ion should make, on the average, just a portion of a cycloid between collisions with neutral particles. It is perhaps well to reiterate here that the basic acceleration process is considered to be first the acquisition of additional momentum from the electric field by the ion, then the transfer of this additional momentum to a neutral particle by collision, and then the equal distribution, on the average, of this additional momentum over the n_3/n_2 neutral particles per ion. The portion of the cycloid that it is desirable for the ion to traverse is one for which, on the average, three conditions are satisfied.

The first of these conditions is that, on the average, at collision the ion's velocity is a specified amount greater than the average forward velocity of the neutral particle. Satisfying this condition allows additional momentum to be imparted to the neutral particle at a specified rate. Part of this additional momentum goes into random motion of the particle and tends to raise the temperature of the gas. On the other hand, the cooling effect associated with acceleration of the plasma tends to lower the temperature. Thus, satisfying this condition results in a partial control over temperature.

The second condition is that the collision occurs when, on the average, the instantaneous direction of motion of the ion is in the direction of the axis of the channel. The reason for this condition is that it is desirable to drive the neutral particles axially along the channel and not toward a wall.

The third condition is that the ion travels on the average along the axial direction—that is, in addition to moving axially at the time of collision, the average velocity of the ion should lie along the direction of the axis and not be directed toward a wall where on contact with an electrode the ion would be neutralized.

Some of these three conditions are mutually contradictory, but usually not seriously so. First, one notes that the ions' paths without collisions can be a series of prolate, common, or curtate cycloids. A cycloid is generated by a point within, on, or without a rolling circle, the center of which moves with a constant velocity of translation $\frac{\mathbf{E}\times\mathbf{B}}{\mathbf{B}\cdot\mathbf{B}}$. If the distance of the generating point from the center of the circle is less than the radius of the circle $\frac{m_2|\mathbf{E}|}{e(\mathbf{B}\cdot\mathbf{B})}$, the point describes a prolate cycloid, as shown in figure 2. If the velocity of the generating point with respect to moving axes fixed in the center of the circle is designated by $\mathbf{w}$, then the instantaneous velocity of the ion with respect to coordinates fixed in the laboratory is

$$\frac{\mathbf{E}\times\mathbf{B}}{\mathbf{B}\cdot\mathbf{B}}+\mathbf{w}$$

What is desired is that, on the average, the ion's path be the result of repeating a small section of this cycloid (fig. 2)—the section between the two indicated points, a section for which the initial and the final directions are nearly the same and which approximates a straight line. This desired result is perhaps more easily seen in the hodograph plane, as in figure 3. The instantaneous velocity of the ion is $\frac{\mathbf{E}\times\mathbf{B}}{\mathbf{B}\cdot\mathbf{B}}+\mathbf{w}$. The neutral particle has the velocity $\mathbf{v}_3$ before

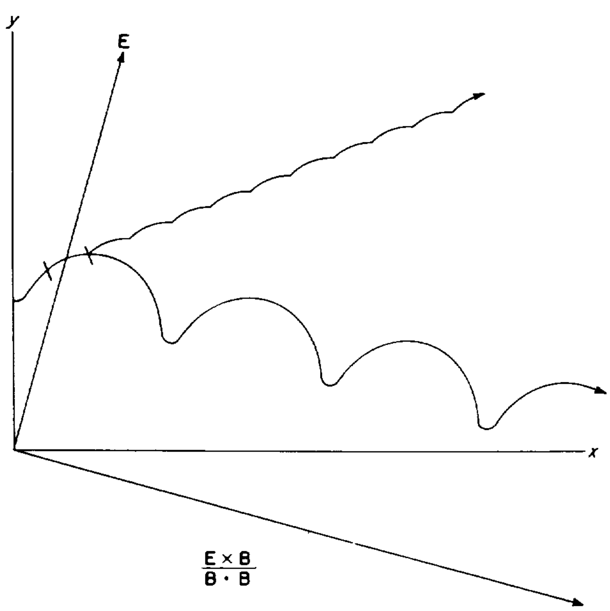

FIGURE 2.—Path of ion without and with collisions.

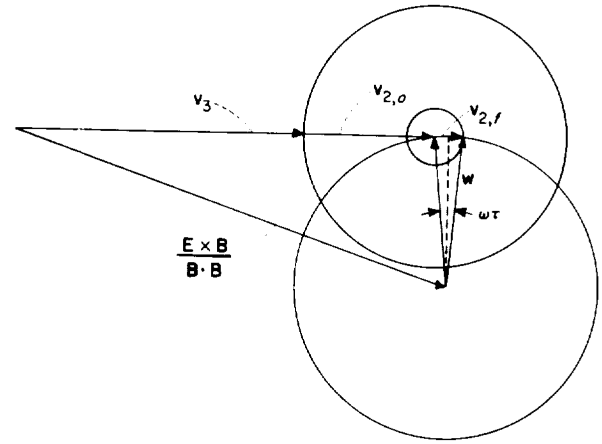

FIGURE 3.—Hodograph of ion-neutral particle collision process.

collision. The ion has the initial velocity $\mathbf{v}_{2,o}$ in its free path after its previous collision and is accelerated to $\mathbf{v}_{2,f}$ before the next collision. At collision, its velocity vector remains on the same small sphere on which it lay when its value was $\mathbf{v}_{2,f}$, and the velocity of the neutral particle remains on the larger sphere. (These spheres have a common center and are represented in fig. 3 by circles.) These velocity vectors lie at the opposite ends of collinear radii of the two spheres, both before and after collision. If the scattering is isotropic in the center-of-mass system, then, on the average, the two velocities after collision lie at the mutual center of the spheres. Thus, on the average, at a collision the ion gives up the velocity increment it gained from the electric field between collisions and returns to the initial velocity in its free path, $\mathbf{v}_{2,o}$. By inspection of figure 3 it can be seen that, since the radii of the two spheres are in inverse ratio to the masses of the particles, the neutral particle on the average gains $\frac{m_2}{m_2+m_3}$ of the difference in velocity that existed before the collision. From figure 3 it can also be seen that the velocity lost by the ion at a collision is the fraction $\frac{1}{\frac{m_2}{m_3}+\frac{1}{2}}$ times the difference between its average velocity $\mathbf{v}_2$ $\left(\text{which is equal to } \frac{\mathbf{v}_{2,o}+\mathbf{v}_{2,f}}{2}\right)$ and the velocity of the neutral particle $\mathbf{v}_3$. This result comes from use of the hodograph and does not account for thermal motions. An unpublished analysis by Adolf Busemann of the Langley Research Center has shown that, if thermal motions of the ion and the neutral particle are taken into account, the above fraction becomes approximately $\frac{1}{\frac{m_2}{m_3}+1}$ or $\frac{m_3}{m_2+m_3}$. Thus, since the ion loses in velocity at a collision what it gained in velocity from the electric field between collisions, one can write that

$$m_2(\mathbf{v}_{2,f}-\mathbf{v}_{2,o})=\frac{m_2m_3}{m_2+m_3}(\mathbf{v}_2-\mathbf{v}_3)=eE_x\tau_{23}$$

Thus, the horizontal component of the electric field is

$$E_x=\frac{m_2m_3}{m_2+m_3}\frac{1}{e\tau_{23}}(\mathbf{v}_2-\mathbf{v}_3)$$

as was derived in reference 2. This result is repeated here only for subsequent use in demonstrating that the expression for E_x derived in a subsequent section agrees within a negligibly small amount with this result from the hodograph.

The analysis of the accelerator on a microscopic or particle basis can be extended much further but to do so is not necessary as the analysis has been carried through on a continuum basis. It may be mentioned, however, that of the three desirable conditions discussed, the first two are adopted in the subsequent analysis. The first one is, in effect, adopted because constant static temperature is specified. Thus, the rate of acceleration is specified to be such that the decrease in temperature due to acceleration is just compensated for by the increase in temperature due to Joule heating. The second condition is adopted in order that the driving force on the neutral particles may be directed parallel to the axis of the channel. The third condition cannot then be met and the average ion velocity vector is directed at a slight angle above the axis, as can be seen from both figures 2 and 3. Equations for determining the magnitude of this angle are derived in a subsequent section.

QUANTITATIVE ANALYSIS FROM MACROSCOPIC POINT OF VIEW

In order to analyze the plasma accelerator from a macroscopic point of view, the equations of motion, of energy balance, and of conservation of mass are written in terms of average velocities. The three equations of motion for the three species in a three-component plasma have been adapted from those given by Schlüter in reference 4 and quoted also in reference 5 (section 48) with the addition of thermal-diffusion terms which, of course, are not needed for the present analysis. Equations of motion for the charged particles were derived in reference 2, but they do not include the effects of Coulomb forces; therefore, Schlüter's equations, which do include these effects, are used herein. Schlüter's equations of motion, all other equations taken from references 5 and 6, and all data have been converted to the rationalized mksq system of units. The equations of motion for the electrons, ions, and neutral particles are, respectively,

$$n_1m_1\mathbf{v}_1(\nabla\mathbf{v}_1)+\nabla p_1+n_1n_3\epsilon_{13}(\mathbf{v}_1-\mathbf{v}_3)+n_1n_2\epsilon_{12}(\mathbf{v}_1-\mathbf{v}_2)=-n_1e(\mathbf{v}_1\times\mathbf{B})-n_1e\mathbf{E}\quad(1)$$

$$n_2m_2\mathbf{v}_2(\nabla\mathbf{v}_2)+\nabla p_2+n_2n_3\epsilon_{23}(\mathbf{v}_2-\mathbf{v}_3)+n_2n_1\epsilon_{21}(\mathbf{v}_2-\mathbf{v}_1)=n_2e(\mathbf{v}_2\times\mathbf{B})+n_2e\mathbf{E}\quad(2)$$

$$n_3m_3\mathbf{v}_3(\nabla\mathbf{v}_3)+\nabla p_3+n_3n_1\epsilon_{31}(\mathbf{v}_3-\mathbf{v}_1)+n_3n_2\epsilon_{32}(\mathbf{v}_3-\mathbf{v}_2)=0\quad(3)$$

Consider first equation (1). The two terms on the right-hand side express the force exerted on the electrons by the applied magnetic and electric fields. The last two terms on the left-hand side are the "friction drags" on the electrons caused by, respectively, encounters with neutral particles and encounters with ions. The drag is set proportional to the velocity difference and proportional to the number density of each of the two species. The proportionality factor is ϵ_{ij}, which thus becomes the "friction coefficient" or interaction parameter between particles of species i and particles of species j. In a subsequent section is discussed the determination of the value of the ϵ terms for (a) encounters of charged and neutral particles from kinetic theory, quantum theory, and experimental values of mobilities of charged particles, and (b) encounters of charged particles from the theory of long-range encounters.

The first two terms on the left-hand side of equation (1) will be neglected in the present analysis. They are the term for the acceleration of the electrons under the net effect of all the forces and the gradient of the partial pressure of the electrons. The neglect of these two terms can be justified by comparison of their magnitudes with the magnitudes of the other terms in the equation. A typical set of values, calculated from the results of the present analysis, shows that the order of magnitude of each of the last two terms on the left-hand side and of the two terms on the right-hand side is about 10^5 to 10^6 newtons per cubic meter, whereas the orders of magnitude of the acceleration term and the pressure gradient are respectively, 10^{-1} and 10 newtons per cubic meter.

One further term that has been neglected is the time-dependent variation, at a fixed location, of electron momentum, since only the steady-state case is considered herein. It may be observed that Schlüter's notation $\frac{d}{dt}$ indicates the substantial derivative $\frac{D}{Dt}$, which is $\frac{\partial}{\partial t}$, $+\mathbf{v}\cdot\nabla$. These two quantities are neglected herein in the equations of motion of electrons and ions, but the second one is retained in the equation of motion of the neutral particles, where, instead of $(\mathbf{v}\,.\,\nabla)\mathbf{v}$, the equivalent expression $\mathbf{v}(\nabla\mathbf{v})$ is used.

The equation of motion of the ions is very similar to that of the electrons. The same simplifications can be made. For a typical case, the acceleration term and the partial pressure gradient have magnitudes of about 10^3 and 10, respectively, and can be neglected in comparison with the other four terms, which are of the order of 10^5 to 10^6 newtons per cubic meter each.

Equation (3) is the equation of motion of the neutral particles. Here again the term containing $\frac{\partial}{\partial t}$ has been omitted because only the steady state is being considered. The acceleration and the pressure gradient terms, the first two terms on the left, are large enough to be retained because they are functions of n_3 and, since the present analysis is made to apply to a slightly ionized gas (whether the ions are of the same original species as the neutral particles or are of some other species used as seeding material), n_3 is much larger than n_1. Thus, in a typical case, the first term in equation (3) may be of the order of about 10^5 newtons per cubic meter, the second term about 10^3, and the third and fourth terms about 10^5 to 10^6.

As is pointed out in reference 5 (section 48) the equation of motion for the center of mass of the plasma can be obtained by adding together the three individual equations of motion (eqs. (1) to (3)) with all terms retained in the equations. This addition yields, for $n_1=n_2$,

$$\rho\mathbf{v}_c(\nabla\mathbf{v}_c)+\nabla_p=\mathbf{j}\times\mathbf{B}$$

In this addition, all the "friction" forces between various components of the plasma have dropped out, inasmuch as these forces cannot cause motion of the center of mass of the plasma—that is, ϵ_{ij} must equal ϵ_{ji}.

For subsequent use, equations (1) and (2), with the first two (relatively small) terms in each neglected, are added. The result is

$$n_1n_3\epsilon_{13}(\mathbf{v}_1-\mathbf{v}_3)+n_2n_3\epsilon_{23}(\mathbf{v}_2-\mathbf{v}_3)=n_1e(\mathbf{v}_2-\mathbf{v}_1)\times\mathbf{B}=\mathbf{j}\times\mathbf{B}$$

for

$$n_1=n_2 \tag{4}$$

and with the conventional definition of $\mathbf{j}$,

$$\mathbf{j}=n_1e(\mathbf{v}_2-\mathbf{v}_1) \tag{5}$$

Previous analyses of the plasma accelerator used the equation of motion of the center of mass, together with equations expressing conservation of mass and of energy of the center of mass. A somewhat different procedure is followed herein, in that there are used, as the basic equations that describe the fluid dynamics of the problem, the equation of motion of the neutral particles, an equation expressing conservation of mass of the neutral particles, and an energy balance equation that is written under the assumption that all of the change in energy of the plasma occurs in the energy of the neutral particles. The self-subsistent equations of motion of the charged particles are used somewhat as auxiliary equations, both to permit solution of the fluid-flow problem and to obtain additional results. Only the case of constant cross-sectional area of the channel is considered; thus, conservation of mass of the neutral particles is expressed by

$$\left.\begin{aligned}&\nabla\cdot(n_3\mathbf{v}_3)=0\\ \text{or}\qquad &n_3v_{3,x}=\text{Constant}=\beta\end{aligned}\right\} \tag{6}$$

Under the assumption that any change in energy occurs only in the energy of the neutral particles, the energy balance equation is

$$n_3 m_3 c_p \mathbf{v}_3 \cdot \nabla T_3 + n_3 m_3 (\mathbf{v}_3 \cdot \mathbf{v}_3)(\nabla \mathbf{v}_3) = \mathbf{E} \cdot \mathbf{j}$$

The further restriction is made that the temperature of the neutral particles is kept constant along the length of the channel. As was pointed out in reference 2, there are justifiable reasons for this restriction. At the present state of development of the art, it seems reasonable to expect that a certain minimum temperature is required to maintain a given degree of thermal ionization in the plasma. At this point it is also appropriate to mention that heat transfer to or from the plasma by conduction, convection, and radiation is neglected herein. The energy balance equation thus becomes

$$n_3 m_3 (\mathbf{v}_3 \cdot \mathbf{v}_3)(\nabla \mathbf{v}_3) = \mathbf{E} \cdot \mathbf{j} \tag{7}$$

under the condition that

$$T_3 = \text{Constant} \tag{8}$$

It may be mentioned that the restriction of constant static temperature simply means that the effects of Joule heating on temperature and of the Lorentz force on temperature are made to be equal and opposite. This restriction could be applied in the present analysis either by deriving an expression for grad T_3 and determining the condition for which it vanishes, as was done in reference 2, or by setting grad T_3 equal to zero in the energy-balance equation and wherever else it appears. Both methods lead to exactly the same result. The latter method was used herein. The general gas law can be used, expressed as

$$p_3 = n_3 k T_3 \tag{9}$$

When the degree of ionization α is small, it can be expressed as

$$\alpha = \frac{n_1}{n_3} = \text{Constant} \tag{10}$$

In order that the plasma will be accelerated along the axis of the channel and not toward a wall of the channel, the Lorentz force should be parallel to the direction of the axis. Thus, with the x-axis of the coordinate system in the same direction as the channel axis, there results the condition

$$v_{1,x} = v_{2,x} \tag{11}$$

The condition that the neutral particles flow in the direction of the x-axis is that

$$v_{3,y} = 0 \tag{12}$$

The Mach number of the flow of neutral particles is given by

$$\gamma M_3^2 = \frac{m_3 v_{3,x}^2}{kT_3} \tag{13}$$

When both x- and y-components are included, equations (1) to (13) constitute 16 simultaneous equations in 16 dependent variables. The equations with both x- and y-components are equations (1), (2), and (5). (The y-component of eq. (3) is tantamount to eq. (11).) The 16 variables are the scalars n_1, n_2, n_3, p_3, and T_3, the two components each of the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, $\mathbf{E}$, and $\mathbf{j}$ and the x-component of M. (For simplicity, the vector notation and the subscript 3 are omitted herein from M, and M is understood to mean $\mathbf{M}_3$.) In order that the number of variables may not exceed the number of equations, the quantity $\mathbf{B}$ is taken to be constant. Specifying that $\mathbf{B}$ rather than E_y shall be constant appears to be advisable, if for no other reason than that the length of the accelerator varies approximately as the square of the final Mach number, as is shown in reference 2; if E_y were held constant and $\mathbf{B}$ varied in such a manner that the static temperature of the plasma remained constant, the length of the accelerator would vary with the fourth power of the final Mach number, as is shown in reference 7. In a subsequent section, ϵ_{13} and ϵ_{23} will be shown to be, for given species, functions only of temperature and, since temperature is constant in the present analysis, so are these ϵ terms. The quantity ϵ_{12} will be shown to be a function of the (constant) temperature and proportional to the logarithm of n_2. It can, therefore, to a sufficiently good approximation, be considered a constant.

The plan of the solution of equations (1) to (13) is first to find expressions for the components of the velocities of the charged particles and for the components of the electric field, all in terms of $\mathbf{v}_3$ (or M). Then these results, together with the other equations, are used for eliminating from equation (3) all dependent variables other than M. The resulting equation is integrated to provide the variation of M (or $\mathbf{v}_3$) with x. Thus, since the other variables can be expressed as functions of $\mathbf{v}_3$, the variation with x of all the dependent variables is obtained.

In order to derive expressions for the components of the velocities of the electrons and of the ions, equations (1) and (2), with the first two terms in each neglected, are used. The vector product of equation (1) and $\mathbf{B}$ is formed. From the resulting equation the quantity $\mathbf{v}_1\times\mathbf{B}$ is eliminated by means of equation (1) and $\mathbf{v}_2\times\mathbf{B}$ is eliminated by means of equation (2). After collection of terms, there results the equation

$$\begin{aligned}[(n_3\epsilon_{13})^2+2n_3\epsilon_{13}n_2\epsilon_{12}+e^2B^2]\mathbf{v}_1 &= -n_2n_3\epsilon_{12}(\epsilon_{23}-\epsilon_{13})\mathbf{v}_2 \\ &\quad +[(n_3\epsilon_{13})^2+n_2n_3\epsilon_{12}(\epsilon_{23}+\epsilon_{13})]\mathbf{v}_3 \\ &\quad -n_3\epsilon_{13}e(\mathbf{v}_3\times\mathbf{B})-n_3\epsilon_{13}e\mathbf{E}+e^2(\mathbf{E}\times\mathbf{B}) \qquad (14)\end{aligned}$$

Similarly, the vector product of equation (2) and $\mathbf{B}$ is formed. From the resulting equation the quantity $\mathbf{v}_2\times\mathbf{B}$ is eliminated by means of equation (2) and $\mathbf{v}_1\times\mathbf{B}$ is eliminated by means of equation (1). After collection of terms, the resulting equation is

$$\begin{aligned}[(n_3\epsilon_{23})^2+2n_1n_3\epsilon_{12}\epsilon_{23}+e^2B^2]\mathbf{v}_2 &= n_1n_3\epsilon_{12}(\epsilon_{23}-\epsilon_{13})\mathbf{v}_1+[(n_3\epsilon_{23})^2 \\ &\quad +n_1n_3\epsilon_{12}(\epsilon_{23}+\epsilon_{13})]\mathbf{v}_3+n_3\epsilon_{23}e\mathbf{E} \\ &\quad +n_3\epsilon_{23}e(\mathbf{v}_3\times\mathbf{B})+e^2(\mathbf{E}\times\mathbf{B}) \qquad (15)\end{aligned}$$

A new dimensionless variable u and new dimensionless constants are defined as follows:

$$u=\frac{n_3\epsilon_{23}}{eB}$$

$$\lambda=\frac{\epsilon_{13}}{\epsilon_{23}}$$

$$\mu=\frac{n_1}{n_3}\frac{\epsilon_{12}}{\epsilon_{23}}$$

$$\lambda u=\frac{n_3\epsilon_{13}}{eB}$$

$$\mu u=\frac{n_1\epsilon_{12}}{eB}$$

and are introduced into equations (14) and (15). Then simultaneous solution for $\mathbf{v}_1$ and $\mathbf{v}_2$ of the two equations results in

$$\begin{aligned}[(u^2+2\mu u^2+1)(\lambda^2u^2+2\mu\lambda u^2+1)+\mu^2u^4(1-\lambda)^2]\mathbf{v}_1 &= [(u^2+2\mu u^2+1)(\lambda^2+\mu+\lambda\mu) \\ &\quad -\mu u^2(1-\lambda)(1+\mu+\lambda\mu)]u^2\mathbf{v}_3 \\ &\quad -[(\lambda+\mu+\lambda\mu)u^2+\lambda]\frac{u}{B}(\mathbf{v}_3\times\mathbf{B}) \\ &\quad -[(\lambda+\mu+\lambda\mu)u^2+\lambda]\frac{u}{B}\mathbf{E} \\ &\quad +[(1+\mu+\lambda\mu)u^2+1]\frac{1}{B^2}(\mathbf{E}\times\mathbf{B}) \qquad (16)\end{aligned}$$

$$\begin{aligned}[(u^2+2\mu u^2+1)(\lambda^2u^2+2\lambda\mu u^2+1)+\mu^2u^2(1-\lambda)^2]\mathbf{v}_2 &= [(1+\mu+\lambda\mu)(\lambda^2u^2+2\lambda\mu u^2+1) \\ &\quad +\mu u^2(1-\lambda)(\lambda^2+\mu+\lambda\mu)]u^2\mathbf{v}_3 \\ &\quad +[\lambda(\lambda+\mu+\lambda\mu)u^2+1]\frac{u}{B}(\mathbf{v}_3\times\mathbf{B}) \\ &\quad +[\lambda(\lambda+\mu+\lambda\mu)u^2+1]\frac{u}{B}\mathbf{E} \\ &\quad +[(\lambda^2+\lambda\mu+\mu)u^2+1]\frac{1}{B^2}(\mathbf{E}\times\mathbf{B}) \qquad (17)\end{aligned}$$

From equations (16) and (17), the x- and y-components of $\mathbf{v}_1$ and $\mathbf{v}_2$ can readily be shown to be given by the following equations:

$$\begin{aligned}[(u^2+2\mu u^2+1)(\lambda^2u^2+2\lambda\mu u^2+1)+\mu^2u^4(1-\lambda)^2]v_{1,x} &= [(u^2+2\mu u^2+1)(\lambda^2+\mu+\lambda\mu) \\ &\quad -\mu u^2(1-\lambda)(1+\mu+\lambda\mu)]u^2v_{3,x} \\ &\quad -[(\lambda+\mu+\lambda\mu)u^2+\lambda]\frac{u}{B}E_x+[(1+\mu+\lambda\mu)u^2+1]\frac{E_y}{B} \qquad (18)\end{aligned}$$

$$\begin{aligned}[(u^2+2\mu u^2+1)(\lambda^2u^2+2\lambda\mu u^2+1)+\mu^2u^4(1-\lambda)^2]v_{1,y} &= [(\lambda+\mu+\lambda\mu)u^2+\lambda]uv_{3,x} \\ &\quad -[(\lambda+\mu+\lambda\mu)u^2+\lambda]\frac{u}{B}E_y \\ &\quad -[(1+\mu+\lambda\mu)u^2-1]\frac{E_x}{B} \qquad (19)\end{aligned}$$

$$\begin{aligned}[(u^2+2\mu u^2+1)(\lambda^2u^2+2\lambda\mu u^2+1)+\mu^2u^4(1-\lambda)^2]v_{2,x} &= [(1+\mu+\lambda\mu)(\lambda^2u^2+2\lambda\mu u^2+1) \\ &\quad +\mu u^2(1-\lambda)(\lambda^2+\mu+\lambda\mu)]u^2v_{3,x} \\ &\quad +[\lambda(\lambda+\mu+\lambda\mu)u^2+1]\frac{u}{B}E_x \\ &\quad +[(\lambda^2+\lambda\mu+\mu)u^2+1]\frac{E_y}{B} \qquad (20)\end{aligned}$$

$$[(u^2+2\mu u^2+1)(\lambda^2u^2+2\lambda\mu u^2+1)+\mu^2u^4(1-\lambda)^2]v_{2,y}$$
$$=-[\lambda(\lambda+\mu+\lambda\mu)u^2+1]uv_{3,x}$$
$$+[\lambda(\lambda+\mu+\lambda\mu)u^2+1]\frac{u}{B}E_y$$
$$-[(\lambda^2+\lambda\mu+\mu)u^2+1]\frac{E_x}{B} \quad (21)$$

Equations (18) to (21) are valid for general application when the accelerations and the pressure gradients of the charged particles are negligible. The next step is to obtain expressions for E_x and E_y in terms of u and $v_{3,x}$ for use in eliminating E_x and E_y from equations (18) to (21). The expressions for E_x and E_y are obtained herein for the special case of a plasma accelerator in which the plasma temperature is kept constant and in which the Lorentz force is parallel to the direction of the axis of the channel. First, an expression for E_y in terms of $v_{3,x}$ (and M) is found in the following manner. The equation just preceding equation (4) is used to introduce $\mathbf{j}\times\mathbf{B}$ into equation (3), equation (9) is used to eliminate ∇p_3 in terms of ∇n_3 and ∇T_3, equation (6) is used to replace ∇n_3 in terms of $\nabla\cdot\mathbf{v}_3$, equation (8) is used to eliminate ∇T_3, and equation (13) is used to introduce M. The result is that equation (3) is transformed into

$$n_3m_3(\mathbf{v}_3\cdot\mathbf{v}_3)\frac{\gamma M^2-1}{\gamma M^2}(\nabla\cdot\mathbf{v}_3)=\mathbf{j}\times\mathbf{B}\cdot\mathbf{v}_3 \quad (22)$$

By use of the condition (eq. (11)) that the Lorentz force has no y-component, equations (22) and (7) become, respectively,

$$n_3m_3v_{3,x}\frac{\gamma M^2-1}{\gamma M^2}\frac{dv_{3,x}}{dx}=j_yB \quad (23)$$

$$n_3m_3v_{3,x}^2\frac{dv_{3,x}}{dx}=E_yj_y \quad (24)$$

From equations (23) and (24) it follows that

$$E_y=\frac{\gamma M^2}{\gamma M^2-1}Bv_{3,x} \quad (25)$$

This relation was also derived in reference 2.

To find an equation for E_x, equation (11) is again used. The left-hand sides of equations (18) and (20) are then equal. By equating the right-hand sides of the equations and effecting considerable simplification, the desired relation is found to be

$$\frac{(\lambda+\mu+\lambda\mu)u^2+1}{1-\lambda}E_x=u(E_y-Bv_{3,x})$$

Then, by use of equation (25),

$$E_x=\frac{(1-\lambda)uBv_{3,x}}{[(\lambda+\mu+\lambda\mu)u^2+1](\gamma M^2-1)} \quad (26)$$

It is of interest to note that another set of conditions can also be used to obtain the same expression for E_x. The conditions are

$$v_{3,y}=\frac{\partial p_3}{\partial y}=0$$

for which, from the y-component of equation (3),

$$\lambda v_{1,y}=-v_{2,y}$$

This relation, used in conjunction with equations (19) and (21), also results in equation (26). The two sets of conditions are, of course, physically equivalent, inasmuch as, if there is no Lorentz force in the y-direction, there also should be no neutral particle velocity in that direction and no change in p_3 in that direction.

When E_x and E_y are eliminated from equations (18) to (21) by the use of equations (26) and (25), the resulting equations for the velocity components of the charged particles are

$$v_{1,x}=v_{2,x}=\frac{1}{\lambda u}v_{2,y}+v_{3,x} \quad (27)$$

$$v_{2,y}=-\lambda v_{1,y}=\frac{\lambda uv_{3,x}}{[(\lambda+\mu+\lambda\mu)u^2+1](\gamma M^2-1)} \quad (28)$$

The last step in solving the equations is to integrate the x-component of equation (3). If equation (6) is used in the first term, equations (9), (8), and (6) are used in the second term, and equation (11) is used in the third and fourth terms, equation (3) can be written as

$$\beta m_3\frac{dv_{3,x}}{dx}-\frac{\beta kT_3}{v_{3,x}^2}\frac{dv_{3,x}}{dx}+n_1n_3(\epsilon_{13}+\epsilon_{23})(v_{3,x}-v_{1,x})=0$$

If equations (10), (13), and (6) are used, then

$$\frac{kT_3}{\alpha\beta(\epsilon_{13}+\epsilon_{23})}(\gamma M^2-1)\frac{dv_{3,x}}{dx}=v_{1,x}-v_{3,x} \quad (29)$$

An expression is needed next for $v_{1,x}-v_{3,x}$ in terms of $v_{3,x}$. Equations (27) and (28) yield the needed relation:

$$v_{1,x}-v_{3,x}=\frac{v_{3,x}}{[(\lambda+\mu+\lambda\mu)u^2+1](\gamma M^2-1)}$$

Thus, since by definition $\lambda=\frac{\epsilon_{13}}{\epsilon_{23}}$, equation (29) can be rewritten as

$$\frac{2\alpha\beta\epsilon_{23}(1+\lambda)}{kT_3}dx = \frac{[(\lambda+\mu+\lambda\mu)u^2+1](\gamma M^2-1)^2}{\gamma M^2}d(\gamma M^2) \quad (30)$$

Inasmuch as u is a variable, it must be expressed in terms of M. By defining a constant c^2 as

$$c^2=\frac{m_3\beta^2\epsilon_{23}^2}{kTe^2B^2} \quad (31)$$

it can be shown that

$$u^2=\left(\frac{n_3\epsilon_{23}}{eB}\right)^2=\frac{c^2}{\gamma M^2} \quad (32)$$

Also ζ can be defined as

$$\zeta=\frac{2\alpha\beta\epsilon_{23}(1+\lambda)}{kT_3}x \quad (33)$$

Then equation (30) becomes

$$d\zeta=\frac{[(\lambda+\mu+\lambda\mu)c^2+\gamma M^2](\gamma M^2-1)^2}{(\gamma M^2)^2}d(\gamma M^2) \quad (34)$$

Integration yields

$$\zeta\Big|_{\zeta_o}^{\zeta}=(\lambda+\mu+\lambda\mu)c^2\left[\gamma M^2-2\log_e(\gamma M^2)-\frac{1}{\gamma M^2}\right]+\frac{1}{2}[(\gamma M^2)^2-4\gamma M^2+2\log_e(\gamma M^2)]\Big|_{M_o}^{M}$$

Under the assumption that $x=0$ when $\gamma M^2=1$, this result becomes

$$\zeta=(\lambda+\mu+\lambda\mu)c^2\left[\gamma M^2-2\log_e(\gamma M^2)-\frac{1}{\gamma M^2}\right]+\frac{1}{2}[(\gamma M^2)^2-4\gamma M^2+2\log_e(\gamma M^2)+3] \quad (35)$$

If equation (35) is divided through by $(\lambda+\mu+\lambda\mu)c^2$ and equations (31) to (33) are used, there results

$$\frac{2\sigma B^2}{\rho_3 v_{3,x}}x=\gamma M^2-2\log_e(\gamma M^2)-\frac{1}{\gamma M^2}+\frac{1}{2(\lambda+\mu+\lambda\mu)u^2}\left[\gamma M^2-4+\frac{2\log_e(\gamma M^2)}{\gamma M^2}+\frac{3}{\gamma M^2}\right] \quad (36)$$

where

$$\sigma=\frac{\alpha(1+\lambda)e^2}{(\lambda+\mu+\lambda\mu)\epsilon_{23}} \quad (37)$$

That the quantity σ is the conductivity for the case of no magnetic field is shown in appendix A. Except for the last term on the right, equation (36) is exactly the same result that is obtained in references 1 and 2, if in those references the constant of integration is so adjusted that x is zero for γM^2 equal to unity.

In order to arrive at the significance of the last term in equation (36), which does not appear in the results of previous analyses, the principal difference between the present and previous analyses must be considered. The main difference lies in the statement of the electrical power input to the accelerator that is used in the energy-balance equation. In the present analysis the electrical power input is $\mathbf{E}\cdot\mathbf{j}$ as given by equation (7). In the previous analyses, the power input $\mathbf{E}\cdot\mathbf{j}$ was assumed to be given by equation (A5) of appendix A, in which the last term is the rate at which work is done by the Lorentz force on the center of mass of the plasma and the rate of Joule heating is taken to be $\frac{j^2}{\sigma}$. Equation (A5), however, is based on equation (A4), an approximate form of Ohm's law. To evaluate Joule heating more accurately requires some algebraic manipulation, but the results are interesting. By equation (A1) of appendix A, Ohm's law is

$$\frac{\mathbf{j}}{\sigma}=\mathbf{E}+\frac{1}{1+\lambda}[(\mathbf{v}_1\times\mathbf{B})+\lambda(\mathbf{v}_2\times\mathbf{B})]$$

Scalar multiplication by $\mathbf{j}$ gives

$$\frac{\mathbf{j}\cdot\mathbf{j}}{\sigma}=\mathbf{E}\cdot\mathbf{j}-\frac{1}{1+\lambda}(\mathbf{j}\times\mathbf{B})\cdot(\mathbf{v}_1+\lambda\mathbf{v}_2)$$

By use of equations (5) and (11),

$$\begin{aligned}\frac{j_y^2}{\sigma}&=E_yj_y-j_yBv_{1,x}\\&=E_yj_y-j_yBv_{3,x}-j_yB(v_{1,x}-v_{3,x})\\&=E_yj_y-j_yBv_{3,x}-n_1eB(v_{2,y}-v_{1,y})(v_{1,x}-v_{3,x})\end{aligned}$$

By use of equation (28), this equation becomes

$$\frac{j_y^2}{\sigma}=E_yj_y-j_yBv_{3,x}-n_1eB\frac{1+\lambda}{\lambda}v_{2,y}(v_{1,x}-v_{3,x})$$

and by use of equation (27),

$$E_y j_y - j_y B v_{3,x} = \frac{j_y^2}{\sigma} + n_1 n_3 [\epsilon_{13}(v_{1,x} - v_{3,x})^2 + \epsilon_{23}(v_{2,x} - v_{3,x})^2] \tag{38}$$

This same result is derived in appendix B by a more physical line of reasoning. Equation (38) shows that the rate of Joule heating, which is the difference between the power input and the rate at which the Lorentz force does work on the neutral particles, is given by the two terms on the right-hand side. The first of these two terms is the conventional expression for Joule heating when there is no magnetic field, and the quantity σ in this term is the scalar conductivity given by equation (37). The second term on the right-hand side can be easily understood from either of two slightly different points of view. First, it is recalled that the condition has been set that the net current density j in the accelerator is in the y-direction. In the x-direction, the components of ion and electron velocities are equal, and thus, although there is no net current in that direction, there are actually two equal and opposite currents along that direction. If these currents are defined relative to the speed of the neutral particles, then, according to equation (38), the Joule heating due to the currents is

$$\frac{n_1^2}{\alpha}\epsilon_{13}(v_1)^2_{rel} + \frac{n_1^2}{\alpha}\epsilon_{23}(v_2)^2_{rel}$$

or

$$\frac{\epsilon_{13}}{\alpha e^2}(j_1)^2_{rel} + \frac{\epsilon_{23}}{\alpha e^2}(j_2)^2_{rel}$$

where the subscript *rel* means relative to $v_{3,x}$. But from equation (37) for the case that Coulomb forces, as accounted for by μ, can be neglected (since there is no relative velocity between ions and electrons in the x-direction),

$$\sigma_1 = \frac{\alpha e^2}{\epsilon_{13}}$$

for electrons and

$$\sigma_2 = \frac{\alpha e^2}{\epsilon_{23}}$$

for ions. Thus the Joule heating rate due to the x-components of velocity of the charged particles is $\frac{j_1^2}{\sigma_1}$ for the electrons and $\frac{j_2^2}{\sigma_2}$ for the ions. From a slightly different point of view, it is seen from equation (38) that the loss (to heat) due to the slip between the driving and the driven elements is a function of the velocity difference, just as it is, for example, in fluid-coupling power transmission devices.

To return to a discussion of the last term in equation (36), it should be clear that this term originates from the heating due to the difference between the x-components of the velocities of the charged particles and of the neutral particles. Thus, if $v_{2,x} - v_{3,x}$ approaches 0, then the last term in equation (36) must also approach zero. That such is the case can be shown in the following manner. The last term in equation (36) is proportional to $\frac{1}{u^2}$, but it is seen from the equation immediately preceding equation (30) that, as u approaches ∞, $v_{1,x} - v_{3,x}$ approaches 0.

There is another way of stating the same result. Equation (41), which is derived in the following section on evaluation of the ϵ terms, shows that,

$$\frac{1}{u} = \frac{m_2 + m_3}{m_3}\omega_2\tau_{23}$$

Thus, the last term in equation (36) vanishes as $\omega_2\tau_{23}$ approaches 0. The last term therefore is negligible when $\omega_2\tau_{23}$ is small enough. "Small enough" turns out to be approximately 10^{-3} radian, as is shown in figure 4. For this figure, the last term in equation (36) was written, by using equation (32), as

$$\frac{1}{2(\lambda+\mu+\lambda\mu)c^2}[(\gamma M^2)^2 - 4\gamma M^2 + 2\log_e(\gamma M^2) + 3]$$

Three values of the coefficient of the quantity within the brackets were used in figure 4. One value, zero, corresponds to the less accurate results obtained in references 1 and 2, in which the additonal Joule heating term discussed in the

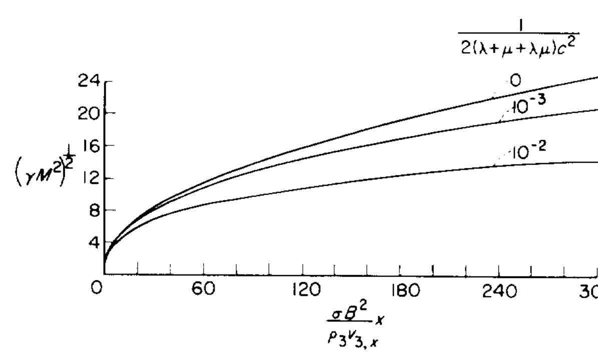

FIGURE 4. Mach number as function of length.

present paper was not taken into account, and thus corresponds to the case of $\omega_2\tau_{23}$ approaching 0. Another value for the coefficient, 10^{-3}, applies to the example conditions discussed subsequently herein, for which $\omega_2\tau_{23}=1.5\times10^{-3}$ radians when the Mach number is 2. The third value for the coefficient, 10^{-2}, corresponds to $\omega_2\tau_{23}=5\times10^{-3}$ radian when the Mach number is 2. As the Mach number increases, however, the density decreases and thus $\omega_2\tau_{23}$ increases. At a Mach number of 12, for example, $\omega_2\tau_{23}$ is six times as large as it is at a Mach number of 2. Figure 4 thus shows that the effect of the additional source of Joule heating on the dimensionless length of accelerator required to reach a given value of Mach number is large for the larger values of Mach number if $\omega_2\tau_{23}$ is not small enough.

A similar kind of result is obtained for the current density j_y. From equation (28) it can easily be shown by use of equations (32), (37), and (41) that

$$j_y=\frac{n_1e(1+\lambda)uv_{3,x}}{[(\lambda+\mu+\lambda\mu)u^2+1](\gamma M^2-1)}$$

$$=\frac{\sigma Bv_{3,x}}{\left[1+\frac{1}{(\lambda+\mu+\lambda\mu)u^2}\right](\gamma M^2-1)}$$

$$=\frac{\sigma Bv_{3,x}}{\left[1+\frac{1}{(\lambda+\mu+\lambda\mu)}\frac{(m_2+m_3)^2}{m_3^2}(\omega_2\tau_{23})^2\right](\gamma M^2-1)} \tag{39}$$

Here again, if $\omega_2\tau_{23}$ is small enough, the equation reduces to that derived in reference 2.

The angle by which the direction of the average ion velocity differs from the axial direction is of interest. The tangent of the angle is $\frac{v_{2,y}}{v_{2,x}}$ and thus, from equations (27) and (28), is

$$\frac{v_{2,y}}{v_{2,x}}=\frac{\lambda u}{1+[(\lambda+\mu+\lambda\mu)u^2+1](\gamma M^2-1)}$$

It is of interest at this point to verify two statements that were made previously. In view of equation (11), the Lorentz force depends on the y-component of $\mathbf{j}$. Furthermore, $v_{1,y}$ can be shown to be much larger than $v_{2,y}$. Thus, the equation immediately preceding equation (4) verifies the statement in the section "Preliminary Background Material" that the accelerating force on the plasma can be thought of either as being the Lorentz force $\mathbf{j}\times\mathbf{B}$, in which $\mathbf{j}$ is due principally to the flow of electrons, or as being due to impacts on neutral particles of charged particles that had been accelerated by the applied fields. Furthermore, use of equation (11) makes it easy to show, by comparing the x-components of the last two terms on the left-hand side of equation (3) and by using the subsequently developed fact that ϵ_{13} is of the order of 1 percent of ϵ_{23}, that the accelerating force on the neutral particles due to electron impacts is only about 1 percent of that due to ion impacts. This result agrees with the description of the acceleration mechanism used in the microscopic analysis, that the ions produce most of the accelerating force on the neutral particles.

It can also be shown that one of the results of the present macroscopic analysis agrees quantitatively with the corresponding result obtained by use of the hodograph. From equations (27), (28), (26), and (41)

$$v_{2,x}-v_{3,x}=\frac{1}{\lambda u}v_{2,y}$$

$$=\frac{v_{3,x}}{[(\lambda+\mu+\lambda\mu)u^2+1](\lambda M^2-1)}$$

$$=\frac{E_x}{(1-\lambda)uB}$$

$$=\frac{(m_2+m_3)\omega_2\tau_{23}E_x}{(1-\lambda)m_3B}$$

$$=\frac{(m_2+m_3)\tau_{23}eE_x}{(1-\lambda)m_3m_2}$$

In the preceding section, the hodograph representation for collisions was used to find that

$$v_{2,x}-v_{3,x}=\frac{(m_2+m_3)\tau_{23}eE_x}{m_2m_3}$$

If the effect of collisions between electrons and neutral particles is neglected, λ vanishes and the two results agree exactly.

EVALUATION OF THE FRICTION FACTORS ϵ_{ij}

The friction factors ϵ_{ij} that appear in equations (1) to (3) are given by equation (61.4) of reference 5 as

$$\epsilon_{ij}=\frac{8}{3}\left(\frac{2}{\pi}kT\frac{m_im_j}{m_i+m_j}\right)^{1/2}Q_{ij} \tag{40}$$

It is worth noting that, since kinetic theory shows the binary collision rate per unit volume between species i and species j to be

$$Z_{ij}=\frac{8}{3}n_in_j\left(\frac{2}{\pi}kT\frac{m_i+m_j}{m_im_j}\right)^{1/2}Q_{ij}$$

then

$$n_in_j\frac{m_i+m_j}{m_im_j}\epsilon_{ij}=Z_{ij}$$

It should be noted that $n_in_j\epsilon_{ij}$, which reference 5 states has the intuitive gas-kinetic meaning of the number of collisions per unit volume per unit time, must be multiplied by $(m_i+m_j)/m_im_j$ to give that number. Inasmuch as the number of collisions per ion per unit time with neutral particles is

$$\frac{m_2+m_3}{m_2m_3}n_3\epsilon_{23}$$

the mean free time is

$$\tau_{23}=\frac{m_2m_3}{(m_2+m_3)n_3\epsilon_{23}}$$

Since the cyclotron frequency of the ion is

$$\omega_2=\frac{eB}{m_2}$$

then

$$\omega_2\tau_{23}=\frac{m_3}{m_2+m_3}\frac{eB}{n_3\epsilon_{23}}$$

and, from the definition of u or equation (32),

$$\omega_2\tau_{23}=\frac{m_3}{m_2+m_3}\frac{1}{u} \tag{41}$$

This result was used in the preceding section.

Equation (40) involves the binary collision rate as calculated from thermal velocities. Although the equation will be used herein to calculate only ϵ_{12} and the temperature correction to experimental data for ϵ_{23}, a check should be made to verify that the collision rate is determined by thermal velocities and not the transport velocities of the particles; in other words, it must be shown that the thermal velocities are much greater than the relative transport velocities. This will be done subsequently for an example set of conditions.

The numerical evaluation of the ϵ terms poses some difficulties. There are three ϵ terms to be evaluated. One of these is ϵ_{13} for the collision of electrons with neutral particles. An apparent difficulty is due to the Ramsauer effect, which causes a large variation in Q_{13} near the lower energies. This variation is illustrated by figure 1.3 of reference 8, for example, in which the collision probability P_c is proportional to the elastic collision cross section Q_{13} of equation (40). The apparent difficulty in evaluating ϵ_{13} lies in the rapid and large variation of Q_{13} with velocity. This difficulty can be circumvented by using experimental or theoretical results on mobilities or drift velocities of electrons. The method is simple. From equation (1), for vanishingly small values of electron acceleration, electron pressure gradient, neutral-particle velocity, and magnetic field, the momentum balance is

$$n_3\epsilon_{13}\mathbf{v}_1=-e\mathbf{E}$$

Thus,

$$\epsilon_{13}=-\frac{e}{n_3}\frac{E}{v_1} \tag{42}$$

The quantity $\frac{v_1}{E}$ is the electron mobility. Data in the form of either mobility or drift velocity can be used for evaluating ϵ_{13}.

For the friction factor between ions and neutral particles ϵ_{23}, no quantum (Ramsauer) effects are important; nevertheless, since Q_{23} may be a function of velocity and data on Q_{23} itself are very scarce, results on mobilities, if they exist, can be used again to evaluate ϵ_{23}:

$$\epsilon_{23}=\frac{e}{n_3}\frac{E}{v_2} \tag{43}$$

The other difficulty lies in evaluating ϵ_{12}, the friction factor due to long-range forces between charged particles, because there appears to be no wholly acceptable theory by which the cross section Q_{12} can be evaluated. Reference 5 (section 63) gives the following formula for Q_{12} which is based on Spitzer's work (for example, ref. 6) and is simplified here for singly ionized ions:

$$Q_{12}=\frac{e^4}{(4\pi\epsilon_okT)^2}\log_e\Lambda \tag{44}$$

The difficulty is in the evaluation of Λ, which contains the limit at which the Coulomb forces can be considered to act. Reference 6 uses the Debye shielding distance for which

$$\Lambda=12\pi\left(\frac{\epsilon_okT}{e^2n_2^{1/3}}\right)^{3/2}$$

Reference 5 (section 63), on the basis of some experimental results on an arc, uses one-fourth the distance between ions and obtains

$$\Lambda = 4\pi \frac{\epsilon_o kT}{e^2 n_2^{1/3}} \tag{46}$$

Numerical evaluation of the ϵ terms for a typical set of conditions is desirable in order to establish orders of magnitude. A nitrogen plasma at 4,000° K, with dissociation neglected and seeded 2 percent with cesium, is assumed as an example. The quantity ϵ_{13} depends on the collision frequency, which depends on the relative velocity of the neutral particles and the electrons due to their thermal motions. On account of the large mass ratio between these two species, however, the relative velocity due to thermal motions is determined almost wholly by the thermal velocity of the electrons. In equation (40), the reduced mass is essentially that of the electron. In this equation, therefore, T can be taken to mean T_1 and Q_{13} can be considered to be a function of T_1. When ϵ_{13} is evaluated from data by means of equation (42), the dependence of ϵ_{13} on T_1 can easily be taken into account by using data obtained at the desired value of T_1, in this case 4,000° K. As is well known, when electrons drift through a gas at a steady velocity in a constant electric field, as in the experiments that measure drift velocity, the temperature of the electrons (and here the term "temperature" is used to indicate the energy of agitational motion of the electrons, which are not in thermal equilibrium with the neutral gas and which do not have a Maxwellian velocity distribution) exceeds the temperature of the gas by a factor that can be large even at moderate values of E/p. The data that are used in equation (42) are taken from figure 1 of reference 10. These data were obtained at a gas temperature of 293° K. The ratio of 4,000 to 293 is 13.6. Figure 7 of reference 11 shows that the temperature of the electrons is 13.6 times that of the natural gas at a value of E/p of 0.5 volt/(cm)(mm Hg) or 0.37 volt-m/newton. Figure 1 of reference 10 then shows that, at this value of E/p, the experimental value of v_1 is 5.1×10^3 m/sec. However, the theoretical Normand value shown in this figure may be preferable to the experimental value. In the experiments, the electrons at 4,000° K apparently excite rotation and vibration of the nitrogen molecules, which are at a much lower temperature. (Evidence for this process is presented and discussed in ref. 9.) This process increases the drift velocity of the electrons but presumably would play a much smaller role in a plasma in which the electrons and the neutral gas were essentially in equilibrium at the same temperature of 4,000° K. Thus, the theoretical curve of drift velocity (curve labeled Normand extrapolated to lower values of E/p) gives perhaps better values. (See fig. 1 of ref. 10.) This method yields a value of v_1 of about 3.3×10^3 m/sec. Thus,

$$\frac{v_1}{E/p}=8.9\times10^3 \text{ newtons/volt-sec}$$

But

$$\frac{v_1}{E/p}=\frac{v_1 n_3 k T_3}{E}$$

thus,

$$\frac{v_1 n_3}{E}=2.2\times10^{24} \text{ m}^{-1}\text{-volt}^{-1}\text{-sec}^{-1}$$

and

$$\epsilon_{13}=\frac{eE}{n_3 v_1}=7.3\times10^{-44} \text{ newton-m}^2\text{-sec}$$

The mobility of cesium ions in nitrogen is given on page 407 of reference 9 at 18° C and 760 mm Hg as

$$\frac{v_2}{E}=2.35\times10^{-4} \text{ m}^2\text{/volt-sec}$$

Thus,

$$\frac{n_3 v_2}{E}=5.9\times10^{21} \text{ m}^{-1}\text{-volt}^{-1}\text{-sec}^{-1}$$

and

$$\epsilon_{23}=\frac{eE}{n_3 v_2}=2.7\times10^{-41} \text{ newton-m}^2\text{-sec}$$

This is the value of ϵ_{23} at 18° C. The temperature variation of $\frac{v_2}{E}$, or of ϵ_{23}, or of Q_{23} in equation (40), has apparently not been determined for cesium ions in nitrogen either experimentally or theoretically. For the present use, therefore, the effect of temperature on thermal velocity only, and not on Q_{23}, is taken into account and yields a value at 4,000° K of

$$\epsilon_{23}=1.0\times10^{-40} \text{ newton-m}^2\text{-sec}$$

(Fortunately, uncertainties in the value of ϵ_{23} have a negligible effect on the value of σ as calculated from eq. (37).)

In order to calculate the friction factor ϵ_{12} for long-range encounters, Λ must be calculated. If the static pressure of the neutral particles is taken as 0.1 atmosphere, the temperature as 4,000° K, and the ionized seed substance as 2-percent mole fraction of the neutral substance, then, from equation (45),

$$\log_e \Lambda = 3.93$$

Actually, this value and the ion density and temperature from which it was calculated are slightly beyond the limiting conditions where the theory of reference 6 might be expected to break down. Equation (46) gives

$$\log_e \Lambda = 2.73$$

With the result from equation (45), the cross section for long-range encounters is, from equation (44),

$$Q_{12} = 6.8 \times 10^{-17} \text{ m}^2$$

Then, from equation (40),

$$\epsilon_{12} = 3.2 \times 10^{-11} \text{ newton-m}^2\text{-sec}$$

Thus, for the assumed example conditions

$$\lambda = \frac{\epsilon_{13}}{\epsilon_{23}} = 7.3 \times 10^{-4}$$

$$\mu = \alpha \frac{\epsilon_{12}}{\epsilon_{23}} = 6.4 \times 10^{-3}$$

From equation (37) the electrical conductivity is

$$\sigma = 720 \text{ mhos/m}$$

The dimensionless variable u, calculated from equation (32) for the example conditions, for B equal to 1 weber per square meter and a pressure of 0.1 atmosphere, is

$$u = \frac{n_3 \epsilon_{23}}{eB} = 115$$

The quantity $\omega_2 \tau_{23}$ for the same conditions is, from equation (41),

$$\omega_2 \tau_{23} = 1.5 \times 10^{-3}$$

The angle between $\mathbf{v}_2$ and $\mathbf{v}_3$, whose tangent is $\frac{v_{2,y}}{v_{2,x}}$, can be calculated under the assumption that at the example conditions the Mach number is 2 and γ is 1.4. Then it can be shown that for $\gamma M^2 = 1$ the angle is 11°, and for a Mach number of 2 or more it is less than 1 minute.

From equations (27) and (28) it can be shown that

$$v_{1,x} - v_{3,x} = v_{2,x} - v_{3,x} = 5.9 \text{ m/sec}$$

$$v_{1,y} = -680 \text{ m/sec}$$

$$v_{2,y} = 0.50 \text{ m/sec}$$

The thermal velocity of the electrons

$$v = \left(\frac{3kT}{m_1}\right)^{1/2} = 4.3 \times 10^5 \text{ m/sec}$$

is enough higher than any of these other velocities to make equation (40) easily applicable.

SUMMARY OF EQUATIONS FROM MACROSCOPIC ANALYSIS

For convenience, the equations that have been derived as a result of the macroscopic analysis are collected and listed in the present section.

For the steady-flow plasma accelerator with constant cross-sectional area, constant static temperature, constant applied magnetic field, Lorentz force directed among the channel axis, small degree of ionization, no friction and no heat transfer, and γM^2 greater than unity, the variation of M (the Mach number of the neutral particles) with distance x along the channel (where x is measured from the place where γM^2 is equal to unity) is given by equation (36) as

$$\frac{2\sigma B^2}{\rho_3 v_{3,x}} x = \gamma M^2 - 2 \log_e (\gamma M^2) - \frac{1}{\gamma M^2} + \frac{1}{2(\lambda+\mu+\lambda\mu)u^2}\left[\gamma M^2 - 4 + \frac{2\log_e(\gamma M^2)}{\gamma M^2} + \frac{3}{\gamma M^2}\right]$$

$$= \gamma M^2 - 2 \log_e (\gamma M^2) - \frac{1}{\gamma M^2} + \frac{1}{2(\lambda+\mu+\lambda\mu)v^2} [(\gamma M^2)^2 - 4\gamma M^2 + 2 \log_e (\gamma M^2) + 3]$$

In equation (36):

(1) The quantity σ is the electrical conductivity of the plasma when $B=0$ and $v_{3,x}=0$ and is given by equation (37) as

$$\sigma = \frac{\alpha e^2 (1+\lambda)}{(\lambda+\mu+\lambda\mu)\epsilon_{23}}$$

(2) The quantity u is given by equations (32) and (41) as

$$\frac{1}{u}=\frac{eB}{n_3\epsilon_{23}}$$

$$=\frac{m_2+m_3}{m_3}\omega_2\tau_{23}$$

and thus when $\omega_2\tau_{23}$ is small enough the last term in equation (36) can be neglected.

(3) From equation (31) the quantity $c^2=\frac{m_3\beta^2\epsilon_{23}^2}{kTe^2B^2}$.

(4) The quantities ϵ_{23}, λ, and μ are "friction factors" or functions of them and can be evaluated from theory or experimental data.

The variation of $v_{3,x}$ with x is found from equation (36) and the following equations (eqs. (13), (12), and (8)):

$$v_{3,x}=\left(\frac{\gamma kT_3}{m_3}\right)^{1/2}M$$

$$v_{3,y}=0$$

$$T_3=\text{Constant}$$

Then the number densities are obtained by use of equations (6), (10), and (4):

$$n_3v_{3,x}=\beta$$

$$n_1=\alpha n_3$$

$$n_2=n_1$$

and p_3 by use of equation (9):

$$p_3=n_3kT_3$$

The quantity j_x is from equation (11)

$$j_x=0$$

The remaining variables are then given by equations (25) to (28) and (39):

$$E_y=\frac{\gamma M^2}{\gamma M^2-1}Bv_{3,x}$$

$$E_x=\frac{(1-\lambda)uBv_{3,x}}{[(\lambda+\mu+\lambda\mu)u^2+1](\gamma M^2-1)}$$

$$v_{2,x}=v_{1,x}=\frac{1}{\lambda u}v_{2,y}+v_{3,x}$$

$$v_{2,y}=-\lambda v_{1,y}=\frac{\lambda uv_{3,x}}{[(\lambda+\mu+\lambda\mu)u^2+1](\gamma M^2-1)}=\frac{\lambda}{1-\lambda}\frac{E_x}{B}$$

$$j_y=\frac{\sigma Bv_{3,x}}{\left[1+\frac{1}{(\lambda+\mu+\lambda\mu)u^2}\right](\gamma M^2-1)}$$

CONCLUDING REMARKS

A theoretical treatment of the steady-flow, linear, crossed-field, direct-current plasma accelerator for inviscid, adiabatic, isothermal, constant-area flow has been developed from the equations of motion of the three components of the plasma. The results are idealized for the case of no wall friction and no transfer of heat and specialized to the case of constant static temperature and constant cross-sectional area. The effect of the ion cyclotron angle $\omega_2\tau_{23}$ (where ω_2 is the ion cyclotron frequency and τ_{23} the ion mean free time) on Joule heating rate and accelerator length is shown to be small only for values of about 10^{-3} radian or less.

Langley Research Center,
National Aeronautics and Space Administration,
Langley Air Force Base Va., *June 1, 1961.*

APPENDIX A

GENERALIZED OHM'S LAW

The generalized form of Ohm's law is easily obtained from the equations of motion of the charged particles when the accelerations and the pressure gradients of the charged particles are neglected. Equation (1) is divided by ϵ_{13}, equation (2) is divided by ϵ_{23}, and the two resulting equations are subtracted and simplified. The result is

$$\frac{\mathbf{j}}{\sigma}=\mathbf{E}+\frac{1}{1+\lambda}\left[(\mathbf{v}_1\times\mathbf{B})+\lambda(\mathbf{v}_2\times\mathbf{B})\right] \tag{A1}$$

where

$$\sigma=\frac{\alpha e^2(1+\lambda)}{(\lambda+\mu+\lambda\mu)\epsilon_{23}} \tag{A2}$$

The quantity σ is thus the scalar conductivity defined by Ohm's law for the case of no magnetic field:

$$\mathbf{j}=\sigma\mathbf{E} \tag{A3}$$

For cases where the accelerations and pressure gradients of the charged particles are negligibly small, equation (A1) is perhaps the most accurate of the generalized statements of Ohm's law in general usage. Less accurate statements are, however, frequently used. In reference 12, for example, the form that was used is exactly equivalent to equation (A1) with λ set equal to zero in the right-hand side of the equation. Inasmuch as $\mathbf{v}_2<\mathbf{v}_1$ and $\lambda<<1$, the approximation used in reference 12 is a very good one. In reference 2, a less accurate approximation was used. As Ohm's law, the equation

$$\frac{\mathbf{j}}{\sigma}=\mathbf{E}+\mathbf{v}_c\times\mathbf{B} \tag{A4}$$

was used so that the power input to the accelerator was

$$\mathbf{E}\cdot\mathbf{j}=\frac{\mathbf{j}\cdot\mathbf{j}}{\sigma}+(\mathbf{j}\times\mathbf{B})\cdot\mathbf{v}_c \tag{A5}$$

APPENDIX B

JOULE HEATING

In the present appendix, the rate of Joule heating in the accelerator is derived by a somewhat different reasoning than that used in the body of this paper.

The rate per unit volume at which the electric field does work on the electrons is the dot product of the force per unit volume exerted by the electric field on the electrons and the velocity of the electrons, that is, the dot product of equation (1) and $\mathbf{v}_1$:

$$n_1 n_3 \epsilon_{13}(\mathbf{v}_1-\mathbf{v}_3)\cdot\mathbf{v}_1+n_1 n_2 \epsilon_{12}(\mathbf{v}_1-\mathbf{v}_2)\cdot\mathbf{v}_1=-n_1 e(\mathbf{E}\cdot\mathbf{v}_1)$$

Likewise, the rate per unit volume at which the field does work on the ions is given by the dot product of equation (2) and $\mathbf{v}_2$:

$$n_2 n_3 \epsilon_{23}(\mathbf{v}_2-\mathbf{v}_3)\cdot\mathbf{v}_2+n_1 n_2 \epsilon_{12}(\mathbf{v}_2-\mathbf{v}_1)\cdot\mathbf{v}_2=n_2 e(\mathbf{E}\cdot\mathbf{v}_2)$$

Adding these two equations yields

$$n_1 n_3 \epsilon_{13}(\mathbf{v}_1-\mathbf{v}_3)\cdot\mathbf{v}_1+n_1 n_3 \epsilon_{23}(\mathbf{v}_2-\mathbf{v}_3)\cdot\mathbf{v}_2 +n_1 n_2 \epsilon_{12}(\mathbf{v}_2-\mathbf{v}_1)\cdot(\mathbf{v}_2-\mathbf{v}_1)=\mathbf{E}\cdot\mathbf{j}$$

The next step is to obtain the rate at which work is done on the neutral particles. Addition of equations (1) and (2) gives an equation for $\mathbf{j}\times\mathbf{B}$ and the dot product of this equation with $\mathbf{v}_3$ gives the rate per unit volume at which work is done on the neutral particles:

$$n_1 n_3 \epsilon_{13}(\mathbf{v}_1-\mathbf{v}_3)\cdot\mathbf{v}_3+n_1 n_3 \epsilon_{23}(\mathbf{v}_2-\mathbf{v}_3)\cdot\mathbf{v}_3=\mathbf{j}\times\mathbf{B}\cdot\mathbf{v}_3$$

Then, by subtraction,

$$\mathbf{E}\cdot\mathbf{j}-\mathbf{j}\times\mathbf{B}\cdot\mathbf{v}_3=n_1 n_3 \epsilon_{23}(\mathbf{v}_1-\mathbf{v}_3)^2 +n_1 n_3 \epsilon_{23}(\mathbf{v}_2-\mathbf{v}_3)^2+n_1 n_2 \epsilon_{12}(\mathbf{v}_2-\mathbf{v}_1)^2$$

By the use of equations (11), (12), (28), (5), and (37), this equation can be transformed to read

$$\begin{aligned} E_y j_y - j_y B v_{3,x} &= n_1 n_1 (1+\lambda)\epsilon_{23}(v_{2,x}-v_{3,x})^2 \\ &\quad + n_1 n_3 \epsilon_{13}(1+\lambda) v_{1,y}^2 \\ &\quad + n_1 n_2 \epsilon_{12}(1+\lambda)^2 v_{1,y}^2 \\ &= n_1 n_3 (1+\lambda)\epsilon_{23}[(v_{2,x}-v_{3,x})^2 \\ &\quad + (\lambda+\mu+\lambda\mu) v_{1,y}^2] \\ &= n_1 n_3 [\epsilon_{13}(v_{1,x}-v_{3,x})^2+\epsilon_{23}(v_{2,x}-v_{3,x})^2] \\ &\quad + \frac{j_y^2}{\sigma} \end{aligned}$$

which is the same as equation (38).

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U.S. GOVERNMENT PRINTING OFFICE: 1962