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# TECHNICAL NOTE

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# A METHOD OF ESTIMATING RESIDUALS IN ORBITAL THEORY

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SUMMARY

The degree of approximation used in determining the orbits of earth satellites is reflected in the residuals (differences between calculated and observed positions). The least-squares procedure generally used to fit theory to observation tends to obscure the significance of theoretical parameters, so that the physical sources of residuals cease to be apparent. A method is outlined herein for estimating the magnitude of the residuals to be expected from an approximate theory presumed to have one missing or incorrect term.

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# A METHOD OF ESTIMATING RESIDUALS IN ORBITAL THEORY

## INTRODUCTION

In determining the orbits of earth satellites, as in many other physical problems, an approximate theory is used. The degree of approximation is reflected in the residuals, i.e., the differences between the calculated and observed positions. Analysis of the residuals provides a measure of our present understanding of the pertinent physical phenomena.

This paper outlines a method of estimating the magnitude of the residuals that are caused by certain inadequacies in present orbital theory. The estimate would be completely straightforward were it not for the fact that a least-squares minimization procedure is generally used to fit the theory to the observation. In this smoothing process, the theoretical parameters tend to lose their physical significance, and the physical sources of residuals cease to be apparent. A simple example will illustrate this point and also provide a model for the calculations to follow.

Suppose that the distance $S$ traveled by an object is observed as a function of time $t$ for some length of time $T$, and that the distance traveled as a function of time can be given rigorously by

$$S = S_0 + Vt + \frac{\dot{V}t^2}{2} . \quad (1)$$

Suppose further that the existence of the $\dot{V}$ term had not been known, and that the observations had been fitted to

$$S' = S_0' + V't . \quad (2)$$

The existence of the $\dot{V}$ term is now proposed, and it must be decided whether the size of the residuals supports this hypothesis. If the mean of the square of the residuals is called $R$, it might first be hypothesized that $R$ is of the order

$$R = \frac{1}{T}\int_0^T (S - S')^2\, dt = \Delta S^2 + \frac{\Delta V^2 T^2}{3} + \frac{\dot{V}^2 T^4}{20} + \Delta S\,\Delta V\, T + \frac{\Delta S\,\dot{V}\,T^2}{3} + \frac{\Delta V\,\dot{V}\,T^3}{4}\,, \qquad (8)$$

where

$$\Delta S = S_0 - S_0'\,, \quad \Delta V = V - V'\,.$$

In the symmetrized form, let

$$\dot{V} = x_0\,, \quad \Delta S = x_1\,, \quad \text{and} \quad \Delta V = x_2\,.$$

Then

$$a_{00} = \frac{T^4}{20}\,, \qquad a_{01} = a_{10} = \frac{T^2}{6}\,,$$

$$a_{11} = 1\,, \qquad a_{02} = a_{20} = \frac{T^3}{8}\,,$$

$$a_{22} = \frac{T^2}{3}\,, \qquad a_{12} = a_{21} = \frac{T}{2}\,;$$

and

$$R_{min} = \dot{V}^2 \frac{\begin{vmatrix} \frac{T^4}{20} & \frac{T^2}{6} & \frac{T^3}{8} \\ \frac{T^2}{6} & 1 & \frac{T}{2} \\ \frac{T^3}{8} & \frac{T}{2} & \frac{T^2}{3} \end{vmatrix}}{\begin{vmatrix} 1 & \frac{T}{2} \\ \frac{T}{2} & \frac{T^2}{3} \end{vmatrix}}$$

$$= \dot{V}^2 \left\{ \frac{T^4}{20} + 12T^{-2}\left[ T^6 \left( \frac{1}{48} - \frac{1}{108} - \frac{1}{64} \right) \right] \right\}$$

$$\approx \dot{V}^2 \left\{ \frac{T^4}{20} - 0.0486\, T^4 \right\}$$

$$\approx 1.4 \times 10^{-3}\, \dot{V}^2 T^4\,. \qquad (9)$$

Note that in this symmetrized formulation, $R_{min}$ may always be written as the difference of two terms. The first term represents the residuals introduced by the missing or incorrect terms in the approximate theory (Equation 3). The second term represents the reduction in the residuals effected by the minimization process. This term measures the extent to which the free parameters can compensate for the incorrect or missing parameter (and in the process separate themselves from their direct physical analogs).

## ERRORS DUE TO DRAG

The primary atmospheric effect (i.e., drag) is a secular shortening of the orbital period. For the satellite 1958 Beta 2, Jacchia* found that the time derivative of the orbital period was sinusoidal with time, with a period of about one month. This suggests that a constant second time derivative of the orbital periods may be used for arcs less than one week, while for longer arcs the sinusoidal variation should be represented.

In this section it is assumed that the approximate theory allows the period to vary linearly with time; and the residuals due to neglecting these terms are estimated.

From the discussion on observables and the minimization of residuals, it is seen that

$$R = (\delta \ell)^2 + (\delta m)^2 = \frac{1}{\beta^2}\left[(\delta\lambda)^2 \cos^2\phi + (\delta\phi)^2\right] \tag{10}$$

If the assumptions are restricted to circular orbits in a central gravitational field, then

$$R = \frac{1}{\beta^2}(\delta f)^2 \, , \tag{11}$$

where $f$ is the true anomaly.

Two calculations are now made:

*(1) $n$ is quadratically time dependent:*

The "true" $f$ is given by

$$f = f_0 + nt + \frac{1}{2}\dot{n}t^2 + \frac{1}{6}\ddot{n}t^3 \; ; \tag{12}$$

while the computed $f$ is taken as

$$f' = f_0' + n't + \frac{1}{2}\dot{n}'t^2 \, , \tag{12a}$$

where $n$ is the mean motion. The minimization of $R$ with respect to $f_0'$, $n'$, $\dot{n}'$ exacts the condition that

$$f_0' = f_0 + \frac{1}{120}\ddot{n}t^3 \, , \tag{13}$$

$$n' = n - \frac{1}{10}\ddot{n}t^2 \, , \tag{13a}$$

$$\dot{n}' = \dot{n} + \frac{1}{2}\ddot{n}t \, , \tag{13b}$$

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*Jacchia, L., Nature 183(4660):526-527, February 21, 1959; and 183(4676):1662-1663, June 13, 1959

$$\delta\phi = x_0 \left[ \frac{jnt\ (1 - \theta^2)^{\frac{1}{2}}\ (4\theta^2 - 1)\ \cos\psi}{\cos\phi} \right]$$

$$+ x_1 \left\{ \frac{\dot{n}t\ (1 - \theta^2)^{\frac{1}{2}} \left[ 1 + \frac{7j(4\theta^2 - 1)}{3} \right] \cos\psi}{\cos\phi} \right\}$$

$$+ x_2 \left[ \frac{-\theta(1 - \theta^2)^{-\frac{1}{2}} \sin\psi + 8jnt\,\theta(1 - \theta^2)^{\frac{1}{2}} \cos\psi}{\cos\phi} \right]$$

$$+ x_4\ \frac{(1 - \theta^2)^{\frac{1}{2}} \cos\psi}{\cos\phi}\ ; \tag{17}$$

and

$$\delta\lambda = x_0 \left[ jnt\,\theta \left( \frac{4\theta^2 - 1}{\cos^2\phi} - 1 \right) \right]$$

$$+ x_1 \left\{ nt \left[ \frac{\theta}{\cos^2\phi} \left( 1 + \frac{7j(4\theta^2 - 1)}{3} \right) - j\theta \right] \right\}$$

$$+ x_2 \left[ jnt \left( \frac{8\theta^2}{\cos^2\phi} - 1 \right) + \frac{\sin\psi\cos\phi}{\cos^2\phi} \right]$$

$$+ x_3$$

$$+ x_4\ \frac{\theta}{\cos^2\phi}\ . \tag{18}$$

With the foregoing, construct

$$R = \frac{1}{\beta^2} \sum_{ij} a_{ij}\ x_i\ x_j = \frac{1}{T} \int_0^T (\delta\ell)^2 + (\delta m^2)\ dt\ . \tag{19}$$

If the time average over an integral number $k$ of orbital periods is taken, it is found that

$$a_{00} = \frac{j^2}{3}\ N^2\theta_1 + O(j^2)\ ,$$

$$a_{11} = \frac{N^2}{3} + \left(\frac{7}{3}\right)^2 a_{00} + \frac{14\,j\,N^2\theta_3}{9}\ ,$$

$$a_{22} = \frac{1}{2(1 - \theta^2)} + j^2 N^2 \theta_4 + O(j)\ ,$$

$$a_{33} = \frac{(1 + \theta^2)}{2}\ ,$$

$$a_{44} = 1\ ;$$

and

$$a_{01} = a_{10} = \frac{jN^2\theta_3}{3} + \frac{7a_{00}}{3} ,$$

$$a_{02} = a_{20} = \frac{j^2N^2\theta_5}{2} + O(j) ,$$

$$a_{03} = a_{30} = \frac{jN\theta}{2}\left[\theta_3 - \frac{1}{2}(1 - \theta^2)\right] ,$$

$$a_{04} = a_{40} = \frac{jN\theta_3}{2} ,$$

$$a_{12} = a_{21} = \frac{7jN^2\theta}{3} + \frac{7j^2N^2\theta_5}{6} + O(j) ,$$

$$a_{13} = a_{31} = \frac{N\theta}{2} + \frac{7jN\theta}{6}\left[\theta_3 - \frac{1}{2}(1 - \theta^2)\right] ,$$

$$a_{14} = a_{41} = \frac{N}{2} + \frac{7jN\theta_3}{6} ,$$

$$a_{23} = a_{32} = \frac{jN}{4}(15\,\theta^2 - 1) ,$$

$$a_{24} = a_{42} = \frac{7jN\theta}{2} ,$$

$$a_{34} = a_{43} = \theta ,$$

where

$$N = 2\pi k ,$$

$$\theta_1 = \theta_3^2 + \frac{\theta^2(1 - \theta^2)}{2} ,$$

$$\theta_3 = 3\theta^3 - 1 ,$$

$$\theta_4 = \frac{97\theta^2 + 1}{6} ,$$

$$\theta_5 = \frac{\theta\left[14\,\theta_3 + (1 - \theta^2)\right]}{3} .$$

In general, it seems necessary to solve for $R_{min}$ numerically. However, there are two special cases of interest that are tractable algebraically:

(1) Setting $x_1 = x_2 = 0$ and thus minimizing only with respect to the initial values of the true anomaly and the longitude of the node, it is found that

$$R_{min} = \frac{x_0^2 j^2 N^2 \theta_1 \beta^{-2}}{12} . \tag{20}$$

This result is consistent with that usually found when compensating for an error term linear with time by correcting the initial position.

(2) Setting $x_3 = x_4 = 0$ and letting $k \geq 100$, then

$$R_{min} \approx \frac{x_0^2 j^2 N^2 \beta^{-2}}{3} \left( \theta_1 - \frac{\theta_3^2 + j^2 N^2 \theta_2 \theta_1}{1 + j^2 N^2 \theta_2} \right) , \tag{21}$$

where

$$\theta_2 = \frac{\left(1 - \theta^2\right)^2}{3} .$$

In general, the correction term here is quite small; this indicates that the mean motion and the inclination cannot compensate for an incorrect $J$.

## CONCLUSION

In this paper a general procedure has been outlined for estimating the magnitude of the residuals expected from an approximate theory presumed to have one missing or incorrect term. If, as is usually the case, a least-squares procedure is used to fit theory to observation, a reasonable estimate of the residuals cannot be obtained by considering only the missing terms. Rather, the remaining free variables may compensate for the missing terms, to reduce the residuals by an order of magnitude.

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