IN-34

TECHNICAL NOTE

D-904

AN INVESTIGATION OF THE NATURAL FREQUENCIES AND MODE SHAPES OF LIQUIDS IN OBLATE SPHEROIDAL TANKS

By H. Wayne Leonard and William C. Walton, Jr.

Langley Research Center
Langley Field, Va.

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

WASHINGTON June 1961

N

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

TECHNICAL NOTE D-904

AN INVESTIGATION OF THE NATURAL FREQUENCIES AND MODE SHAPES OF LIQUIDS IN OBLATE SPHEROIDAL TANKS

By H. Wayne Leonard and William C. Walton, Jr.

L
1
2
5
4

SUMMARY

An experimental investigation was conducted to gain some understanding of the character of the free vibration modes of liquids in oblate spheroidal tanks applicable in missile and space vehicle systems. Measured natural frequencies were obtained for the lowest three or four antisymmetric modes of oscillation as a function of the liquid depth for three orientations of each of several such tanks of different size and oblateness. The orientations considered were such that: (a) the equator of the spheroid was horizontal and oscillations were along a diameter of the circular liquid surface; (b) the equator of the spheroid was vertical and the oscillations were along the minor axis of the elliptical liquid surface; and (c) the equator of the spheroid was vertical and the oscillations were along the major axis of the elliptical liquid surface. The frequency data are presented as dimensionless parameters developed for each orientation to permit the application of the experimental results to the prediction of the natural frequencies of tanks of different size and oblateness. Photographs were made of representative surface wave or mode shapes for each orientation.

INTRODUCTION

Recent investigations in the field of liquid fuel sloshing have shown the need for basic research to broaden the knowledge of the natural frequencies of liquids contained in tanks of various shapes, sizes, and geometries. The natural frequencies of liquids in spheres and right circular cylinders (ref. 1) and in toroids (ref. 2) have been experimentally investigated and compared with theory wherever practical. Other configurations such as circular ring tanks (ref. 3) and conical tanks (ref. 4) have been examined analytically. A rather comprehensive review and bibliography concerning the behavior of liquids in their containers is given in reference 5. Information pertaining to the natural frequencies of liquids contained in oblate spheroids is of immediate importance since this type of tank will be employed in various stages of boost systems currently being built.

The purpose of this paper is to report the results of an experimental investigation of the natural frequencies and mode shapes of liquids contained in oblate spheroidal tanks. The natural frequencies and mode shapes of the liquid motions were obtained for tanks having major axes of 6, 13, and 26 inches, ratios of minor to major axes of 0.5, 0.75, and 1.0, liquid depths ranging from empty to full, and for different orientations with respect to the direction of oscillation. The natural frequencies are presented as dimensionless parameters to permit application of the results to tanks within the range of sizes and geometrical configurations of practical interest.

L1264

SYMBOLS

a — semimajor spheroidal axis, in.

b — semiminor spheroidal axis, in.

f — measured natural frequency, cps

g — acceleration due to gravity, in./sec^2

h — liquid depth in spheroid, in.

h_c — liquid depth in equivalent circular cylinder (see eq. (1)), $\frac{1}{3}\frac{h(3b - h)}{2b - h}$, in.

h_e — liquid depth in equivalent elliptic cylinder (see eq. (3)), $\frac{1}{6}h\left(1 + \frac{4a - h}{2a - h}\right)$, in.

J_1' — first derivative of the Bessel function of the first order and first kind

k — constant proportional to positive parametric zeros of the first derivatives of the Mathieu functions

L — characteristic length

n — an integer designating the mode of liquid oscillation

r — radius of equilibrium circular liquid surface, in.

α — semimajor axis of elliptic equilibrium surface, in.

ϵ_n — nth root of $J_1'(\epsilon_n) = 0$

$\theta_{l,n}$ — equivalent elliptic cylinder frequency parameter for longitudinal orientations, $\omega_n\sqrt{\frac{\alpha}{g}\frac{1}{k_{l,n}}\frac{1}{\tanh\left(\frac{h_e}{\alpha}k_{l,n}\right)}}$

$\theta_{t,n}$ — equivalent elliptic cylinder frequency parameter for transverse orientations, $\omega_n\sqrt{\frac{\alpha}{g}\frac{1}{k_{t,n}}\frac{1}{\tanh\left(\frac{h_e}{\alpha}k_{t,n}\right)}}$

L1264

λ_n — frequency parameter for longitudinal orientations, $\omega_n\sqrt{\frac{a}{g}}$

σ_n — equivalent circular cylinder frequency parameter for horizontal orientations, $\omega_n\sqrt{\frac{r}{g}\frac{1}{\epsilon_n}\frac{1}{\tanh\left(\frac{h_c}{r}\epsilon_n\right)}}$

ψ_n — frequency parameter for transverse orientations, $\omega_n\sqrt{\frac{b}{g}}$

ω_n — measured natural frequency of oscillation of nth mode, radians/sec

Ω_n — calculated natural frequency of oscillation of nth mode, radians/sec

Subscripts:

l — longitudinal

t — transverse

n — mode

APPARATUS AND TEST PROCEDURE

Apparatus

Description of models.- The models of spheroidal tanks studied in these tests consisted of nine configurations, each of which was oriented in three ways with respect to the direction of the liquid oscillations. These orientations are shown in the sketch in figure 1 which also includes the tabulated dimensions of the models tested. Orientations of the tanks for the various liquid modes are listed as follows:

Orientation	Plane of spheroid equator	Direction of applied oscillation with respect to plane of spheroid equator
Horizontal	Horizontal	Parallel
Longitudinal	Vertical	Parallel
Transverse	Vertical	Perpendicular

In all modes, the direction of oscillation was in a plane parallel to the equilibrium liquid surface.

I
1
2
6
4

All models were constructed of clear Plexiglas to permit visual observations of the liquid motions. In all cases water was used as the liquid.

Mechanical shaker.- The models were mounted on a support platform which was suspended in pendulum fashion from overhead beams. Oscillations of the models were induced by means of a mechanical shaker which was connected directly to the support platform as shown in figure 2. The mechanical shaker, described fully in reference 6, is essentially a slider-crank mechanism driven by a variable-speed motor. A tachometer, also shown in figure 2, was connected directly to the shaft of the drive motor and provided a means for obtaining the excitation frequency.

Test Procedure

The testing technique involved inducing translatory oscillation of the models over a range of frequencies to obtain the natural frequencies of the contained liquid. The procedure was repeated over the full range of liquid depths. The mode in question was induced by the mechanical shaker and, upon full development of the wave form, the platform motion was stopped and the frequencies were obtained by observing the time required for a given number of low-amplitude oscillations during the decay of the wave form. Representative surface wave shapes for three orientations of a half-filled spheroid are shown in figures 3 to 5. Data were taken for all modes detected with sufficient clarity for their definition.

DATA REDUCTION

A sample of the test results, showing some of the data taken on spheroid number 4, is given in figure 6. This figure shows the variation in the natural frequencies of the first three liquid modes, in cycles per second, with fullness ratio (h/2b for horizontal orientations and h/2a for transverse and longitudinal orientations) for the three mutually perpendicular orientations of the tank.

In order to apply the test results to oblate spheroidal tanks of general geometry and size, some method of rendering the data dimensionless is desired. This problem is greatly simplified if simple, exact, closed form analytical expressions are available for the natural frequencies of the liquid. A ratio of the measured natural frequencies to the frequencies predicted by such expressions yields the ideal dimensionless parameter.

L
1
2
6
4

Since no simple, closed form, analytical expressions exist for the natural liquid frequencies in oblate spheroidal tanks, some alternate nondimensionalization procedure must be found. These procedures and the parameters selected thereby are presented in the following sections for each orientation. It should be noted that the parameters selected are not unique. Several parameters were investigated and those yielding the best results were selected for use.

Horizontal Orientations

The frequency parameter for the horizontal modes was selected as the ratio of the experimentally determined natural liquid frequencies ω_n to the natural frequencies of a liquid contained in an upright circular cylinder having a radius r equal to the radius of the liquid surface in the spheroid and a liquid depth $h_c = \frac{1}{3}\frac{h(3b - h)}{(2b - h)}$ which will yield a cylindrical volume equal to the volume of the liquid contained in the spheroid. The exact expression for the liquid frequencies Ω_n in such a cylinder given by reference 7 is in radians per second

$$\Omega_n = \sqrt{\frac{g}{r}\epsilon_n \tanh\left(\frac{h_c}{r}\epsilon_n\right)} \qquad (1)$$

where n is the mode of liquid oscillation, g is the acceleration due to gravity, and ϵ_n is the nth zero of J_1'. The resulting frequency parameter is

$$\sigma_n = \frac{\omega_n}{\Omega_n} = \omega_n\sqrt{\frac{r}{g}\frac{1}{\epsilon_n}\frac{1}{\tanh\left(\frac{h_c}{r}\epsilon_n\right)}} \qquad (2)$$

Transverse and Longitudinal Orientations

An equivalent cylinder-type frequency parameter similar in appearance to σ_n (eq. (2)) can be constructed for the transverse and longitudinal orientations, but the numerical evaluation of its ingredients is complicated. The parameter is the ratio of the experimentally determined natural frequencies to the corresponding frequencies calculated for liquid contained in an upright elliptic cylinder whose cross section is congruent to the equilibrium liquid surface in the spheroid. As in the derivation of σ_n, the liquid depth h_e assigned to the hypothetical cylinder is adjusted so that the volume of liquid contained in the cylinder is equal to the liquid volume in the spheroid and is therefore given by $h_e = \frac{1}{6} h\left(1 + \frac{4a - h}{2a - h}\right)$ where h is the liquid depth in the spheroid and a is the semimajor spheroid axis. The natural frequency of the nth mode of the liquid in such a cylinder in radians per second is given in reference 8 as

$$\Omega_{l,n} = \sqrt{\frac{g}{\alpha} k_{l,n} \tanh\left(\frac{h_e}{\alpha} k_{l,n}\right)} \tag{3a}$$

for longitudinal orientations and as

$$\Omega_{t,n} = \sqrt{\frac{g}{\alpha} k_{t,n} \tanh\left(\frac{h_e}{\alpha} k_{t,n}\right)} \tag{3b}$$

for transverse orientations.

In equation (3), α is defined as the semimajor axis of the cylinder section and the constants $k_{l,n}$ and $k_{t,n}$ are proportional to the positive parametric zeros of the first derivatives of the Mathieu functions as discussed in reference 8. (For the convenience of the reader, approximate values of $k_{l,1}$ and $k_{t,1}$ determined by the methods of reference 9 are plotted as functions of eccentricity in figure 7.) The resulting frequency parameters are

$$\theta_{l,n} = \frac{\omega_n}{\Omega_n} = \omega_n\sqrt{\frac{\alpha}{g} \frac{1}{k_{l,n}} \frac{1}{\tanh\left(\frac{h_e}{\alpha} k_{l,n}\right)}} \tag{4a}$$

for longitudinal orientations and

$$\theta_{t,n} = \frac{\omega_n}{\Omega_n} = \omega_n\sqrt{\frac{\alpha}{g} \frac{1}{k_{t,n}} \frac{1}{\tanh\left(\frac{h_e}{\alpha} k_{t,n}\right)}} \tag{4b}$$

for transverse orientations.

From a dimensional analysis based on the linearized theory of liquid oscillations (ref. 7), it may be seen that a parameter of the form

$$\omega_n\sqrt{\frac{L}{g}}$$

where L is some characteristic length, is sufficient to nondimensionalize the natural frequencies of liquids in tanks having equal eccentricities. For the transverse orientations, the characteristic length was chosen as the semiminor axis b, and the parameter selected is

$$\psi_n = \omega_n\sqrt{\frac{b}{g}} \qquad (5)$$

A similar parameter was developed for longitudinal orientations. In this case the characteristic length was selected as the semimajor spheroidal axis a and the frequency parameter is

$$\lambda_n = \omega_n\sqrt{\frac{a}{g}} \qquad (6)$$

DATA PRESENTATION AND DISCUSSION OF RESULTS

The frequency parameters developed in the preceding section are presented in terms of the fullness ratio $\left(\frac{h}{2b}\right.$ for horizontal orientations and $\frac{h}{2a}$ for transverse and longitudinal orientations$\left.\right)$ in figures 8 to 12. Data for the higher orientations in some of the smaller models were limited because of the difficulty in defining the orientations in these tanks.

Horizontal Orientations

The experimental frequency data from nine horizontal spheroids is presented in terms of the equivalent circular cylinder frequency parameter in figure 8. Values of the frequency parameter are given as a function of the tank fullness ratio $\frac{h}{2b}$ for the first five liquid modes. With the exception of slight scatter at the near-full condition for the first mode, the frequency parameter is approximately unity for all modes of all spheroids examined. It appears, therefore, that the frequency parameter is practically independent of tank size and eccentricity and further that the equivalent cylinder method is accurate in predicting the natural liquid frequencies in any horizontal spheroid subjected to lateral oscillations.

Transverse and Longitudinal Orientations

Relationships between the natural frequency of the fundamental liquid mode and the liquid depth in nine spheroids are presented in figure 9 for both transverse and longitudinal orientations. The data are presented in terms of the equivalent elliptic cylinder parameter θ_1 as a function of the fullness ratio $\frac{h}{2a}$. The figure indicates that the equivalent elliptic cylinder method yields a parameter which, although not unity throughout the depth range, is essentially independent of both tank size and eccentricity.

Natural-frequency—depth relationships for the first three transverse modes of nine spheroids are presented in terms of the frequency parameter in figures 10 and 11. Values of the frequency parameter ψ_n are given as a function of the tank fullness ratio $\frac{h}{2a}$ for the first three modes. The figures indicate that, for a given eccentricity, the values of the frequency parameter may be approximated by a single curve for a given modal number with the exception of slight size-effect scatter in the second mode at the near-empty condition for ratios of $\frac{b}{a}$ of 0.5 and 0.75.

The scatter evidenced at the near-empty condition is more pronounced for the smallest tank (2a = 6.2 inches) in both figures 10(a) and 10(b). This scatter is possibly due to slight irregularities in the tank due to modeling accuracy limitations. It should be noted at this point that the mode designated as the "second mode" in figure 11 corresponds to the "third mode" in figure 10, and that "second mode" in figure 10 has no counterpart in figure 11. This fact is readily apparent by comparing the surface wave shapes in the spheroidal tank as shown in figure 4(b) with the surface wave shapes in the sphere of reference 1. A normal mode of oscillation is present in the tanks having elliptical equilibrium surface planforms which does not exist in tanks having circular equilibrium surface shapes. This fact is very significant in any comparison of the data of figures 10 and 11.

The results obtained from the oscillation of nine longitudinally oriented spheroids are presented in figures 11 and 12 in terms of the frequency parameter λ_n. Values of the frequency parameter are given as a function of the fullness ratio $\frac{h}{2a}$. As with the transverse orientations, the values of the frequency parameter for a given mode and tank eccentricity may be closely approximated by a single curve except at the near-empty condition where some size effect is noted in the first two modes. Again the scatter is more pronounced for the smallest model and is possibly due to irregularities in the tank shape. As was the case for the transverse orientations, a normal mode of oscillation of the liquid not present in spherical tanks was obtained for this orientation of the spheroidal tanks. This mode, whose surface wave form is very similar in appearance to that of the second longitudinal mode of a horizontal cylinder (ref. 1), corresponds to the second mode in figures 12(a), 12(b), and 5(b).

CONCLUDING REMARKS

An experimental investigation was conducted to determine the natural antisymmetric mode shapes and frequencies of a body of liquid contained in an oblate spheroidal tank. Measured natural frequencies were obtained for the lowest three or four modes of oscillation as a function of liquid depth for oscillations along the three principal axes of each of several spheroids of different size and ellipticity. All oscillations were in a plane parallel to the equilibrium surface.

The parameter developed for oscillations of the liquid when the equilibrium surface is circular (horizontal orientation) appears to include the effects of both ellipticity and size. This parameter was developed by comparing the measured natural frequencies with those in a hypothetical equivalent circular cylinder having a cross section congruent to the equilibrium surface and containing a volume of liquid equal to the volume of liquid in the spheroid. This method appears to yield excellent correlation between the frequencies of a liquid in any horizontally oriented oblate spheroid.

Two sets of parameters were developed for orientations of the tank yielding an elliptical equilibrium surface. The first set, developed from an equivalent elliptical cylinder, appears to correlate both size and ellipticity in rendering dimensionless the fundamental natural frequency for oscillations both parallel to (longitudinal orientations) and perpendicular to (transverse orientations) the plane of the spheroid equator.

The second set of parameters, developed from a dimensional analysis of the linearized theory of liquid oscillations, are essentially independent of tank size but exhibit a marked dependence on eccentricity.

It is felt that the parameters presented may be used to correlate the frequencies of liquids in oblate spheroidal tanks within the realm of practical interest.

Langley Research Center,
National Aeronautics and Space Administration,
Langley Field, Va., April 11, 1961.

REFERENCES

1. McCarty, John Locke, and Stephens, David G.: Investigation of the Natural Frequencies of Fluids in Spherical and Cylindrical Tanks. NASA TN D-252, 1960.

2. McCarty, John Locke, Leonard, H. Wayne, and Walton, William C., Jr.: Experimental Investigation of the Natural Frequencies of Liquids in Toroidal Tanks. NASA TN D-531, 1960.

3. Bauer, Helmut F.: Theory of the Fluid Oscillations in a Circular Cylindrical Ring Tank Partially Filled With Liquid. NASA TN D-557, 1960.

4. Green, J. W., and Landau, H. J.: A Third Note on Conical Sloshing. PA/M-553/3, Space Tech. Labs., Inc., Aug. 19, 1957.

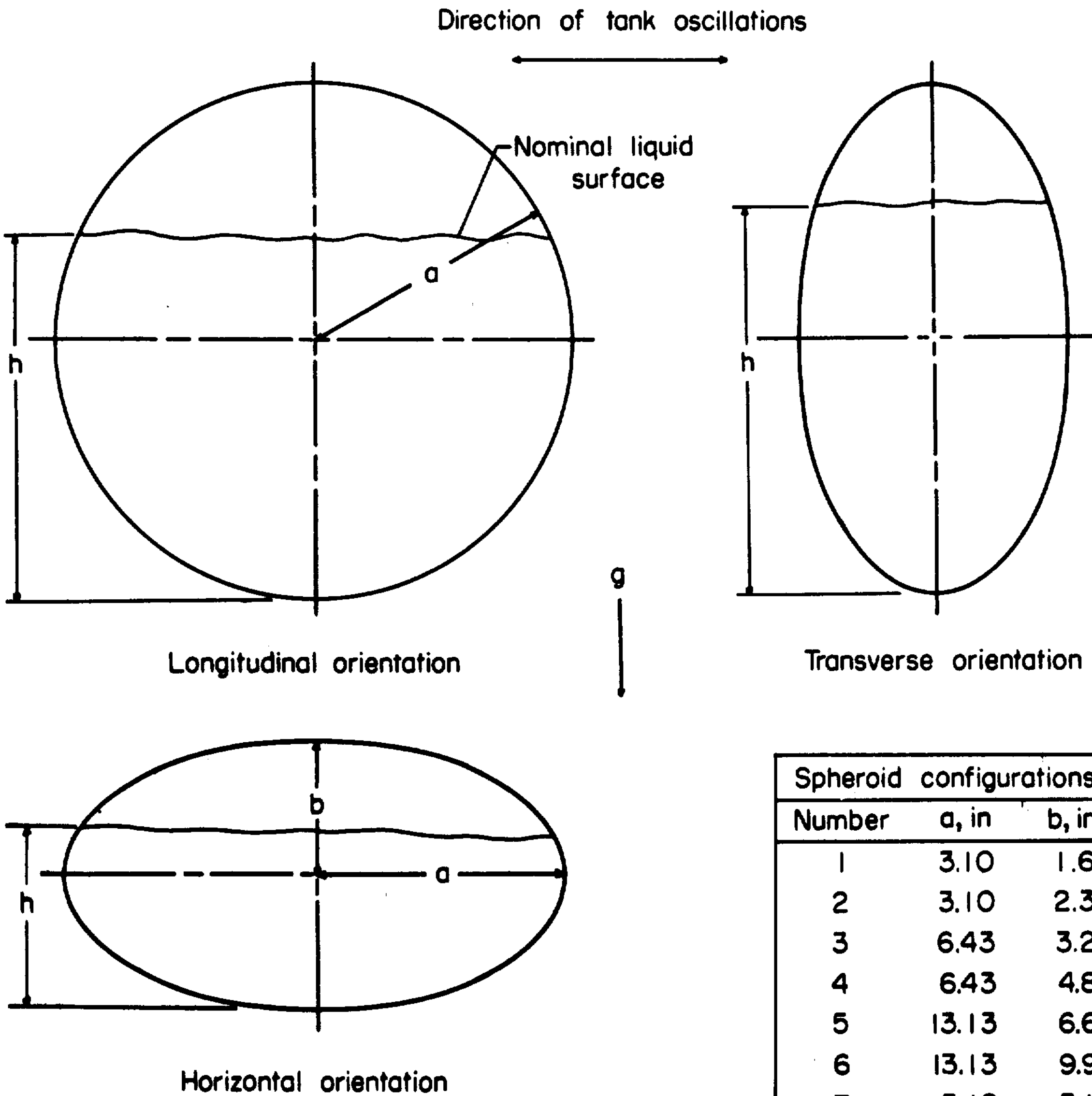
5. Cooper, R. M.: Dynamics of Liquids in Moving Containers. ARS Jour., vol. 30, no. 8, Aug. 1960.

6. Silveira, Milton A., Maglieri, Domenic J., and Brooks, George W.: Results of an Experimental Investigation of Small Viscous Dampers. NACA TN 4257, 1958.

7. Lamb, Horace: Hydrodynamics. Sixth ed., Dover Publications, 1945.

8. Chu, Wen-Hwa: Sloshing of Liquids in Cylindrical Tanks of Elliptic Cross-Section. Tech. Rep. No. 2 (Contract DA-23-072-ORD-1251), Southwest Res. Inst., Sept. 1958.

9. McLachlan, N. W.: Theory and Application of Mathieu Functions. Oxford Univ. Press, 1947.



Spheroid configurations		
Number	a, in	b, in
1	3.10	1.6
2	3.10	2.3
3	6.43	3.2
4	6.43	4.8
5	13.13	6.6
6	13.13	9.9
7	3.10	3.10
8	6.43	6.43
9	13.13	13.13

Figure 1. - Orientations and dimensions of spheroidal tanks studied.

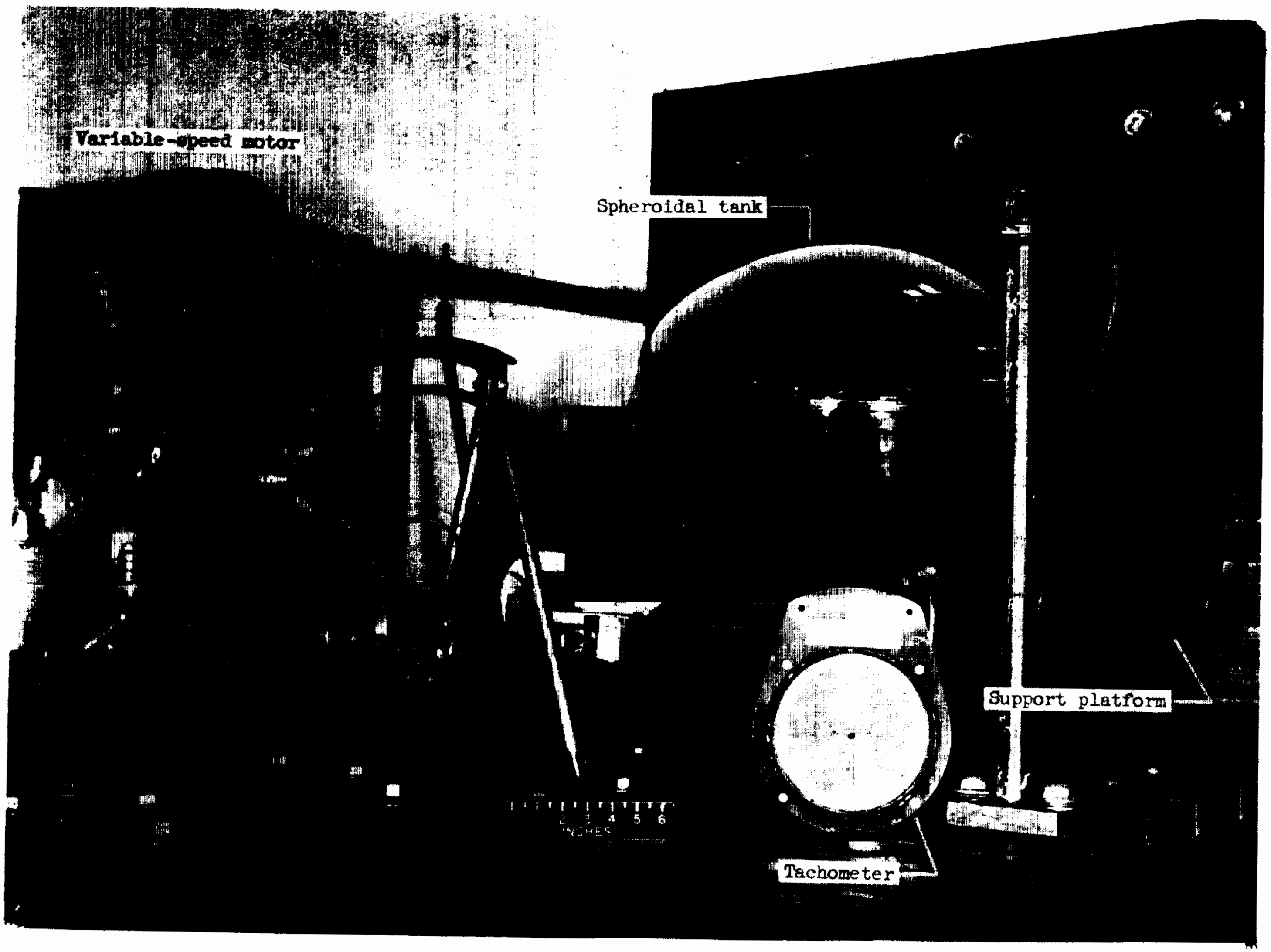
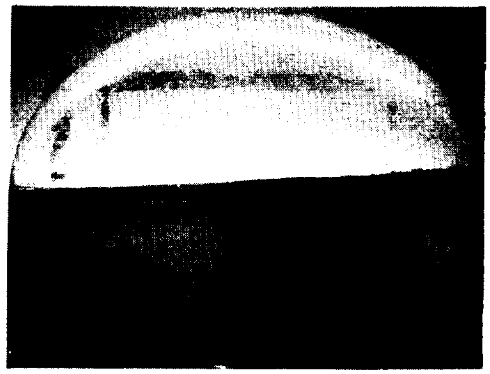
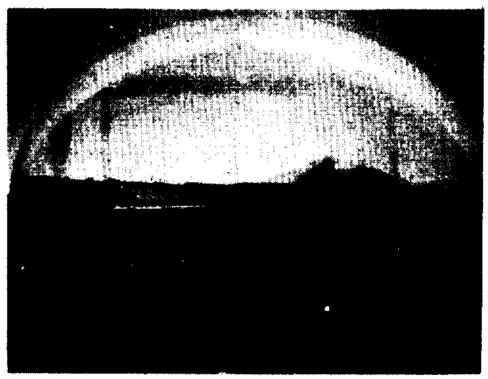


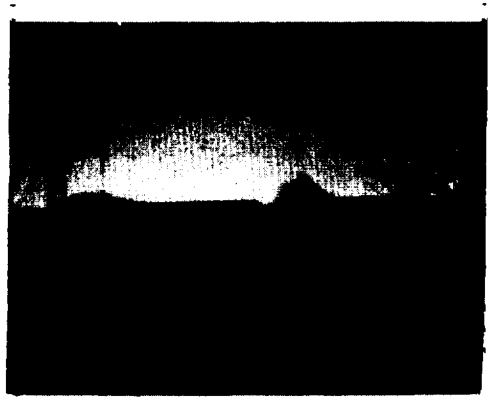
Figure 2.- Test apparatus for mechanically exciting spheroidal tanks. L-60-3493.1



(a) First mode.



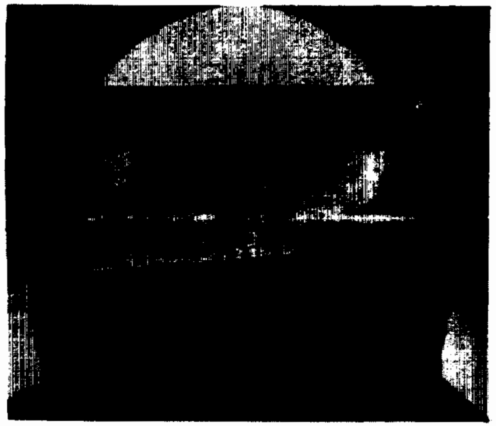
(b) Second mode.



(c) Third mode.

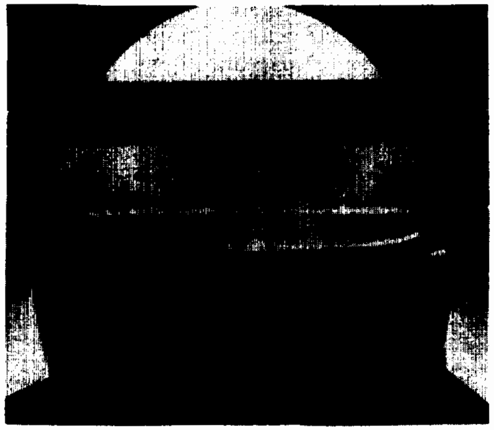
L-61-2151

Figure 3.- Surface wave shapes for first three natural modes of liquid in a spheroid. Horizontal orientation.



(a) First mode.

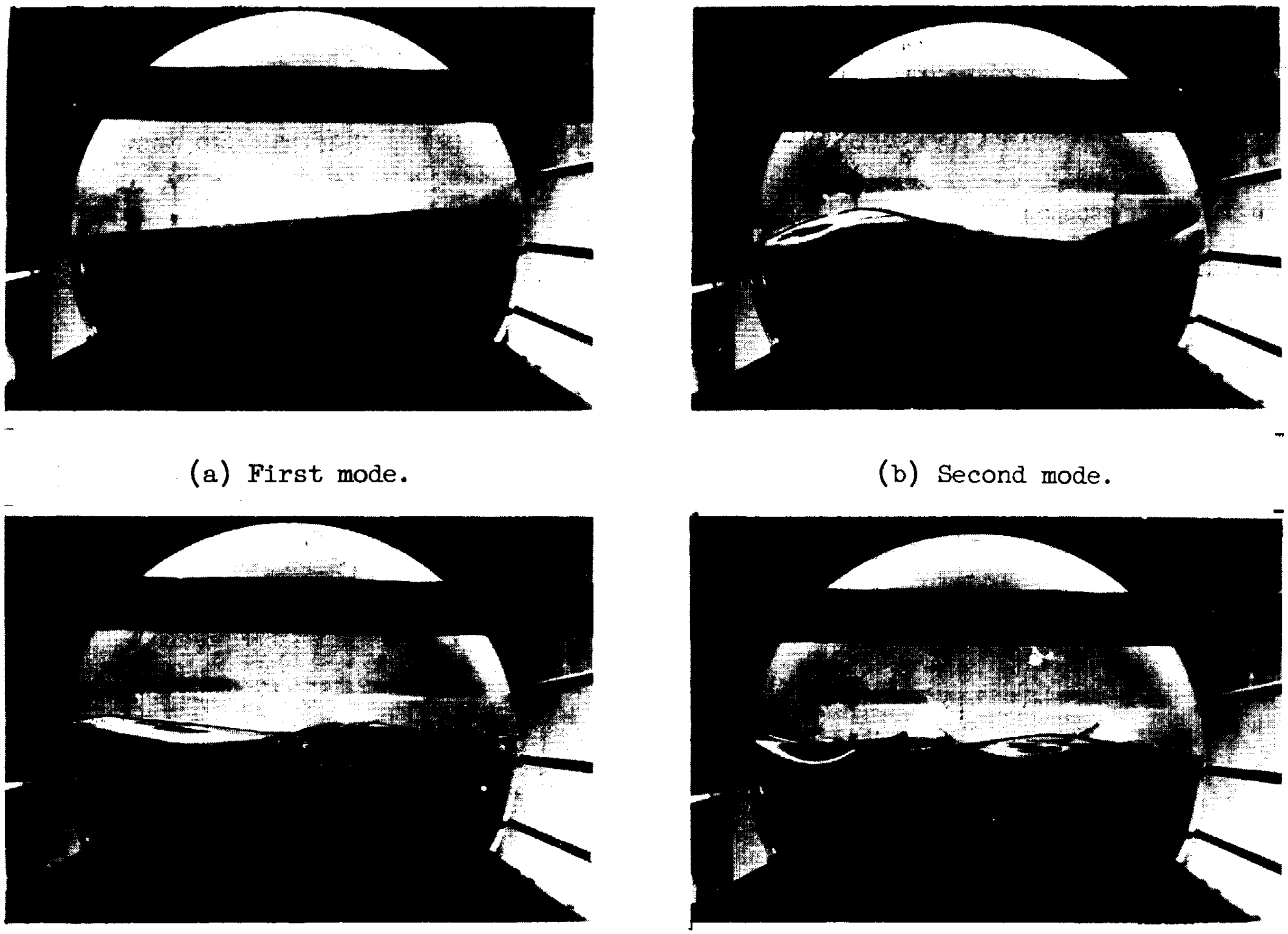
(b) Second mode.



(c) Third mode.

L-61-2152

Figure 4.- Surface wave shapes for first three natural modes of liquid in a spheroid. Transverse orientation.



(a) First mode.

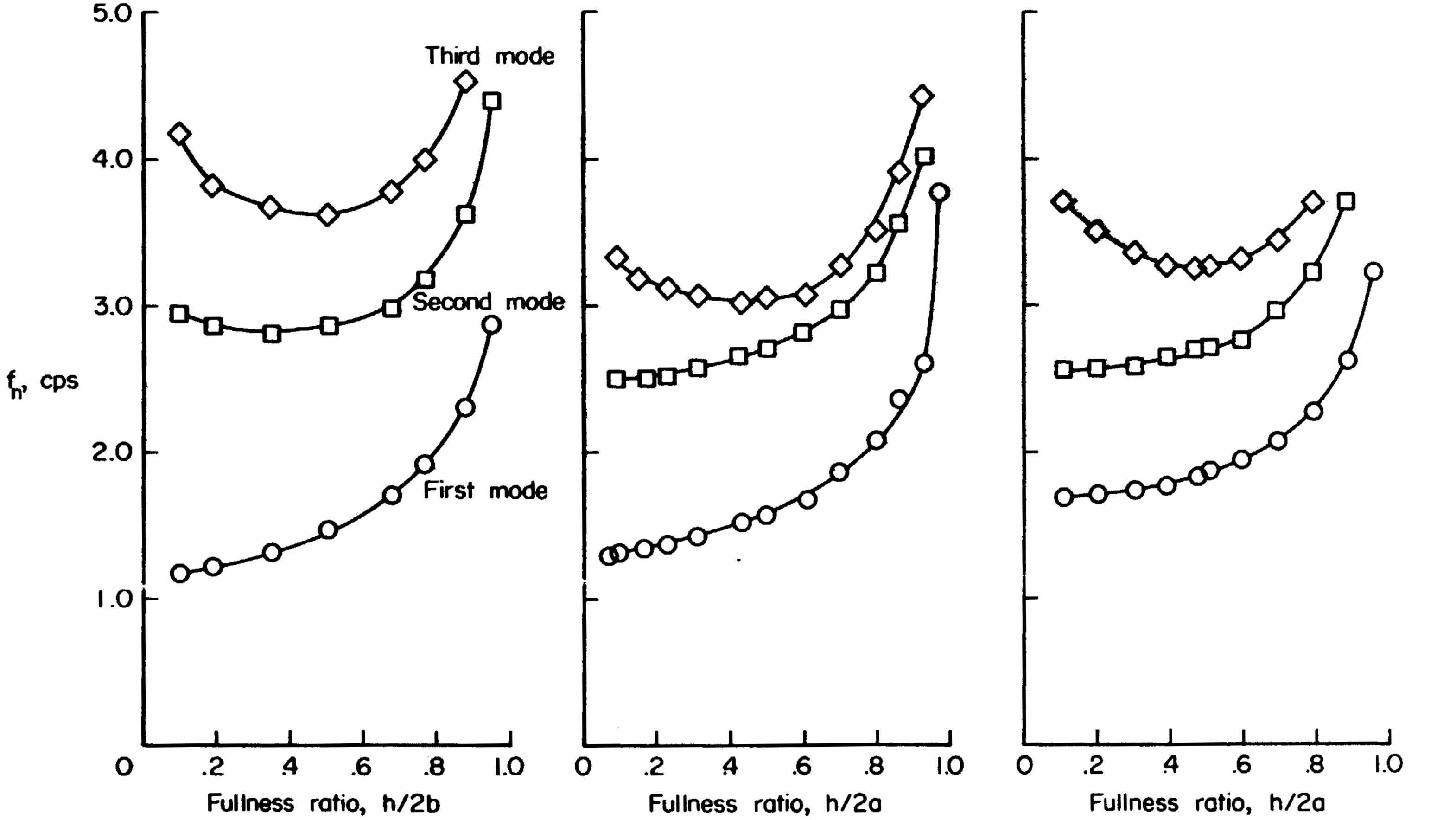
(b) Second mode.

(c) Third mode.

(d) Fourth mode.

L-61-2153

Figure 5.- Surface wave shapes for first four natural modes of liquid in a spheroid. Longitudinal orientation.



(a) Horizontal orientation. (b) Longitudinal orientation. (c) Transverse orientation.

Figure 6.- Variations of the natural frequencies of liquid with fullness ratio for three orientations of a spheroidal tank. Spheroid 4.

L-1264

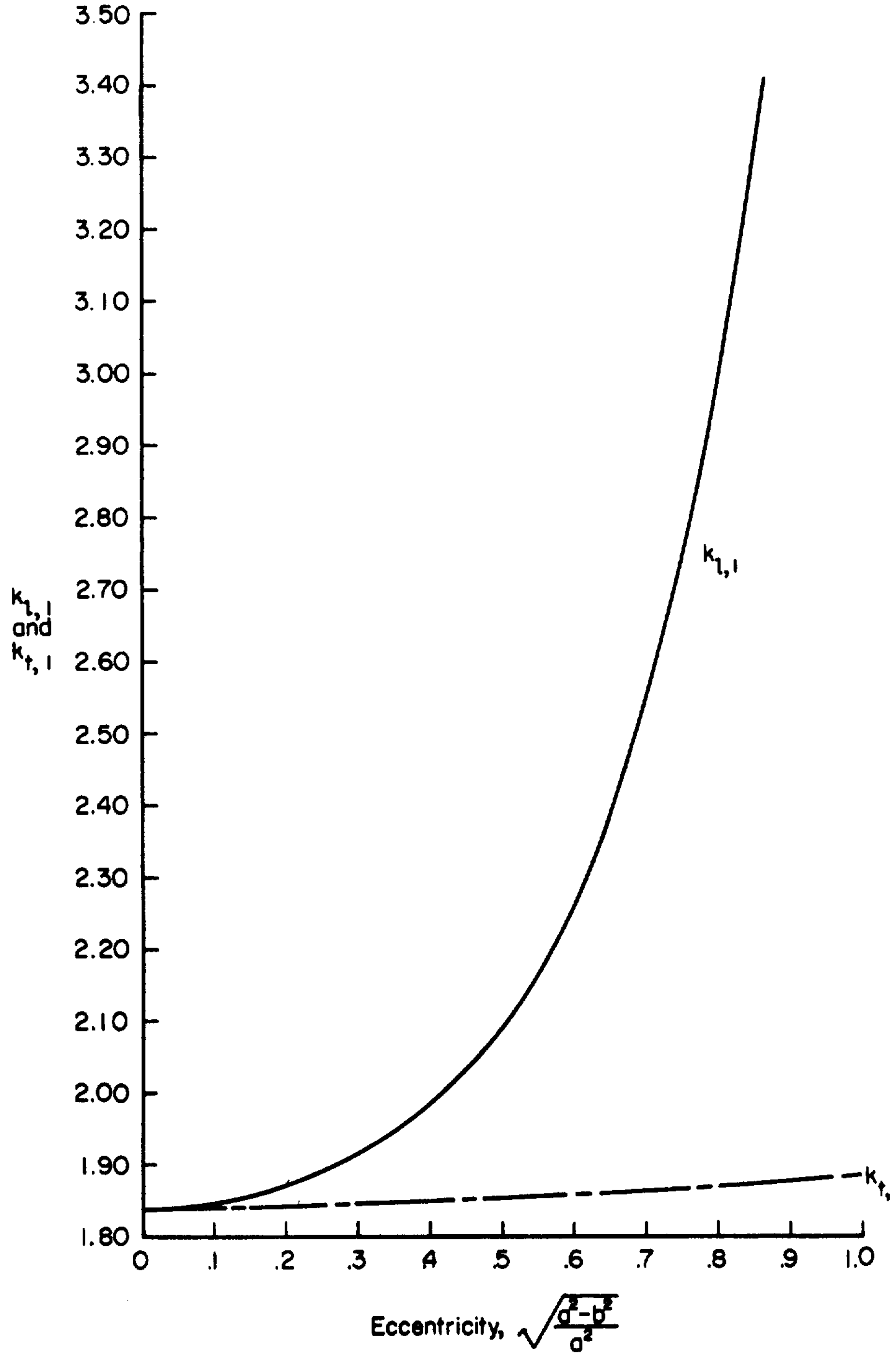
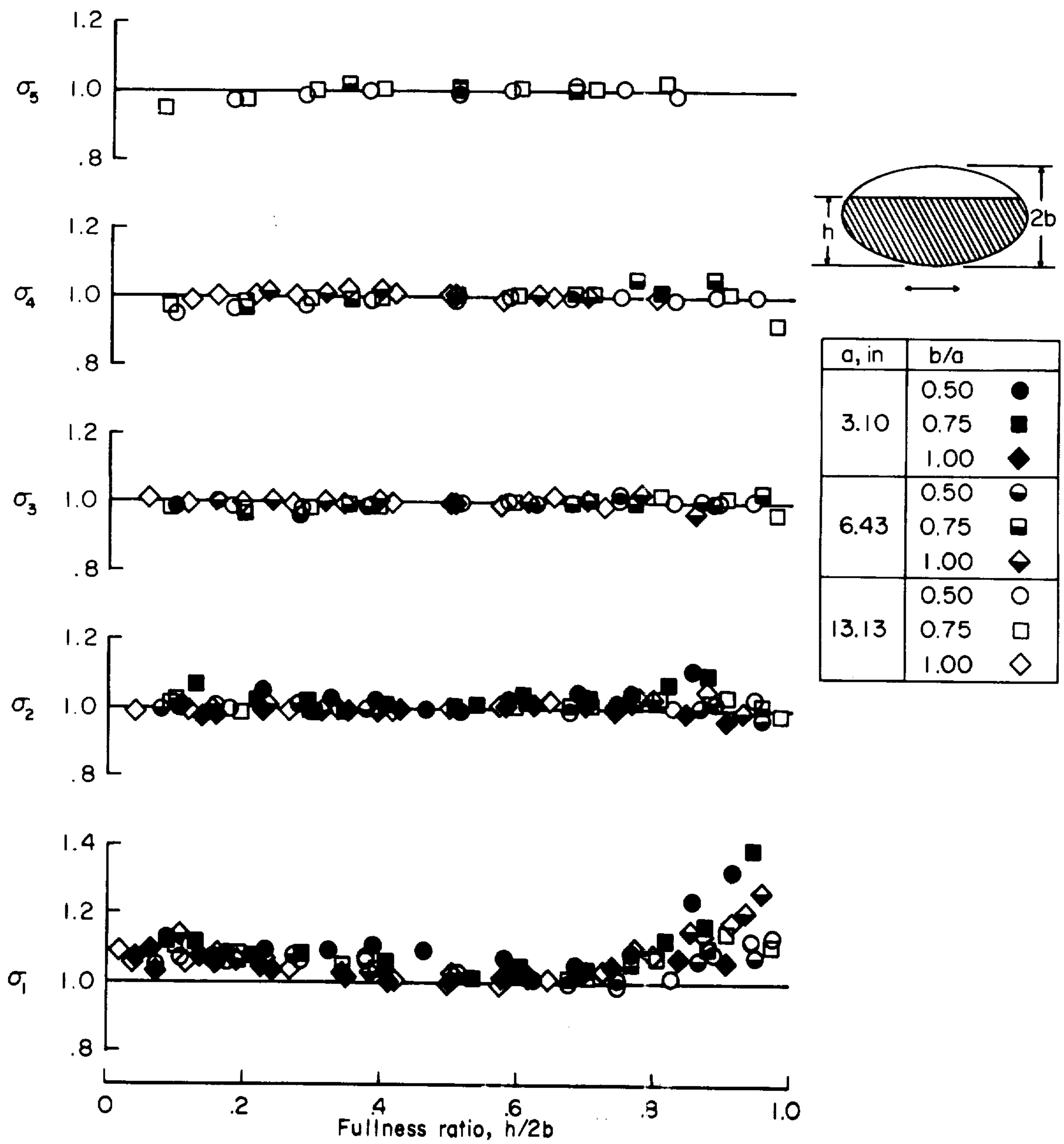


Figure 7.- Variation of $k_{l,1}$ and $k_{t,1}$ with eccentricity.

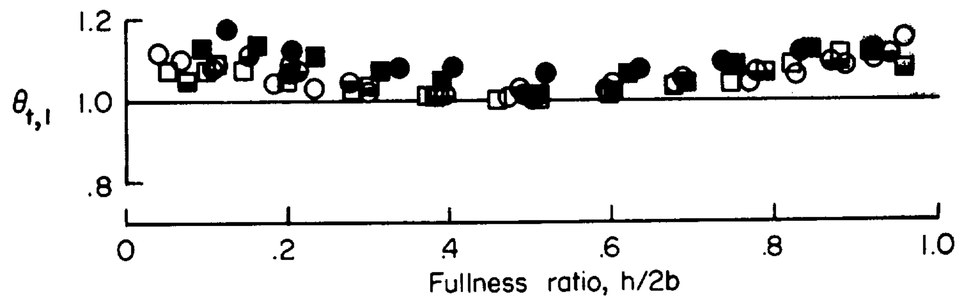


L-1264

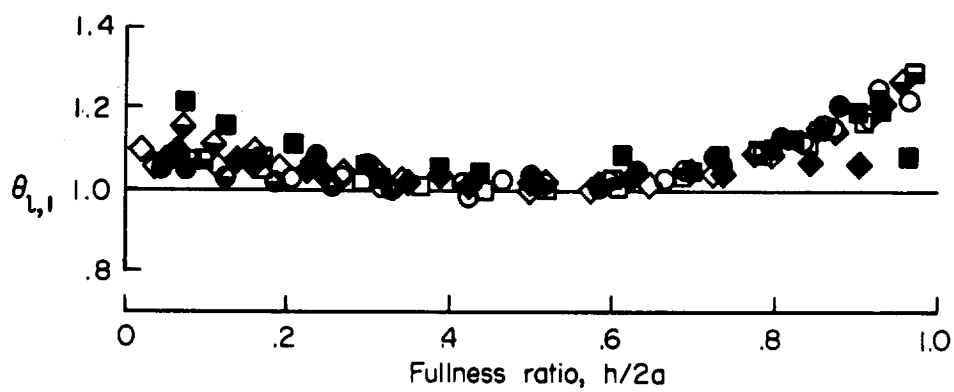
Figure 8.- Variation of frequency parameter $\left(\sigma_n = \omega_n\sqrt{\frac{r}{g}\,\frac{1}{\epsilon_n}\,\frac{1}{\tanh\left(\frac{h_c}{r}\,\epsilon_n\right)}}\right)$ with fullness ratio for liquid in spheroids. Horizontal orientation.

L-1264

a, in	b/a	
3.10	0.50	●
	0.75	■
	1.00	◆
6.43	0.50	◒
	0.75	⬓
	1.00	⬙
13.13	0.50	○
	0.75	□
	1.00	◇

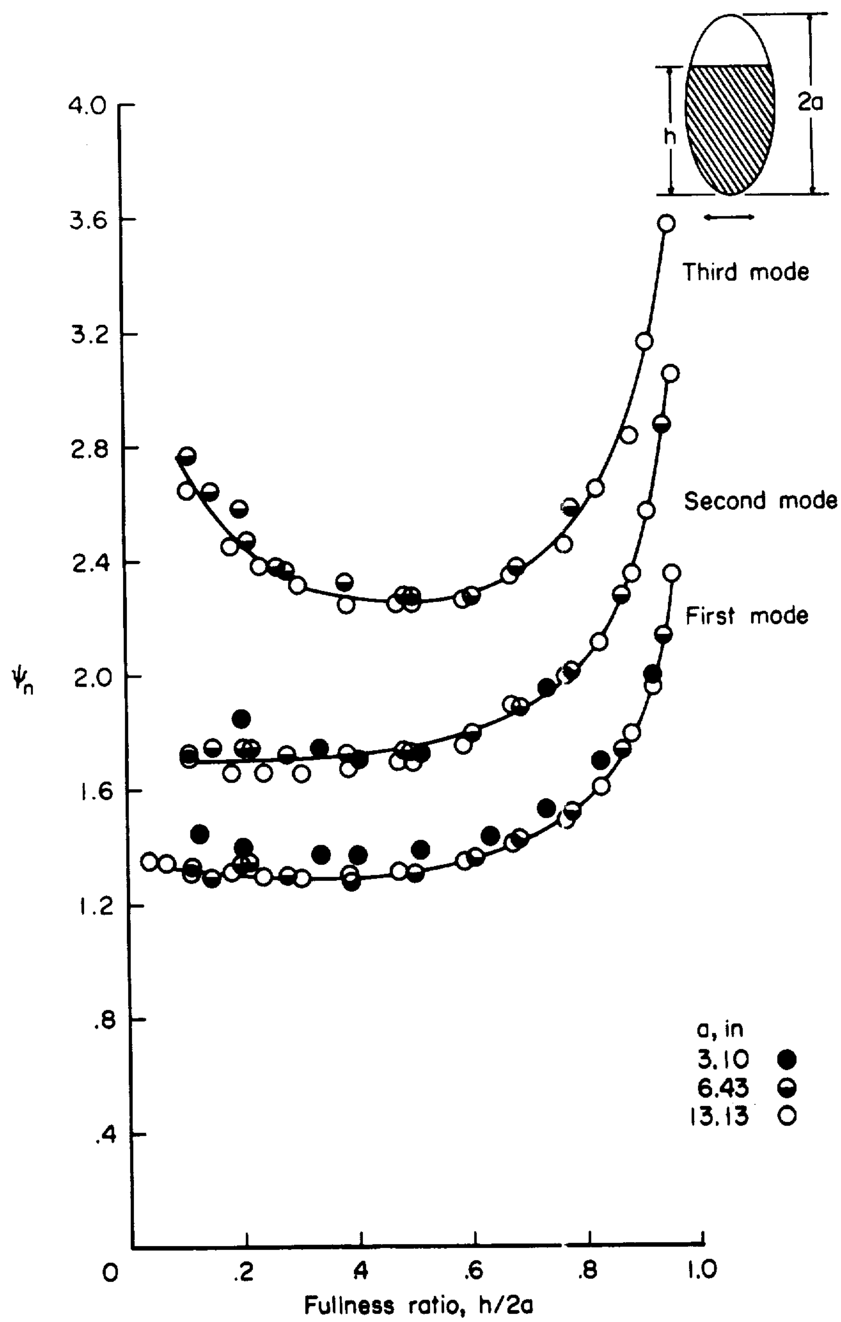


(a) Transverse orientations. $\theta_{t,n} = \omega_{t,n}\sqrt{\frac{a}{g}\frac{1}{k_{t,n}}\frac{1}{\tanh\left(\frac{h_e}{a}k_{t,n}\right)}}$.



(b) Longitudinal orientations. $\theta_{l,n} = \omega_{t,n}\sqrt{\frac{a}{g}\frac{1}{k_{l,n}}\frac{1}{\tanh\left(\frac{h_e}{a}k_{l,n}\right)}}$.

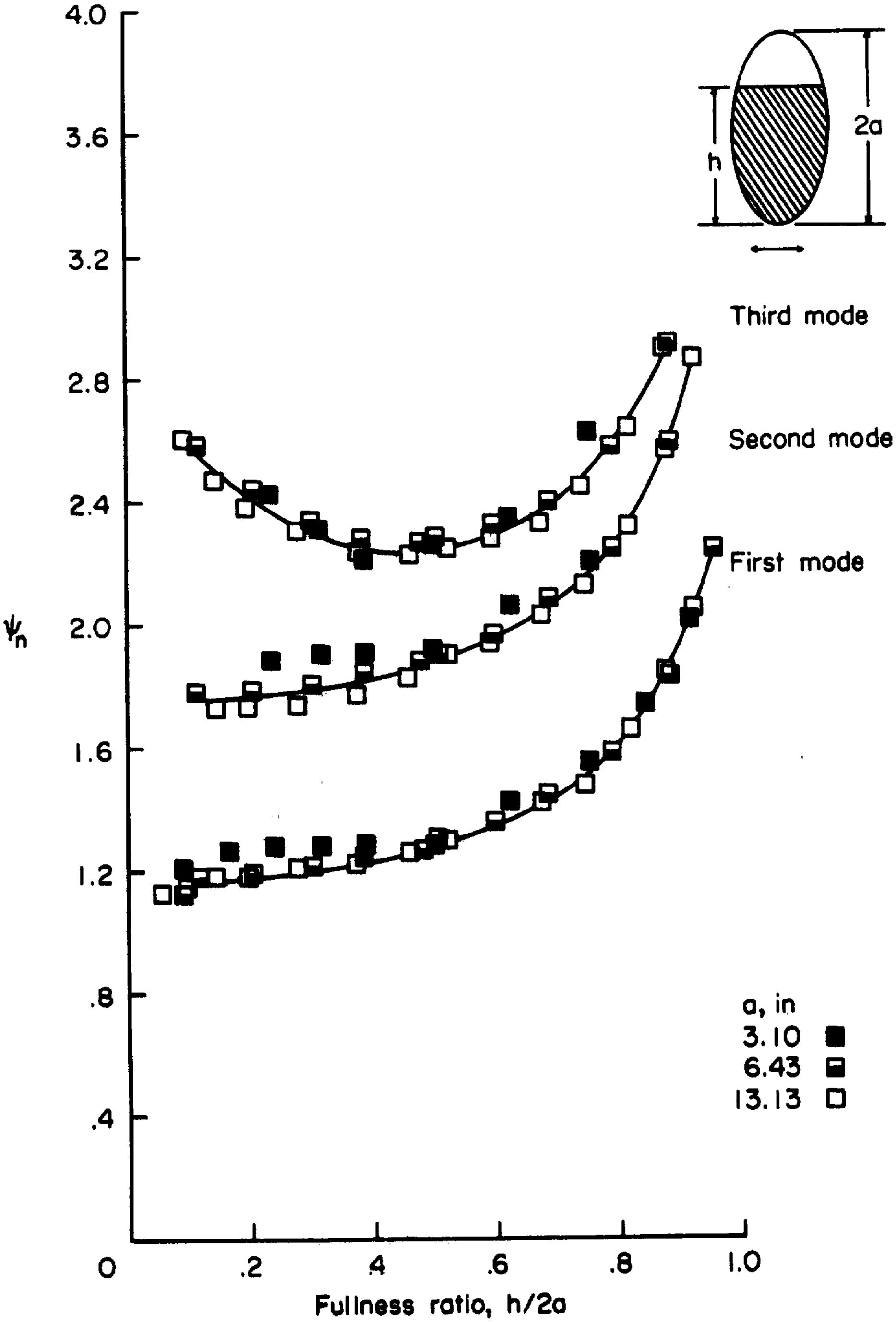
Figure 9.- Variation of liquid frequency parameter with fullness ratio for liquid in spheroids.



L-1264

(a) b/a = 0.50.

Figure 10.- Variation of frequency parameter $\left(\psi_n = \omega_n\sqrt{\frac{b}{g}}\right)$ with fullness ratio for liquid in spheroids. Transverse orientation.



(b) b/a = 0.75.

Figure 10.- Concluded.

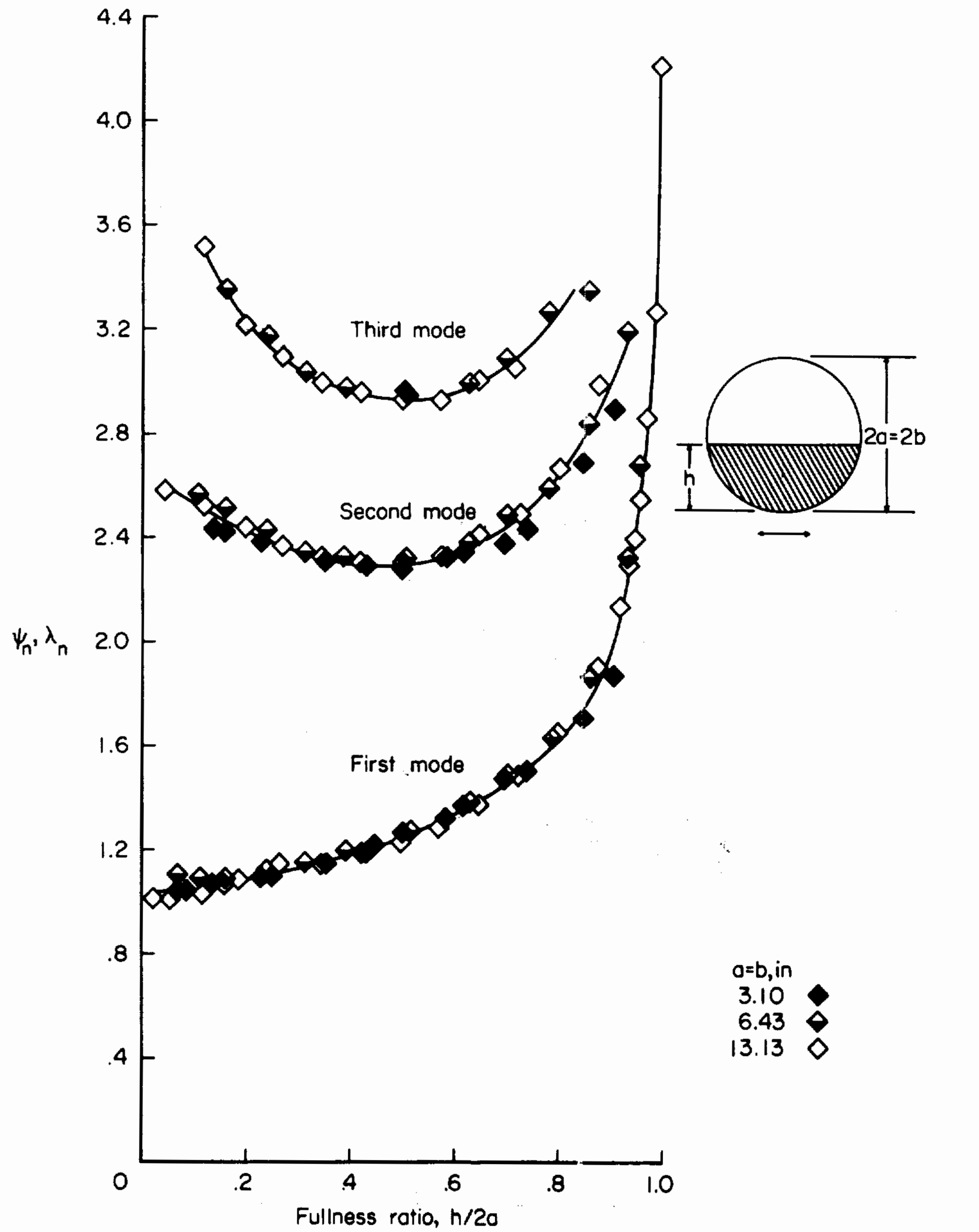
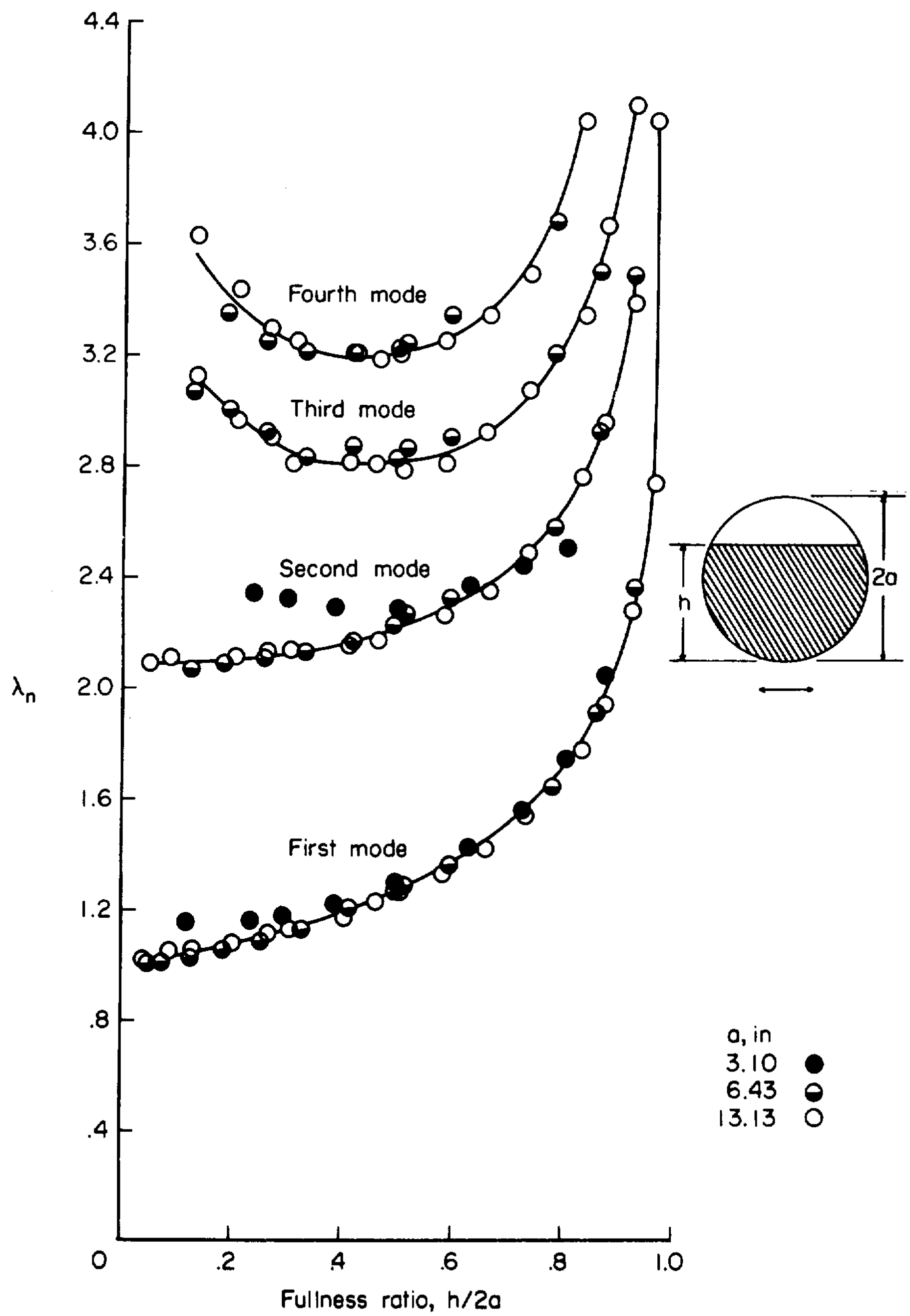


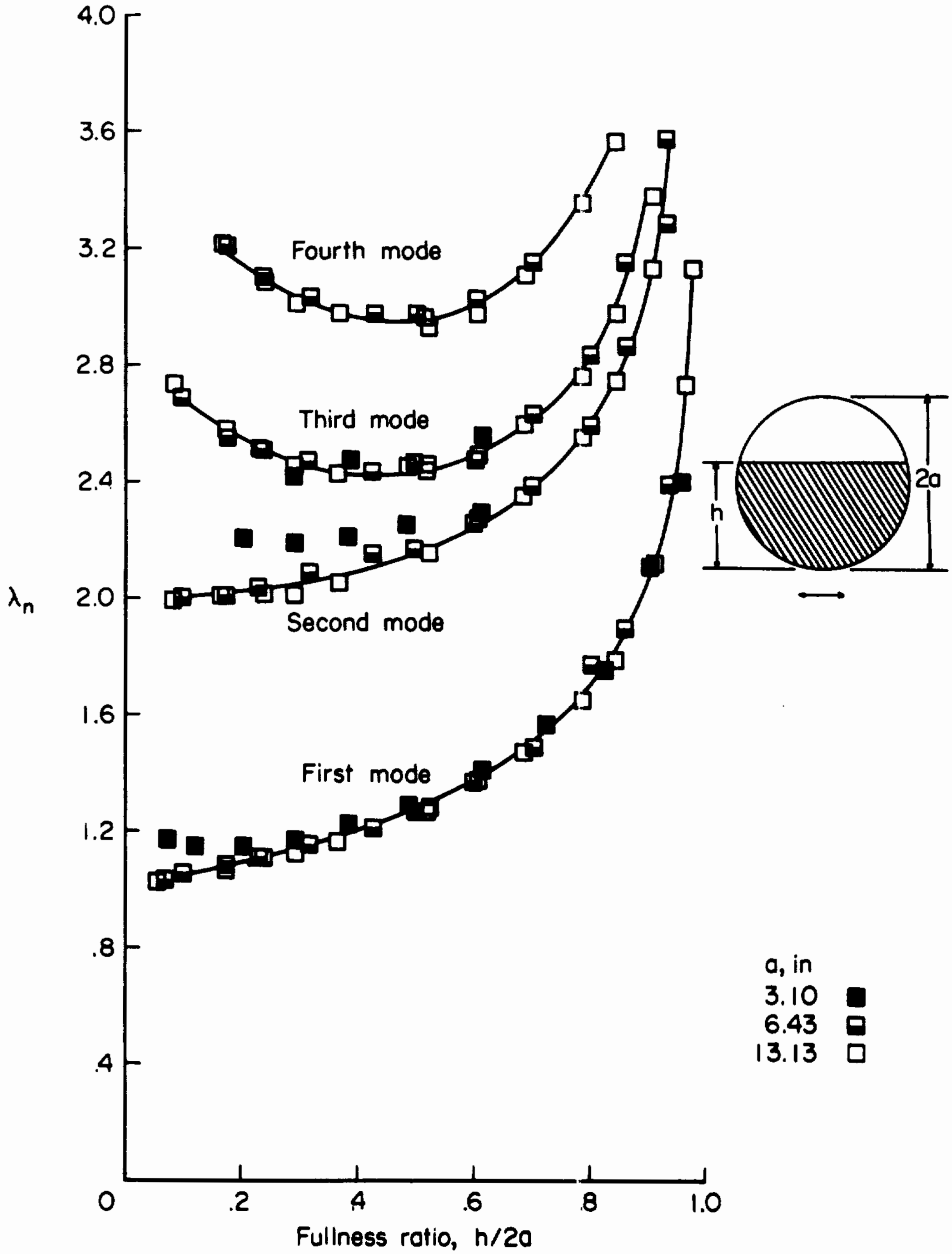
Figure 11.- Variation of frequency parameter $\left(\psi_n = \omega_n\sqrt{\frac{b}{g}} = \lambda_n = \omega_n\sqrt{\frac{a}{g}}\right)$ with fullness ratio for liquids in spheroids. Longitudinal and transverse orientations b/a = 1.0; sphere.

L-1207+



(a) b/a = 0.50.

Figure 12.- Variation of frequency parameter $\left(\lambda_n = \omega_n\sqrt{\frac{a}{g}}\right)$ with fullness ratio for liquid in spheroids. Longitudinal orientations.



(b) b/a = 0.75.

Figure 12.- Concluded.