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ON FULLY DEVELOPED CHANNEL FLOWS: SOME SOLUTIONS AND LIMITATIONS, AND EFFECTS OF COMPRESSIBILITY, VARIABLE PROPERTIES, AND BODY FORCES [1]

By Stephen H. Maslen

SUMMARY

An examination of the effects of compressibility, variable properties, and body forces on fully developed laminar flows has indicated several limitations on such streams.

In the absence of a pressure gradient, but presence of a body force (e.g., gravity), an exact fully developed gas flow results. For a liquid this follows also for the case of a constant streamwise pressure gradient. These motions are exact in the sense of a Couette flow. In the liquid case two solutions (not a new result) can occur for the same boundary conditions. An approximate analytic solution was found which agrees closely with machine calculations.

In the case of approximately exact flows, it turns out that for large temperature variations across the channel the effects of convection (due to, say, a wall temperature gradient) and frictional heating must be negligible. In such a case the energy and momentum equations are separated, and the solutions are readily obtained. If the temperature variations are small, then both convection effects and frictional heating can consistently be considered. This case becomes the constant-property incompressible case (or quasi-incompressible case for free-convection flows) considered by many authors.

Finally, there is a brief discussion of cases wherein streamwise variations of all quantities are allowed but only in such form that the independent variables are separable. For the case where the streamwise velocity varies inversely as the square root of distance along the channel, a solution is given.

INTRODUCTION

Among all possible fluid flows, one of the most useful is the fully developed (i.e., independent of streamwise distance) channel flow. The flow is taken to be the motion generated by a constant pressure gradient (the familiar Poiseuille flow) or by a body force (ref. 1). In either case one usually considers only an incompressible or quasi-incompressible flow with fixed fluid properties. This is in marked contrast to the case of Couette flow where two parallel walls move with respect to each other. In such a case there is no need to limit oneself to a perfect gas or to any particular variation of the transport properties (ref. 2 is a case in point).

The crucial difference between the Couette and the Poiseuille flows is that the former admits a stream wherein nothing depends on the streamwise distance, while the latter requires that the pressure vary in the flow direction. Hence, to some small degree, at least, the other fluid properties will also vary in that direction if the state equation involves the pressure. The present study is an examination of the general circumstances under which there can be a fully developed laminar flow past fixed boundaries.

One special problem considered is an unusual situation found by Ostrach (refs. 1 and 3 to 6). He discusses the flow of a fluid in a two-dimensional channel under the influence of gravity. Incompressible flow is assumed except as is required to generate a varying body force, and the fluid transport properties are assumed not to vary. Under these assumptions, the surprising result is found that there are two solutions to the flow in question for a certain range of values of the flow parameters. The first corresponds roughly to the neglect of frictional heating, while the other is

[1] Supersedes NACA Technical Note 4319 by Stephen H. Maslen, 1958.

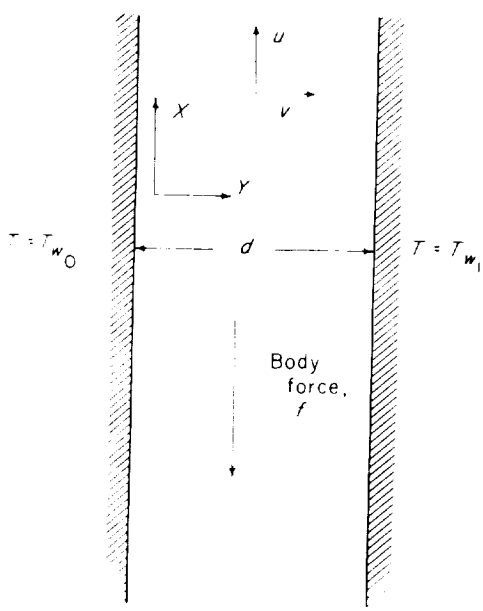
near the (nontrivial) solution for homogeneous boundary conditions.

There may be some doubt as to the stability of one of the solutions, presumably the second one. In any case, a question arises about the effect of considering a real fluid having variable viscosity and thermal conductivity as well as being truly compressible.

Accordingly, the present paper treats the consequences of such generalizations. However, to reiterate, one serious restriction is made on all the flows considered herein: The flow is always fully developed, with the result that the effects of conditions near either end of the channel are ignored.

ANALYSIS

Consider a two-dimensional flow of a viscous compressible fluid acting under the influence of an axial body force such as gravity. Variable viscosity and thermal conductivity are admitted. The configuration is shown in the following sketch:



The equations of motion are

$$(\rho u)_X+(\rho v)_Y=0 \tag{1}$$

$$\rho(uu_X+vu_Y)+P_X=-\rho f+2(\mu u_X)_X+[\mu(u_Y+v_X)]_Y-\frac{2}{3}[\mu(u_X+v_Y)]_X \tag{2}$$

$$\rho(uv_X+vv_Y)+P_Y=2(\mu v_Y)_Y+[\mu(u_Y+v_X)]_X-\frac{2}{3}[\mu(u_X+v_Y)]_Y \tag{3}$$

$$\rho c_v(uT_X+vT_Y)+P(u_X+v_Y)=(kT_X)_X+(kT_Y)_Y+\mu[2u_X^2+u_Y^2+v_X^2+2v_Y^2+2u_Yv_X-\frac{2}{3}(u_X+v_Y)^2] \tag{4}$$

(Symbols are defined in appendix A.) Consider, in addition, two possible forms of a state equation, one applying for a gas, and the other for a liquid:

$$P=\rho RT \qquad \text{(Gas)} \tag{5a}$$

$$\rho=\bar{\rho}[1-\bar{\beta}(T-\bar{T})] \qquad \text{(Liquid)} \tag{5b}$$

In equation (5b), $\bar{\beta}$ is the (small) volumetric expansion coefficient, and $\bar{\rho}$ and $\bar{T}$ are reference values. The significant difference between the two state equations is that the second is independent of pressure.

The boundary conditions on the channel walls are

$$\left.\begin{aligned} u(X,0)=u(X,d)=v(X,0)=v(X,d)=0 \\ T(X,0)=T_{w_0}(X) \\ T(X,d)=T_{w_1}(X) \end{aligned}\right\} \tag{6}$$

where d is the distance between the channel walls. The temperature boundary conditions could, of course, be replaced entirely or in part by a heat-transfer condition but, for the purpose of this report, such a change is unimportant.

Equations (1) to (6) are sufficient to define the fully developed flow in a channel provided the viscosity and conductivity variations with temperature are known, and provided further that the forced-flow pressure gradient, if any, is specified. In seeking solutions of these equations for flows in a very long channel (i.e., fully developed), three approaches are considered: first, exact solutions entirely independent of distance along the channel (X); second, solutions approximately independent of X; and third, solutions wherein the variables are separable. In each case the results can be expected to differ according to which of the state equations applies. In this connection, it is important to observe that the viscosity and thermal conductivity vary differently in liquids and gases. In particular, the viscosity rises with temperature for a gas and falls for a liquid.

The phrase "exact solution" should perhaps be defined. In this report it is understood to mean a solution which satisfies equations (1) to (6) rigorously. However, no consideration is given to conditions near the ends of the channel. There are two relatively simple circumstances under which such exact solutions can be found. For a gas nothing can vary with X, not even the pressure. For a liquid this restriction is moderated to the extent that only the gradient of the pressure need be independent of X. This relaxed condition occurs because of the pressure-independent state equation (eq. (5b)) for a liquid.

EXACT FLOW OF GAS

If nothing depends on X, equation (1) to (5a) and (6) become

$$v=0 \tag{7a}$$

$$(\mu u_Y)_Y=\rho f \tag{7b}$$

$$P_Y=0 \tag{7c}$$

$$(kT_Y)_Y=-\mu u_Y^2 \tag{7d}$$

$$P=\rho RT \tag{7e}$$

$$\left.\begin{aligned} u(0)=u(d)&=0\\ T(0)&=T_{w_0}\\ T(d)&=T_{w_1}\end{aligned}\right\} \tag{8}$$

Assume that the viscosity and thermal conductivity vary as powers of temperature, and also, for convenience, change the independent variable (Y).

Thus, suppose

$$\left.\begin{aligned}\mu&=aT^{n_1}\\ k&=bT^{n_2}\end{aligned}\right\} \tag{9}$$

and let

$$\left.\begin{aligned}\eta&=-1+\frac{2\int_0^Y \frac{\mathrm{d}Y}{\mu/\mu_{w_0}}}{B\equiv\int_0^d \frac{\mathrm{d}Y}{\mu/\mu_{w_0}}}\\ \text{or}\quad \frac{2Y}{B}&=\int_{-1}^{\eta}\frac{\mu\mathrm{d}\eta}{\mu_{w_0}}\end{aligned}\right\} \tag{10}$$

where the constant B is as yet undetermined. Then equations (7) become

$$\left.\begin{aligned} u_{\eta\eta}&=\left(\frac{aB^2fP}{4R\mu_{w_0}^2}\right)T^{n_1-1}\\ P&=\text{Constant}\\ \left(\frac{k}{\mu}T_\eta\right)_\eta&=-u_\eta^2\end{aligned}\right\} \tag{11}$$

If the viscosity varies linearly with temperature ($n_1=1$), then these equations are separated. If n_2 is also unity, the solution is

$$u=\left(\frac{aB^2fP}{8R\mu_{w_0}^2}\right)(\eta^2-1) \tag{12}$$

$$T=\frac{T_{w_0}+T_{w_1}}{2}+\eta\left(\frac{T_{w_1}-T_{w_0}}{2}\right)-\frac{a}{b}\left(\frac{aB^2Pf}{4R\mu_{w_0}^2}\right)^2\left(\frac{\eta^4-1}{12}\right) \tag{13}$$

Finally, B can be found from the second form of equation (10). Thus,

$$\frac{2\mathrm{d}T_{w_0}}{B}=\int_{-1}^{+1}T\mathrm{d}\eta=(T_{w_0}+T_{w_1})+\frac{2}{15}\frac{a}{b}\left(\frac{aB^2Pf}{4R\mu_0^2}\right)^2$$

or

$$\frac{1}{240}\left(\frac{BT_m}{\mathrm{d}T_{w_0}}\right)^5\left[\left(\frac{\rho_m^2fd^3}{k_m\mu_m}\frac{fd}{T_m}\right)\right]+\left(\frac{BT_m}{\mathrm{d}T_{w_0}}\right)=1 \tag{14}$$

where

$$T_m=\frac{T_{w_0}+T_{w_1}}{2} \tag{14a}$$

The quantity in brackets is essentially the parameter K defined in reference 1. This is always positive. Under these circumstances equation (13) has only one real root, that root being such that

$$0<\frac{BT_m}{\mathrm{d}T_{w_0}}<1$$

For example, for air under standard conditions, if $d=3$ and f is gravitational acceleration, equation (14) yields $BT_m/\mathrm{d}T_{w_0}=0.904$. A convenient standard for comparison of the present solution with more approximate results is given by the mass flow. This is

$$\int_0^d \rho u\mathrm{d}Y=\frac{BP}{2RT_{w_0}}\int_{-1}^{+1}u\,\mathrm{d}\eta=-\left(\frac{BT_m}{\mathrm{d}T_{w_0}}\right)^3\left(\frac{\rho_m^2fd^3}{12\mu_m}\right) \tag{15}$$

The corresponding result for incompressible flow with constant fluid properties, or for compressible flow with variable properties, but neglecting frictional heating, is

$$\int_0^d \rho u \, dY = -\left(\frac{\rho_m^2 f d^3}{12\mu_m}\right)$$

For the case cited prior to equation (15), where $BT_m/dT_{w_0}=0.904$, the actual mass flow is about 25 percent below the incompressible value. Unfortunately, the mean flow velocity is about 800 feet per second. This high velocity would probably preclude the possibility of laminar flow even existing. If an example leading to slower flow were considered, the difference between the two results would have been small. This demonstrates that the variation of viscosity is unimportant.

A final comment: If the viscosity is not assumed to vary linearly with the temperature, the momentum and energy equations cannot be separated. As this circumstance (lack of separation) led Ostrach (refs. 1 and 4 to 6) to find two solutions for the flow rather than one, it is perhaps worth examining further. Define

$$H=\int_0^T \frac{k}{\mu}\, dT \tag{16}$$

Then equations (11) become

$$u_{\eta\eta}=\frac{aB^2Pf}{4R\mu_{w_0}^2}\left(\frac{aH(n_2-n_1+1)}{b}\right)^{-(1-n_1)/(1-n_1+n_2)} \tag{17a}$$

$$P=\text{Constant}$$

$$H_{\eta\eta}=-u_\eta^2 \tag{17b}$$

For gases, one expects that $n_1<1$ and $n_2>0$; thus the right side of equation (17a) is a decreasing function of H. In this case there can be at most one solution of the problems. The argument goes this way. Suppose one solution is known. If a second solution has larger H, then by equation (17a), $u_{\eta\eta}$ is smaller. Hence, for a reasonable profile, u_η is reduced. Then, by equation (17b), $H_{\eta\eta}$ is of lesser magnitude. Hence for a reasonable case, H must also be small, which is a contradiction.

It should be emphasized, however, that this case of no pressure gradient whatever has no connection with the work reported in references 1 and 4 to 6.

This completes the solution for the exact fully developed flow of a gas. There are two generalizations which can readily be made. These involve the addition of a body force transverse to the channel and the addition of heat sources in the fluid. The solutions are given in appendix B.

EXACT FLOW OF LIQUID

Here it is assumed that there is a pressure gradient such that, at least, P_X and P_Y are independent of X. No other X-dependence is admitted. Then the system is that given in equations (7) and (8), except for the X-momentum equation, which can conveniently be written as

$$(\mu u_Y)_Y=-\overline{\rho\beta}(T-T^*)f + (P_X+\overline{\rho}f)-\overline{\rho\beta}f(T-T^*) \tag{18}$$

where

$$T^*=\frac{P_X+\overline{\rho}(1+\overline{\beta T})f}{\overline{\rho\beta}f} \tag{19}$$

It is worth observing here that, if the reference point is changed in equation (5b) (i.e., a new $\overline{T}$), this has no effect whatever on the value of T^*. This is because the state equation (eq. (5b)) is really of the form $\rho=A-BT$, where A and B are fixed. In that case equation (19) is really

$$T^*=\frac{P_X+Af}{Bf}$$

Hence T^* is a function only of P_X, f, and the material. Then if η and H, defined by equations (9) and (16), respectively, replace y and T, the momentum equation becomes

$$u_{\eta\eta}=-\left(\frac{B^2\overline{\rho\beta}f}{4\mu_{w_0}^2}\right)\mu(T-T^*) \tag{20}$$

while the appropriate energy equation is, again, equation (17b).

Now the forcing term (the right side of eq. (20)) should be considered. According to its definition (eq. (16)), H is an increasing function of T. Hence, at least for constant viscosity, the forcing term is an increasing function of H. If μ and k are constant, then the forcing function is linear in H and, for this case, it has been shown (refs. 1 and 6) that two solutions occur under certain conditions. In the present situation things are not that simple,

and the results depend on how μ and k vary. For many liquids the conductivity varies only moderately over a fairly wide range of temperature, while the viscosity may change severalfold. Two cases, water and liquid sodium, are illustrated in tables I and II. In the present discussion the

TABLE I.—VISCOSITY AND CONDUCTIVITY OF LIQUID SODIUM

Temperature, T, °C	Conductivity, k, watts/(cm)(°C)		Viscosity, μ, centipoises	
	Experiment (ref. 9)	$\frac{1117}{1170+T}$	Experiment (ref. 9)	Eq. (21) (a)
200	0.815	0.815	0.450	0.450
300	.759	.760	.345	.350
400	.712	.712	.284	.286
500	.668	.668	.243	.242
600	.627	.636	.210	.210
700	.590	.598	.186	.185
800	.547	.567	.165	.166
900			.150	.150

[a] $T_a=150$°C; $s=158$ (centipoises)(°C).

TABLE II.—VISCOSITY OF WATER UNDER ATMOSPHERIC CONDITIONS

Temperature, T, °C	Viscosity, μ, centipoises	
	Experiment (ref. 9)	Eq. (21) (a)
0	1.79	1.79
20	1.01	.90
40	.66	.60
60	.48	.45
80	.36	.36
100	.28	.30

[a] $T_a=20$° C; $s=36$ (centipoises) (°C).

variation of conductivity is neglected. The viscosity can be written to good approximation as

$$\mu=\frac{s}{T+T_a} \tag{21}$$

where s and T_a are constants (for water $s=0.36$ (centipoise) (°C), and $T_a=20$° C if T is in °C). This expression is compared in tables I and II with experimental values. It is to be expected that $(T+T_a)>0$ in the range where the fluid remains a liquid. Under these circumstances, equation (16) yields

$$H=\frac{k}{2s}T(T+2T_a) \tag{22}$$

and equation (20) becomes

$$u_{\eta\eta}=-\left(\frac{B^2\overline{\rho\beta}fs}{4\mu_{w_0}^2}\right)\left[-\sqrt{\frac{kT_a^2+2sH(T^*)}{kT_a^2+2sH(T)}}\right] \tag{23}$$

From equation (17b), it is seen that, if u_η^2 is large, H must vary more or less parabolically upward across the channel. Then it follows (eq. (23)) that, when $H-H(T^*)$ has a large magnitude, the forcing term increases only slowly with H, while for small $H-H(T^*)$, the forcing term is linear in H. The latter reduces the problem to the usual free-convection situation (ref. 1), while the former (large $H-H(T^*)$) approaches the usual Poiseuille case, wherein the forcing term is constant. This circumstance at least restricts the range of flow parameters for which two solutions, as found in reference 1, can exist.

Stated more explicitly, if $(T-T_{w_0})/(T_{w_0}+T_a)$ remains small, the viscosity is essentially constant and the system becomes that solved in reference 1.

Had the variation of conductivity been allowed for, a small modification of the foregoing argument might occur. If the conductivity drops as the temperature rises, the forcing term would move toward a more linear variation with H.

In general, the solution of the system given by equations (23), (17b), and (8) is not simple. However, after two limiting cases are discussed, the general case can be described. The first such case is that of small frictional heating; the second is for small temperature variations, and therefore viscosity can be considered constant.

SMALL FRICTIONAL HEATING

First consider the case of small frictional heating. The formulation involving η and H (eqs. (23) and (17b)) is not convenient for this case. Hence, consider equations (18), (7d), and (8). The viscosity is defined by equation (21), and the conductivity is assumed constant. When the frictional heating is small so that the right side of equation (7d) is negligible, one obtains

$$T=T_{w_0}+\frac{Y}{d}(T_{w_1}-T_{w_0}) \tag{24}$$

Then equation (18) can be integrated to give

$$u=-\frac{\overline{\rho\beta}fd^2}{24s}(T_{w_1}-T_{w_0})^2\left[3\left(\frac{Y}{d}\right)^4+(4\alpha_2+8\alpha_1)\left(\frac{Y}{d}\right)^3+(12\alpha_1\alpha_2-\alpha_3)\left(\frac{Y}{d}\right)^2-2\alpha_2\alpha_3\left(\frac{Y}{d}\right)\right] \tag{25}$$

where

$$\left.\begin{aligned}\alpha_1&=\frac{T_{w_0}-T^*}{T_{w_1}-T_{w_0}}\\ \alpha_2&=\frac{T_{w_0}+T_a}{T_{w_1}-T_{w_0}}\\ \alpha_3&=\frac{12\alpha_1\alpha_2+4(\alpha_2+2\alpha_1)+3}{(1+2\alpha_2)}\end{aligned}\right\}\tag{26}$$

and T^* is defined by equation (19). From these, the wall shear is

$$\left.\begin{aligned}(\mu u_Y)_{Y=0}&=\bar{\rho}f\bar{\beta}d(T_{w_1}-T_{w_0})(\alpha_3/12)\\ (\mu u_Y)_{Y=d}&=\bar{\rho}f\bar{\beta}d(T_{w_1}-T_{w_0})(\alpha_3/12-\alpha_1-1/2)\end{aligned}\right\}\tag{27}$$

The net mass flow is

$$\int_0^d \rho u\,\mathrm{d}Y=\frac{\bar{\rho}^2d^3f\bar{\beta}}{360s}(T_{w_1}-T_{w_0})^2\times\left[1+\frac{5(1+2\alpha_1)(1+6\alpha_2+6\alpha_2^2)}{1+2\alpha_2}\right]\tag{28}$$

These results can be compared with those for constant fluid properties. (This case is given on p. 10 of ref. 1.) In such a case, the temperature is again given by equation (24). The velocity distribution, wall shear, and mass flow are given, respectively, by

$$u=-\frac{\bar{\rho}\bar{\beta}fd(T_{w_1}-T_{w_0})^2}{12s}(1+2\alpha_2)\times\left[\frac{Y}{d}\left(\frac{Y}{d}-1\right)\left(\frac{Y}{d}+1+3\alpha_1\right)\right]\tag{29}$$

$$\left.\begin{aligned}(\mu u_Y)_{Y=0}&=\bar{\rho}f\bar{\beta}d(T_{w_1}-T_{w_0})\ [(1+3\alpha_1)/6]\\ (\mu u_Y)_{Y=d}&=\bar{\rho}f\bar{\beta}d(T_{w_1}-T_{w_0})\ [-(2+3\alpha_1)/6]\end{aligned}\right\}\tag{30}$$

$$\int_0^d \rho u\mathrm{d}Y=\frac{\bar{\rho}^2d^3f\bar{\beta}}{48s}(T_{w_1}-T_{w_0})^2[(1+2\alpha_1)(1+2\alpha_2)]\tag{31}$$

In these last equations the viscosity has been taken as that corresponding to the average temperature; that is,

$$\mu=\frac{2s}{T_{w_1}+T_{w_0}+2T_a}\tag{32}$$

A comparison of the results obtained for the cases of constant and variable properties is given in figure . The fluid is liquid sodium, and the temperatures of the two walls (100° and 900° C) differ enough that the viscosity varies by a factor of about 4. In spite of this there is no significant difference between the results for the two cases. That is, the effect of variable viscosity is unimportant even though the temperature variations are large.

It is interesting to observe the case when $T^*=(T_{w_0}+T_{w_1})/2$, the average fluid temperature. Then $\alpha_1=-1/2$, $\alpha_3=-1$, and the shear is not only the same at each wall, but is the same in the constant- and variable-viscosity cases. However, the velocity profiles differ slightly; and, while the mass flow is zero for the constant-property case, it is not for variable properties.

SMALL TEMPERATURE VARIATION

In the case where the frictional heating is considered but where the temperature variations are small, the fluid properties can be considered constant. This case has been solved by Ostrach in some detail (refs. 1 and 6), by machine methods. However, another method of getting the same results is now presented that has the advantage of giving the parametric dependence simply. The same method is applied later to the general case (large temperature variations). However, the justification of the procedure is most convincingly displayed by comparison with the aforementioned machine solutions.

In this case of small temperature differences, it is again convenient to work from equations (18), (7d), and (8). If the temperature changes are small enough that $(T-T_{w_0})/(T_{w_0}+T_a)$ is small everywhere, the viscosity and conductivity can be considered constant.

The following dimensionless quantities are defined:

$$\left.\begin{aligned}U&=\frac{\bar{\rho}\bar{\beta}fd^2}{4k_m}u\\ \tau&=(T-T^*)\frac{\bar{\rho}^2\bar{\beta}^2f^2d^4}{16k_m\mu_m}\equiv\frac{K(T-T^*)}{16(T_{w_0}-T^*)}\\ \eta&=2\frac{Y}{d}-1\end{aligned}\right\}\tag{33}$$

where

$$K=\frac{(\bar{\rho}\bar{\beta}fd^2)^2(T_{w_0}-T^*)}{k_m\mu_m}\tag{34}$$

is the parameter defined in reference 1. In terms of the new variables, equations (18), (7d), and (8) become

$$U_{\eta\eta} = -\tau \tag{35}$$

$$\tau_{\eta\eta} = -U_{\eta}^{2} \tag{36}$$

$$\left.\begin{aligned} U(\pm 1) &= 0 \\ \tau(-1) &= K/16 \\ \tau(+1) &= \frac{K}{16}\left(\frac{T_{w_1}-T^*}{T_{w_0}-T^*}\right) \equiv mK/16, \text{say} \end{aligned}\right\} \tag{37}$$

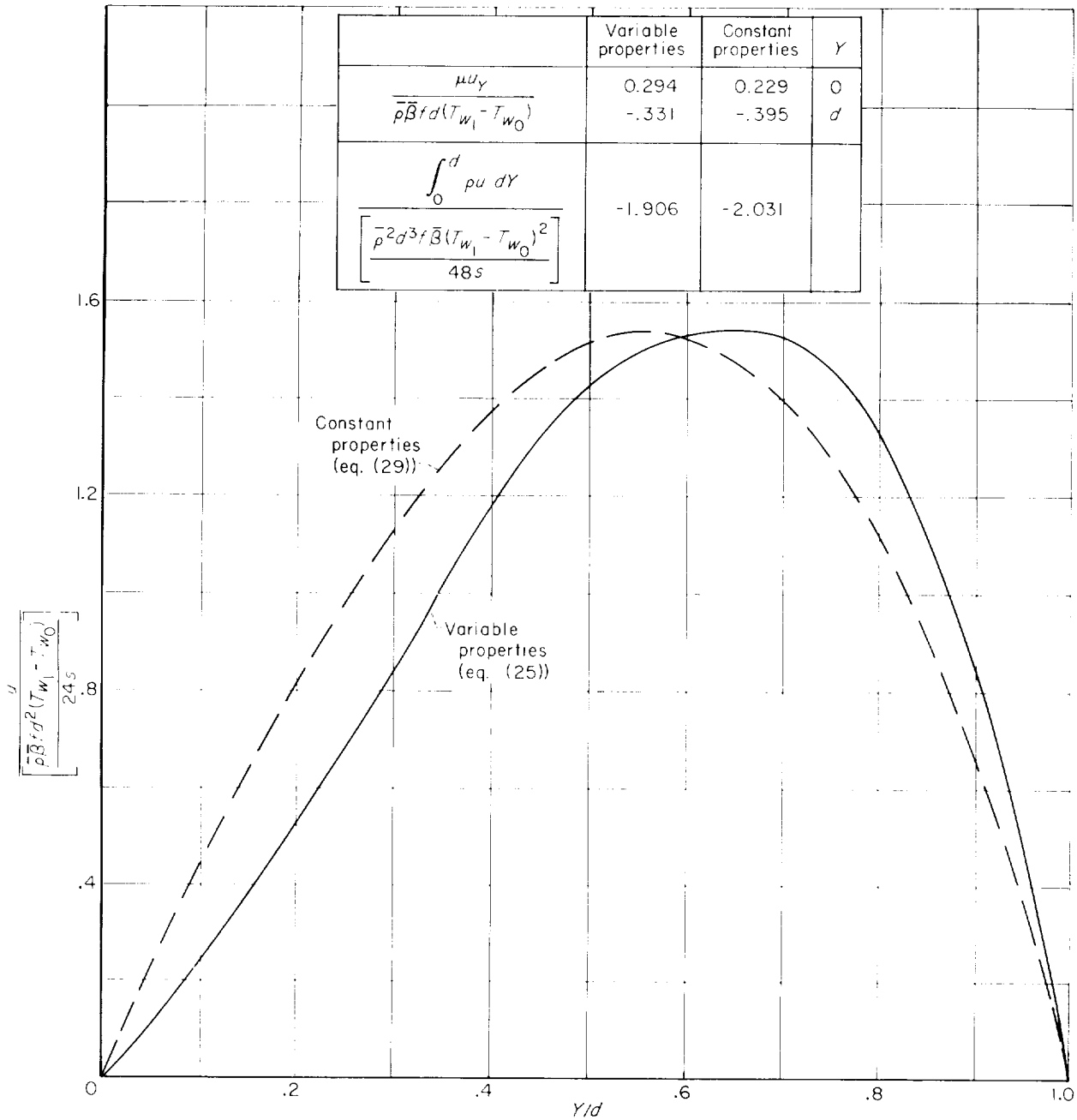


FIGURE 1.—Effect of property variation on velocity profiles. T_{w_0}, 100° C; T_{w_1}, 900° C; T^*, 0° C; liquid sodium.

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An approximate solution of this system can be found by iteration. The velocity profiles are usually parabolic. Thus, suppose

$$U^{(0)}=A(1-\eta^2) \tag{38}$$

where A is an undetermined parameter. Then equation (36) yields, subject to the boundary conditions,

$$\tau=\frac{K}{32}[(m+1)+\eta(m-1)]+\frac{A^2(1-\eta^4)}{3} \tag{39}$$

If this is put into equation (35), the resulting velocity distribution is

$$U^{(1)}=-\frac{K}{32}\left[\frac{m+1}{2}(\eta^2-1)+\frac{(m-1)}{6}(\eta^3-\eta)\right]-\frac{A^2}{6}\left[(\eta^2-1)-\frac{(\eta^6-1)}{15}\right] \tag{40}$$

This process could be continued, assuming convergence, but is stopped at this point. If equations (39) and (40) are put into equation (36) and the result is integrated across the channel, a quadratic equation for the unknown parameter A^2 follows. This is

$$A^4-2A^2\left\{\frac{17,325}{808}\left[1-\frac{K(m+1)}{210}\right]\right\}+\frac{1,155}{103,424}\times[K(m+1)]^2\left[1-\frac{m}{4(m+1)^2}\right]=0 \tag{41}$$

Real solutions exist if

$$\frac{210}{1-\sqrt{\frac{707}{2,640}\left[4-\frac{m}{(m+1)^2}\right]}}<K(m+1)<\frac{210}{1+\sqrt{\frac{707}{2,640}\left[4-\frac{m}{(m+1)^2}\right]}} \tag{42}$$

The boundaries defined by equation (42) are plotted in figure 2 for $K>0$. Some limiting values found in reference 6 by machine methods are shown for comparison. Agreement is excellent.

The two solutions can be examined generally in the following manner. If $K(m+1)$ is moderate (say, in the range 0 to 40), then equation (41) yields, approximately, the two results

$$\left.\begin{aligned} A^2 &\approx [0.016K(m+1)]^2 \\ \text{or}\quad &\approx 42 \end{aligned}\right\} \tag{41a}$$

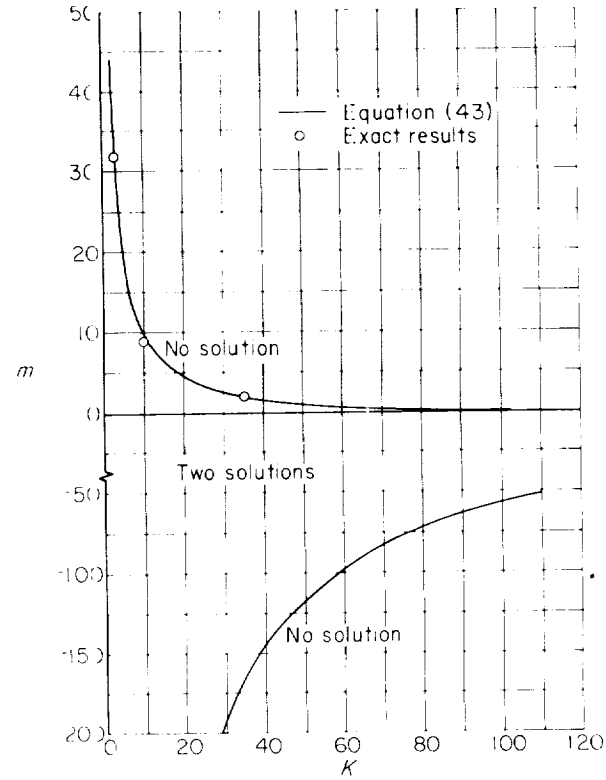

FIGURE 2.—Regions of existence of solutions of equations (35) to (37).

The mass flow and heat transfer to the walls are, respectively,

$$\left.\begin{aligned} \int_0^d \rho u\,dY &\approx \frac{k_m}{\beta f d}\frac{K(m+1)}{24} \\ \text{or}\quad &\approx \frac{k_m}{\beta f d}16 \end{aligned}\right\} \tag{43}$$

$$\left.\begin{aligned} \Delta(k_m T_Y)&\equiv k_m[(T_Y)_{Y=0^+}-(T_Y)_{Y=d}] \\ &\approx \frac{256k_m(T_{w_0}-T^*)}{3Kd}[0.016K(m+1)]^2 \\ \text{or}\quad &\approx \frac{256k_m(T_{w_0}-T^*)}{3Kd}40 \end{aligned}\right\} \tag{44}$$

The first (smaller) solution is one in which there is negligible frictional heating and, hence, heat transfer, while the second is quite the opposite. The second case is, as is pointed out in reference 4, one of regenerative heating. There is a large amount of heat transfer to the walls (eq. (44)); this heat is supplied by frictional heating of the fluid occasioned by large mass flow (eq. (48))

and the resultant high shear. Notice that even when $T_{w_1}=T_{w_0}=T^*$, so that the problem is an homogeneous one ($K=0$ in eq. (37), although $K/(T_{w_0}-T^*)$ in eq. (33) is not zero), this second solution does not vanish. For that matter, the second solution is virtually independent of K and m, provided K and m take on moderate values.

The range of validity of these results is limited by the condition that $(T-T_{w_0})/(T_{w_0}+T^*)$ be small so that the viscosity variation is negligible. In the case of the second solution (large A^2), the maximum of T occurs near the center of the channel, and thus equations (38), (44), and (46a) yield

$$\frac{T_{max}-T_{w_0}}{T_{w_0}+T_a}\approx\frac{200}{K}\frac{T_{w_0}-T^*}{T_{w_0}+T_a}$$

If, for example, $K=10$, then $T_{w_0}-T^*$ is limited to a few degrees. This implies also that the wall temperatures must be virtually equal. On the other hand, for applications of the first solution (small A^2), the only restriction is that $(T_{w_0}-T_{w_1})/(T_{w_0}+T_a)$ be small. However, if the viscosity is that for some suitably defined average, the error due to larger temperature variations should be unimportant. This conjecture is based on the example discussed after equation (32).

Before giving a numerical example, it is worthwhile to examine the order of magnitude of the numbers one obtains in physical problems for the present case of small temperature differences. From equations (33), (34), (40), and (41a), one has, very roughly,

$$\left.\begin{aligned}
u(\eta=0)&\approx\frac{m+1}{16}\sqrt{k\left[\frac{k(T_{w_0}-T^*)}{\mu}\right]} && \text{(Small solution)}\\
\text{or}\quad &\approx 25\sqrt{\frac{k(T_{w_0}-T^*)}{\mu K}} && \text{(Large solution)}\\
\text{or}\quad T(\eta=0)-T_{w_0}&\approx\left(\frac{m-1}{2}\right)(T_{w_0}-T^*) && \text{(Small solution)}\\
&\approx\left(\frac{210}{K}-1\right)(T_{w_0}-T^*) && \text{(Large solution)}
\end{aligned}\right\}\qquad(45)$$

For example, for water at 0° C, if $K=25$, $T_{w_0}-T^*=0.25°$ C, and $m=1$, the numbers for the various cases are

	Small	Large
$u(\eta=0)$, ft/sec	25	200
$T(\eta=0)-T_{w_0}$, °C	0	2

This case corresponds to a channel width of 14.6 inches. A more general idea of the orders of the numbers involved in the second solution can be obtained as follows. From equation (33),

$$\frac{u^2}{T-T^*}=\frac{k}{\mu}\frac{U^2}{\tau}\qquad(46)$$

However, from equations (39), (40), and (41a) the maximums of U and τ for the second solution usually occur near $\eta=0$ and have the respective values of about $6\frac{1}{2}$ and 14. Hence, for the second solution,

$$\left(\frac{u^2}{T-T^*}\right)_{\eta=0}\approx 3k/\mu\qquad(46a)$$

Actually there is no combination of K and m such that equations (41), (39), and (40) yield $U^2_{max}/\tau_{max}<1$. For water, $3k/\mu$ is of the order 10^4(ft/sec)^2/°C, and hence if $T-T^*$ is 1° C, the velocity maximum is 100 feet per second. For liquid sodium, $3k/\mu$ is about 10^7(ft/sec)^2/°C; thus a temperature difference of only 1° C corresponds to a maximum velocity of 3,000 feet per second. It therefore appears that, if the velocity is to be kept moderate to maintain laminar flow, the temperature variation must be small. Hence, the assumption of constant fluid properties is a good one.

To compare the present method of calculating the velocity and temperature profiles with the more exact solutions obtained by machine methods (ref. 1), a single example is shown in figure 3. Remember that this is a constant-fluid-property situation. The example is that of water flowing in a channel 14.7 inches wide and for which $T_{w_0}-T^*=1/10°$ C. The wall temperatures are 20.0° and 20.1° C. This leads to $K=10$. The

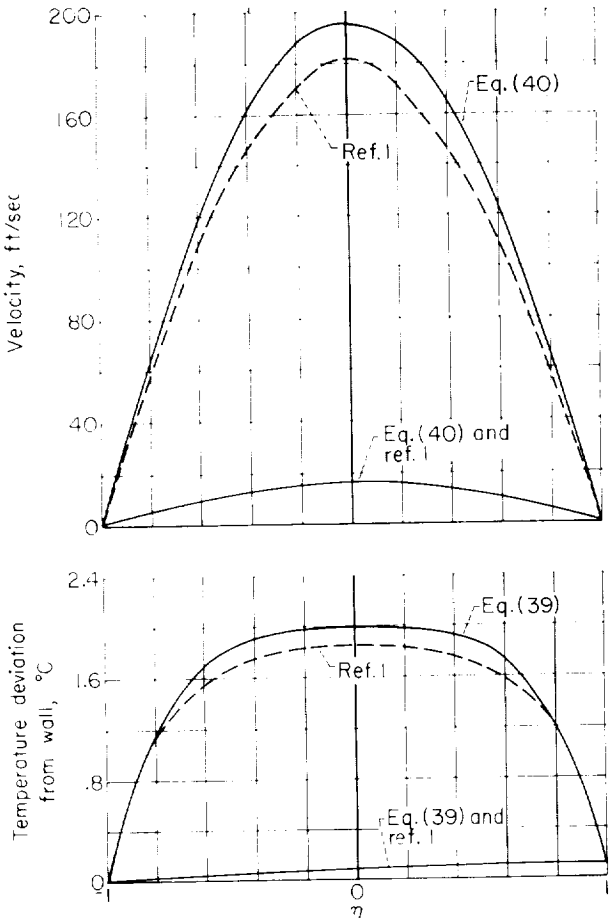

FIGURE 3. Velocity and temperature profiles. m, 2; k, 10; T_{w_0} T^*, 1/10° C; water.

agreement is within 10 percent for the second solution and, not surprisingly, virtually exact for the smaller one.

Before ending the discussion of exactly fully developed flows, it should be observed that the iterative procedure used to get solutions here can be applied in the other cases considered in references 1, 4, and 6, namely those involving wall temperature gradients and heat sources in the fluid.

As is stated earlier, the present iterative procedure can be applied directly to the original problem wherein the frictional heating is considered and large temperature changes are contemplated. The solution for such a flow is given in appendix C for the case of equal wall temperatures. The only difference from the case just discussed is that some of the integrals are rather involved and the equation for the amplitude is more complicated. However, as is observed previously, if the velocities are to be kept moderate in the second solution, the temperature variation will be negligible.

APPROXIMATE SOLUTIONS

The results thus far presented have the beauty of being exact within the limitations of fully developed flow. However, several cases arise in which such a limited view is unacceptable. The simplest such case is the flow of a gas with a pressure gradient, the ordinary Poiseuille flow. Another case of some interest is that involving a wall temperature gradient. The extent to which such flows can be considered fully developed is examined in what follows.

Solutions of equations (1) to (6) are sought. Again, a long channel is assumed and end effects are neglected. In such a case, any gradients in the X-direction (in the flow direction) must be small. Hence, write the variables as follows:

$$\left.\begin{aligned}
P&=\overline{P}[1+\epsilon g(x)+\ldots]\\
\rho&=\overline{\rho}[\rho_0(y)+\delta\rho_1(x,y)+\ldots]\\
T&=\overline{T}[T_0(y)+\delta T_1(x,y)+\ldots]\\
\mu&=\overline{\mu}[\mu_0(y)+\delta\mu_1(x,y)+\ldots]\\
k&=\overline{k}[k_0(y)+\delta k_1(x,y)+\ldots]\\
u&=\overline{u}[u_0(y)+\delta u_1(x,y)+\ldots]\\
v&=\frac{d}{L}\overline{u}[0+\delta v_1(x,y)+\ldots]\\
x&=X/L\\
y&=Y/d
\end{aligned}\right\}\qquad(47)$$

where d/L is small, d being the wall spacing and L being a length of flow, as yet undefined. The other two parameters, ϵ and δ, are small but unrelated at this time. For a gas δ must be at least as large as ϵ for the state equation (eq. (5)) to make sense, while for a liquid the number δ is determined by the temperature boundary conditions. The barred quantities are, except for $\overline{u}$, given parameters chosen so that ρ_0, k_0, μ_0, and so forth, are of unit order. The value of $\overline{u}$ is initially unknown because there is no characteristic velocity for internal flows of this kind.

Had the term $g(x)$ in the pressure (the first of eqs. (47)) been considered as a function of y also, then added terms would be introduced because of the y-momentum equation (eq. (3)). However, these are, analogously to the usual boundary-layer analysis, of higher order than what is retained in the other equations of motion.

If equations (47) are put into equations (1), (2), and (4) and only the dominant terms of each kind are retained, there follows

$$\rho_0(u_{1,x}+v_{1,y})+\rho_{0,y}v_1+\rho_{1,x}u_0=0 \tag{48}$$

$$\frac{\delta\overline{\rho u}^2}{L}[\rho_0(u_0u_{1,x}+v_1u_{0,y})]+\left\{\frac{\overline{P}\epsilon g_x(x)}{L}+\bar{\rho}f(\rho_0+\delta\rho_1)\right\}=\frac{\overline{\mu u}}{d^2}[(\mu_0u_{0,y})_y] \tag{49}$$

$$\frac{\delta\bar{\rho}c_p\overline{uT}}{L}[\rho_0(u_0T_{1,x}+v_1T_{0,y})]+\frac{\delta\overline{Pu}}{L}(u_{1,x}+v_{1,y})=\frac{\overline{kT}}{d^2}[(k_0T_{0,y})_y+\delta(k_0T_{1,y}+k_1T_{0,y})_y]+\frac{\overline{\mu u}^2}{d^2}(\mu_0u_{0,y}^2) \tag{50}$$

Finally the state equations (eqs. (5)) yield

$$\left.\begin{aligned}\overline{P}&=\bar{\rho}R\overline{T}\\ \rho_0T_0&=1\\ \rho_0T_1+\rho_1T_0&=(\epsilon/\delta)g(x)\end{aligned}\right\}\text{(Gas)} \tag{51}$$

$$\left.\begin{aligned}\rho_0-1&=-\overline{\beta T}(T_0-1)\\ \rho_1&=-\overline{\beta T}T_1\end{aligned}\right\}\text{(Liquid)} \tag{52}$$

Now observe that the terms in the braces in equation (49) describe the driving forces of the flow and must therefore be of the same order of magnitude as the viscous terms (which contain the highest derivatives). The following four cases will be considered in turn:

Case	Body force	Large temperature variation (T_0 variable)
I	Yes	Yes
II	No	Yes
III	Yes	No
IV	No	No

CASE I—T_0 VARIABLE, FREE CONVECTION

For case I, terms of order δ can be neglected as compared to corresponding zero-order terms. Thus, ρ_1 can be set equal to zero in equation (49), as can k_1 and $T_{1,y}$ in equation (50). Assuming ρ_0 and $g_x(x)$ to be of unit order, the body force and pressure gradients must be of similar size so that

$$\frac{\overline{P}\epsilon}{L}=\bar{\rho}f \tag{53}$$

Actually, it is only necessary that $\overline{P}\epsilon/L\leq\bar{\rho}f$ in order to have the body force matter. However, it can be assumed that equation (53) holds, and ϵ can be discarded later if it is small enough.

Then, using equations (51) and (52), the driving term in equation (49) is

$$\left.\begin{aligned}&\frac{\overline{P}\epsilon}{L}g_x(x)+\bar{\rho}f(\rho_0)=\bar{\rho}f\left[1+g_x(x)-\frac{T_0-1}{T_0}\right]\quad\text{(Gas)}\\ &\text{or}\\ &\qquad=\bar{\rho}f\overline{\beta T}\left[\frac{1+g_x(x)}{\overline{\beta T}}-(T_0-1)\right]\quad\text{(Liquid)}\end{aligned}\right\} \tag{54}$$

For a mixed flow, $g_x(x)=0(1)$ for a gas or $g_x(x)=-1+0(\overline{\beta T})$ for a liquid. A pure free-convection flow might arbitrarily be defined as one for which $g_x(x)=-1$, but, for convenience, any flow involving body forces is henceforth referred to as a free-convection flow.

Definitions of the barred reference values in equations (47) are all straightforward, except that for $\bar{u}$, and a selection of values can readily be made a priori. However, $\bar{u}$ must be chosen such that u_0 is of unit order, and there is no way of knowing ahead of time how big the flow will be. Hence, for the moment, let us beg the question and define simply

$$C_2\frac{\overline{\mu u}}{d^2}=\bar{\rho}f\overline{\beta T} \tag{55}$$

where C_2 is unknown and, for a gas, $\overline{\beta T}\equiv 1$. It is shown later that C_2 is a number in the range 10 to 50. Then equations (48), (49), and (50) can be written

$$\rho_0(u_{1,x}+v_{1,y})+\rho_{0,y}v_1+\rho_{1,x}u_0=0 \tag{56}$$

$$\left.\begin{aligned}&(\mu_0u_{0,y})_y+C_2\left\{\frac{T_0-1}{T_0}-[1+g_x(x)]\right\}\\ &\qquad=\frac{\delta\bar{u}^2C_2}{fL\overline{\beta T}}[\rho_0(u_0u_{1,x}+v_1u_{0,y})]\quad\text{(Gas)}\\ &(\mu_0u_{0,y})_y+C_2\{T_0-1-[1+g_x(x)]/\overline{\beta T}\}\\ &\qquad=\frac{\delta\bar{u}^2C_2}{fL\overline{\beta T}}[\rho_0(u_0u_{1,x}+v_1u_{0,y})]\quad\text{(Liquid)}\end{aligned}\right\} \tag{57}$$

$$\left.\begin{aligned}(k_0T_{0,y})_y=&\frac{\bar{\mu}c_v}{k}\frac{\delta\bar{u}^2C_2}{fL\overline{\beta T}}\left\{[\rho_0(u_0T_{1,x}+v_1T_{0,y})]+\frac{\bar{P}}{c_v\overline{\rho T}}\right.\\&\left.\times\left[\rho_0(u_0T_{1,x}+v_1T_{0,y})-u_0\frac{\epsilon}{\delta}g_x(x)\right]\right\}-\frac{\overline{\mu u}^2}{\overline{kT}}(\mu_0u_{0,y}^2)\\&\text{(Gas)}\\(k_0T_{0,y})_y=&\frac{\bar{\mu}c_v}{k}\frac{\delta\bar{u}^2C_2}{fL\overline{\beta T}}\left\{[\rho_0(u_0T_{1,x}+v_1T_{0,y})]+\frac{\bar{P}}{c_v\overline{\rho T}}\right.\\&\left.\times\left[\frac{\overline{\beta T}}{\rho_0}(u_0T_{1,x}+v_1T_{0,y})\right]\right\}-\frac{\overline{\mu u}^2}{\overline{kT}}(\mu_0u_{0,y}^2)\ \text{(Liquid)}\end{aligned}\right\}\tag{58}$$

The system whose solution is sought is given by equations (56), (57), (58), and (51) or (52) plus the boundary conditions (eq. (6)). Assuming that $g_x(x)$ is given, the unknown dependent variables remaining are seven in number, u_0, u_1, v_1, ρ_0, ρ_1, T_0, and T_1. With five equations and seven dependent variables, some further restriction must be made. The difficulty arises mainly in the terms u_1 and v_1. For a gas it appears ρ_1, u_1, and v_1 are of the same order as T_1 and hence that the inertia term in equation (57) is of the same size as the convection term in equation (58). Hence, it follows that, in order for the solution to be determined, both of these terms must be negligible. The same result follows for liquids, although perhaps not so obviously. In this case (eq. (51)), ρ_1 is very small, of the order of $\overline{\beta T}$, where $\bar{\beta}$ is the volumetric expansion coefficient. However, for liquids the viscosity is a very strong function of temperature, and thus $\mu_1=0(T_1)$. Hence, there is no particular reason to assume that u_1 and v_1 are not the same size as T_1. Accordingly, if these terms matter, the present formulation is useless.

For these terms to be negligible, two courses are open. One is to have everything x-independent as in the exact solutions described earlier. The second is that the parameter $C_2\delta\bar{u}^2/fL\overline{\beta T}$ be small. This is not simple. For example, if equation (55) is used and a channel 1 inch wide and having a characteristic length L of 10 feet is assumed, then under standard conditions and gravitational acceleration,

$$\left.\begin{aligned}C_2\delta\bar{u}^2/fL\overline{\beta T}&=66{,}000\ \delta/C_2 &&\text{(For water)}\\&=6{,}400\ \delta/C_2 &&\text{(For air)}\end{aligned}\right\}\tag{59}$$

Consider $C_2=1$. For the air case, δ must be as large as ϵ, which is about 1/3,000 here. For water δ can be chosen by an applied wall temperature gradient. In either case it seems difficult to make $C_2\delta\bar{u}^2/fL\overline{\beta T}$ small, in fact, unless C_2 is a large number.

If this question is ignored for the moment, it can be observed for a gas that, if $\delta=\epsilon$ (and δ must be as large as ϵ), the coefficients of the inertia or convection terms and of the dissipation terms are

$$\left.\begin{aligned}&C_2\delta\bar{u}^2/fL\overline{\beta T}=C_2\epsilon\bar{u}^2/fL=C_2\rho\bar{u}^2/\bar{P}\\\text{and}\quad&\frac{\overline{\mu u}^2}{\overline{kT}}=\frac{R\overline{Pr}}{c_p}(\overline{\rho u}^2/\bar{P})\end{aligned}\right\}\tag{60}$$

(where $\overline{Pr}$ is a Prandtl number), which are both essentially the squares of a Mach number and are the same size if $C_2=1$. Hence, for the gas case, at least, the frictional heating must also be negligible because, as is shown later, $C_2>1$.

For a liquid it can be seen, by trying some cases, that the frictional heating must again be very small provided δ/ϵ is not virtually zero.

Now to return to the question of a value for C_2. Suppose that equation (55) is used together with its subsequent consequences. Then, for the sake of an example, let $g_x(x)=-1$ and assume a gas. The differential equations become

$$(\mu_0u_{0,y})_y=C_2(1-T_0)/T_0$$

$$(k_0T_{0,y})_y=0$$

The boundary conditions are

$$u_0(0)=u_0(1)=0$$

$$T_0(0)=T_{w_0}$$

$$T_0(1)=T_{w_1}$$

To make the point about orders of magnitude, consider that μ_0 and k_0 are proportional to the temperature (i.e., $\mu_0=k_0=T_0$). Then the equations can be solved very readily. Maximum speeds

have been computed for several cases and are shown in the following table:

T_{w_0}	T_{w_1}	$\left\lvert\frac{u_0}{C_2}\right\rvert_{max}$
1/4	1	0.12
1/2	3/2	.01
1	2	.03

Then $|u_0|_{max}$ is of unit order if C_2 is, say, 40. Thus for the conditions cited in connection with equation (59),

$$C_2\delta\overline{u}^2/fL\overline{\beta T}=1{,}650\delta \qquad \text{(For water)}$$

or

$$=160\delta \qquad \text{(For air)}$$

which, particularly for the air case, can readily be made small. In a case (such as eqs. (35) to (37)) where two solutions occur, the same conclusion about C_2 follows, although the argument is rather tortuous.

Finally, then, the inertia terms must be negligible for the flow to be fully developed. In that case the thermal convection and frictional dissipation are negligible. The equations for fully developed free-convection flow in a channel are (eq. (55) is assumed to apply with $C_2=1$)

$$\left.\begin{aligned}(\mu_0 u_{0,y})_y&=\frac{1-T_0}{T_0}+1+g_x\\(k_0T_{0,y})_y&=0\end{aligned}\right\}\qquad\text{(Gas)}\qquad(61a)$$

$$\left.\begin{aligned}(\mu_0 u_{0,y})_y&=1-T_0+\frac{1+g_x}{\beta T}\\(k_0T_{0,y})_y&=0\end{aligned}\right\}\qquad\text{(Liquid)}\qquad(61b)$$

$C_2=1$ is used here and also in equation (55) because C_2 was introduced only as an aid in determining what matters in the equations of motion. With these small terms eliminated, C_2 can be dropped.

It should be remembered that these equations correspond to cases where the temperature variations across the channel can be large. The solutions are valid provided only that $\delta u_{max}^2/fL\overline{\beta T}$ is small. For consistency, of course, $g_x(x)$ must be constant.

The system is readily solved by first integrating the energy equation. The result in parametric form is

$$y=\frac{\int_{T_{w_0}}^{T}k_0\,\mathrm{d}T}{\int_{T_{w_0}}^{T_{w_1}}k_0\,\mathrm{d}T}\qquad(62)$$

$$u=\frac{\int_{T_{w_0}}^{T_{w_1}}\left(\frac{k_0}{\mu_0}\int_{T_{w_0}}^{T_1}k_0R\,\mathrm{d}T_2\right)\mathrm{d}T_1}{\left[\int_{T_{w_0}}^{T_{w_1}}k_0\,\mathrm{d}T\right]^2}$$

$$\times\left\{\frac{\int_{T_{w_0}}^{T}\left(\frac{k_0}{\mu_0}\int_{T_{w_0}}^{T_1}k_0R\,\mathrm{d}T_2\right)\mathrm{d}T_1}{\int_{T_{w_0}}^{T_{w_1}}\left(\frac{k_0}{\mu_0}\int_{T_{w_0}}^{T_1}k_0R\,\mathrm{d}T_2\right)\mathrm{d}T_1}-\frac{\int_{T_{w_0}}^{T}\frac{k_0}{\mu_0}\,\mathrm{d}T}{\int_{T_{w_0}}^{T_{w_1}}\frac{k_0}{\mu_0}\,\mathrm{d}T}\right\}$$

where R is the right side of the first of equations (61a) or (61b).

This then is the solution of a free-convection flow wherein large temperature variations across the channel are admitted. The effect of longitudinal wall temperature variations would presumably be allowed for by considering that these were local profiles, by a sort of strip theory. Results of the kind in reference 4 would apply to case III, discussed later on, wherein small temperature variations are assumed throughout.

CASE II—T_0 VARIABLE, FORCED CONVECTION

This is a flow in which the body force is considered to be negligible. In such a case equation (53) no longer applies, but one assumes that $f=0$ and that ϵ is given. Without loss, take $g_x(x)\equiv 1$. Equation (55) is replaced by

$$\frac{\overline{P}\epsilon}{L}=\frac{\overline{\mu u}}{d^2}\qquad(63)$$

Then equations (61) and (62) with their accompanying conditions hold, but with the right sides of the first of equations (61a) and (61b) replaced by unity. The solution is that given in equations (62) but with $R=1$.

The analysis thus far given applies to the case where large temperature differences are allowed ($T_0\neq 1$). If only small differences are permitted, a somewhat different formulation results.

CASE III—$T_0 \equiv 1$, FREE CONVECTION

If $T_0 \equiv 1$, then $\rho_0 = \mu_0 = 1$. Also, equation (53) still applies, and hence the driving term in equation (49) is (compare with eq. (54))

$$\left.\begin{aligned} &\frac{\overline{P}\epsilon}{L} g_x(x) + \overline{\rho} f(1+\delta\rho_1) \\ &\quad = \overline{\rho} f \delta \left[-T_1 + \frac{\epsilon g}{\delta} + \frac{1+g_x(x)}{\delta} \right] \quad \text{(Gas)} \\ &\quad = \overline{\rho} f \delta \overline{\beta T} \left[-T_1 + \frac{1+g_x(x)}{\delta \overline{\beta T}} \right] \quad \text{(Liquid)} \end{aligned}\right\} \tag{64}$$

Then, for a free-convection flow, it is required that

$$g_x(x) = -1 + 0(\delta \overline{\beta T}) \tag{65}$$

Under such a condition the driving and viscous forces are of the same order provided (compare with eq. (55))

$$\overline{\rho} f \delta \overline{\beta T} = \overline{\mu u}/d^2 \tag{66}$$

If these last equations and equations (51) and (52) are used, equations (48), (49), and (50) become

$$u_{1,x} + v_{1,y} = -\rho_{1,x} u_0 \tag{67}$$

$$u_{0,yy} = T_1 - \frac{\epsilon g(x)}{\delta} - \frac{1+g_x(x)}{\delta \overline{\beta T}} = \frac{\overline{u}^2}{f L \overline{\beta T}} (u_0 u_{1,x} + v_1 u_{0,y}) \tag{68}$$

$$T_{1,yy} = \frac{\overline{u}^2}{\delta f L \overline{\beta T}} \frac{\overline{\mu} c_r}{k} \left\{ u_0 T_{1,x} + \frac{\overline{P}}{\overline{\rho T} c_r} \left[\overline{\beta T} u_0 T_{1,x} - \frac{\epsilon}{\delta} u_0 g_x(x) \right] \right\} - \frac{\overline{\mu u}^2}{\delta \overline{k T}} u_{0,y}^2 \tag{69}$$

where the terms involving ϵ/δ do not appear if the fluid is a liquid.

In distinction to the result for the case of large temperature variations, in equation (68),

$$\frac{\overline{u}^2}{f L \overline{\beta T}} = 66{,}000\delta^2 \qquad \text{(For water)}$$

or

$$= 6{,}400\delta^2 \qquad \text{(For air)}$$

for the circumstances of equation (59). In the case for air, if $\delta = \epsilon$, this is a very small number and should, indeed, be negligible. The convection term in equation (69) is of order δ or larger, as is the frictional heating term, and should be retained.

Then equations (68) and (69) are

$$u_{0,yy} + 0 = -T_1 + \frac{\epsilon}{\delta} g(x) + \frac{1+g_x(x)}{\delta \overline{\beta T}} \tag{70}$$

$$T_{1,yy} = \frac{\overline{\mu u}^2}{\delta \overline{k T}} \left\{ -u_{0,y}^2 \left[+ \frac{c_r}{f L \overline{\beta}} \left(1 + \frac{\overline{P\beta}}{\overline{\rho} c_r} \right) T_{1,x} - g_x \right] u_0 \right\} \tag{71}$$

where again the terms involving ϵ/δ in equation (70) and g_x in equation (71) appear only for a gas.

Now to determine what form the temperature variations can take. It was assumed at the outset that u_0 is independent of x. For this case it is readily shown that the most general forms allowable for the temperature and pressure gradient are

$$T_1 = a_1 x + a_2 + T_2(y) \tag{72}$$

$$g_x(x) = -1 + \delta \overline{\beta T} [(\epsilon/\delta + a_1)x + a_2] \tag{73}$$

Then equations (68) and (69) become

$$u_{0,yy} = -T_2 \tag{70a}$$

$$T_{2,yy} = \frac{\overline{\mu u}^2}{\delta \overline{k T}} \left\{ -u_{0,y}^2 + u_0 \left[1 + a_1 \frac{c_r}{f L \overline{\beta}} \left(1 + \frac{\overline{P\beta}}{\overline{\rho} c_r} \right) \right] \right\} \tag{71a}$$

where the term $\overline{\mu u}^2/\delta \overline{k T}\{u_0[1]\}$ disappears for a liquid. These equations require small temperature variations but admit substantial mass-flow rates.

Then, in terms of u_0 and $T_2(y)$, equations (70a) and (71a) are a pair of ordinary differential equations and are nonlinear only if the frictional heating is important. Solutions in the linear case are quite simple (ref. 4) and in the nonlinear case can be found by the iterative method given earlier following equation (37). Ostrach discusses this system extensively in reference 4, where, among other things, some machine solutions are given.

CASE IV—$T_0 \equiv 1$, FORCED CONVECTION

For case IV equation (63) still applies, and equations (70) and (71) are replaced by

$$u_{0,yy} = g_x(x) \tag{74}$$

$$T_{1,yy} = \frac{\overline{\mu u}^2}{\overline{k T}} \left[-u_{0,y}^2 + \frac{\delta \overline{\rho T} c_r}{\overline{P\epsilon}} \left(1 + \frac{\overline{P\beta}}{\overline{\rho} c_r} \right) u_0 T_{1,x} + g_x u_0 \right] \tag{75}$$

The temperature variation must have the form

$$T_1=(a_1+a_2y)x+T_2(y)$$

while $g_x(x)$ is a constant. These equations are easily solved because the dependent variables are separated. The result is, of course, the familiar Poiseuille flow.

OTHER GAS FLOWS

The flows discussed thus far have included only those cases wherein the velocity is essentially independent of distance along the channel. If this restriction is lifted, the problem becomes vastly more complicated. Therefore, only one class of solutions is examined here. These solutions are ones in which the independent variables are separable. Such a flow is out of the question for liquids unless it can be assumed that the temperature is a function of y only. This is because of the form of the state equation (eq. (5b)). It can be shown that this limitation to T independent of x leaves only the fully developed cases discussed previously.

First recall that, to have a fully developed flow in any sense, the channel must be very long and the dependence on x must be much weaker than that on y. Then equation (1) to (5) can be approximated as

$$(\rho u)_X+(\rho v)_Y=0 \tag{76}$$

$$\rho(uu_X+vu_Y)+P_X=-\rho f+(\mu u_Y)_Y \tag{77}$$

$$P_Y=0 \tag{78}$$

$$\rho c_v(uT_X+vT_Y)+P(u_X+v_Y)=(kT_Y)_Y+\mu u_Y^2 \tag{79}$$

$$P=\rho RT \qquad \text{(Gas)} \tag{80a}$$

$$\rho=\bar{\rho}\,[1-\bar{\beta}(T-\bar{T})] \qquad \text{(Liquid)} \tag{80b}$$

These equations can be derived formally in the same manner as they are derived for external boundary layers. The only difference is that the Reynolds number of boundary-layer analysis is replaced by a ratio $(L/d)^2$, where L is a characteristic flow length and d is the channel width.

If the fluid is a gas (eq. (80a)) and the body force is negligible, and the viscosity and thermal conductivity each vary with the temperature as

$$\mu=aT^{\zeta} \tag{81}$$

$$k=bT^{\zeta}$$

then the permissible separated forms are

$$\left.\begin{aligned}
u/u_1(Y)&=(X/L)^{\theta} &&\text{or}\quad e^{\theta X/L}\\
v/v_1(Y)&=(X/L)^{\theta-1} &&\text{or}\quad e^{\theta X/L}\\
\rho/\rho_1(Y)&=(X/L)^{\theta(2\zeta-1)+1} &&\text{or}\quad e^{(2\zeta-1)\theta X/L}\\
T/T_1(Y)&=(X/L)^{2\theta} &&\text{or}\quad e^{2\theta X/L}\\
P/P_1&=(X/L)^{\theta(2\zeta+1)+1} &&\text{or}\quad e^{(2\zeta+1)\theta X/L}
\end{aligned}\right\} \tag{82}$$

where θ is an arbitrary constant.

It is interesting, though perhaps irrelevant, that exactly the same variations of free-stream velocity are allowed for similar solutions of the external-boundary-layer equations (ref. 7).

The exponential form in equations (82) is valid for the complete Navier-Stokes equations, while the other form depends critically on the assumption of a very long channel.

Several other somewhat unrelated comments about this result are perhaps in order. First, for a liquid the requirement previously stated, that $\partial T/\partial X=0$, leads to the condition that nothing varies with X. This case has already been examined. Second, for a gas, if the viscosity and conductivity do not have the same variations with temperature (eqs. (81)), only the trivial X-independent separation results. The X-independent solution corresponds in equations (82) to the exponential variation with $\theta=0$ and was discussed starting with equations (7). Finally, if the body force is important ($-\rho f$ in eq. (77)), the forms given in equations (82) apply, but only with $\theta=0$ (exponential and uninteresting) or $\theta=\frac{1}{2}$ (power of X).

The forms given in equations (82) have two other properties of interest. The through-flow Mach number, which is proportional to $u/\sqrt{T}$, is independent of X. Also, unless there is flow through the channel walls, all the solutions except $\theta=0$ (exponential) and $\theta=-\frac{1}{2}\zeta$ (power of X) must be flows with zero net mass flow. This is because the mass flow is

$$\left.\begin{aligned}
&\int_0^d \rho u\,\mathrm{d}Y=(X/L)^{2\zeta\theta+1}\int_0^d \rho_1(Y)u_1(Y)\,\mathrm{d}Y\\
\text{or}\quad &\qquad\qquad =e^{2\zeta\theta X/L}\int_0^d \rho_1(Y)u_1(Y)\,\mathrm{d}Y
\end{aligned}\right\} \tag{83}$$

which must not vary with X unless there is flow through the walls.

If equations (82) are put into equations (76) to (80a), the result is

$$\left.\begin{aligned}
&\left(\frac{2\zeta\theta+\lambda}{L}\right)\rho_1 u_1+(\rho_1 v_1)_Y=0\\
&\rho_1\left(\frac{\theta}{L}u_1^2+v_1 u_{1,Y}\right)+\left[\frac{\theta(2\zeta+1)+\lambda}{L}\right]P_1\\
&\qquad=(\mu_1 u_{1,Y})_Y-\rho_1 f P_{1,Y}=0\\
&c_v\rho_1\left(\frac{2\theta}{L}u_1 T_1+v_1 T_{1,Y}\right)+P_1\left(\frac{\theta}{L}u_1+v_{1,Y}\right)\\
&\qquad=(k_1 T_{1,Y})_Y+\mu_1 u_{1,Y}^2 P_1=\rho_1 R T_1
\end{aligned}\right\}\tag{84}$$

where $\lambda=0$ if the exponential variation is used and $\lambda=1$ if the variation is as a power of x. Also, the term $\rho_1 f$ can appear only if, as already mentioned, $\theta=0$ (exponential) or $\theta=1/2$ (power of x); k_1 and μ_1 have the obvious definitions.

Now a new space variable, η, is introduced from equation (10). Then equations (84) become

$$\left\{\begin{matrix}BP_1(2\zeta\theta+\lambda)\\2RL[T_1(-1)]^\zeta\end{matrix}\right\}u_1+T_1^{1-\zeta}(\rho_1 v_1)_\eta=0\tag{85}$$

$$\begin{aligned}
\left(\frac{P_1\theta}{RL}\right)u_1^2+&\left\{\frac{2[T_1(-1)]^\zeta}{B}\right\}T_1^{1-\zeta}(\rho_1 v_1)u_{1,\eta}\\
&+\left\{\left[\frac{\theta(2\zeta+1)+\lambda}{L}\right]P_1\right\}T_1\\
&=\frac{4a[T_1(-1)]^{2\zeta}}{B^2}T_1^{1-\zeta}u_{1,\eta\eta}-\frac{P_1 f}{R}
\end{aligned}\tag{86}$$

$$\begin{aligned}
&\left\{\left[\left(\frac{2c_p}{R}-1-2\zeta\right)\frac{\theta}{L}-\frac{\lambda}{L}\right]P_1\right\}u_1 T_1\\
&\qquad+\frac{2c_p}{B}[T_1(-1)]^\zeta(\rho_1 v_1)T_1^{1-\zeta}T_{1,\eta}\\
&\qquad=\frac{4b[T_1(-1)]^{2\zeta}}{B^2}T_1^{1-\zeta}\left(T_{1,\eta\eta}+\frac{a}{b}u_{1,\eta}^2\right)
\end{aligned}\tag{87}$$

$$P_1=\rho_1 R T_1\tag{88}$$

Observe that equations (85), (86), and (87) form a set of three ordinary differential equations for three variables (u_1, T_1, and $\rho_1 v_1$). One curious feature of the system is that it is of fifth order and there are, in general, six boundary conditions to be applied (one each to u_1, v_1, and T_1 at each wall). Hence some restriction must be placed on the combinations of boundary conditions for given values of the parameters, particularly P_1 and θ. Actually this is no different from what happens in analysis of the external laminar boundary layer. In that case, however, the boundary condition on v (or v_1) at the outside of the boundary layer is not satisfied, nor is there any particular reason for it to be. For channel flows such an omission is probably not allowable. In general, this means simply that there must be a flow through at least one wall, and this flow cannot be prescribed if similarity is to be maintained.

If, on the other hand, one examines the case where $2\zeta\theta+\lambda=0$ (eq. (85)), it is permissible to set $v_1=0$. Then the system is of fourth order with four boundary conditions. The condition $2\zeta\theta+\lambda=0$ corresponds exactly to the two cases mentioned in connection with equation (83). These are the only ones in which v_1 can vanish at the walls. Of the two cases defined by $2\zeta\theta+\lambda=0$, namely, $\lambda=0$, $\theta=0$ (exponential x-variation) and $\lambda=1$, $\zeta\theta=-1/2$ (x^θ-variation), the first has been solved earlier. In the second case, for linear viscosity-temperature variation, equations (86) and (87) become ($-2\theta=\zeta=\lambda=1$)

$$u_1^2+RT_1=-\left[\frac{8aRL}{P_1 B^2}T_1^2(-1)\right]u_{1,\eta\eta}\tag{89}$$

$$\begin{aligned}
\left[\frac{P_1 B^2}{8Lb}T_1^2(-1)\left(1-\frac{2c_p}{R}\right)\right]u_1(RT_1)&=(RT_1)_{\eta\eta}\\
&\quad+\frac{aR}{b}u_{1,\eta}^2
\end{aligned}\tag{90}$$

where the boundary conditions are

$$\left.\begin{aligned}
&u_1(\pm1)=0\\
&T_1(-1)=\text{Constant}\\
&T_1(+1)=\text{Another constant}
\end{aligned}\right\}\tag{91}$$

The solution of this system is more difficult to obtain than is the solution of equations (35) to (37), although a similar procedure can be followed.

If the flow is fairly slow, with the Mach number limited to, say, a few tenths, the profiles are, to good accuracy

$$\left.\begin{aligned} T_1 &= \frac{1}{2}\left[T_1(+1)+T_1(-1)\right]+\frac{\eta}{2}\left[T_1(+1)-T_1(-1)\right] \\ u_1 &= \left\{\frac{P_1 d^2}{aL\left[T_1(+1)+T_1(-1)\right]}\right\} \\ &\quad\times\left\{\frac{1-\eta^2}{4}+\left[\frac{T_1(+1)-T_1(-1)}{T_1(+1)+T_1(-1)}\right]\frac{\eta(1-\eta^2)}{12}\right\} \end{aligned}\right\} \tag{92}$$

where use has been made of equation (10) to determine B. It may be observed that the distance L can be defined by

$$L=-\left(\frac{P}{\frac{\partial P}{\partial X}}\right)_{X=L}$$

which follows directly from the last of equations (82). If the wall temperatures are equal, the velocity profile is exactly the familiar Poiseuille one.

A final remark: It can be seen that the system described by equations (90) and (91) will probably admit pairs of solutions just as the free-convection flow of equations (35) to (37) does.

CONCLUDING REMARKS

When fully developed channel flows are considered, the cases that can be solved exactly are very limited. For a gas a constant pressure is required, or at least one which does not vary in a streamwise direction. This case is analogous to Couette flow in that no approximations need be made in arriving at a relatively simple mathematical problem. In the case of a liquid, one can solve the exact case of constant pressure gradient in the streamwise direction. For both the gas and the liquid, the wall temperatures must be constant. In the gas case nothing astonishing happens. However, in the liquid case a surprising result arises. There appear to be (except for certain singular cases) either two or no solutions for the flow. This result, which has been discussed extensively by Ostrach, has one solution for which frictional heating is negligibly small. The second is one in which the frictional heating is large, and thus the temperature is raised and the buoyancy effect is increased. In the present report an approximate analytic solution of this problem is given. The results agree very well with Ostrach's machine calculations. Although an analysis is given for the case of variable viscosity, it turns out that for the cases of interest, wherein the fluid velocity is kept within reason, the temperature variations are small and there is no reason to consider variable viscosity or conductivity.

These so-called exact solutions, particularly in the case of a gas, do not cover all the flows of interest. Hence, consideration is given to cases in which there are streamwise temperature and pressure gradients but in which the flow velocity is virtually independent of distance along the channel. For a gas the mere presence of a pressure gradient requires a temperature gradient, while for a liquid the presence or absence of a streamwise temperature variation is governed by the wall temperature conditions.

In these cases one of two situations occurs. If the temperature variation across the channel is of the order of the temperature level, then in order that "channel flow" be maintained, the convection terms in the energy equation must be negligible. This implies that the mean flow Mach number is small and also that the frictional heating is negligible. For such circumstances the equations are separated and can readily be integrated for any case of interest. Only one solution exists.

On the other hand, if the temperature variations are small, more complicated effects occur. This situation of very small temperature changes admits very large flow velocities (see the discussion following eqs. (46)). Then both the frictional heating and the thermal convection effects can be significant. In such cases (when the frictional heating matters) two solutions can occur. These flows qualify as quasi-incompressible in that the only place where compressibility effects matter is in the buoyancy term in the streamwise momentum equation.

The forced flow perhaps deserves an added remark. When the temperature varies only slightly across the channel, the velocity profile must be the usual parabolic one. When the variation in temperature is large, the profiles can still be found in closed form but are more complicated.

If streamwise variations of velocity are allowed, the flow is more complex. A description is given of the circumstances under which the independent variables are separable. These forms can yield new results only for gas flows and show that the streamwise variation must be either as a power of x (streamwise coordinate) or exponentially with x. With two exceptions, only one of which admits x-variations, these flows require that the body force be negligible. The exponential cases apply to the full Navier-Stokes equations, while the other ones require an expansion of the equations of motion in terms of the width-to-length ratio of the channel. For all these cases, the streamwise Mach number is independent of x. All but one of these possible flows lead to difficulties with boundary conditions and require a flow through the walls. The lone exception has streamwise velocity proportional to $1/\sqrt{x}$. For small flow Mach numbers the solution is similar to that for Poiseuille flow but allows for temperature variations across the channel.

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APPENDIX A

SYMBOLS

A,A_1,A_2	parameters defined in eqs. (41) and (C5)
a,b	parameters defined in eqs. (9)
B	parameter defined in eqs. (10)
C	parameter defined in eq. (C9)
c_p,c_v	specific heats
d	wall spacing
f,f_1	body forces in X- and Y-directions, respectively, considered positive in the minus X- and Y-directions
$g(x)$	pressure perturbation, eq. (47)
H,H_1	temperature functions, eqs. (16) or (22) and (C1)
K	parameter defined in eq. (34)
k	thermal conductivity
L	characteristic length, eq. (47)
m	parameter defined in eq. (37)
P	pressure
R	gas constant, eq. (5a)
s,T_a	parameters defined in eq. (21)
T	temperature
T^*	reference temperature defined in eq. (19)
U,U_1	dimensionless velocities in eqs. (33) and (C1)
u,v	velocity components in X- and Y-directions, respectively
X,Y	Cartesian coordinates, X being in the main flow direction
x,y	X/L and Y/d, respectively
$\alpha_1,\alpha_2,\alpha_3$	parameters defined in eqs. (26)
$\bar{\beta}$	volumetric expansion coefficient (see eq. (5b))
δ,ϵ	small parameters introduced in eqs. (47)
η	dimensionless distance across channel in eq. (10)
ζ,θ,λ	parameters defined in eqs. (81) and (82) and after eq. (84)
μ	viscosity
ρ	density
τ	dimensionless temperature difference in eq. (33)
Subscripts:	
m	mean value corresponding to average of wall temperatures
w_0	wall conditions at $Y=0$
w_1	wall conditions at $Y=d$
X, Y, x, y, η	partial derivative with respect to that variable
0, 1	zero-order and first-order solution in eq. (47)
Superscripts:	
(0), (1)	first two approximations
bars	reference values in eqs. (5b) or (47)

APPENDIX B

A GENERALIZATION OF EQUATIONS (7)

If a transverse body force f_1 and a distribution of heat sources ρQ, where Q is a constant, is included, equations (7) become

$$\left.\begin{aligned} v&=0\\ (\mu u_Y)_Y&=\rho f\\ P_Y&=-\rho f_1\\ (kT_Y)_Y&=-\mu u_Y^2-\rho Q \end{aligned}\right\} \qquad \text{(B1)}$$

These equations can be solved in exactly the same manner as equations (7). The results are, for linear variation of viscosity and conductivity with temperature, $f_1\neq 0$, and $Q=0$,

$$\left.\begin{aligned} P&=P_0e^{-\sigma\eta}\\ u&=\frac{fP_0R}{2af_1^2}\left[(\eta-1)e^{\sigma}-(\eta+1)e^{-\sigma}+2e^{-\sigma\eta}\right]\\ T&=\frac{T_{w_1}+T_{w_0}}{2}+\eta\left(\frac{T_{w_1}-T_{w_0}}{2}\right)\\ &\quad-\frac{a}{b}\left(\frac{fP_0R}{2af_1^2}\right)^2\left\{\frac{\eta^2}{2}(e^{\sigma}-e^{-\sigma})^2-\frac{4}{\alpha}(e^{\sigma}-e^{-\sigma})e^{-\sigma\eta}\right.\\ &\quad+e^{-2\sigma\eta}+\frac{2}{\alpha}(e^{2\sigma}-e^{-2\sigma})-(e^{2\sigma}+e^{-2\sigma})+1\\ &\quad\left.+\eta\left[\frac{e^{2\sigma}-e^{-2\sigma}}{2}-\frac{2}{\alpha}(e^{\sigma}-e^{-\sigma})^2\right]\right\} \end{aligned}\right\} \qquad \text{(B2)}$$

where $c=Bf_1/2RT_{w_0}$ is defined by

$$\sigma=\frac{2T_mR}{df_1}-\frac{aR}{bdf_1}\left(\frac{fP_0R}{2af_1^2}\right)^2\left[(e^{\sigma}-e^{-\sigma})^2(1/3-4/\sigma^2)+\frac{7}{22}(e^{2\sigma}-e^{-2\sigma})+2\right]$$

and, for $f_1=0$ and $Q\neq 0$,

$$\left.\begin{aligned} P&=\text{Constant}\\ u&=\frac{aB^2fP}{8R\mu_{w_0}^2}(\eta^2-1)\\ T&=\frac{T_{w_0}+T_{w_1}}{2}+\eta\left(\frac{T_{w_1}-T_{w_0}}{2}\right)-\frac{a}{b}\left(\frac{aB^2fP}{4R\mu_{w_0}^2}\right)^2\left(\frac{\eta^4-1}{12}\right)\\ &\quad+\frac{PQB^2}{8bRT_{w_0}^2}(1-\eta^2) \end{aligned}\right\} \qquad \text{(B3)}$$

where

$$\frac{1}{240}\left(\frac{BT_m}{dT_{w_0}}\right)^5\left[\left(\frac{\rho_m^2fd^3}{\mu_mk_m}\right)\frac{fd}{T_m}\right]+\left(\frac{BT_m}{dT_{w_0}}\right)^3\left(\frac{\rho_mQd^2}{12k_mT_m}\right)+\left(\frac{BT_m}{dT_{w_0}}\right)=1$$

If Q is positive, there is only one real root of this equation, that root being such that

$$0<\frac{BT_m}{dT_{w_0}}<1$$

If Q is sufficiently negative, there can be three real positive roots.

Other solutions can readily be obtained for the case where neither f_1 nor Q vanishes or where other distributions of heat sources occur.

APPENDIX C

EXACT SOLUTION FOR LIQUID

The problem at hand is to solve equations (27) and (17b), subject to equations (8). Define

$$\left.\begin{aligned} H_1 &= \frac{kT_a^2+2sH(T)}{kT_a^2+2sH(T^*)} = \left(\frac{T+T_a}{T^*+T_a}\right)^2 \\ U_1 &= u\sqrt{\frac{2s}{kT_a^2+2sH^*}} = \frac{u\sqrt{2s/k}}{T^*+T_a} \end{aligned}\right\} \tag{C1}$$

In terms of these variables the problem is

$$U_{1,\eta\eta} = -\left[\frac{B^2\overline{\rho\beta}fs\sqrt{2s/k}}{4(T^*+T_a)}\right]\left(1-\frac{1}{\sqrt{H_1}}\right) \tag{C2}$$

$$H_{1,\eta\eta} = -U^2_{1,\eta} \tag{C3}$$

$$\left.\begin{aligned} U_1(\pm 1) &= 0 \\ H_1(\pm 1) &= [(T_{w_0}+T_a)/(T^*+T_a)]^2 \end{aligned}\right\} \tag{C4}$$

where, to keep the problem from getting out of hand, equal wall temperatures are assumed. For convenience, define

$$\left.\begin{aligned} A_1 &= [(T_{w_0}+T_a)/(T^*+T_a)]^2 \\ A_2 &= \frac{B^2\overline{\rho\beta}fs\sqrt{2s/k}}{4(T^*+T_a)} \end{aligned}\right\} \tag{C5}$$

It should be remembered that the parameter B, and hence A_2, is as yet undetermined (recall eq. (10)).

The solution is found in exactly the same manner as the solution of equations (35), (36), and (37). Thus assume

$$U^{(0)} = A(1-\eta^2)$$

Then equations (C3) and (C4) yield

$$H_1 = A_1 + A^2(1-\eta^4)/3 \tag{C6}$$

If this is put into equation (C2), that expression becomes

$$U^{(1)}_{1,\eta\eta} = -A_2\left(1-\frac{\sqrt{3}/A}{\sqrt{1+\frac{3A_1}{A^2}-\eta^4}}\right) \tag{C7}$$

If this is integrated and the appropriate boundary conditions are satisfied, the result is

$$\frac{U^{(1)}_1}{A_2} = \frac{1-\eta^2}{2} + \sqrt{3}/A\left(\eta\int_0^\eta \frac{d\eta}{\sqrt{C^4-\eta^4}} - \int_0^1 \frac{d\eta}{\sqrt{C^4-\eta^4}} + \int_\eta^1 \frac{\eta\, d\eta}{\sqrt{C^4-\eta^4}}\right) \tag{C8}$$

where

$$C^4 = 1+3A_1/A^2 \tag{C9}$$

If equations (C8) and (C6) are put back into equation (C3) and the result is integrated across the channel, the result is

$$\frac{8A^2}{3A_2^2} = \frac{2}{3} - \frac{2\sqrt{3}}{A}\left[\int_0^1 \frac{(1-\eta^2)d\eta}{\sqrt{C^4-\eta^4}}\right] + \frac{6}{A^2}\int_0^1\left(\int_0^1 \frac{d\eta}{\sqrt{C^4-\eta^4}}\right)^2 d\eta \tag{C10}$$

Before using this equation to determine A, something has to be done about B, which is as yet undetermined. If equations (21), (C1), and (C6) are used, the second of equations (10) yields

$$\frac{dA(T^*+T_a)}{\sqrt{3}sB} = \int_0^1 \frac{d\eta}{\sqrt{C^4-\eta^4}} \tag{C11}$$

If this is put into the first of equations (C5), eliminating B, and the result is put into equation (C10), the defining equation for A is, finally,

$$A^2-3A\sqrt{3}\int_0^1\frac{(1-\eta^2)\mathrm{d}\eta}{\sqrt{C^4-\eta^4}}+9\int_0^1\left(\int_0^\eta\frac{\mathrm{d}\eta_1}{\sqrt{C^4-\eta^4}}\right)^2\mathrm{d}\eta-\frac{288sk\int_0^1\left(\frac{\mathrm{d}\eta}{\sqrt{C^4-\eta^4}}\right)^4}{[\rho\beta fd^2(T^*+T_a)]^2}=0 \qquad \text{(C12)}$$

This can be solved to very good accuracy by

$$A=\frac{3\sqrt{3}}{2}\int_0^1\frac{(1-\eta^2)\mathrm{d}\eta}{\sqrt{C^4-\eta^4}}+\frac{2\sqrt{72}sk\left(\int_0^1\frac{\mathrm{d}\eta}{\sqrt{C^4-\eta^4}}\right)^2}{\rho\beta fd^2(T^*+T_a)} \qquad \text{(C13)}$$

The second root of equation (C12) is never a significant one. This equation can readily be solved for A in terms of C (which is itself a function of A; see eq. (C9)) and the parameter

$$\frac{[\rho\beta fd^2(T^*+T_a)]^2}{sk}\equiv\frac{K}{\sqrt{A_1}(\sqrt{A_1}-1)}$$

where K is defined in equation (34). As in the case of constant properties, two possible values for A are again found.

The various integrals appearing in equations (C8) and (C13) can be evaluated as follows (ref. 8):

$$\left.\begin{aligned}
\int_0^\eta\frac{\mathrm{d}\eta}{\sqrt{C^4-\eta^4}}&=\frac{1}{C\sqrt{2}}\left[F\left(\sin^{-1}\sqrt{\frac{2\eta^2}{\eta^2+C^2}},1/\sqrt{2}\right)\right.\\
&\left.\equiv F(\pi/2,1/\sqrt{2})-F(\cos^{-1}\eta/C,1/\sqrt{2})\right]\\
\int_\eta^1\frac{\eta\,\mathrm{d}\eta}{\sqrt{C^4-\eta^4}}&=\frac{1}{2}[\sin^{-1}(1/C^2)-\sin^{-1}(\eta^2/C^2)]\\
\int_0^1\frac{\eta^2\mathrm{d}\eta}{\sqrt{C^4-\eta^4}}&=C\sqrt{2}\left\{E\left(\sin^{-1}\sqrt{\frac{2}{1+C^2}},1/\sqrt{2}\right)\right.\\
&\left.-\frac{1}{2}F\left[\sin^{-1}\sqrt{2/(1+C^2)},1/\sqrt{2}\right]-\sqrt{\frac{C^2-1}{C^2+1}}\right\}
\end{aligned}\right\} \qquad \text{(C14)}$$

where $F(\varphi,1/\sqrt{2})$ and $E(\varphi,1/\sqrt{2})$ are the respective elliptic integrals of the first and second kind of amplitude φ and modulus $1/\sqrt{2}$. Two of these integrals can be approximated as follows:

$$\int_0^\eta\frac{\mathrm{d}\eta}{\sqrt{C^4-\eta^4}}=\frac{1}{C^2}(\eta+\eta^5/10C^4)$$

which is correct to 1 percent if $C>1.3$, and

$$\int_0^1\frac{(1-\eta)\mathrm{d}\eta}{\sqrt{C^4-\eta^4}}=\frac{2}{3C^2}\left(1+\frac{3}{70C^4}+\frac{1}{88C^8}\right)$$

which is correct to 1 percent if $C>1.0$.

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