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MEMORANDUM

THE SYNTHESIS OF OPTIMUM HOMING MISSILE GUIDANCE SYSTEMS WITH STATISTICAL INPUTS

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SUMMARY

An analytical approach is presented which is applicable to the optimization of homing navigation guidance systems which are forced to operate in the presence of radar noise. The two primary objectives are to establish theoretical minimum miss distance performance and a method of synthesizing the optimum control system. The factors considered are: (1) target evasive maneuver, (2) radar glint noise, (3) missile maneuverability, and (4) the inherent time-varying character of the kinematics.

Two aspects of the problem are considered. In the first, consideration is given only to minimization of the miss distance. The solution given cannot be achieved in practice because the required accelerations are too large. In the second, results are extended to the practical case where the limited acceleration capabilities of the missile are considered by placing a realistic restriction on the mean-square acceleration so that system operation is confined to the linear range.

Although the exact analytical solution of the latter problem does not appear feasible, approximate solutions utilizing time-varying control systems can be found. One of these solutions - a range multiplication type control system - is studied in detail. It is shown that the minimum obtainable miss distance with a realistic restriction on acceleration is close to the absolute minimum for unlimited missile maneuverability. Furthermore, it is shown that there is an equivalence in performance between the homing and beam-rider type guidance systems. Consideration is given to the effect of changes in target acceleration, noise magnitude, and missile acceleration on the minimum miss distance.

INTRODUCTION

The wide application of homing guidance principles in missile systems poses an important question to the missile designer of what performance can be expected of a homing missile attacking a target. Of even greater importance is knowing how to synthesize a missile control system to achieve this optimum performance. The purpose of this report is to present a method whereby answers to these questions can be obtained.

The factors which affect missile system performance are so numerous, diverse, and often interrelated that all of them cannot be handled simultaneously. Nevertheless, there are certain aspects which are considered so essential to the realization of meaningful results that they cannot be ignored. The basic factors which it is believed must be included in the present homing problem are: (1) target maneuvers, (2) glint noise, (3) missile maneuverability, and (4) the inherent time-varying character of the kinematics. There are other aspects of the problem which are not considered; these either are not of fundamental importance to the problem or they must be handled by independent means. In this category are such factors as the three-dimensional aspect of the problem and the radar blind range of active radars. Thus the problem we wish to solve can be stated quite simply as follows: synthesize a homing guidance system which will produce minimum miss distance when operating under known conditions of target maneuver, glint noise, missile maneuverability, and time-varying kinematics.

There are two possible approaches. The usual solution of this problem is based on a cut-and-try approach, in which different control systems are simulated on an analog computer and parameters are varied in an attempt to find the best performance. There are so many possibilities to be investigated, however, that there is little chance of finding the best of all systems in this manner, although such studies can yield insight into the character of the problem.

A second approach is to leave the control system unspecified at the outset and to determine both the form and the coefficients of the system by a synthesis method. Such an approach is exemplified by the pioneering work of Wiener. Three principal assumptions made in the Wiener theory are: (1) the inputs can be represented by stationary random time series, (2) the system is linear, and (3) the system is time invariant. There are many problems which can be formulated without violating these assumptions, and to these problems the Wiener theory is applicable. An important example is the beam-rider guidance problem; the optimization of such a system has been reported on in reference 1. There are, however, many problems which cannot be described without violating one or more of these assumptions. In the homing problem to be considered here, previous theory cannot be applied because the problem is essentially a time-varying one. The reason is that guidance of the homing missile is accomplished by employing line-of-sight measurements which involve a time-varying range between missile and target. Consequently, the purpose of this report will be to consider the optimization of the homing system by including the time-varying range simultaneously with the target maneuver, noise, and missile maneuvering capability. We will be concerned with both the synthesis of the optimum homing system and the establishment of minimum theoretical miss distances.

The first part of the report provides the background and formulation of the problem. The second part treats the synthesis of the optimum homing system when the effect of practical missile maneuvering capabilities is temporarily ignored. This step provides a natural transition to the next section in which the finite missile maneuverability is taken into account. Finally, the method is illustrated in the last two sections where specific systems are derived.

IMPORTANT SYMBOLS

a_T	acceleration of target perpendicular to reference line
c_o	impulse response of optimum control system
k	twice the average switching rate of target acceleration, 1/sec
k_a	aerodynamic gain, $\frac{ft/sec^2}{radian}$
L	Laplace transform operation
N	noise magnitude or zero frequency noise spectral density, $\frac{ft^2}{radian/sec}$
R	range between missile and target, ft
s	variable in the Laplace transform
T	time at collision point, sec
T_N	time constant of the noise spectrum shaping filter, sec
V_M	missile velocity magnitude, ft/sec
V_R	relative velocity between missile and target in the direction of the line of sight, ft/sec
V_T	target velocity magnitude, ft/sec
w_o	impulse response of optimum control system and aerodynamics combination
y_M	missile displacement perpendicular to reference line, ft
y_N	apparent target displacement from true target center due to noise, ft
y_T	target displacement perpendicular to reference line, ft
γ_d	perturbation in missile velocity vector, radians
δ	control-surface deflection, radians

ϵ	error between target and missile position, y_T-y_M, ft
ϵ_N	component of error ϵ due to noise, ft
ϵ_T	component of error ϵ due to target maneuver, ft
λ_a	perturbation in apparent line of sight between missile and target, radians
Φ	spectral density of total input signal
Φ_N	spectral density of noise displacement y_N, $\frac{ft^2}{radian/sec}$
Φ_T	spectral density of target displacement y_T, $\frac{ft^2}{radian/sec}$
ω	angular frequency, radians/sec
$\widetilde{(\ )}$	ensemble average of ( )

PROBLEM FORMULATION

Geometry

It is widely accepted that the ideal attack course is one for which the line of sight from missile to target does not rotate. Such a course is called constant true bearing, or CTB. This course is especially desirable because it is the only course which never requires lateral acceleration of the missile greater than that of the target. However, the ideal CTB course cannot be achieved in practice because of time lags in the missile system. Furthermore, information about the target is obscured by noise so that the real target position is never known precisely. For these reasons the best any missile can be expected to do is to fly a reasonable approximation to CTB. Thus in formulating the problem analytically, one is led naturally to a representation of the target and missile maneuvers as perturbations from a CTB course.

In the derivation of the homing system equations three simplifying basic assumptions are made which can be summarized as follows:

1. Coplanar attack.- Target and missile maneuvers are assumed to occur entirely in a plane, which will be taken as the horizontal plane for simplicity. An alternative assumption - which amounts to the same thing - is that the pitch and yaw control channels operate without mutual interaction, so that the components of motion in one plane can be considered independently of motions in the other plane. One or the other of these assumptions can be expected to apply reasonably well to a large number of realistic cases.

2. Constant missile and target velocity magnitudes V_M and V_T.- This assumption is usually a reasonable approximation to most real cases where velocities are not constant. This is because, as will be shown, only the last few seconds of flight need be considered in determination of miss distance, during which time velocity magnitude changes are usually small.

3. Small angle approximation.- The perturbations of the line-of-sight angle and missile and target heading angles are assumed to be small enough to permit linearization of the geometry. The validity of this assumption will be examined in detail later in the report.

If the target does not maneuver, the second of these assumptions reduces the CTB course to a straight line. Since the target flight path is also a straight line under these circumstances, the entire attack situation can then be represented by a simple coplanar diagram of straight lines, as indicated in figure 1. This diagram makes a convenient reference for the homing problem equations. The line of sight is designated as the x axis, with the origin at the missile, and the entire reference system moves with constant velocity magnitude, V_M. Target and missile maneuvers and noise are then regarded as small perturbations from this coordinate system, occurring entirely within the plane of the straight-line CTB diagram. The complete geometry can thus be represented as in figure 1, where it can be seen that the angle between the apparent and reference lines of sight is

$$\lambda_a = \tan^{-1} \frac{y_T - y_M + y_N}{R} \tag{1a}$$

Since range is approximately equal to $x_T - x_M$ and since λ_a is assumed small, this equation is simplified to

$$\lambda_a = \frac{y_T - y_M + y_N}{x_T - x_M} \tag{1b}$$

Also, in figure 1 it can be seen that

$$\dot{x}_T - \dot{x}_M = -V_T \cos(\varphi_o + \varphi_d) - V_M \cos(\gamma_o + \gamma_d) \tag{2a}$$

which can be simplified because of the assumption of small perturbations to

$$\dot{x}_T - \dot{x}_M = -V_T \cos \varphi_o - V_M \cos \gamma_o = -V_R \tag{2b}$$

Likewise

$$\dot{\epsilon} = \dot{y}_T - \dot{y}_M = V_T \sin(\varphi_o + \varphi_d) - V_M \sin(\gamma_o + \gamma_d) \tag{3a}$$

which becomes (because $V_T \sin \varphi_o - V_M \sin \gamma_o = C$)

$$\dot{\epsilon} = \varphi_d V_T \cos \varphi_o - \gamma_d V_M \cos \gamma_o \tag{3b}$$

From equations (1b), (2b), and (3b), it is possible to construct a block diagram of the homing kinematics as shown in figure 2. Note that the quantity $x_T - x_M$ is replaced by

$$-\int V_R dt = -V_R t + R_o = V_R\left(\frac{R_o}{V_R} - t\right) = V_R(T-t)$$

where R_o is defined as the range at $t = 0$ when the problem is considered to begin, and $T-t$ is the time to go, or time remaining until collision.

Control System and Aerodynamics

The portion of the block diagram of figure 2 which closes the kinematics loop described above consists of the control system and aerodynamics, the function of which is to develop a γ_d in response to the input angle λ_a. It should be noted that the exact form of control is left unspecified since it is to be determined by the optimization procedure. However, two conditions are imposed in this study: (1) physical realizability of the over-all system, and (2) limited output capability (i.e., limited control deflection) of the control system. There are many other conditions which could be imposed but need not be considered here because they are of minor importance. One of these conditions is the blind range of the radar, which although an inherent characteristic of all active radars, is assumed to be zero in this study; that is, it is assumed that the seeker is able to detect the target and provide guidance information all the way to the end of flight. The reasonableness of this assumption is discussed in later sections.

Idealized missile aerodynamics are assumed in this report for mathematical convenience; that is, the transfer function relating control motion δ to the missile displacement perpendicular to the velocity vector V_M is taken to be simply k_a/s^2, where k_a is the aerodynamic gain and $1/s^2$ represents two kinematic integrations. This assumption is considered reasonable because specific aerodynamics are expected to have little effect on the over-all missile performance.[1] Furthermore, generality is not compromised since arbitrary aerodynamic factors can be included if desired. It will be noted that with ideal aerodynamics, limited control surface deflections are equivalent to limited acceleration capabilities, and either concept can be used to describe the limited capabilities of the missile.

[1]Reference 2 shows that if a control system is optimized for a specific set of aerodynamic factors, the final effect on miss distance is small provided that the aerodynamic factors are reasonably fast (i.e., reasonably high in undamped natural frequencies). Although reference 2 is applicable to only the beam-rider guidance system, it can be seen from later portions of this report that the same conclusion will apply to the homing problem.

Target Evasive Maneuver and Noise Inputs

Since the design of a system depends on the characteristics of the inputs to be encountered, it is necessary at this point to discuss these characteristics. For the homing study, target maneuvers and glint noise are the only inputs considered, and it is assumed that they are uncorrelated. They can therefore be shown as independent inputs in figure 2 and can be defined independently. The properties of the inputs used in this report are the same as those employed in the beam-rider study of reference 1 and are reviewed in the following paragraphs.

The glint noise, y_N, can be described best in statistical terms. Many experimental determinations of glint noise indicate that, typically, it has a Gaussian distribution and a spectral density[2] of the form

$$\Phi_N = \frac{N}{{T_N}^2\omega^2+1} \tag{4}$$

As in reference 1, a value of $N = 15$ feet squared per radian per second and $T_N = 0.0265$ second is used for the example employed later in the report. Such values are typical of a large bomber target.

The type of target maneuver upon which to base the system design can never be determined with certainty, since the target quite obviously may maneuver in many different ways. The best evasive maneuver would result if the target pilot possessed unlimited knowledge about the attacking missile and could therefore always maneuver in the optimum manner to avoid being hit. Such a concept is possibly somewhat unreasonable because of the difficulty in obtaining and properly utilizing all the information necessary to execute such a maneuver. A more reasonable assumption is that the target pilot knows only that he is being fired at and therefore executes some evasive maneuver. The concept which is used in this report is to picture the target evasive maneuver as a stationary random process in which the target turns at its maximum possible rate alternately in opposite directions without regard to what the attacking missile is doing. The length of time between switches is assumed to be randomly distributed according to a Poisson distribution $(1/\bar{l})\exp(-l/\bar{l})$, where l represents the length of the interval and $\bar{l}$ the average interval length. It can be shown (ref. 3) that the spectral density of the target displacement is then

$$\Phi_T = \frac{k{a_T}^2}{\pi\omega^4(\omega^2+k^2)} \tag{5}$$

where the quantity a_T represents the magnitude of acceleration perpendicular to the line of sight, and k is twice the average switching rate, or $k = 2/\bar{l}$. For the example situation to be used later in the report,

[2]The definition of spectral density used here is the same as in reference 1, except that the ensemble average $\widetilde{\theta^2}$ is used instead of the time average $\overline{\theta^2}$, that is, $\widetilde{\theta^2(t)} = \int_{-\infty}^{\infty} \Phi_\theta(\omega)d\omega$.

the target is assumed to maneuver with ±1g acceleration with an average period of 5 seconds, which gives $a_T = 32.2$ ft/sec^2 and k = 0.4 switch/sec.

There are several important comments to be made concerning this type of input. First of all, a statistical description of the target maneuver process is a desirable one, since target motions cannot be described as unique functions of time. Secondly it is clear that the maneuver is a severe one and puts the system to a good test; it is often found that systems designed according to theories based on either no maneuver or very weak maneuvers have unacceptably poor performance in the presence of a more severe maneuver. Another consideration not generally realized is that the stationary process described above is also applicable to certain important nonstationary processes. In any real problem it is apparent that the inputs are distinctly nonstationary. For instance, they are nonstationary because the target motion and noise do not exist for an infinitely long time into the past. However, the nonstationary character of the input is due to the strict mathematical definition. It is clear that in the practical case it makes little difference to the missile so far as miss distance is concerned, whether a process is considered to persist over an infinite or a finite period so long as the process begins at a time before the end of the attack by an amount equal to or greater than the system response time. (Of course, the process may terminate any time after the attack is over without affecting the results.) In other words, an infinite period is, for practical purposes, simply one which is longer than the system response time. Thus when the system response times are short, results obtained by means of the stationary input apply directly to an important class of nonstationary problems. The results presented herein are in this category.

Optimization Criterion

Mean-square miss distance is chosen here as the criterion for the evaluation and optimization of system performance. Although a probability of kill criterion would be more desirable to use, this concept is unwieldy in application, and is closely related to mean-square miss distance anyway.

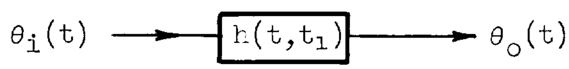
It should be noted that miss distance as used here is defined in terms of the radar center of the target and the center of gravity of the missile; that is, miss distance is the closest approach of the missile center of gravity to the center of the target. Since there is only one miss distance associated with any one attack, mean-square miss distance implies a large (theoretically infinite) number of attacks against targets with identical statistical maneuver and noise properties.

SYNTHESIS OF THE OPTIMUM HOMING SYSTEM

General Considerations

In the optimization problem to be considered in this report there are certain concepts which are so intimately associated with the solution that it will be necessary to discuss these concepts in the following paragraphs.

Mean-square output.- For systems characterized by linear constant-coefficient differential equations the relationship between the output and given analytical or statistical inputs is well known and particularly simple. For time-varying systems, however, the relationship is similar but not so simple. The general linear time-varying system may be conveniently described (see sketch) in terms of its time-varying impulse response



$h(t,t_1)$, which is the response at time t due to an impulse put in at time t_1; that is, if $\theta_i(t)$ is an impulse $u_o(t-t_1)$ then the output $\theta_o(t)$ is defined as the impulse response $h(t,t_1)$. This response function can be used to find the output due to any arbitrary input. However, when the input is a random process, the output can be described only by its statistical properties. One of the most useful properties is the mean-square ensemble average. For the particular case when the input θ_i has a constant spectral density of unity magnitude, the mean-square ensemble output $\overline{\theta_o^2(t)}$ is given by the equation[3]

$$\overline{\theta_o^2(t)} = 2\pi \int_{-\infty}^{t} h^2(t,t_1)\,dt_1 \qquad (6)$$

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[3] The form of this equation is a consequence of the definition given in footnote 2. That is the relation $\overline{\theta_i^2(t)} = \varphi_{ii}(0) = \int_{-\infty}^{\infty} \Phi_{ii}(\omega)\,d\omega$ (for a stationary process) implies the transform pair $\varphi_{ii}(\tau) = \int_{-\infty}^{\infty} \Phi_{ii}(\omega)e^{i\omega\tau}\,d\omega$ and $\Phi_{ii}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \varphi_{ii}(\tau)e^{-i\omega\tau}\,d\tau$. Thus for unit spectral density input, $\varphi_{ii}(\tau) = 2\pi u_o(\tau)$. It then follows from the relation between input and output correlation functions

$$\varphi_{oo}(t,t') = \int_{-\infty}^{t} \int_{-\infty}^{t'} h(t,t_1)h(t',t_1')\varphi_{ii}(t_1'-t_1)\,dt_1'\,dt_1$$

that for unit spectral density input,

$$\overline{\theta_o^2(t)} = \varphi_{oo}(t,t) = 2\pi \int_{-\infty}^{t} h^2(t,t_1)\,dt_1$$

In this report the mean-square values of several different quantities will be of interest. By appropriate interpretations of the quantities $\theta_o(t)$ and $h(t,t_1)$ each of these can be obtained from equation (6).

The preceding discussion is applicable only when the input is white noise. However, spectrums of physical phenomena are never flat. Since we are interested in mean-square values, which depend only on the input spectrum, it follows that if the input is replaced by another random function with an identical spectral density, the mean-square output will be the same. Such a random function can be readily generated by simply passing white noise through an appropriate shaping filter. By this means an equivalent system can be derived which has the same mean-square output as the actual system and furthermore has a white noise input. The output can now be obtained from equation (6) if $h(t,t_1)$ is defined as the impulse response of the equivalent system, which is a series combination of the appropriate spectrum-shaping filter and the actual system.

Adjoint theory.- To evaluate equation (6) it is necessary to obtain the impulse response $h(t,t_1)$. This is usually done by solving the differential equation describing the system being studied. For linear time-invariant systems this can be done relatively simply by solving the system differential equation by classical methods; however, for time-varying systems analytical solutions are not generally possible. Nevertheless, solutions can be obtained by the use of an analog computer. The system is simulated on the computer, an impulse $u_o(t-t_1)$ is introduced at time t_1, and the computer generates the system response $h(t,t_1)$ for that particular value of t_1. To obtain the system response for all t_1 it is apparent that a great many runs are required. In this way there results a family of curves $h(t,t_1)$ plotted against t for various values of t_1. However, to evaluate equation (6), curves of $h(t,t_1)$ plotted against t_1 for various values of t are required. Such curves must be obtained by cross-plotting the computer runs, which is obviously a somewhat unwieldy procedure.

The difficulties apparent in this method can be largely overcome by utilizing adjoint theory. This is because the adjoint system can be used to generate $h(t,t_1)$ as a function of t_1 for a particular value of t. Thus the entire operation indicated in equation (6), that is, generation of $h(t,t_1)$, squaring, and integrating, can be performed in a single computer run.

A description of the adjoint theory is given in reference 4 and need not be repeated here. However, to fully understand the homing problem it will be necessary to review certain essentials of the theory. To begin with, it must be pointed out that the theory is applicable only to linear systems, either time-varying or non-time-varying. Within this restriction any system may be replaced by its corresponding adjoint system. There are two ways in which the adjoint system may be constructed. In one method the original differential equation is transformed by the application of simple rules into the corresponding adjoint differential equation from which the adjoint system block diagram can be constructed. The other method, which is used in this report, is to construct the adjoint system block diagram directly from the original block diagram. Whichever method is used, the effect of the transformation is, in a sense, a reversal of the time scale of the problem. In the real time problem we are concerned with a particular time interval from $t = 0$ when the problem begins to $t = T$ when the problem may be considered to end because it has no useful meaning beyond this time. For example, in the homing problem $t = 0$ might be defined as the time of launching and $t = T$ as the time when the missile reaches the target. In the adjoint, the time interval is identical but the time scale is viewed in the reverse sense. This is readily seen from the definition of adjoint time, which is $\tau = T-t$. Thus, the beginning and end of the interval in adjoint time, $\tau = 0$ and $\tau = T$, correspond respectively to the end and beginning of the interval in real time, $t = T$ and $t = 0$.

The rules for deriving the adjoint circuit can be summarized as follows:

1. The input and output of each box are interchanged and the direction of signal flow reversed.

2. The input and output of the over-all system are interchanged. That is, whereas the impulse is introduced into the original block diagram at point A and the output obtained at point B, in the adjoint circuit the impulse is applied at point B and the output is obtained at point A.

3. A time-varying element in the real time problem, which is represented by $F(t)$ during the interval of interest, is replaced in the adjoint problem by $f(\tau)$. This function is made to be simply $F(T-\tau)$. In other words $f(\tau)$ is the same as $F(t)$ but runs backward starting at $F(T)$ and ending at $F(0)$.

In terms of the adjoint circuit and adjoint time, equation (6) for the mean-square output can now be rewritten as follows

$$\widetilde{\theta_o^2(t_2)} = \int_{-\infty}^{t_2} h^2(t_2,t_1)dt_1 = \int_{T-t_2}^{\infty} h^2(t_2,T-\tau)d\tau$$

$$= \int_{0}^{\infty} h^2(t_2,T-\tau)d\tau \qquad (7)$$

where the h^2 has been redefined to include the 2π factor. Thus h now represents the impulse response of the series combination of the spectrum-shaping filter, the actual system, and a $\sqrt{2\pi}$ factor. Equation (7) may now be rewritten in terms of the time τ_1 at which the impulse is introduced into the adjoint circuit. In view of the reciprocity which exists between the real-time and adjoint-time systems it is seen that τ_1 is simply $T-t_2$. Thus by eliminating t_2 we have

$$\widetilde{\theta_o^2(t_2)} = \int_0^\infty h^2(T-\tau_1, T-\tau)d\tau$$

$$= \int_0^\infty h^2(\tau, \tau_1)d\tau \qquad (8)$$

where $h(\tau,\tau_1)$ is the response of the adjoint system to an impulse introduced at τ_1 plotted as a function of τ (but is not of the same functional form as $h(T-\tau_1,T-\tau)$). It should be noted that in this latter form the t_2 on the left side of (8) is contained implicitly in the τ_1 on the right side. Also, the lower limit has been replaced by zero in equation (8), since the impulse response is zero for $\tau < T-t_2$. This limit corresponds to the obvious fact that in the original system there can be no response at t_2 due to an input which occurs after t_2.

The infinite upper limit on the integral in equation (8) needs some further explanation. Since we are interested only in the time interval $\tau = T-t_2$ to T, corresponding to the real time interval $t = 0$ to t_2, it is apparent that the integration has no meaning beyond $\tau = T$. However, if the impulse response has died out by the time τ reaches the value T, then the ∞ upper limit is valid. In terms of the physical problem this is equivalent to the fact, mentioned in the problem formulation, that inputs occurring before the time of interest, t_2, by an amount equal to or greater than the missile response time can have no effect on the mean-square output at t_2. For this reason it is implied in equation (8) that the initial time to go T must be at least as great as the impulse response time of the system. This is a reasonable assumption in most cases. For instance, it will be seen in the typical example introduced later that the impulse response does not extend beyond a few seconds.

Solution With No Control Motion Restriction

In this section will be considered the solution for the optimum homing control system which satisfies only the minimum miss-distance criterion. No attempt is made to account for saturation effects due to limits on the control motion. For this reason the solution is not expected to be of practical interest. However, the solution of this simpler problem can be used to lead to the solution of the more complex problem in which the finite missile maneuverability is considered.

Adjoint block diagram.- The first step in the solution is to derive the adjoint block diagram. For this purpose it will be desirable to redraw figure 2 in a more suitable form as given by figure 3. It should be noted that in view of the discussion of the previous section the inputs y_N and y_T

have been replaced by filtered white noise. The spectrum-shaping filters shown are readily derived from the respective spectral densities of y_N and y_T as given in equations (4) and (5). So that the block diagram will be consistent with equation (8) the input to the block diagram is shown as white noise of unit spectral density which is then multiplied by $\sqrt{2\pi}$. The quantities (y_N), (y_T), and (ϵ) appearing in figure 3 are shown in parentheses because they are not the same as the true quantities except in terms of their mean-square values. All of the system except for the range-varying kinematic gain is lumped together and identified by its impulse response $w(\tau,\xi)$, or equivalently its transform $W(s,t)$, which is considered a function of t in order to include time-varying control systems. The transfer function $W(s,t)$ thus contains the kinematic gains $1/V_R$ and $V_M \cos \gamma_o$, the idealized aerodynamics, and the control system. It is now possible to construct the adjoint block diagram corresponding to figure 3 by means of the rules given previously; the result is given in figure 4. Several points should be noted concerning this figure. First, the asterisk superscript denotes that $W^*(s,t)$ is the adjoint of $W(s,t)$. Second, the impulse input required to obtain the system impulse response is shown introduced at the point corresponding to the error, $\epsilon = y_T - y_M$, in the original system. Third, corresponding to the two inputs which appear in the original system, there are two outputs which appear in the adjoint, one related to the error due to noise and the other to the error due to target maneuver.

Adjoint equations.- It is now desired to write the expression for the mean-square error $\widetilde{\epsilon^2}$ in terms of the unknown control system $w(\tau,\xi)$. From figure 4 it can be seen that the error is composed of two parts, one due to noise and the other due to target maneuver. By utilizing equation (8) the mean-square value of the components $\widetilde{\epsilon_N^2}$ and $\widetilde{\epsilon_T^2}$ can be written in terms of the corresponding impulse responses E_{Nf} and E_{Tf} shown in figure 4. Since the noise and target maneuvers are uncorrelated, the total mean-square error is given by

$$\widetilde{\epsilon^2(t_2)} = \int_0^\infty [E_{Nf}^2(\tau,\tau_1) + E_{Tf}^2(\tau,\tau_1)]d\tau \tag{9}$$

In order to find the optimum control system, it is now necessary to express equation (9) in terms of the unknown control system $w(\tau,\xi)$. It is shown in appendix A that the mean-square error $\widetilde{\epsilon^2}$ can be related to the control system $w(\tau,\xi)$ by the following two expressions:

$$\widetilde{\epsilon^2(t_2)} = \int_0^\infty \left(\left\{ \int_0^\infty h(\tau-x,\tau_1) h_N(\tau,\tau-x) dx \right\}^2 + \left\{ \int_0^\infty [u_o(\tau-x-\tau_1) - h(\tau-x,\tau_1)] h_T(\tau,\tau-x) dx \right\}^2 \right) d\tau \tag{10}$$

$$h(\tau,\tau_1) = \frac{w(\tau,\tau_1)}{\tau} - \frac{1}{\tau}\int_0^\tau w(\tau,\xi)h(\xi,\tau_1)d\xi \qquad (11)$$

In equation (10) it is seen that the mean-square error consists of two parts, the first due to noise and the second due to target maneuver. The known factors in this equation are h_N and h_T, the impulse responses of the spectrum-shaping filters multiplied by $\sqrt{2\pi}$, and u_o, the unit impulse. The only unknown is $h(\tau,\tau_1)$ which can be interpreted as the impulse response of the adjoint homing system. The second equation, (11), is an integral equation in which the response $h(\tau,\tau_1)$ is related to the control system $w(\tau,\xi)$. Thus it is clear that by the pair of equations (10) and (11) the mean-square error is expressed in terms of the unknown control system $w(\tau,\xi)$ by means of the intermediate response $h(\tau,\tau_1)$.

Solution.- It is now necessary to determine the optimum control system $w(\tau,\xi)$ which will minimize the mean-square error. No convenient means is known for treating such general time-varying expressions as equations (10) and (11). However, rather than determining the minimum error for all times, we may greatly simplify the equations by confining our attention to the error at zero range, or $t = T$, that is, the miss distance. (In fig. 4 this corresponds to making the input $u_o(\tau)$, so that the output becomes $\epsilon^2(T)$.) In this case the equations can be solved as shown in appendix A. Since we have confined our interest to one instant of time, the time-varying nature of $h(\tau,\tau_1)$ is arbitrary. Consequently, there are a great many time-varying control systems $w(\tau,\tau_1)$ which are optimum. As discussed in appendix A it will only be of interest here to examine the simplest of such systems, the constant-coefficient control system. The solution for this case is

$$W_o(s) = -\frac{\frac{d}{ds}H_o(s)}{1-H_o(s)} \qquad (12)$$

where

$$H_o(\omega) = \frac{1}{2\pi\Phi^+(\omega)}\int_0^\infty e^{-i\omega t}\int_{-\infty}^\infty \frac{\Phi_T(\alpha)e^{i\alpha t}}{\Phi^-(\alpha)}\,d\alpha\,dt \qquad (13)$$

Note that $\Phi = \Phi_T + \Phi_N = \Phi^+\Phi^-$ where the + superscript refers to the upper half-plane poles and zeroes while the - refers to the lower half-plane poles and zeroes.

For the example case to be considered in this report, for which the numerical values describing the input spectra are given in the problem formulation section, the optimum control system $W_o(s)$ is found to be a rational function of the general form

$$W_o(s) = 2\,\frac{A(s)}{sB(s)} = 2\,\frac{a_3s^3+a_2s^2+a_1s+1}{s(b_4s^4+b_3s^3+b_2s^2+b_1s+1)} \tag{14}$$

Numerical evaluation gives

$$W_o(s) = 2\,\frac{(3.58s+1)(0.373s^2+0.886s+1)}{s(2.50s+1)(0.490s^2+0.727s+1)(0.687s+1)} \tag{15}$$

An important conclusion can now be drawn regarding the relative performance of homing and beam-rider systems. This conclusion is that the two systems have precisely the same optimum miss distance performance for any given attack situation. The reason for this will be indicated in the following discussion.

It will be recalled that the optimization of the beam-rider system as treated in reference 5 was based on minimizing the time average of the squared error. However, following the theory in the present report it would be equally possible to minimize the mean-square ensemble error of miss distance. The result would be to yield an expression for $H_o(\omega)$ which is identical in <u>form</u> to that given in equation (13) for the homing system. (It should <u>not be</u> inferred that the control systems are therefore of the same form, however, because of the basic difference between the kinematics of homing and beam-rider guidance.) It is clear that the <u>numerical</u> values of these two H_o's will be identical in any given attack situation only if the expressions for Φ_T and Φ_N are the same for both systems. Under these circumstances the mean-square ensemble miss distances will be identical. That the Φ's <u>are</u> the same is readily seen for the tail-chase situation, but it is not immediately apparent for other attack aspects. Although it was not intended in the beam-rider study to deal with the general case, it can be shown that under certain circumstances the beam-rider problem is amenable to generalization of the same type employed in the homing problem. One such circumstance is when the tracking radar is sufficiently far from the target that rotations of the line of sight are small. Another is when the tracking radar is sufficiently far from the target that rotations of the line of sight are small. In either case, the target motions and noise may be regarded as perturbations about a reference line which moves at constant velocity without rotation. This is precisely the same situation assumed for the homing system geometry so that inputs and therefore miss distances of the two systems are identical. In other words, in any given attack situation the homing and beam-rider systems are actually solving the same problem, and it should not be surprising to find that they have the same optimum performance. A somewhat different situation develops when missile acceleration limiting is considered, but this question will be deferred to the next section.

The performance of the homing system can now be shown. Note that for the beam-rider system the error ϵ is both stationary and ergodic. Thus, as a consequence of the ensemble average equivalence just shown between the unrestricted homing and beam-rider problems, it is possible without further calculation to utilize certain of the beam-rider results directly in the homing problem. To illustrate, for the example attack situation a curve of minimum obtainable miss distance versus noise magnitude for the beam rider (taken from ref. 5) is plotted as the lowest solid curve in figure 5; the corresponding curve for the homing system, identical to that for the beam rider, is shown as the lower dotted curve. This curve represents the theoretical lower limit of miss distance, and is useful in making comparisons with the performance of other homing systems.

It is interesting to compare these results with the performance which would be obtained by disregarding noise theory in the design. As an example, the uppermost curve in figure 5 shows the noise performance of a system optimized for a target motion with no noise present. A comparison of this curve with the optimum curve shows that a significant reduction in miss distance should be possible.

Solution With Control Motion Restriction

Synthesis of the optimum control system, taking into account missile acceleration saturation, is considered in this section. The importance of this phase of the homing problem has been discussed previously, and can be graphically illustrated in this section.

Adjoint block diagram.- As a first step in the optimization procedure for the restricted control motion case, it will be necessary to reconstruct the adjoint block diagram of figure 4 in a form suitable for evaluation of mean-square control motion. To obtain mean-square control motion, an impulse input must be introduced at the point in the adjoint system which corresponds to the control motion in the real time system. Thus, the idealized aerodynamics, k_a/s^2, are separated from the remainder of the system. This remainder is designated $C^*(s,t)$, again showing C^* as a function of time to include the possibility of a time-varying control system. The desired adjoint block diagram is shown in figure 6. It can be seen that mean-square miss distance can also be evaluated with this same block diagram by merely changing the location of the input impulse. These alternatives are both shown in figure 6. As in the unrestricted problem, $\overline{\epsilon^2(T)}$, or mean-square miss, is obtained at the output by introducing an impulse at point A at $\tau = 0$. As for the control motion, however, we are interested in not only the end point, $t = T$, but all other times t_2 prior to $t = T$ as well. This quantity, $\overline{\delta^2(t_2)}$, can be obtained at the output by introducing the impulse $u_0[\tau-(T-t_2)]$ at point B (i.e., at $\tau = \tau_1 \equiv T-t_2$).

Impracticability of unrestricted solution.- It is now easy to show the difficulty with the unrestricted solution previously obtained. The constant-coefficient control system W_o^* given in equation (14) is split into aerodynamic and control system parts and inserted into the block diagram of figure 6. Because of the form of $C^*(s,t)$, an impulse introduced at point B would result in an infinite output of the homing system adjoint. This means that infinite missile accelerations would be called for[4] when this system operates in the presence of glint noise, and serious limiting would occur. This result is also obtained from time-varying W_o.

It can be shown that the infinite acceleration which would be required can be readily reduced. Since the called-for acceleration spectrum extends to somewhat higher frequencies than does the input spectrum, a certain amount of additional filtering can be added to the control system to result in finite accelerations without significantly altering the miss distance performance. However, in order to reduce the magnitude of acceleration to practical levels, so much filtering must be added that it becomes an appreciable part of the desired system filtering. The miss is thereby increased. The problem thus amounts to compromising between increasing the miss distance and decreasing the called-for acceleration, and is well suited to analytical attack. Such an analytical approach was taken in the beam-rider study of reference 1. Here the miss distance was minimized with a restriction on the available control motion. This restriction was chosen so that the probability of the control surface physically limiting is small and the system therefore operates essentially as a linear one. This is the approach which will be adopted in the remainder of this report. It should be noted in particular that other types of limiting which might exist in the guidance system are not considered in the theoretical formulation of the problem. This is because it is believed that acceleration limiting is predominant and that imposing a suitable restriction on this quantity will satisfactorily reduce the other types of limiting. The validity of this assumption will be treated in a later section.

Adjoint equations.- The problem we wish to solve can now be stated in terms of finding that control system transfer function, $C^*(s,t)$, which will minimize the mean-square miss distance, $\overline{\epsilon^2(T)}$, with a restriction on the mean-square ensemble average of the called-for control motion, $\overline{\delta^2(t_2)}$. From physical reasoning, it is clear that to obtain an optimum system, this mean-square control motion should be as large as possible at all times throughout flight; that is, it should remain constant at the maximum permissible value. Formulated in terms of the adjoint block diagram, this means that we want to find the system which will do both of the following:

[4]Strictly speaking infinite acceleration is called for only when the noise spectrum is white (i.e., $T_N = 0$). However, since T_N is generally quite small, the called-for acceleration is extremely large, essentially infinite for practical purposes.

1. When an impulse is introduced at point A at time $\tau = 0$, produce an output which is a minimum.

2. When an impulse is introduced at point B, produce an output of a specified magnitude independent of the time at which the impulse is introduced.

Mathematically this amounts to minimizing the quantity

$$\overline{\epsilon^2(T)} + \rho\overline{\delta^2(t_2)} \tag{16}$$

for all values of t_2, where ρ is a Lagrangian multiplier.

The expressions for both the mean-square error and the mean-square control motion can now be written in terms of the unknown control system $c(\tau,\xi)$. These equations are shown in appendix A to be:

Mean-square error

$$\overline{\epsilon^2(t_2)} = \int_0^\infty \left(\left\{\int_0^\infty h(\tau-x,\tau_1)h_N(\tau,\tau-x)dx\right\}^2 + \left\{\int_0^\infty [u_o(\tau-x-\tau_1)-h(\tau-x,\tau_1)]h_T(\tau,\tau-x)dx\right\}^2\right)d\tau \tag{17}$$

$$h(\tau,\tau_1) = \frac{k_a}{\tau}\iint c(\tau,\tau_1)d\tau\ d\tau - \frac{k_a}{\tau}\int_0^\tau h(\xi,\tau_1)\iint c(\tau,\xi)d\tau\ d\tau\ d\xi \tag{18}$$

Mean-square control motion

$$\overline{\delta^2(t_2)} = \int_0^\infty \left(\left\{\int_0^\infty g(\tau-x,\tau_1)h_N(\tau,\tau-x)dx\right\}^2 + \left\{\int_0^\infty [u_o(\tau-x-\tau_1)-g(\tau-x,\tau_1)]h_T(\tau,\tau-x)dx\right\}^2\right)d\tau \tag{19}$$

$$g(\tau,\tau_1) = \frac{c(\tau,\tau_1)}{\tau} - \frac{k_a}{\tau}\int_0^\tau g(\xi,\tau_1)\iint c(\tau,\xi)d\tau\ d\tau\ d\xi \tag{20}$$

As can be seen in equations (17) and (18) the mean-square error is related to the control system $c(\tau,\xi)$ through the intermediate response $h(\tau,\tau_1)$. Similarly in (19) and (20), the mean-square control motion is related to $c(\tau,\xi)$ by means of $g(\tau,\tau_1)$. Thus, it is clear that the error and control motion are related to the unknown control system $c(\tau,\xi)$ by means of the four equations above.

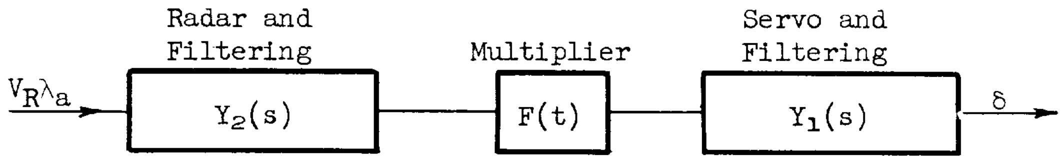
The optimization problem is now one of solving the above equations (17) through (20) for the optimum control system $c(\tau,\xi)$ so as to minimize the miss distance with the desired control motion restriction. These equations are so formidable that exact minimization appears impractical. Even when the problem is simplified to minimizing miss distance with a control motion restriction at the end of the attack only, no solution has been found. As a consequence of the difficulties involved, some approximate method must be found to synthesize the optimum control system.

Approximate solution approach.- It will now be necessary to review some of the physical reasoning which plays an important part in the optimization of the homing system. To begin with, the desirability of the missile following a CTB course has been emphasized previously. Such a course, however, can never be followed in practice because of the noise and the system time lags. Consequently, some rms miss is inevitable. When the missile has unlimited acceleration capabilities, it should be able to do the best possible job in approximating the CTB course and the minimum miss distance should result. This minimum has been shown in a previous section to be identical for both the beam-rider and homing systems. However, it is clear that any real missile with limited acceleration capabilities cannot achieve this ultimate performance because the noise produces an infinite called-for acceleration which the missile cannot achieve. Thus the CTB course cannot be followed as closely as with unlimited acceleration capability, and the minimum miss distance is increased. In this situation the smallest miss distance can be obtained by including in the optimization theory a restriction on the acceleration capability of the missile. By this procedure, it was established in the case of the beam-rider study that the miss distance is increased only a small amount and that the missile still flies quite close to the CTB course. Actually, miss distance is quite closely related to the relative displacements of the ends of the line of sight from the CTB reference, that is, the quantity ϵ which is y_T-y_M. For the beam-rider system these quantities are directly related so that minimizing miss distance is entirely equivalent to minimizing these line-of-sight deviations from CTB. For the homing system, however, the same equivalence is not valid. In this case the quantity to be minimized is the miss distance, but because of the kinematic time-varying range it is related in a complicated fashion to the deviations from CTB. Nevertheless, it is clear that small miss distance inherently requires a course not far from CTB; otherwise maneuvers of the target during the terminal phase could not be adequately corrected for by the missile and the miss distance would necessarily be increased. As a result of these arguments we would naturally expect that for the acceleration-limited homing system to achieve minimum miss distance, it would have to fly close to a CTB course. Following such a course does not appear to place any undue demands on the homing system. We would expect that as long as the missile has the same acceleration capability as in the beam-rider system, the homing missile should be capable of following the CTB course just as well as the beam-rider missile, and should therefore be able to obtain the same miss distance performance. Thus, as a good approximation it can be said that the performance of the optimum homing system with a realistic restriction on missile maneuverability is identical to that of the optimum constant-coefficient system. The verification of this approximation must be deferred to later sections (see p. 29). By means of this approximation, however, it is now possible to investigate the synthesis of the homing system.

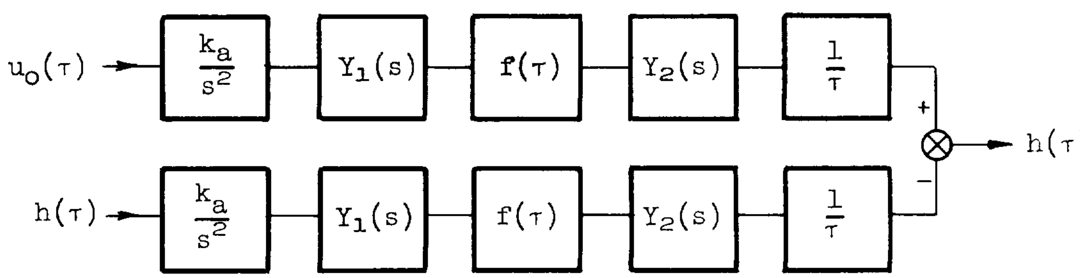
The three requirements which the homing system must satisfy can now be enumerated: (1) the miss distance performance should be the same as for the constant-coefficient beam-rider system, (2) the rms control motion should remain constant throughout flight, and (3) the control system should be a reasonable one, that is, one which can be readily mechanized without being overly complicated.

Let us consider how each of these three requirements can be satisfied. To satisfy the first requirement, that the miss distance performance be the same as for the constant-coefficient or beam-rider guidance system, it is clear from equation (17) that it is only necessary for the impulse response of the adjoint system $h(\tau)$ to be the same as $y(\tau)$ which is the impulse response for the optimum constant-coefficient beam-rider system. Then it can be seen that the control system $c(\tau,\xi)$ which gives the desired $h(\tau)$ can be found by solving the integral equation (18). It has not been found feasible to solve this equation directly. The difficulty appears to be in the manner of representing the control systems by a single impulse response $c(\tau,\xi)$. Since the impulse response of even a physically very simple time-varying system may be quite complicated, we would expect that the solution for the optimum control system would be even more unwieldy. In addition, there is no assurance that the solution would satisfy the third requirement, that is, that the system be a simple and reasonable one. Consequently, it has been necessary to find another representation.

A more suitable representation which has been found is based on splitting up the control system into several parts, each with its own impulse response. Since control systems employing multiplying elements are the most common and easily constructed, it is desirable to consider control systems of this type. A simple form is one which consists of two parts, a time-varying multiplying element and a non-time-varying transfer function. The question arises as to where the multiplier should be placed. The entire control system can be visualized as a single complicated filter constructed of a number of elements, among which are the radar at the input and a control-positioning servo at the output. Practically speaking, the multiplier cannot be located at the input of the system because at least some elements of the radar must precede it. Similarly, it is not practicable to place the multiplier at the system output. Thus, the only reasonable configuration for the real time control system is as shown in the following sketch where the multiplier is designated as $F(t)$ to indicate the time variation. The adjoint diagram of the complete homing system would then appear as in figure 7.



The above representation of the control system automatically satisfies the third requirement that the system not be too complex. Furthermore, as will now be shown, this representation of the control system enables the first requirement to be satisfied, that is, the desired control system to be synthesized so as to achieve the desired system impulse response. An equation which replaces equation (18) can now be derived - one that relates the response $h(\tau)$ not to $c(\tau,\xi)$ as in (18) but rather to the control system components as it has been split up. For this purpose, figure 7 for the adjoint homing system is redrawn. From this diagram, an



equation relating the homing system impulse response $h(\tau)$, and the control system components can be derived:

$$h(\tau) = \frac{1}{\tau} \; L^{-1}\left(Y_2(s)\, L\left\{f(\tau)\; L^{-1}\left[\frac{k_a Y_1(s)}{s^2}\right]\right\}\right) -$$

$$\frac{1}{\tau}\, L^{-1}\left(Y_2(s)\, L\left\{f(\tau) \int_0^{\tau} h(\tau-x)\left[\iint k_a y_1(x)\,dx\;dx\right]dx\right\}\right) \qquad (21)$$

Here the symbol L represents the usual Laplace transform operation.

Multiplying this equation by τ and taking the Laplace transform we get

$$-\frac{dH(s)}{ds} = Y_2(s)\,L\left\{f(\tau)\,L^{-1}\left[\frac{k_a Y_1(s)}{s^2}\right] - \right.$$

$$\left. f(\tau)\int_0^{\tau} h(\tau-x)\left[\iint k_a y_1(x)\,dx\;dx\right]dx\right\} \qquad (22)$$

Under certain conditions this equation can be solved; that is, for certain time-varying functions $f(\tau)$, this equation can be solved for the remainder of the control system $Y_1(s)$ and $Y_2(s)$. Thus, the desired impulse response $h(\tau)$ and therefore the desired miss distance performance can be achieved. Since there are a great many solutions of equation (22), a whole class of optimum time-varying homing systems can be generated, all of which have the desired miss distance performance.

The remaining requirement, satisfying the desired control-motion restriction, can be achieved as follows. Of all the homing systems which are solutions of equation (22), it is clear that each system will have different control motion requirements throughout flight. Thus, from this whole class of systems, the one or more having the desired restriction can be chosen. It would be expected that when the system with constant rms control requirements throughout the flight is chosen, the magnitude of rms control motion would be identical with that for the constant-coefficient system.

It is apparent that the success of the approach outlined is dependent on a suitable initial choice for the time-varying element $f(\tau)$. The types of time-varying elements which appear most promising are functions-of-range multipliers such as: (1) range multiplication, (2) a power of range multiplication, (3) exponential-type multipliers, or (4) combinations of these types. Other types of elements such as a variable time-constant filter do not appear as feasible in satisfying the three requirements for the homing system. It should also be pointed out that the non-time-varying control system can be regarded as a special case of the above systems.

Example situation.- In the following sections certain systems of interest will be examined in detail in terms of a particular example situation. This will be taken, for convenience of comparison, to be the same as that of the beam-rider example used in reference 1 except that idealized aerodynamics will be assumed here instead of the specific form of aerodynamics employed in reference 1.

The example to be used is based on a tail-chase situation; although this attack may seem somewhat trivial for the homing problem it will be shown later to be easily generalized to more interesting situations. The noise and target maneuvers are assumed to have spectral densities Φ_N and Φ_T as defined by equations (4) and (5), the specific numbers used being considered typical of what would be expected in an attack against a large bomber. The magnitudes of the bomber and missile velocities are taken to be $V_T = 880$ ft/sec and $V_M = 2640$ ft/sec, respectively, which gives a relative closing velocity magnitude of $V_R = 1760$ ft/sec. The ideal aerodynamics are represented by a control effectiveness constant, $1/T_d$, of 0.4792 radian/sec, giving an over-all aerodynamic gain $k_a = V_M/T_d$, of 1265 ft/sec^2 per radian.

For these conditions the optimum over-all transfer function $Y_o(s)$ has been determined using the method of reference 1 so as to restrict the rms control motion, $\sqrt{\overline{\delta^2}}$, to an arbitrary but typical value of 0.128 radian.[5] Because we are assuming idealized aerodynamics, this is equivalent to restricting the rms acceleration to 5 g's. The solution for the optimum transfer function is found to be of the form:

$$Y_o(s) = \frac{T_\alpha^2 s^2 + 2\zeta_\alpha T_\alpha s + 1}{(T_\beta s+1)(T_\gamma^2 s^2 + 2\zeta_\gamma T_\gamma s+1)(T_u^2 s^2 + 2\zeta_u T_u s+1)} \qquad (23)$$

where the constants are all more or less complicated funtions of the problem parameters. For the particular values of the parameters given above, there is obtained

$$Y_o(s) = \frac{1.282s^2+1.774s+1}{(0.6868s+1)(0.490s^2+0.714s+1)(0.0694s^2+0.3726s+1)} \qquad (24)$$

The impulse response of the system represented by this particular Y_o is plotted in figure 8 for reference purposes. It is of interest to note that this response approaches zero rather rapidly after a time interval of a few seconds. Thus, only these last few seconds of the attack need be considered in the analysis.

The miss distance performance for this constant-coefficient case has been found to be 21.4 feet, which represents the optimum performance for the given situation. Since the noise magnitude is one of the most important quantities and most subject to wide variations, the minimum miss distance performance for varying amounts of noise is given for the beam rider in figure 5 by the middle solid curve. The operating point for the example case is shown. From the discussion of the previous sections, the minimum miss distance performance of all optimum time-varying homing systems is indicated by the dotted curve.

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[5]The 0.128 radian restriction implies that the actual limits on control motion are enough greater than this value that limiting will occur infrequently enough for the system operation to be essentially linear.

Proportional Navigation

Constant-coefficient or time-invariant control systems are of particular interest in homing navigation because of their simplicity and widespread use. Proportional navigation control systems are principal examples. However, proportional navigation systems are not only time invariant but also have a particular form, namely that the steady-state relationship between $\dot{\lambda}_a$ and $\dot{\gamma}_d$ is a constant. (This is equivalent to a factorable s in the numerator of the control system transfer function C(s).) In the analysis which follows only the condition of time invariance will be imposed on the control system, with no restriction on the particular form. The synthesis procedure will then prescribe the optimum form of control, and it will be seen that this form is precisely that of proportional navigation. The merits of this type control will then be examined.

From the discussion of the previous section it is clear that the optimum homing system should have about the same performance as the optimum beam-rider system. However, with a constant-coefficient control system it is not expected that both the optimum miss distance and control motion characteristics can be achieved simultaneously. For this reason we defer considerations of control motion requirements for the moment and simply assume that the system is linear. Then in order to achieve the minimum miss distance, we desire to find the optimum constant-coefficient control system C(s) as shown in figure 6. This should be done so as to make the homing system impulse response $h(\tau)$, in equations (17) and (18), identical with the optimum beam rider, or constant-coefficient, impulse response $y_o(\tau)$, which is the inverse transform of (23). It is clear that in figure 7 a time-invariant control system is represented by $f(\tau) = 1$. Now the optimum control system can be found by solving equation (22). If $f(\tau) = 1$ and $h(\tau) = y_o(\tau)$ are substituted in equation (22),

$$-\frac{dY_o(s)}{ds} = Y_2(s)\,L\left\{L^{-1}\left[\frac{k_a Y_1(s)}{s^2}\right] - \int_0^{\tau} y_o(\tau-x)\left[\iint k_a y_1(x)\,dx\,dx\right]dx\right\} \qquad (25)$$

Carrying out these operations reduces this equation to

$$-\frac{dY_o(s)}{ds} = \frac{k_a Y_1(s) Y_2(s)}{s^2}\left[1-Y_o(s)\right] \qquad (26)$$

where, again, $Y_o(s)$ is the optimum beam-rider transfer function. Now from a comparison of figures 6 and 7, it is clear that the time-invariant control system $C_o(s) = Y_1(s)Y_2(s)$. Making this substitution in (26) we can then solve for the control system:

$$C_o(s) = - \frac{\frac{s^2}{k_a} \frac{d}{ds} [Y_o(s)]}{1-Y_o(s)} \qquad (27)$$

It is interesting to examine the general form for $C_o(s)$. By substituting equation (23) in (27) it is found that

$$C_o(s) = \frac{s^2}{k_a} \frac{2A(s)}{sB(s)} = \frac{2}{k_a} \frac{sA(s)}{B(s)} \qquad (28)$$

where $A(s)$ and $B(s)$ are polynomials ending with unity as the last term. For the particular example situation used in this report, evaluation of equation (28) gives

$$C_o(s) = 0.00158 \frac{s(3.52s+1)(0.243s^2+0.808s+1)(0.181s^2+0.678s+1)}{(2.5s+1)(0.49s^2+0.727s+1)(0.687s+1)(0.0694s^2+0.372s+1)(0.0322s^2+0.254s+1)}$$

(29)

The miss distance performance with this control system is, of course, the same as the beam-rider system, or 21.4 feet.

The interesting thing to be noted about $C_o(s)$ is that its form is that of a proportional navigation control system; that is, the transfer function from λ_a input to $\dot{\gamma}_d$ output contains (1) a factorable s which can be regarded as representing the differentiation of λ_a performed by the missile radar, (2) a number of lead and lag terms, and (3) a constant gain. The form of proportional navigation is obtained here without imposing any requirement other than that the control system be time invariant. It will be noted that the considerations involved here are quite different from those which are usually considered in homing studies. However, it should not be concluded on this basis that proportional navigation is optimum since acceleration requirements have not yet been considered.

It should be further noted that the method given here will always result in a guidance system obeying the proportional navigation law. This is because the constant-coefficient transfer function, Y_o, has coefficients which are not all independent. It can be shown that in equation (23)

$$2\zeta_\alpha T_\alpha = T_\beta + 2\zeta_\gamma T_\gamma + 2\zeta_u T_u$$

Physically, this means that the open-loop transfer function associated with Y_o always contains a $1/s^2$ factor which enables the missile to follow an accelerating target with a constant steady-state lag. As a consequence, in (27) the factor $[dY_o(s)/ds]/[1-Y_o(s)]$ will have a factorable s in its denominator, and therefore the homing control system derived from such Y_o's will automatically be of the proportional navigation type. Furthermore, this type of control was also obtained in the unrestricted homing missile problem, which was treated completely independently of the beam-rider problem. Thus, it is clear that the requirement of proportional navigation is dependent ultimately on the nature of the target maneuver; that is, the fact that the target executes a switching acceleration maneuver.

Now we will consider the magnitude of the acceleration or, equivalently, the control motions, demanded of the missile during its flight for the proportional navigation control system just derived. This consideration is important since the 21.4-foot miss distance will be achieved only if the assumption of no limiting is not violated. The control motion requirements can be obtained most conveniently from the computer. With the $C_o(s)$ of equation (29) inserted in figure 6, measurements were made using the procedures outlined previously. Results of such measurements are given in figure 9 where the rms δ is plotted as a function of range, or time to go. This curve might be called the δ characteristic of the system. Any point on the curve may be interpreted as the root-mean-square ensemble average of control motion at that particular range. The desired amount of control motion, which is identical to the beam-rider value, is also shown in figure 9 for comparison.

Examination of figure 9 reveals the following information: (1) The mean-square control motion is finite at all ranges, in contrast to what would be found in the unrestricted case, where the δ characteristic is infinite at all ranges. (2) The desired restriction on control motion (i.e., $\sqrt{\overline{\delta^2}} = 0.128$ radian) has not been achieved, since less than the design control motion is used at intermediate and long ranges while considerably more than the desired amount of control motion is called for at short ranges. In the actual operation of such a system it can be seen that the control motion would limit at the short ranges. Consequently, the minimum miss distance indicated at the operating point in figure 5 would not be achieved.

A compensating effect in the utilization of control motions is evident here. This arises from the fact that there is associated with the prescribed miss distance performance a certain average effort which the system must expend. The beam rider operates in the most efficient manner, using maximum available control motion uniformly during its flight. But the proportional navigation system utilizes less than the required effort for part of the time and in order to compensate must try to make up the deficiency by calling for more than the average effort near the end of flight. In order to make more efficient use of the control motion capabilities in the homing system, it is apparent that a time-varying control system is desired.

Range Multiplication Control

In the previous section it has been illustrated that a time-varying control system is required in order to satisfy the desired control restriction throughout flight. There are several forms of time-varying control which appear capable of satisfying this restriction as has been mentioned. However, since range multiplication is of wide interest and since we will see that it provides the approximate desired control restriction, the other forms of control previously suggested will not be examined here. It is important to note that the use of range multiplication type control does not automatically make the system optimum. The multiplier is only a small part of the whole system and the remainder of the system must be chosen to achieve the design objectives.

The complete range multiplication system is represented in figure 7 if we let $f(\tau) = \tau$. In this case the two filters $Y_1(s)$ and $Y_2(s)$ must be found so as to achieve minimum miss distance. To do this acceleration requirements are deferred for the moment, assuming the system to be linear, and the filters are chosen so as to give an impulse response $h(\tau)$ identical with the optimum impulse response of the constant-coefficient system $y_o(\tau)$. The solution can be obtained from equation (22). We have

$$-\frac{dY_o(s)}{ds} = Y_2(s)L\left\{\tau L^{-1}\left[\frac{k_aY_1(s)}{s^2}\right] - \tau\int_0^\tau y_o(\tau-x)\left[\iint k_a y_1(x)dx\ dx\right]dx\right\} \tag{30}$$

Carrying out these operations, we get

$$-\frac{dY_o(s)}{ds} = Y_2(s)\ \frac{d}{ds}\left[-\frac{k_aY_1(s)}{s^2} + \frac{k_aY_o(s)Y_1(s)}{s^2}\right]$$

or

$$\frac{dY_o(s)}{ds} = k_aY_2(s)\frac{d}{ds}\left\{\frac{Y_1(s)[1-Y_o(s)]}{s^2}\right\} \tag{31}$$

Now there are a great many control systems, that is, combinations of $Y_1(s)$ and $Y_2(s)$ which will satisfy equation (31). A unique solution is possible by choosing either $Y_1(s)$ or $Y_2(s)$ and determining the other. However, if the latter choice is made, it is clear that expression (31) will be a differential equation in $Y_1(s)$, the solution of which involves unusual functions for a control system - functions which at best would have to be approximated in some manner. On the other hand if $Y_1(s)$ is chosen, the solution for $Y_2(s)$ is readily obtained. From relation (31),

$$Y_2(s) = \frac{\frac{d}{ds}\left[Y_o(s)\right]}{\frac{d}{ds}\left\{\frac{k_a Y_1(s)}{s^2}\left[1-Y_o(s)\right]\right\}} \tag{32}$$

The choice for the servo $Y_1(s)$ is not completely arbitrary. For example, if the servo is chosen to be quite fast (ideally unity) and no filtering is provided in the amplifying system driving the servo, the required $Y_2(s)$, radar and associated filtering, is unstable. The considerations involved in choosing $Y_1(s)$ so as to result in a feasible system are outlined in appendix B. For the example attack situation, the simplest practical form for $Y_1(s)$ is found to be

$$Y_1(s) = \frac{bs+1}{as+1} = \frac{1.05s+1}{2.53s+1} \tag{33}$$

Application of equation (32) using equations (24) and (33) then gives the required radar and associated filtering, Y_2, as

$$Y_2(s) = 0.00338\,\frac{s(2.50s+1)(0.243s^2+0.808s+1)(0.181s^2+0.678s+1)}{(2.53s+1)(0.446s^2+1.20s+1)(0.110s^2+0.476s+1)(0.0195s^2+0.196s+1)} \tag{34}$$

The form of this transfer function is particularly interesting. It is seen here that the radar and filtering, Y_2, contains a factorable s in the numerator which can be considered to correspond to the differentiation performed by the radar. In this respect the situation is similar to that of the proportional navigation system discussed earlier. The method used here will <u>always</u> produce a Y_2 with a factorable s. It can be shown, again, from the equations presented in appendix B that this result is due ultimately to the fact that the target executes a switching acceleration maneuver.

Let us now examine the performance of this system. First of all it is clear that the miss distance is identical with that for the optimum constant-coefficient system (beam-rider). This value, as indicated by the operating point in figure 5, is 21.4 feet. As for the control motion requirements figure 9 illustrates the δ characteristic for the range multiplication system. The difference between this curve and that of the proportional navigation system is striking. It can be seen that the demands on the missile are now essentially uniform throughout flight, and at the desired design level which is also indicated in figure 9. The control curve has not only been flattened but its curvature has actually been reversed, so that there is a little too much restriction at short ranges and not quite enough at the intermediate and long ranges. The compensating effect of control motion at different times during the flight is again evident here just as in the proportional navigation case. The ideal system would be one in which the control motion requirements are constant at the design value throughout the flight. Thus it is clear that the range multiplier system is not quite an optimum form of control and that possibly one of the other multiplying functions would be better. However, the range multiplier control is so close to optimum that little improvement could be expected. This is because during the portion of flight where the δ characteristic exceeds the design value, it does so only slightly. The amount of limiting would be so small that the assumption of linearity would still be valid and the expected miss distance would be achieved.

It is possible, if so desired, to modify the range multiplier slightly to obtain a perfectly flat δ characteristic as given by the dotted curve in figure 9. Such a modification is shown in figure 10, and it is seen that this altered multiplier differs only slightly from the original. The miss distance performance for the modified system is unaltered as a consequence of the compensating control motion effect described earlier.

It should be pointed out that in practice it is not necessary to adhere to the range multiplication scheme at long ranges. The reason is that the operation of the system at times long before the collision point has little effect on miss distance even when the controls are used inefficiently. This is because, as discussed earlier and indicated in figure 8, the impulse response dies out rather rapidly so that inputs which occur before the collision point by an amount greater than the duration of the impulse response have no effect on the miss. Thus the δ characteristic need only be flat for the last few seconds of flight and may fall off at longer ranges.

The results obtained for this system indicate very clearly the following points: (1) the requirement for an optimum system, that the rms control motion be of a specified constant magnitude at all ranges, is satisfied; (2) the requirement for an optimum system, that the line-of-sight rotation throughout flight be small, is satisfied; although small line-of-sight rotations have been merely an assumption throughout the analysis, verification of this assumption will be given in the next section; and (3) the miss distance is not much greater than the absolute minimum obtainable with unlimited maneuverability. Thus it is apparent that the system synthesized represents a good approximation to the homing guidance optimum, and the principal objectives of optimization have been achieved.

It is now possible to show that the analogy between the beam-rider and homing problems, already established for the unrestricted case extends equally well to the restricted case. We have already concluded (p. 19) that we would expect the same over-all performance from these two systems when control motion limitations are present. Now in this section we can see the proof of this, the two systems having not only the same miss distance but also the same maneuverability requirements throughout the flight. This is illustrated in figure 9, where the control requirements for the beam-rider system are shown for comparison with those of the homing system. As a result of this equivalence between systems, we may utilize certain miss distance studies made previously for the beam-rider system. For example, the results of reference 2 may be used to relate minimum miss distance to all of the factors which place an inherent limitation on its minimum value. Such factors are the target acceleration and average switching rate of acceleration, radar noise magnitude, and missile acceleration capabilities. This relationship is as follows:

$$\sqrt{\overline{\epsilon^2}} = 9.20 + 5.02 \frac{Na_T}{a_M} + \frac{1.34}{(32.2)^2} a_T^2 + \frac{7.94}{\overline{l}} \qquad (35)$$

A discussion of these effects is given in reference 2.

ADDITIONAL CONSIDERATIONS

It is desirable at this point to discuss certain aspects of the problem not previously considered. This discussion is divided into several sections, in the first of which the initial assumptions made in the analysis will be considered. In the second section, the effect of blind range which was neglected in the original statement of the problem is evaluated. The last section contains a discussion on the effect of attack situations other than the tail chase used in the example case.

Initial Assumptions

It will be recalled that the assumption which has been used in order to linearize the homing system geometry is that the deviations of the missile flight path from a straight-line CTB course are small. Thus the perturbation angles, λ_a (deviation in the apparent line-of-sight angle) and γ_d (deviation in the missile heading from the straight-line CTB course) are assumed small. The validity of this assumption has been verified by actual measurement on the analog computer. It must be noted, however, that λ_a varies with range, being very small at the larger ranges and increasing to a theoretically infinite value at zero range. Nevertheless, the rms λ_a is found to be sufficiently small over almost all of the flight, the small angle assumption thus being violated only at such short ranges that the effect on miss distance is negligible.

Another assumption which has been made is that control deflection is the most critical saturating quantity, in the sense that a restriction applied to this quantity would automatically prevent limiting of other quantities of interest. Thus the system is assumed to be linear, and computer measurements have verified the reasonableness of this assumption. Of particular interest is the called-for rate of control motion because it is related to the servo power. The rms control motion rate was found to be about 2 radians/sec; since the rate limit of present missile servos is several times this value, rate limiting is certainly inconsequential. Similarly, the called-for voltage levels in various parts of the control system were measured and found to be easily attainable.

Blind Range

Since we have assumed in this report that the blind range of the radar is zero, the results are, strictly speaking, valid only for semiactive radars. (Most radars are of this type.) For active radars, however, this assumption is only an approximation. To evaluate the effects of neglecting blind range, measurements of miss distance for the range multiplication system were made by means of the analog computer. The blind range was simulated by the opening of a switch at the appropriate location in the radar. The results are presented in figure 11. For a representative blind range figure of 300 feet it can be seen that blind range effects are nearly negligible. Even for larger values of blind range miss distance increases only a few feet. Since the miss distance of a system designed specifically for a large blind range would also increase from the zero blind range case, it is concluded that the zero blind range optimum system is essentially optimum even for a fairly large blind range.

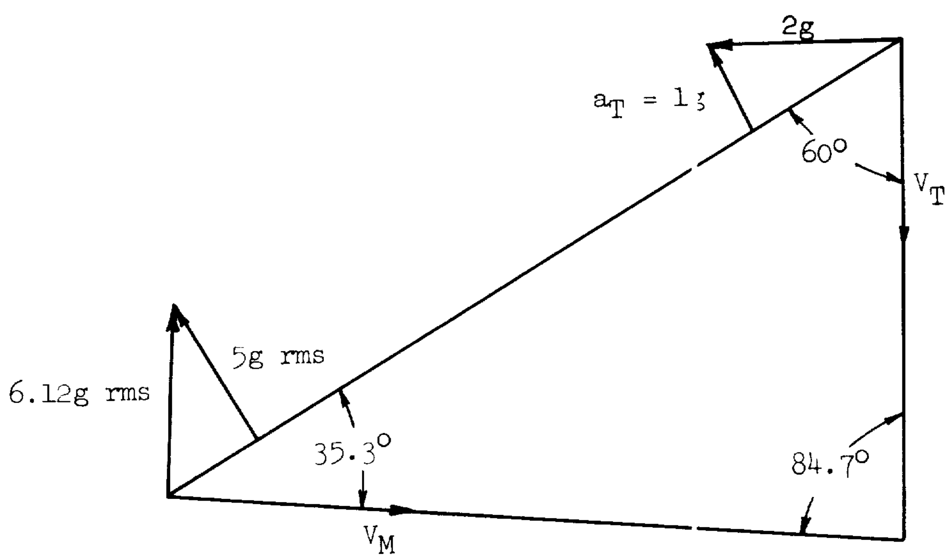
Other Attack Situations

Since the optimization of the homing system given in this report has been illustrated by means of a particular example situation, it is of considerable interest to examine the application to other attack situations.

In the first place, the tail chase which has been used here is quite common with present-day missiles. This is because so many considerations enter into the problem that the possible attack aspects are severely limited to something approximating the tail chase. From geometrical considerations it can be seen that the optimum system is particularly insensitive to changes in attack aspect from the true tail chase. Even as much as 30° off the tail is essentially the same as a tail chase.

In the second place, if the exact solution is desired for some attack aspect other than the tail chase it can be readily obtained by the methods already presented; that is, (1) specify the problem parameters - attack aspect, target acceleration and noise spectrum for that aspect, and the missile-to-target speed ratio; (2) compute the CTB course and determine the components of available missile and target acceleration perpendicular to the line of sight; (3) determine the equivalent optimum constant-coefficient (beam-rider) system, $Y_o(s)$; and (4) determine the optimum range-multiplication control system for the homing missile. It will be seen that different numerical values of the parameters of the optimum control system result for almost every different situation. Fortunately, however, it is found that the parameters of the optimum control system are not especially sensitive to changes in the problem parameters. It is for this reason that designing the system for some sort of average of the situations expected to be encountered would result in near optimum performance for the whole range of conditions without altering the system.

The third comment regarding attack aspect is that any given solution applies exactly to many different situations if the problem parameters are reinterpreted. In other words, given a solution, one can work backward and find a whole series of problems for which the given solution is valid. To illustrate this idea for the previous tail-chase solution, let us consider the attack shown in the following sketch. The attack is nearly



from the beam, and the missile-to-target speed advantage is 1.5 to 1. If the missile speed, V_M, is assumed to be 2640 ft/sec as in the report example, the target speed, V_T, would be 1760 ft/sec. It can be seen that to have the same a_T for which the solution is valid (i.e., ±1g), the target must now be assumed to maneuver with ±2g acceleration. Similarly the missile must have a 6.12g rms capability in order that the component of acceleration perpendicular to the line of sight will be 5g rms as in the example solution. On the other hand, if the missile has only a 5g capability, the same solution holds if all quantities are scaled down appropriately (i.e., target acceleration, ±1.63 g's; magnitude of noise spectrum, N, 10.0 ft^2/radian/sec; and miss distance, 17.5 ft).

CONCLUDING REMARKS

The synthesis method presented in this report is an approximate method - one that has been shown to yield a homing system with near optimum performance. As is typical of such methods, there are a number of ways to realize the near-optimum system. One of these, utilizing range multiplication between the radar and control actuator servo, has been given. Possibly other types of multipliers might prove to have advantages. It is also possible that with further work an exact solution could be found. Besides these possibilities, there are other factors which might influence the specific system design. For example, in situations where launching errors are important it may be desirable to make modifications in order to reduce launching error in a minimum time. In other situations, modifications may be necessary because of requirements of simplicity on certain parts of the system or the necessity of using certain fixed and unalterable elements in the system. Other alterations would be required in order to supply artificial damping to the missile. Since there are many ways in which optimum performance can be achieved, it should be possible to include such requirements without sacrificing the performance. However, an investigation of such factors is beyond the scope of this report.

It seems appropriate to point out that the example given in this report is for a specific attack situation with particular numerical constants. Thus, there is no indication of what effect changes in the problem parameters will have on the over-all system performance. For example one might be interested in the effect of target acceleration and noise on the minimum obtainable miss distance, or in how far to go in building greater maneuverability into the missile. It would appear such effects could only be evaluated by carrying through the procedure for each attack situation. Fortunately, however, the results of reference 2 can be used for this purpose because of the equivalence between beam rider and homing guidance.

One of the significant by-products of this study has been the development of the equivalence between the beam-rider and homing guidance problems. As we have seen, both systems attempt to solve the same basic problem although in a different fashion. There are other - apparently different - systems which attempt to solve the same problem such as the automatic interceptor, or antimissile missiles. These problems might well be cast in a similar mold. In fact generalization to include wide classes of weapons systems would be a valuable objective of future work.

The most urgent extension of the present work would appear to be to the critical high-altitude problem. Under conditions of higher speeds and altitudes of potential targets, the acceleration advantage of the missile over the target is lost, and the present theory is inadequate. The reason is that the problem becomes a nonlinear one. Extensions of the theory should be made to nonlinear aspects of the problem so as to include such conditions.

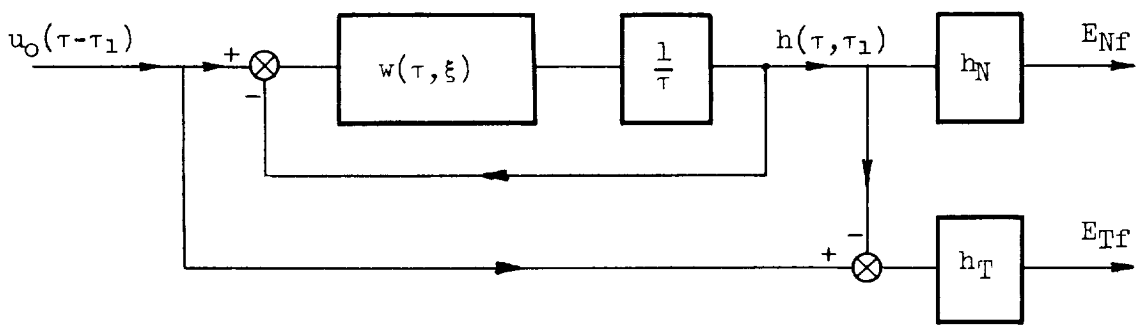
Ames Research Center
National Aeronautics and Space Administration
Moffett Field, Calif., Nov. 13, 1958

APPENDIX A

DETERMINATION OF THE OPTIMUM HOMING CONTROL SYSTEM

No Restrictions

In this section optimization of the unrestricted homing guidance system will be considered. Reference should be made to the text in the report for a discussion of the problem. As a starting point here, it will be desirable to redraw figure 4 in the following form:



In response to the impulse introduced into the input, the output consists of two parts shown above as $E_{Nf}(\tau,\tau_1)$ and $E_{Tf}(\tau,\tau_1)$. These quantities are the filtered noise and target error signals. Since they are uncorrelated, the error is composed of only these two parts, one due to noise and the other due to target maneuver. If we identify each of the signals E_{Nf} and E_{Tf} with the h in equation (8), the expression for mean-square error can be seen to be:

$$\widetilde{\epsilon^2(t_2)} = \int_0^\infty [E_{Nf}^2(\tau,\tau_1) + E_{Tf}^2(\tau,\tau_1)]d\tau \tag{A1}$$

The assumptions involved in writing this expression are given in the text proper. From the above figure it can be seen that each of the output signals is given by

$$E_{Nf}(\tau,\tau_1) = \int_{-\infty}^{\tau} h(\xi,\tau_1)h_N(\tau,\xi)d\xi \tag{A2}$$

$$E_{Tf}(\tau,\tau_1) = \int_{-\infty}^{\tau} [u_o(\xi-\tau_1)-h(\xi,\tau_1)]h_T(\tau,\xi)d\xi \tag{A3}$$

By changing to a new variable $x = \tau-\xi$, we get

$$E_{Nf}(\tau,\tau_1) = \int_0^\infty h(\tau-x,\tau_1)h_N(\tau,\tau-x)\,dx \qquad \text{(A4)}$$

$$E_{Tf}(\tau,\tau_1) = \int_0^\infty [u_o(\tau-x-\tau_1)-h(\tau-x,\tau_1)]h_T(\tau,\tau-x)\,dx \qquad \text{(A5)}$$

Now equation (A1) can be written as

$$\overline{\epsilon^2(t_2)} = \int_0^\infty\Bigg(\left\{\int_0^\infty h(\tau-x,\tau_1)h_N(\tau,\tau-x)\,dx\right\}^2 + \left\{\int_0^\infty [u_o(\tau-x-\tau_1)-h(\tau-x,\tau_1)]h_T(\tau,\tau-x)\,dx\right\}^2\Bigg)\,d\tau \qquad \text{(A6)}$$

Equation (A6) is a general form which can also be used when control-motion restriction is considered; the only difference would be a reinterpretation of the symbols h and ϵ. In the present case, however, we are concerned only with minimizing miss distance which is the value of error at time T. Also the noise and target motion are assumed to be stationary. Thus the miss distance becomes

$$\overline{\epsilon^2(T)} = \int_0^\infty\Bigg(\left\{\int_0^\infty h(\tau-x)h_N(x)\,dx\right\}^2 + \left\{\int_0^\infty [u_o(\tau-x)-h(\tau-x)]h_T(x)\,dx\right\}^2\Bigg)\,d\tau \qquad \text{(A7)}$$

By expanding (A7), we obtain

$$\overline{\epsilon^2(T)} = \int_0^\infty\int_0^\infty\int_0^\infty\Big\{h_N(x)h_N(y)h(\tau-x)h(\tau-y) + h_T(x)h_T(y)[u_o(\tau-x)u_o(\tau-y) - u_o(\tau-x)h(\tau-y) - u_o(\tau-y)h(\tau-x) + h(\tau-x)h(\tau-y)]\Big\}\,dx\,dy\,d\tau \qquad \text{(A8)}$$

The particular h which minimizes the miss distance can now be found by variational methods. One considers $h_o(\tau)$ to be the desired optimum $h(\tau)$, and introduces a variation $h_1(\tau)$; that is, $h(\tau) = h_o(\tau) + \eta h_1(\tau)$, where η is a real parameter. After substitution of this relationship in equation (A8), the necessary condition for a minimum of (A8) is that $\lim_{\eta \to 0} \partial \epsilon^2(T)/\partial \eta = 0$. By carrying out this calculation, it can be shown that

$$\int_0^\infty \int_0^\infty \int_0^\infty \Big\{ h_N(x)h_N(y)[h_1(\tau-x)h_o(\tau-y) + h_1(\tau-y)h_o(\tau-x)] - h_T(x)h_T(y)[u_o(\tau-x)h_1(\tau-y) + u_o(\tau-y)h_1(\tau-x) - h_o(\tau-y)h_1(\tau-x) - h_o(\tau-x)h_1(\tau-y)] \Big\} \, dx \, dy \, d\tau = 0 \tag{A9}$$

From the symmetry in the above expression, (A9) can be reduced to

$$\int_0^\infty h_1(\tau-x)\Big[\int_0^\infty\int_0^\infty h_N(x)h_N(y)h_o(\tau-y)dy\,dx - \int_0^\infty\int_0^\infty h_T(x)h_T(y)u_o(\tau-y)dy\,dx + \int_0^\infty\int_0^\infty h_T(x)h_T(y)h_o(\tau-y)dy\,dx\Big]d\tau = 0 \tag{A10}$$

Since $h_1(\tau-x)$ is arbitrary and is zero for $\tau < x$, the minimizing condition is

$$\int_0^\infty\int_0^\infty [h_N(x)h_N(y)h_o(\tau-y) - h_T(x)h_T(y)u_o(\tau-y) + h_T(x)h_T(y)h_o(\tau-y)]dy\,dx = m(\tau-x) \tag{A11}$$

where $m(\tau-x) = 0$ for $\tau > x$. The optimum, h_o, can be found by taking the Laplace transformation of equation (A11) to give

$$\int_0^\infty\int_0^\infty\int_0^\infty \Big\{h_N(x)h_N(y)h_o[(\tau-x) + x-y] - h_T(x)h_T(y)u_o[(\tau-x) + x-y] +$$

$$h_T(x)h_T(y)h_o[(\tau-x) + x-y]\Big\} e^{sx}e^{-sy}e^{-s[(\tau-x)+x-y]}dy\,dx\,d(\tau-x) = M(s) \tag{A12}$$

where $M(s)$ has no poles in the left half-plane. Now by interchanging the order of integration one can reduce equation (A12) to the following:

$$M(s) = \int_0^\infty h_N(x)e^{sx}dx\int_0^\infty h_N(y)e^{-sy}dy\int_0^\infty h_o[(\tau-x)+x-y]e^{-s[(\tau-x)+x-y]}d[(\tau-x)+x-y] -$$

$$\int_0^\infty h_T(x)e^{sx}dx\int_0^\infty h_T(y)e^{-sy}dy\int_0^\infty u_o[(\tau-x)+x-y]e^{-s[(\tau-x)+x-y]}d[(\tau-x)+x-y] +$$

$$\int_0^\infty h_T(x)e^{sx}dx\int_0^\infty h_T(y)e^{-sy}dy\int_0^\infty h_o[(\tau-x)+x-y]\, e^{-s[(\tau-x)+x-y]}d[(\tau-x)+x-y] \tag{A13}$$

Thus

$$M(s) = H_N(-s)H_N(s)H_o(s) - H_T(-s)H_T(s) + H_T(-s)H_T(s)H_o(s) \tag{A14}$$

Since $H_N(-s)H_N(s) = \Phi_N(s)$ and $H_T(-s)H_T(s) = \Phi_T(s)$,

$$M(s) = \Phi_N(s)H_o(s) - \Phi_T(s) + \Phi_T(s)H_o(s) \tag{A15}$$

This equation is the transform of the Wiener-Hopf equation and the solution (e.g., see ref. 4) is

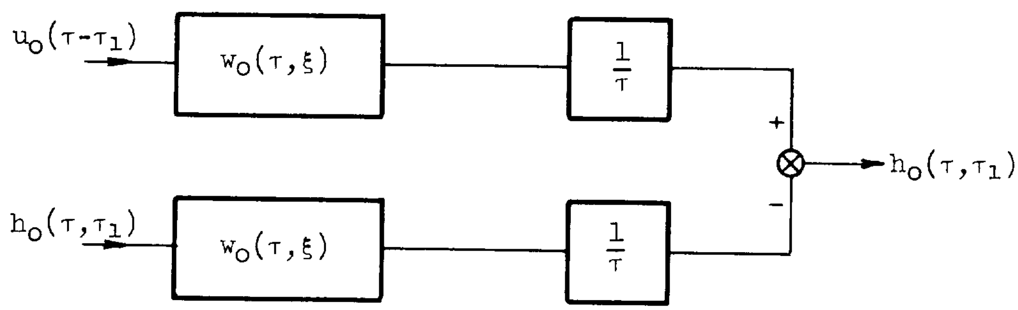
$$H_o(\omega) = \frac{1}{2\pi\Phi^+(\omega)}\int_0^\infty e^{-i\omega t}\int_{-\infty}^\infty \frac{\Phi_T(\alpha)e^{i\alpha t}}{\Phi^-(\alpha)}\,d\alpha\,dt \tag{A16}$$

where

$$\Phi = \Phi_T + \Phi_N = \Phi^+\Phi^-$$

The functions Φ^+ and Φ^- are, of course, the factors of Φ with poles and zeros in the upper and lower half-planes, respectively.

It is now possible to determine the control system $W(s,t)$ so as to achieve the desired impulse response $h_o(\tau)$ just found. Redrawing the figure at the beginning of the appendix A as



it can be seen that the output response $h_o(\tau,\tau_1)$ is related to the control system $w_o(\tau,\xi)$ by the integral equation

$$h_o(\tau,\tau_1) = \frac{w_o(\tau,\tau_1)}{\tau} - \frac{1}{\tau}\int_0^{\tau} w_o(\tau,\xi)h_o(\xi,\tau_1)d\xi \qquad \text{(A17)}$$

As in preceding paragraphs we may let $\tau_1 = 0$ to obtain

$$\tau h_o(\tau) = w_o(\tau) - \int_0^{\tau} w_o(\tau,\xi)h_o(\xi)d\xi$$

If we were concerned with the analysis problem (i.e., the determination of the output $h(\tau)$ for a given control system $w(\tau,\xi)$), the standard methods of integral equation theory would be applicable. Here, however, we are interested in the synthesis problem - the synthesis of $w_o(\tau,\xi)$ to achieve a desired output $h_o(\tau)$. It has not been found feasible to solve this equation directly except in the special case when the control system is non-time varying. The reason seems to be related to the mode representing the control system. Even time-varying systems which are physically simple are quite complicated to represent in the form of an impulse response. Synthesizing the control system by solving equation (A17) seems even more difficult. Nevertheless, a method which exploits simpler representations of the control system will be presented in the text. It will be seen from the discussion there that there are a great many solutions of equation (A17). These are of interest primarily when the control restriction problem is considered. For present purposes, let us consider only the simplest of these control systems - one with constant coefficients.

Then equation (A17) can be written as

$$\tau h_o(\tau) = w_o(\tau) - \int_0^\tau w_o(\tau-\xi)h_o(\xi)d\xi \qquad \text{(A18)}$$

This equation is a special Volterra integral equation of the second kind, and can be solved by taking the Laplace transform of equation (A18) to obtain

$$W_o(s) = - \frac{\frac{d}{ds}[H_o(s)]}{1-H_o(s)} \qquad \text{(A19)}$$

With Control Motion Restriction

In order to minimize the mean-square error with a restriction on control motion, it is necessary to express these quantities in terms of the unknown control system. For this purpose, as explained in the text, the control system must be represented as shown in figure 6. Note that we are now concerned with the control system $c(\tau,\xi)$, which does not include the aerodynamics, rather than $w(\tau,\xi)$ as in the preceding discussion. From a comparison of figures 4 and 6 it is clear that these responses are related as follows

$$w(\tau,\xi) = \iint k_a c(\tau,\xi)d\tau\, d\tau \qquad \text{(A20)}$$

where the constants of integration will be assumed zero.

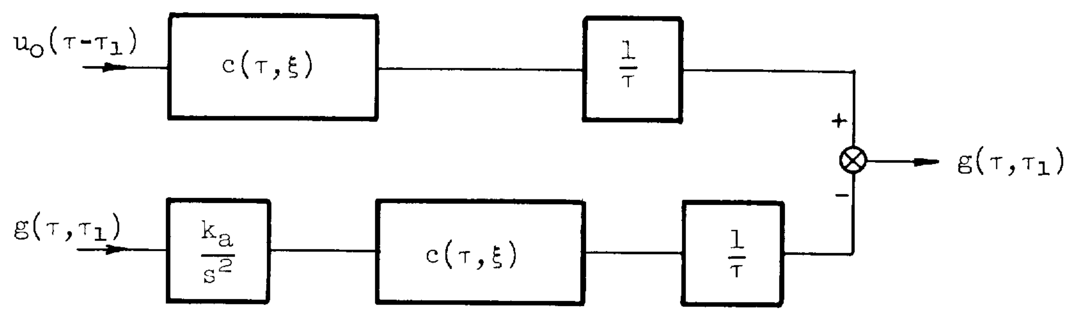
From figure 6 two equations for miss distance can be written. In fact, the first of these equations is identical with (A6). The second, an integral equation, is obtained from (A17) by eliminating $w(\tau,\xi)$ by means of (A20). For mean-square error:

$$\overline{\epsilon^2(t_2)} = \int_0^\infty\left(\left\{\int_0^\infty h(\tau-x,\tau_1)h_N(\tau,\tau-x)dx\right\}^2 + \right.$$

$$\left.\left\{\int_0^\infty [u_o(\tau-x-\tau_1)-h(\tau-x,\tau_1)]h_T(\tau,\tau-x)dx\right\}^2\right)d\tau \qquad \text{(A21)}$$

$$h(\tau,\tau_1) = \frac{k_a}{T}\iint c(\tau,\tau_1)d\tau\, d\tau - \frac{k_a}{T}\int_0^{\tau} h(\xi,\tau_1)\left[\iint c(\tau,\xi)d\tau\, d\tau\right] d\xi \tag{A22}$$

Two similar equations for the mean-square control motion may also be written in terms of the control system $c(\tau,\xi)$. The derivation of the first of these equations would follow along the same lines as for (A6). However, the error ϵ should be replaced by δ and the response $h(\tau,\tau_1)$ by its corresponding response $g(\tau,\tau_1)$. The response $g(\tau,\tau_1)$ represents, of course, the output of the homing system adjoint of figure 6. Thus, the first equation is given by (A23) below. For the second equation, we can redraw figure 6 as follows



From this sketch the response $g(\tau,\tau_1)$ and the control system $c(\tau,\xi)$ can be seen to be related by the expression given in (A24). For mean-square control motion:

$$\widetilde{\delta^2(t_2)} = \int_0^{\infty}\left(\left\{\int_0^{\infty} g(\tau-x,\tau_1)h_N(\tau,\tau-x)dx\right\}^2 + \left\{\int_0^{\infty}[u_o(\tau-x-\tau_1)-g(\tau-x,\tau_1)]h_T(\tau,\tau-x)dx\right\}^2\right) d\tau \tag{A23}$$

$$g(\tau,\tau_1) = \frac{c(\tau,\tau_1)}{T} - \frac{k_a}{T}\int_0^{\tau} g(\xi,\tau_1)\left[\iint c(\tau,\xi)d\tau\, d\tau\right]d\xi \tag{A24}$$

APPENDIX B

DETERMINATION OF THE OPTIMUM RANGE MULTIPLICATION HOMING CONTROL SYSTEM WITH CONTROL MOTION RESTRICTION

It will be the purpose of this appendix to indicate the factors which govern the design of the control system. As pointed out in the text, the most feasible design is obtained by expressing $Y_2(s)$, the radar and filtering, in terms of $Y_1(s)$, the servo and filtering. The solution has been shown in equation (32) to be

$$Y_2(s) = \frac{\frac{d}{ds} Y_o(s)}{\frac{d}{ds}\left\{\frac{k_a Y_1(s)}{s^2}\left[1-Y_o(s)\right]\right\}} \qquad \text{(B1)}$$

By expanding (B1) we obtain

$$Y_2(s) = \frac{\frac{s^2}{k_a}}{\left[\frac{dY_1(s)}{ds} - \frac{2Y_1(s)}{s}\right]\left[\frac{1-Y_o(s)}{dY_o(s)/ds}\right] - Y_1(s)} \qquad \text{(B2)}$$

Now we can represent $Y_1(s)$ as $C(s)/D(s)$. Also $-[dY_o(s)/ds]/[1-Y_o(s)]$ can be represented by the form given in equation (28) (i.e., $2A(s)/sB(s)$). Note that $A(s)$, $B(s)$, $C(s)$, and $D(s)$ are assumed to be polynomials which end with unity as the last term. With these substitutions equation (B2) becomes

$$Y_2(s) = \frac{1}{k_a}\,\frac{s^2D^2(s)A(s)}{-\frac{sB(s)}{2}\left[D(s)\frac{dC(s)}{ds} - C(s)\frac{dD(s)}{ds}\right] + B(s)C(s)D(s) - C(s)A(s)D(s)} \qquad \text{(B3)}$$

Now if we can take the simplest case of $Y_1(s) = 1$, (B3) reduces to

$$Y_2(s) = \frac{1}{k_a}\left[\frac{s^2A(s)}{B(s)-A(s)}\right] \qquad \text{(B4)}$$

For the specific $A(s)$ and $B(s)$ applicable to the example attack situation (see eqs. (28) and (29)) it can be shown that the denominator polynomial Y_2 has some negative coefficients. Thus an unstable Y_2 would be required.

The next simplest case is that for which $Y_1(s) = 1/D(s)$ where $D(s) = as+1$. Equation (B3) then reduces to

$$Y_2(s) = \frac{2}{k_a}\left\{\frac{s^2D^2(s)A(s)}{as[3B(s)-2A(s)] + 2[B(s)-A(s)]}\right\} \tag{B5}$$

This equation would appear to be satisfactory. In the first place although the same factor $B(s)-A(s)$ occurs as in the previous case, $Y_2(s)$ will be stable here providing the time constant of the servo, a, is large enough. Secondly, it will be noted that only one factorable s would occur in the numerator and would represent the differentiation performed by the radar. Thus this $Y_2(s)$ would appear to be satisfactory. However, it is found that another consideration - the signal/noise ratio at the radar output - would appear to become a critical factor with this system. For this reason no further consideration will be given to this system. Furthermore, the design objectives can be accomplished in another way.

The next case to be considered is one in which $Y_1(s) = bs+1/as+1$. It is easy to show that the general form of $Y_2(s)$ will be

$$Y_2(s) = k_1 \frac{s(\alpha_7 s^7 + \alpha_6 s^6 + \dots + 1)}{(\beta_9 s^9 + \beta_8 s^8 + \dots + 1)} \tag{B6}$$

where k_1 is a constant. The high order of this system can be reduced by properly choosing the numerical values of $Y_1(s)$. From (B3) it is seen that if $D(s)$ is chosen to be a factor of $B(s)$, that is, $B(s) = D(s)\,B_1(s)$, the order of $Y_2(s)$ in equation (B6) would be reduced by one in both numerator and denominator. Further, $C(s)$ can be chosen so that a factor of

$$-\frac{s}{2}\left[D(s)\,\frac{dC(s)}{ds} - C(s)\,\frac{dD(s)}{ds}\right] + C(s)D(s)$$

is the same as a factor of $A(s)$. That is

$$-\frac{s}{2}\left[D(s)\,\frac{dC(s)}{ds} - C(s)\,\frac{dD(s)}{ds}\right] + C(s)D(s) = G(s)F(s)$$

and $A(s) = A_1(s)F(s)$. The order of $Y_2(s)$ is reduced again by one. The result of both these reductions is to give a $Y_2(s)$ of the form

$$Y_2(s) = k_2 \frac{s(\gamma_5 s^5 + \gamma_4 s^4 + \dots + 1)}{(\lambda_7 s^7 + \lambda_6 s^6 + \dots + 1)} \tag{B7}$$

where k_2 is a constant. Following this procedure in numerical terms, a suitable $Y_1(s)$ is found to be

$$Y_1(s) = \frac{bs + 1}{as + 1} = \frac{1.05s + 1}{2.5s + 1} \tag{B8}$$

and the accompanying $Y_2(s)$ is

$$Y_2(s) = 0.00338 \frac{s(2.50s+1)(0.243s^2+0.808s+1)(0.181s^2+0.678s+1)}{(2.53s+1)(0.446s^2+1.20s+1)(0.110s^2+0.476s+1)(0.0195s^2+0.196s+1)} \tag{B9}$$

REFERENCES

1. Stewart, Elwood C.: Application of Statistical Theory to Beam-Rider Guidance in the Presence of Noise. II - Modified Wiener Filter Theory. NACA TN 4278, 1958.

2. Stewart, Elwood C., Druding, Frank, and Nishiura, Togo: The Effects of Target and Missile Characteristics on Theoretical Minimum Miss Distance for a Beam-Rider Guidance System in the Presence of Noise. NASA MEMO 2-12-59A, 1959.

3. Lawson, James L., and Uhlenbeck, George E., eds.: Threshold Signals. McGraw-Hill Book Company, Inc., 1950.

4. Laning, J. Halcombe, Jr., and Battin, Richard H.: Random Processes in Automatic Control. McGraw-Hill Book Company, Inc., 1956.

5. Stewart, Elwood C.: Application of Statistical Theory to Beam-Rider Guidance in the Presence of Noise. I - Wiener Filter Theory. NACA RM A55E11, 1955.

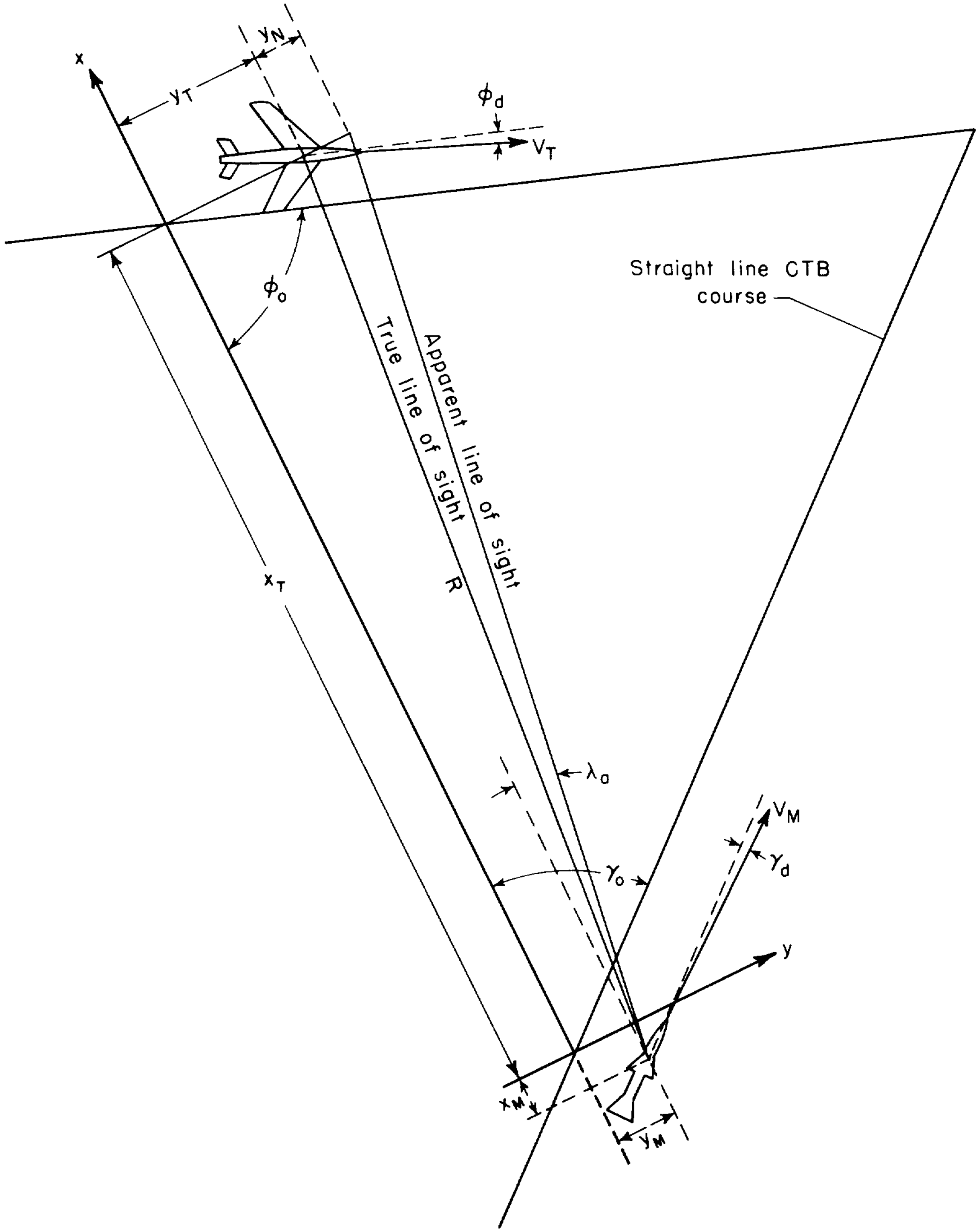


Figure 1.- Geometry of the coplanar homing problem.

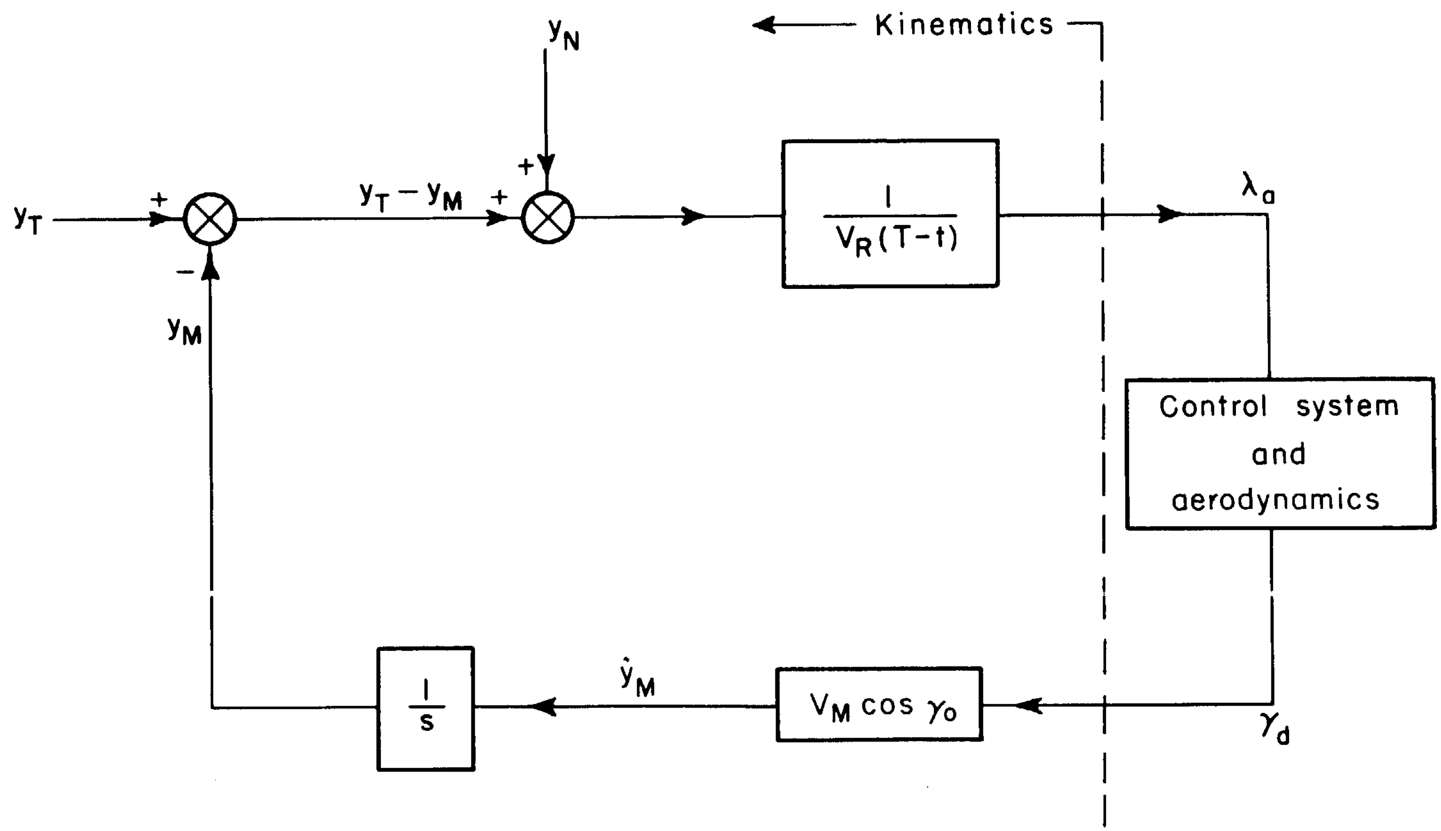


Figure 2.- Block diagram of homing system.

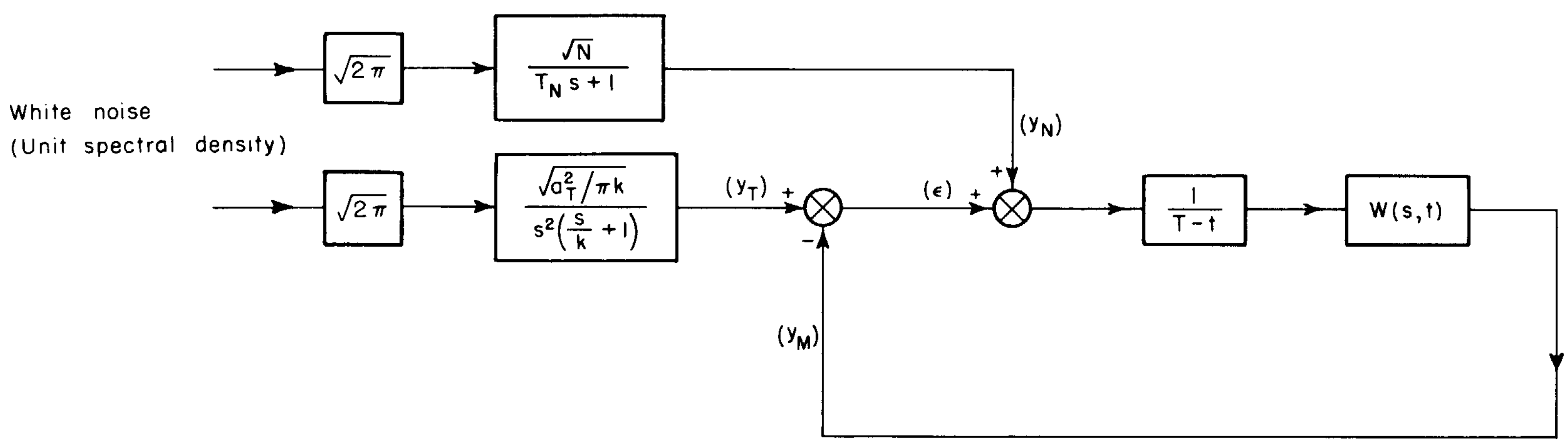


Figure 3.- Block diagram of simplified homing system.

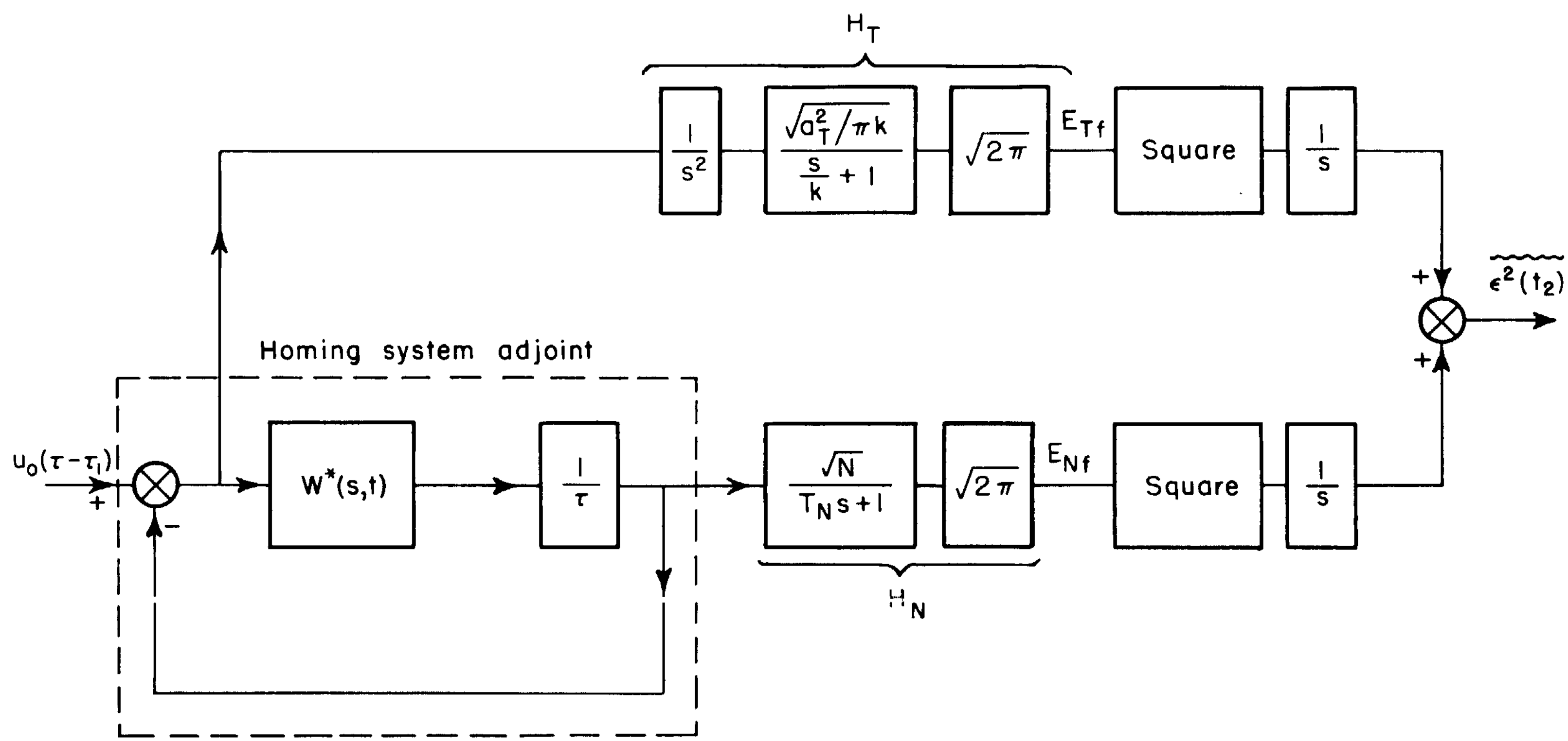


Figure 4.- Adjoint block diagram of simplified homing system; control motion restriction not considered.

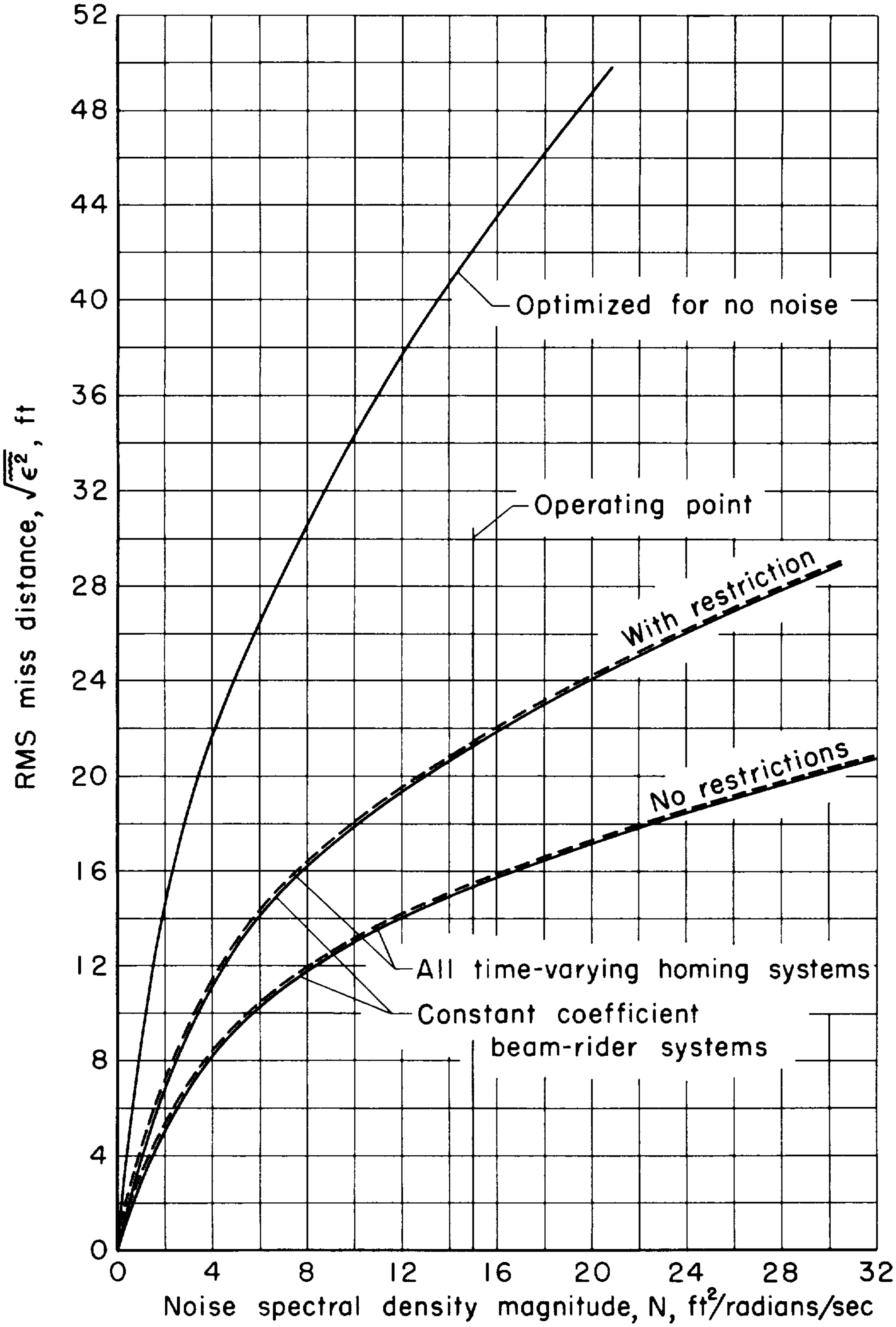


Figure 5.- Minimum miss distance performance.

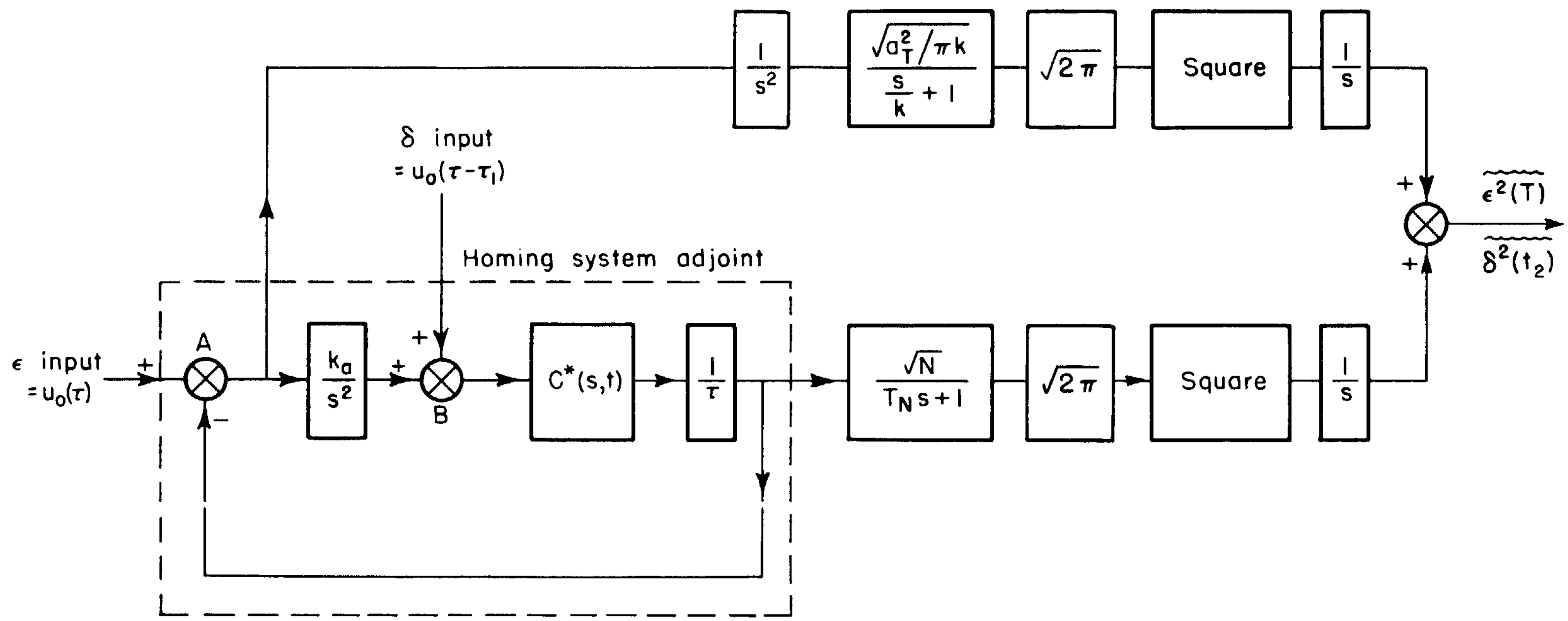


Figure 6.- Adjoint block diagram of complete homing system; control motion restriction considered.

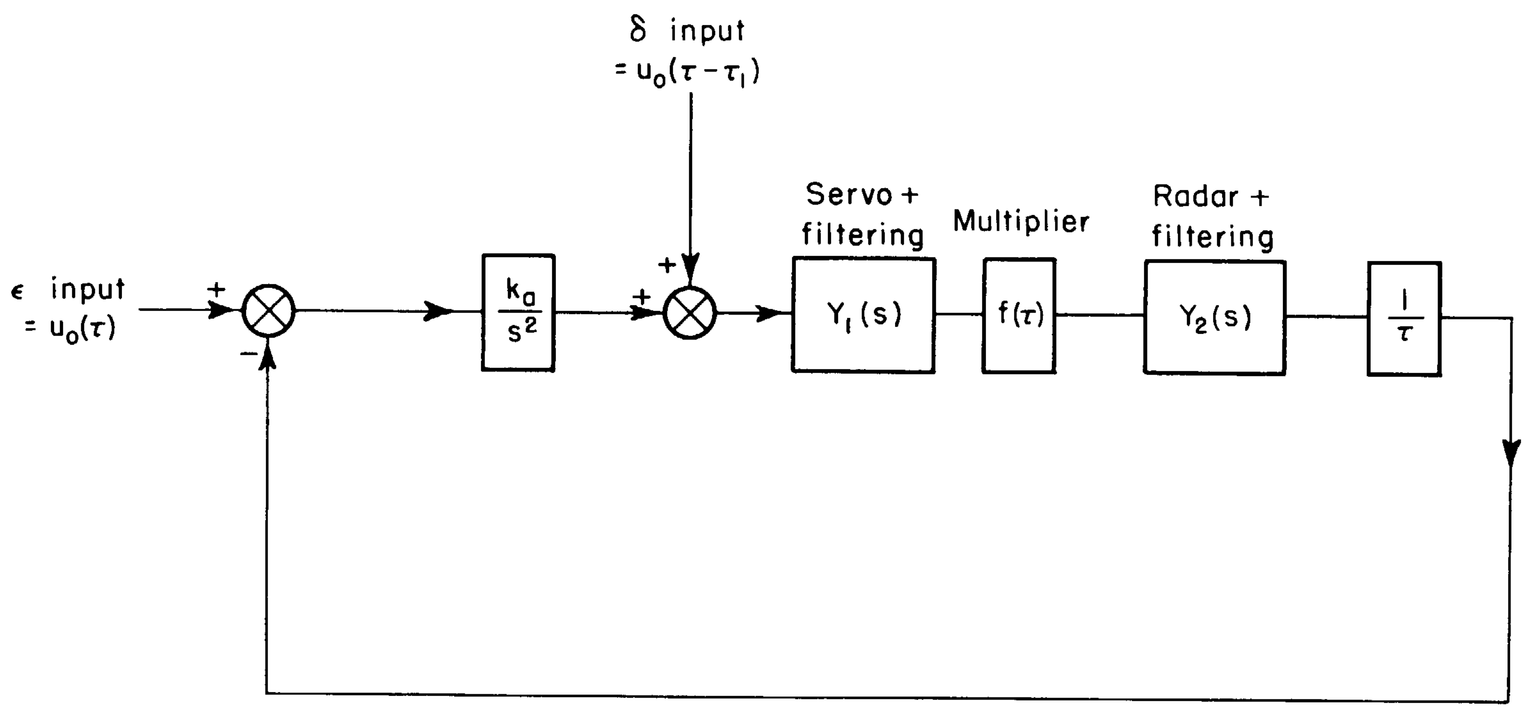


Figure 7.- Adjoint block diagram of multiplication-control homing system.

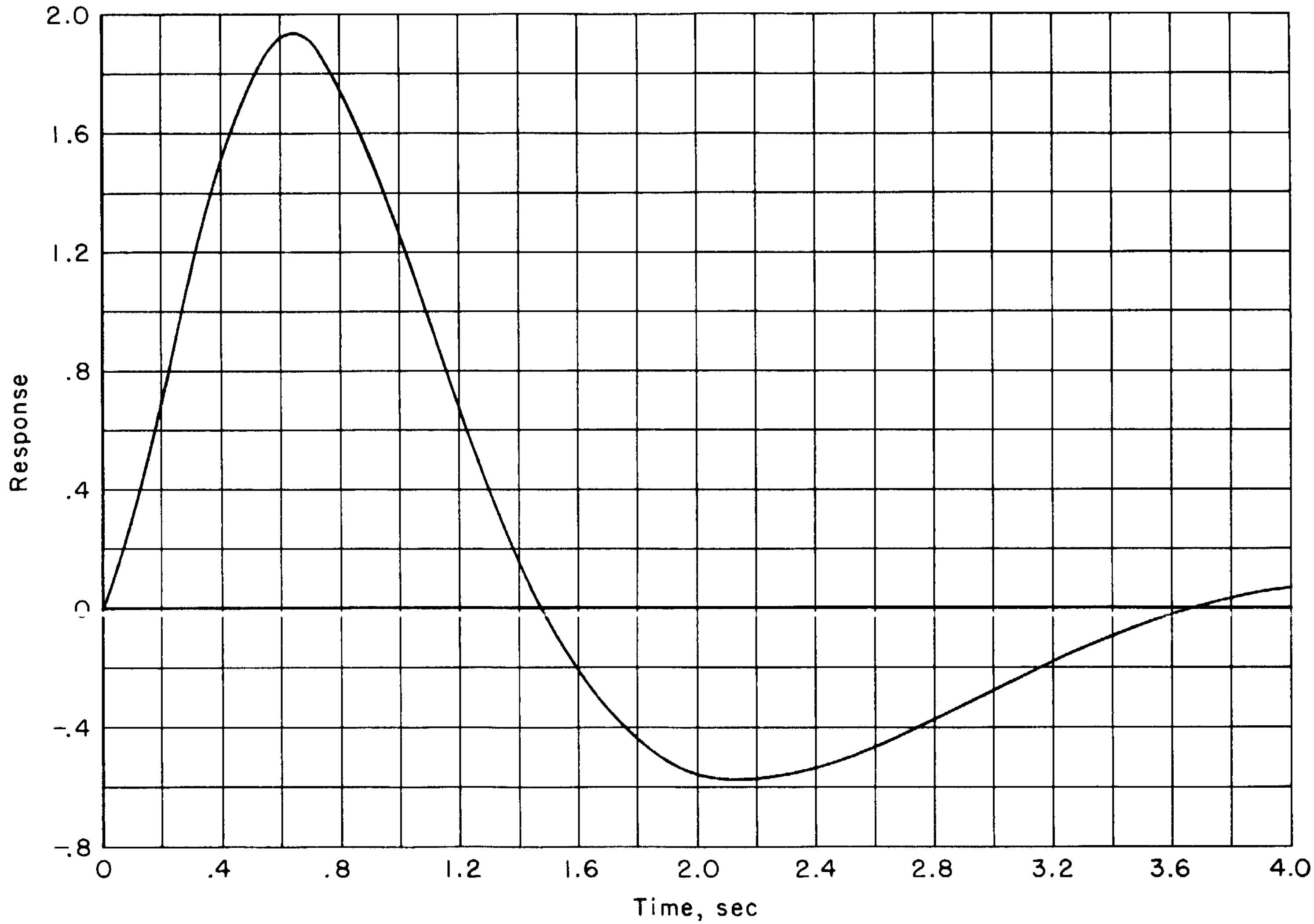


Figure 8.- Optimum unit impulse response for example case.

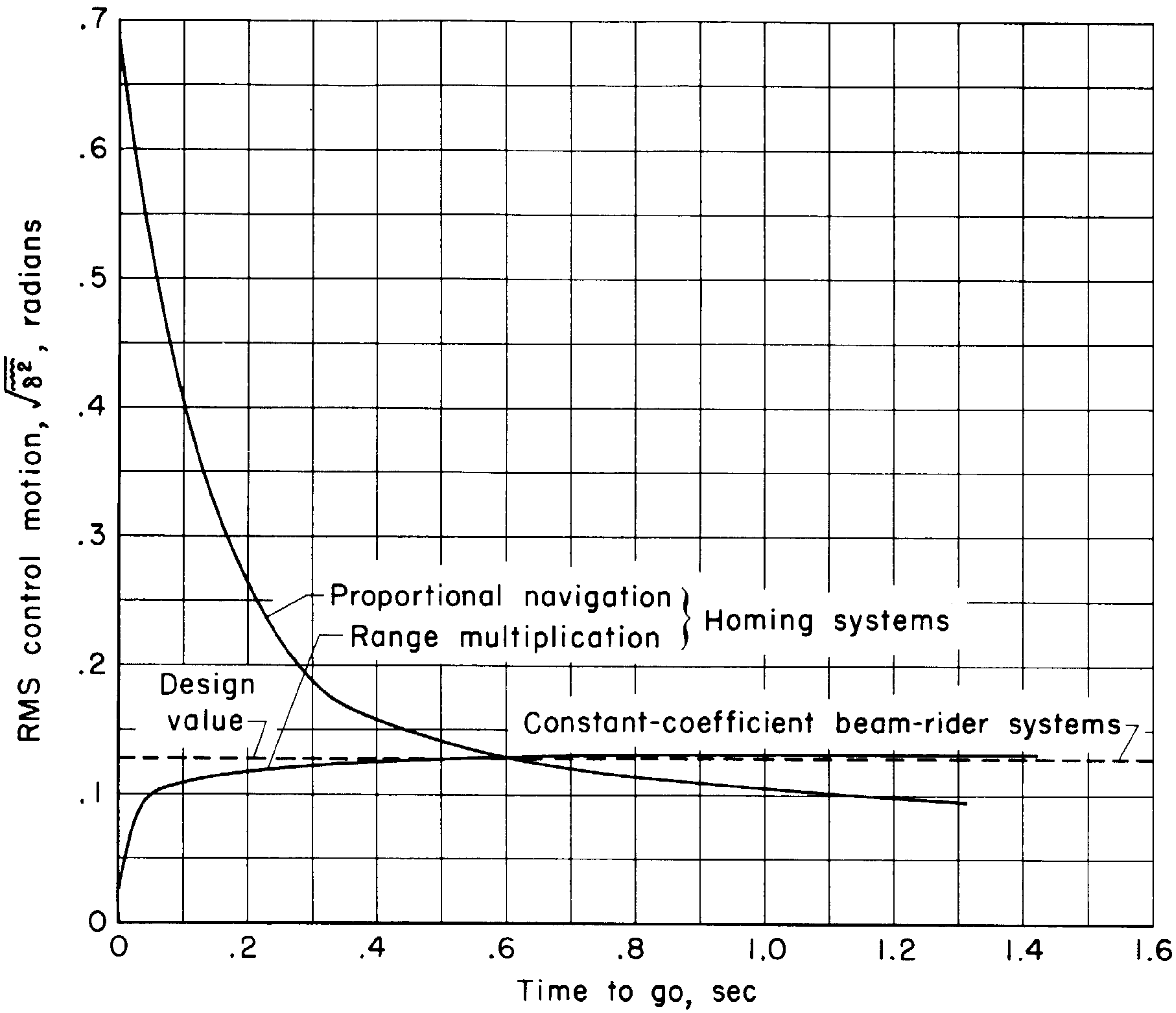


Figure 9.- Control motion requirements for homing systems.

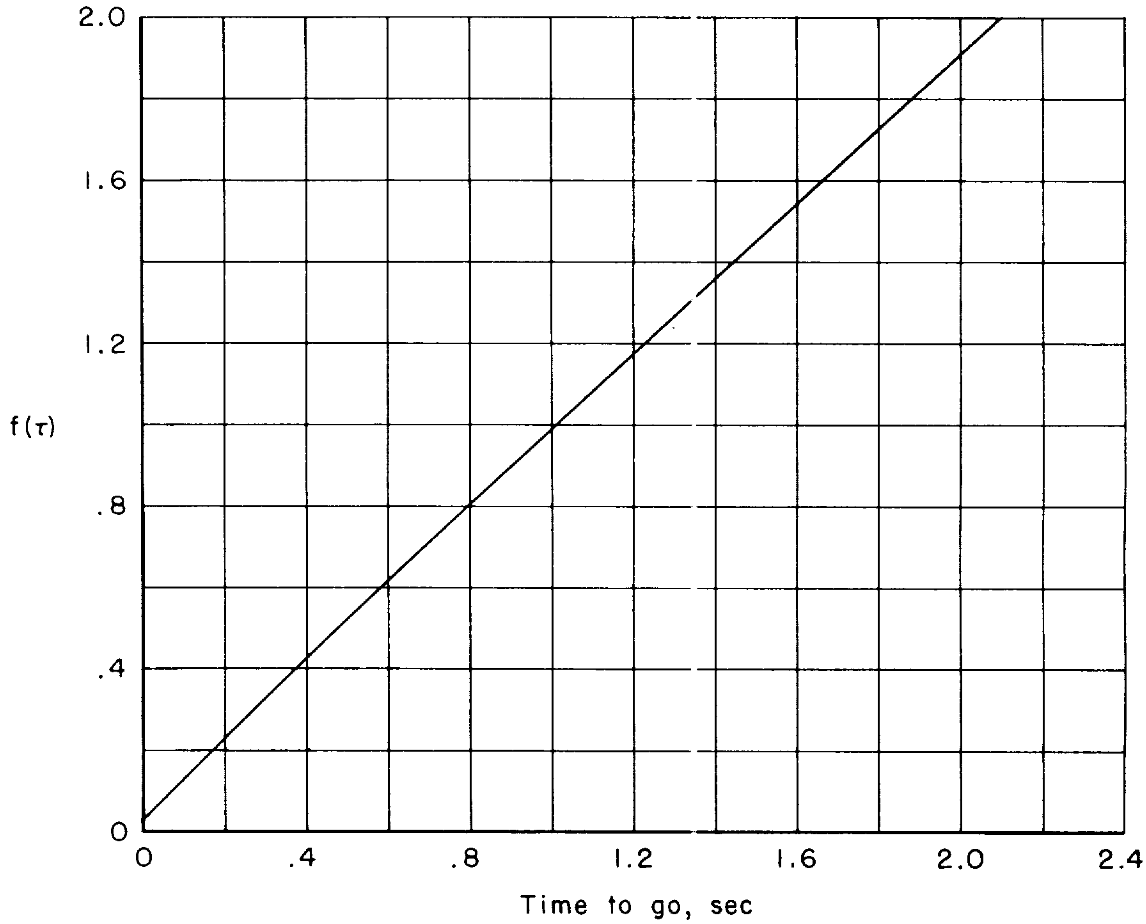


Figure 10.- Modified range multiplier.

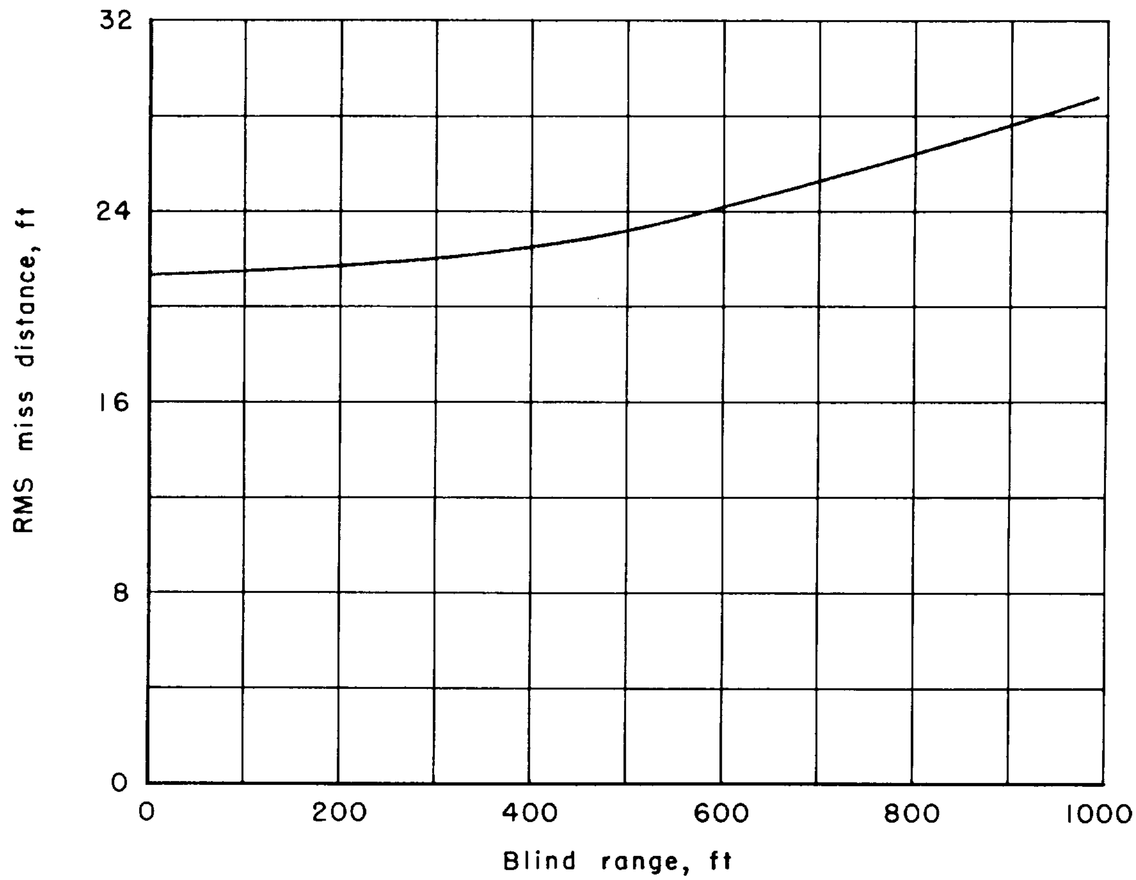


Figure 11.- Effect of blind range on miss distance for example case.