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Abstract

Robust control system analysis and design is based on an uncertainty description, called a linear fractional transformation (LFT), which separates the uncertain (or varying) part of the system from the nominal system. These models are also useful in the design of gain-scheduled control systems based on Linear Parameter Varying (LPV) methods. Low-order LFT models are difficult to form for problems involving nonlinear parameter variations. This paper presents a numerical computational method for constructing an LFT model from a given LPV model. The method is developed for multivariate polynomial problems, and uses simple matrix computations to obtain an exact low-order LFT representation of the given LPV system without the use of model reduction. Although the method is developed for multivariate polynomial problems, multivariate rational problems can also be solved using this method by reformulating the rational problem into a polynomial form.

1.0 Introduction

Formulation of linear fractional transformation (LFT) models of systems involving nonlinear parameter variations is of interest for robust control system analysis and design, as well as for control of linear parameter varying (LPV) systems. Moreover, the LFT models should be of low order for efficient computation during analysis and design. A matrix singular value decomposition (svd) approach was presented in 1985 in references [1] and [2] for computing LFT's for problems involving linear parameter variations. However, construction of low-order LFT models for problems involving nonlinear parameter dependencies is very difficult, because it is equivalent to a multidimensional minimal state-space realization problem for which there is no general theory. The approach that has been taken to date for solving nonlinear parameter-dependent problems is to successively decompose the system until all components are linear, and then to compute an LFT for each linear component based on the result presented in [1] and [2]. The LFT's associated with each system component are then combined using LFT properties to form the LFT model of the full system. Model reduction is usually required using this approach, because unnecessary repetitions of the varying parameters usually result. A decomposition method for LFT modeling of nonlinear parameter-dependent systems was first presented in reference [3], and later refined in reference [4]. This latter paper presented a special decomposition approach which reduces the number of unnecessary repetitions of the varying parameters, although model reduction is still employed to reduce the dimension of the resulting LFT model of the full system.

The approach presented in this paper is an extension of the computational approach of references [1] and [2] for nonlinear parameter-dependent systems, and is based on reference [5]. Specifically, the computational approach is developed for multivariate matrix polynomial problems, although multivariate rational problems can be solved using this approach by reformulating the rational problem to be in a multivariate polynomial form. Reference [6] presents a method for doing this. The LFT modeling approach presented in this paper requires no matrix decompositions for multivariate polynomial problems, and achieves a low-order LFT model directly - i.e., without the use of model reduction. Moreover, the computations are based on simple matrix operations, including the svd and solving linear matrix equations.

2.0 LFT Modeling Problem Definition

The LFT modeling problem to be addressed in this paper is defined below. It is assumned that the problem to be solved is in a multivariate matrix polynomial form. However, as shown in reference [6], multivariate rational problems can be reformulated as multivariate polynomial problems and solved using this approach. The problem is stated as follows.

Given: A linear parameter varying (LPV) model of a nonlinear parameter-dependent system, as represented by the following equation

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \mathbf{y} \end{bmatrix} = \begin{bmatrix} \mathbf{A}(\delta) & \mathbf{B}(\delta) \\ \mathbf{C}(\delta) & \mathbf{D}(\delta) \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{u} \end{bmatrix} = \mathbf{S}(\delta) \begin{bmatrix} \mathbf{x} \\ \mathbf{u} \end{bmatrix} \tag{2.1a}$$

$$\delta = [\delta_1, \delta_2, \cdots, \delta_m] \in \mathbf{R}^m \tag{2.1b}$$

where $S(\delta)$ has been separated into nominal and varying components, and the varying (or uncertain) component, $S_\Delta(\delta)$, has been formulated as an LFT problem given by the following equation

$$\mathbf{S}_\Delta(\delta) = \mathbf{P}_{21}\Delta(\mathbf{I} - \mathbf{P}_{11}\Delta)^{-1}\mathbf{P}_{12} = \mathbf{P}_{21}(\mathbf{I} - \Delta\mathbf{P}_{11})^{-1}\Delta\mathbf{P}_{12} \tag{2.2}$$

in which each element of $S_\Delta(\delta)$ is a multivariate polynomial function of the varying parameters, δ

Find: A low-order state-space uncertainty model that satisfies equation (2.2) and is characterized by the constant matrices P_{21}, P_{12}, and P_{11} and the uncertainty matrix $\Delta(\delta)$, as depicted below in Figure 1.

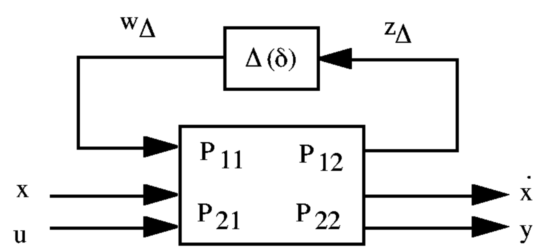

Figure 1. LFT Model of the Uncertain System

The P_{22} matrix represents the nominal part of the system, and is characterized by the nominal A, B, C, and D system matrices. The $S_\Delta(\delta)$ matrix of equation (2.2) is a known matrix of multivariate polynomials based on the LPV model for the system. Formulation of this matrix was discussed in reference [6]. The LFT model equations associated with Figure 1 are given below.

$$\mathbf{z}_\Delta = \mathbf{P}_{11}\mathbf{w}_\Delta + \mathbf{P}_{12} \begin{bmatrix} \mathbf{x} \\ \mathbf{u} \end{bmatrix} \tag{2.3a}$$

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \mathbf{y} \end{bmatrix} = \mathbf{P}_{21}\mathbf{w}_\Delta + \mathbf{P}_{22} \begin{bmatrix} \mathbf{x} \\ \mathbf{u} \end{bmatrix} \tag{2.3b}$$

$$\mathbf{w}_\Delta = \Delta\, \mathbf{z}_\Delta \qquad (2.3c)$$

where: $$\Delta(\delta) = \mathrm{diag}\,[\, \delta_1 \mathbf{I}_{n_1}, \delta_2 \mathbf{I}_{n_2}, \ldots, \delta_m \mathbf{I}_{n_m} \,] \in \mathbf{R}^{\, n_\Delta \times n_\Delta} \qquad (2.4a)$$

$$n_\Delta = \sum_{i=1}^{m} n_i \quad , \quad n_i = \dim(\, \mathbf{I}_i\,) \qquad (2.4b)$$

The LFT modeling problem consists of solving equation (2.2) for P_{21}, P_{12}, and P_{11} over some low-order Δ matrix (as defined by equation (2.4)). This is equivalent to a multidimensional minimal state-space realization problem over the m varying parameters in δ. Unfortunately, there is no existing minimal realization theory for general multidimensional systems (i.e., for $m \geq 3$) that can be used in solving this problem. In fact, there are no general minimality tests for multidimensional systems given a realization. This paper presents a numerical computational approach for solving equation (2.2) for P_{21}, P_{12}, and P_{11} such that the resulting Δ matrix is of low order. These results are summarized in Section 3.

3.0 Main Results: LFT Model Computation

As discussed in Section 2, the LFT problem to be solved is given by the following equation:

$$S_\Delta(\delta) = P_{21}(I - \Delta(\delta)P_{11})^{-1}\, \Delta(\delta)P_{12} \;\; , \quad S_\Delta(\delta) \in \mathcal{P}^{\, n_{rows} \times n_{cols}} \qquad (3.1)$$

The term $S_\Delta(\delta)$ is a known matrix function of the normalized uncertain parameters in δ, and P_{21}, P_{12}, and P_{11}, are the unknown matrix variables to be determined. The dimension of $\Delta(\delta)$ must also be determined in constructing the LFT model such that the resulting dimension is low-order. It is assumed that the functional form of the elements of $S_\Delta(\delta)$ is multivariate polynomial. However, as discussed in Section 2, rational problems can also be solved by reformulation of the rational problem (see Reference [6]).

3.1 Numerical LFT Solution Approach

As can be seen in equation (3.1), solving for the matrices P_{21}, P_{12}, P_{11} and $\Delta(\delta)$ involves the inversion of the matrix $[I - \Delta(\delta)P_{11}]$. For multivariate polynomial problems, this matrix inversion can be exactly replaced by a finite series and an associated nilpotency condition. This is expressed in the following equations.

$$(\, I - \Delta(\delta)P_{11}\,)^{-1} = I + (\Delta(\delta)P_{11}) + (\Delta(\delta)P_{11})^2 + \ldots + (\Delta(\delta)P_{11})^r \qquad (3.2)$$

$$[\Delta(\delta)P_{11}]^{r+1} = 0 \qquad (3.3)$$

Substituting equation (3.2) into equation (3.1) results in the following equation for $S_\Delta(\delta)$.

$$S_\Delta(\delta) = P_{21}\Delta(\delta)P_{12} + P_{21}[\, \Delta(\delta)P_{11} + (\Delta(\delta)P_{11})^2 + \ldots (\Delta(\delta)P_{11})^r \,]\Delta(\delta)P_{12} \qquad (3.4)$$

The first term on the right side of equation (3.4), i.e. $P_{21}\Delta P_{12}$, represents the linear uncertain components of $S_\Delta(\delta)$, and the second term adds in the nonlinear terms. For the case of multivariate polynomial uncertainties, the nonlinear terms of $S_\Delta(\delta)$ consist of crossterms of the δ parameters and n^{th}-order terms. Thus, the order (r) of the highest term in the series of equation (3.4) is determined by the degree of the highest term appearing in $S_\Delta(\delta)$, where crossterm degree can be defined as follows.

$$\text{degree}\left(\delta_1^{\xi_1}\, \delta_2^{\xi_2}\, \delta_3^{\xi_3} \ldots \delta_i^{\xi_i} \right) = \left(\xi_1 + \xi_2 + \ldots + \xi_i \right) - 1 \quad ; \quad i \leq m \tag{3.5}$$

Then, the exponent r in equation (3.4) can be defined by the following inequality.

$$r \leq \left(\eta_1 + \eta_2 + \ldots + \eta_m \right) - 1 \tag{3.6}$$

where η_i is the maximum degree of δ_i in $S_\Delta(\delta)$.

Since the uncertain system matrix, $S_\Delta(\delta)$, has as its elements multivariate polynomial functions of δ, it can be easily expanded in a similar manner as the right side of equation (3.4), i.e.:

$$\mathbf{S}_\Delta(\delta) = \mathbf{S}_{\Delta_\mathbf{0}}(\delta) + \mathbf{S}_{\Delta_\mathbf{1}}(\delta) + \cdots + \mathbf{S}_{\Delta_\mathbf{r}}(\delta) \tag{3.7}$$

Then like terms from equations (3.4) and (3.7) can be equated as follows.

$$\mathbf{S}_{\Delta_\mathbf{i}}(\delta) = \mathbf{P_{21}}(\Delta(\delta)\mathbf{P}_{11})^{\mathbf{i}}\Delta(\delta)\mathbf{P_{12}} \quad , \quad i = 0, 1, \ldots, r \tag{3.8}$$

The uncertainty modeling problem therefore requires that equations (3.8) be solved for P_{21}, P_{12}, P_{11}, and $\Delta(\delta)$ such that the nilpotency condition of equation (3.3) is satisfied.

In order to evaluate equations (3.8) and (3.3) in more detail, consider an expanded definition of P_{11}, P_{12}, and P_{21} containing partitioned submatrices associated with the $\delta_i I_{n_i}$ blocks of the Δ matrix given in equation (2.4a), as shown below.

$$\mathbf{P}_{11} = \begin{bmatrix} \mathbf{P}_{11_{\delta_1\delta_1}} & \mathbf{P}_{11_{\delta_1\delta_2}} & \cdots & \mathbf{P}_{11_{\delta_1\delta_m}} \\ \mathbf{P}_{11_{\delta_2\delta_1}} & \mathbf{P}_{11_{\delta_2\delta_2}} & \cdots & \mathbf{P}_{11_{\delta_2\delta_m}} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{P}_{11_{\delta_m\delta_1}} & \mathbf{P}_{11_{\delta_m\delta_2}} & \cdots & \mathbf{P}_{11_{\delta_m\delta_m}} \end{bmatrix} \tag{3.9}$$

$$\mathbf{P}_{12} = \begin{bmatrix} \mathbf{P}_{12_{\delta_1}} \\ \mathbf{P}_{12_{\delta_2}} \\ \vdots \\ \mathbf{P}_{12_{\delta_m}} \end{bmatrix} \tag{3.10}$$

$$\mathbf{P}_{21} = \begin{bmatrix} \mathbf{P}_{21_{\delta_1}} & \mathbf{P}_{21_{\delta_2}} & \cdots & \mathbf{P}_{21_{\delta_m}} \end{bmatrix} \tag{3.11}$$

where: $$\mathbf{P}_{11_{\delta_i \delta_j}} \in R^{n_i \times n_j}, \ \mathbf{P}_{12_{\delta_i}} \in R^{n_i \times n_{cols}}, \ \mathbf{P}_{21_{\delta_i}} \in R^{n_{rows} \times n_i} \tag{3.12}$$

Equation (2.4a) is repeated here for convenience.

$$\Delta = \begin{bmatrix} \delta_1 \mathbf{I}_{n_1} & \delta_2 \mathbf{I}_{n_2} & \cdots & \delta_m \mathbf{I}_{n_m} \end{bmatrix} \tag{3.13}$$

Substituting equations (3.9) - (3.13) into equations (3.8) and (3.3) leads to a set of extremely complicated equations to solve. In order to satisfy the nilpotency condition of equation (3.3), the matrix P_{11} must itself be nilpotent. Allowing P_{11} to have a pre-defined nilpotent structure provides a means of somewhat simplifying these equations while assisting in satisfying the nilpotency condition of equation (3.3). The following Lemma establishes a general nilpotency structure that will be used throughout this paper.

Lemma 3.1

Let $\mathbf{A} \in \mathbf{R}^{n \times n}$ be a quasi-triangular partitioned matrix whose main-diagonal blocks are nilpotent, as defined below.

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} & \cdots & \mathbf{A}_{1m} \\ \mathbf{0} & \mathbf{A}_{22} & \cdots & \mathbf{A}_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{A}_{mm} \end{bmatrix} \tag{3.14a}$$

$$\mathbf{A}_{ii} \in \mathbf{R}^{n_i \times n_i}, \ \mathbf{A}_{ii}^{\eta_i} = 0, \ \eta_i \le n_i \ , \ i = 1, 2, \dots, m \tag{3.14b}$$

Then matrix A is a nilpotent matrix with index of nilpotency, η, as defined below.

$$\mathbf{A}^{\eta} = 0 \ , \qquad \eta = \sum_{i=1}^{m} \eta_i \le n \tag{3.15}$$

Proof:

Nilpotency of matrix A is clearly established by considering the eigenvalues of A. Since A is upper triangular, its eigenvalues are comprised of the eigenvalues of its main-diagonal blocks. Since each main-diagonal block is itself nilpotent, the eigenvalues of each must be zero (see Reference [7]). Hence, the eigenavalues of A must be zero and A must therefore be nilpotent. The index of nilpotency, η, of matrix A is established by the following.

Let: $r = \eta_1 + \eta_2 + \dots + \eta_m$

$$\Rightarrow \qquad \mathbf{A}^{r} = \mathbf{A}^{\eta_1 + \eta_2 + \dots + \eta_n} = \mathbf{A}^{\eta_1} \mathbf{A}^{\eta_2} \cdots \mathbf{A}^{\eta_m}$$

Then, each matrix A^{η_i} contains a zero diagonal block corresponding to A_{ii}, since η_i is its index of nilpotency. It can therefore be shown that multiplication of these matrices to obatin A^r for $r = \eta_1 + \eta_2 + ... + \eta_m$ results in the zero matrix, since each main-diagonal block is zero. However, if $r < \eta_1 + \eta_2 + ... + \eta_m$ then one of the main-diagonal blocks will not be zero, hence A^r will not equal zero. Thus, the nilpotency index for A must be equal to $r = \eta_1 + \eta_2 + ... + \eta_m$. As can be verified in Reference [8], the nilpotency index for any matrix must be less than or equal to its dimension (i.e., n for matrix A). This can also be verified by the following.

$$\eta_i \le n_i \quad \text{for every } i = 1, 2, \dots, m$$
$$\Rightarrow \quad \eta = \sum_{i=1}^{m} \eta_i \le \sum_{i=1}^{m} n_i = n$$

Thus, equation (3.15) is satisfied.

QED

Note that the quasi-triangular structure defined by Lemma 3.1 is sufficient but not necessary for nilpotency. Other special structures can also be found. In fact, nilpotent matrices can be fully populated with nonzero elements. However, assuming some special structure for P_{11} simplifies the solution of equations (3.8) and (3.3). For implementation purposes, allowing the special structure to be more general than upper-quasi-triangular may result in a less conservative (i.e., lower order) P-Δ model for some problems. However, for purposes of this paper, Lemma 3.1 will be used to fix the structure of P_{11} so that the solution can be clearly derived.

The quasi-triangular structure defined by Lemma 3.1 can be used in expanding equations (3.8) and (3.3). Thus, let P_{11} be defined to have the following upper quasi-triangular structure.

$$\mathbf{P}_{11} = \begin{bmatrix} \mathbf{P}_{11_{\delta_1\delta_1}} & \mathbf{P}_{11_{\delta_1\delta_2}} & \cdots & \mathbf{P}_{11_{\delta_1\delta_m}} \\ 0 & \mathbf{P}_{11_{\delta_2\delta_2}} & \cdots & \mathbf{P}_{11_{\delta_2\delta_m}} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{P}_{11_{\delta_m\delta_m}} \end{bmatrix} \tag{3.16}$$

where: $$(\mathbf{P}_{ii_{\delta_i\delta_i}})^{\eta_i} = 0, \quad \eta_i \le n_i \;, \quad i = 1, 2, \dots, m \tag{3.17}$$

Then substitution of equations (3.10), (3.11) and (3.16) into equations (3.8) yields the following set of equations.

Linear Terms:

$$\mathbf{P}_{21_{\delta_i}} \mathbf{P}_{12_{\delta_i}} = \mathbf{S}_{\Delta_{0_{\delta_i}}} \;, \quad i = 1, 2, \dots, m \tag{3.18}$$

ξ^{th} **-Degree Terms**:

$$\mathbf{P}_{21_{\delta_i}} (\mathbf{P}_{11_{\delta_i \delta_i}})^{\xi-1} \mathbf{P}_{12_{\delta_i}} = \mathbf{S}_{\Delta_{\xi-1}(\delta_i)^{\xi}} \quad , \quad i = 1, 2, \ldots , m \qquad (3.19)$$

Crossterms:

$$\mathbf{P}_{21_{\delta_{i_1}}} (\mathbf{P}_{11_{\delta_{i_1}\delta_{i_1}}})^{\xi_{i_1}-1} \mathbf{P}_{11_{\delta_{i_1}\delta_{i_2}}} (\mathbf{P}_{11_{\delta_{i_2}\delta_{i_2}}})^{\xi_{i_2}-1} \cdots \mathbf{P}_{11_{\delta_{i_{n_T-1}}\delta_{i_{n_T}}}} (\mathbf{P}_{11_{\delta_{i_{n_T}}\delta_{i_{n_T}}}})^{\xi_{i_{n_T}}-1} \mathbf{P}_{12_{\delta_{i_{n_T}}}}$$

$$= \mathbf{S}_{\Delta_{\xi-1}(\delta_{i_1})^{\xi_{i_1}} (\delta_{i_2})^{\xi_{i_2}} \cdots (\delta_{i_{n_T}})^{\xi_{i_{n_T}}}} \qquad (3.20)$$

where:

$$\xi = \xi_{i_1} + \xi_{i_2} + \ldots + \xi_{i_{n_T}}$$
$$i_1 = 1, 2, \ldots , m - (n_T - 1)$$
$$i_2 = i_1 + 1, i_1 + 2, \ldots , m - (n_T - 2)$$
$$\vdots$$
$$i_{n_T} = i_1 + (n_T - 1), \ldots , m$$

n_T = number of parameters in the crossterm

Note that the $\mathbf{S}_\Delta$ terms on the right-hand side of equations (3.18) - (3.20) are the known constant matrix coefficients associated with the indicated parameter terms in $S_\Delta(\delta)$. Moreover, depending on the number of parameters and the degree of each appearing in $S_\Delta(\delta)$, there can be literally hundreds of $\mathbf{S}_\Delta$ coefficient terms - and hence equations to be solved.

3.2 Numerical LFT Model Solution

This section presents a numerical approach for solving all equations of the form defined by equations (3.18) - (3.20) such that the nilpotency condition of equation (3.3) is satisfied and the resulting P-Δ model is of low-order. The results of this section are divided into three sub-sections. The first sub-section presents a solution for P_{21}, P_{12}, and the main-diagonal blocks of P_{11}; the second sub-section presents a solution for the off-diagonal blocks of P_{11}; and the third sub-section presents results relating to nilpotency and reducibility of the resulting model.

3.2.1 Simultaneous Solution of P_{21}, P_{12}, and P_{11} Main-Diagonal Blocks for each δ_i Parameter

The P_{21}, P_{12}, and P_{11} main-diagonal blocks are solved simultaneously for each uncertain parameter δ_i using the linear and ξ^{th}-degree terms defined by equations (3.18) and (3.19). Moreover, the solution is accomplished such that the resulting main-diagonal blocks of P_{11} are nilpotent with the appropriate index of nilpotency - as required by equation (3.17). This solution is accomplished numerically with a matrix singular value decomposition (svd) by recognizing that this part of the problem is equivalent to a 1-D state-space (minimal) realization problem and by appropriately defining an equivalent Hankel matrix. The solution is accomplished for each δ_i parameter as shown by the following theorem (which is based on Theorem 6-4, pages 268 - 272, of reference [9]).

<u>Theorem 3.1</u>

Consider the linear and ζ^{th}-degree terms of $S_\Delta(\delta) \in \mathcal{P}^{n_{rows}\times n_{cols}}$, which can be expanded as follows

$$S_{\Delta_{L,\zeta}}(\delta) = [S_{\Delta_{0_{\delta_i}}}]\,\delta_i + [S_{\Delta_{1_{\delta_i^2}}}]\,\delta_i^2 + \dots + [S_{\Delta_{\eta_i-1_{\delta_i^{\eta_i}}}}]\,\delta_i^{\eta_i} \tag{3.21a}$$

$$\Rightarrow \quad \mathbf{S}_{\Delta_{L,\zeta}} = \sum_{n=1}^{\eta_i} [\mathbf{S}_{\Delta_{n-1_{\delta_i^n}}}]\,\delta_i^n \tag{3.21b}$$

and use the constant coefficient matrices of equation (3.21) to form the Hankel matrices defined below

$$\bar{\mathbf{S}}_{\Delta_{0_{\delta_i}}} = \begin{bmatrix} \mathbf{S}_{\Delta_{0_{\delta_i}}} & \mathbf{S}_{\Delta_{1(\delta_i)^2}} & \mathbf{S}_{\Delta_{2(\delta_i)^3}} & \cdots & \mathbf{S}_{\Delta_{\eta_i-1(\delta_i)^{\eta_i}}} \\ \mathbf{S}_{\Delta_{1(\delta_i)^2}} & \mathbf{S}_{\Delta_{2(\delta_i)^3}} & \vdots & ⋰ & \mathbf{0} \\ \mathbf{S}_{\Delta_{2(\delta_i)^3}} & \vdots & \mathbf{S}_{\Delta_{\eta_i-1(\delta_i)^{\eta_i}}} & ⋰ & \vdots \\ \vdots & \mathbf{S}_{\Delta_{\eta_i-1(\delta_i)^{\eta_i}}} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{S}_{\Delta_{\eta_i-1(\delta_i)^{\eta_i}}} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \end{bmatrix} \tag{3.22}$$

$$\bar{\mathbf{S}}_{\Delta_{1_{\delta_i}}} = \begin{bmatrix} \mathbf{S}_{\Delta_{1(\delta_i)^2}} & \mathbf{S}_{\Delta_{2(\delta_i)^3}} & \cdots & \mathbf{S}_{\Delta_{\eta_i-1(\delta_i)^{\eta_i}}} & \mathbf{0} \\ \mathbf{S}_{\Delta_{2(\delta_i)^3}} & \vdots & ⋰ & \mathbf{0} & \mathbf{0} \\ \vdots & \mathbf{S}_{\Delta_{\eta_i-1(\delta_i)^{\eta_i}}} & ⋰ & \vdots & \vdots \\ \mathbf{S}_{\Delta_{\eta_i-1(\delta_i)^{\eta_i}}} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \end{bmatrix} \tag{3.23}$$

Using a matrix svd, factor equation (3.22) as follows

$$\bar{\mathbf{S}}_{\Delta_{0_{\delta_i}}} = \mathbf{U}_{\delta_i}\Sigma_{\delta_i}\mathbf{V}_{\delta_i}^T = (\mathbf{U}_{\delta_i}\Sigma_{\delta_i}^{1/2})(\Sigma_{\delta_i}^{1/2}\mathbf{V}_{\delta_i}^T) = \bar{\mathbf{P}}_{21_{\delta_i}}\bar{\mathbf{P}}_{12_{\delta_i}} \tag{3.24}$$

where: $\qquad \mathrm{rank}(\overline{\mathbf{S}}_{\Delta_0}{}_{\delta_i}) = \mathrm{rank}(\overline{\mathbf{P}}_{21}{}_{\delta_i}) = \mathrm{rank}(\overline{\mathbf{P}}_{12}{}_{\delta_i})$

Then the matrices $P_{21_{\delta_i}}$, $P_{12_{\delta_i}}$, and $P_{11_{\delta_i\delta_i}}$ form an irreducible realization of $S_{\Delta_{L,\zeta}}(\delta)$ as definedby equation (3.21), where:

$$\mathbf{P}_{21_{\delta_i}} = \begin{bmatrix} \mathbf{I}_{n_{rows}} & \mathbf{0} \end{bmatrix} \overline{\mathbf{P}}_{21_{\delta_i}} \tag{3.25}$$

$$\mathbf{P}_{12_{\delta_i}} = \overline{\mathbf{P}}_{12_{\delta_i}} \begin{bmatrix} \mathbf{I}_{n_{cols}} \\ \mathbf{0} \end{bmatrix} \tag{3.26}$$

$$\mathbf{P}_{11_{\delta_i\delta_i}} = (\overline{\mathbf{P}}_{21_{\delta_i}})^{\dagger} \overline{\mathbf{S}}_{\Delta_1}{}_{\delta_i} (\overline{\mathbf{P}}_{12_{\delta_i}})^{\dagger} \tag{3.27}$$

and the notation $(\mathbf{A})^{\dagger}$ designates the pseudoinverse of matrix $\mathbf{A}$.

<u>Proof</u>:

From equation (3.24), define the following:

$$(\overline{\mathbf{S}}_{\Delta_0}{}_{\delta_i})^{\dagger} = (\overline{\mathbf{P}}_{12_{\delta_i}})^{\dagger} (\overline{\mathbf{P}}_{21_{\delta_i}})^{\dagger} \tag{3.28}$$

Then it is easy to show that:

$$\begin{aligned} \overline{\mathbf{S}}_{\Delta_0}{}_{\delta_i} (\overline{\mathbf{S}}_{\Delta_0}{}_{\delta_i})^{\dagger} \overline{\mathbf{S}}_{\Delta_0}{}_{\delta_i} &= \overline{\mathbf{P}}_{21_{\delta_i}} \overline{\mathbf{P}}_{12_{\delta_i}} (\overline{\mathbf{P}}_{12_{\delta_i}})^{\dagger} (\overline{\mathbf{P}}_{21_{\delta_i}})^{\dagger} \overline{\mathbf{P}}_{21_{\delta_i}} \overline{\mathbf{P}}_{12_{\delta_i}} \\ &= \overline{\mathbf{P}}_{21_{\delta_i}} \overline{\mathbf{P}}_{12_{\delta_i}} = \overline{\mathbf{S}}_{\Delta_0}{}_{\delta_i} \end{aligned} \tag{3.29}$$

Define the following relationship between the Hankel matrices of equations (3.22) and (3.23):

$$\overline{\mathbf{S}}_{\Delta_1}{}_{\delta_i} = \mathbf{M}_{\delta_i} \overline{\mathbf{S}}_{\Delta_0}{}_{\delta_i} = \overline{\mathbf{S}}_{\Delta_0}{}_{\delta_i} \mathbf{N}_{\delta_i} \tag{3.30}$$

which generalizes to:

$$\mathbf{M}_{\delta_i}{}^{n} \overline{\mathbf{S}}_{\Delta_0}{}_{\delta_i} = \overline{\mathbf{S}}_{\Delta_0}{}_{\delta_i} \mathbf{N}_{\delta_i}{}^{n} \quad ; \quad n = 0, 1, 2, \ldots \tag{3.31}$$

where:

$$\mathbf{M}_{\delta_i} = \begin{bmatrix} \mathbf{0} & \mathbf{I}_{n_{rows}} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_{n_{rows}} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \mathbf{I}_{n_{rows}} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \end{bmatrix} \qquad (3.32)$$

$$\mathbf{N}_{\delta_i} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{I}_{n_{cols}} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{n_{cols}} & \ddots & \vdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \mathbf{0} & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{I}_{n_{cols}} & \mathbf{0} \end{bmatrix} \qquad (3.33)$$

Consider the following:

$$(\mathbf{P}_{11_{\delta_i\delta_i}})^2 = [(\overline{\mathbf{P}}_{21_{\delta_i}})^\dagger \overline{\mathbf{S}}_{\Delta_{1_{\delta_i}}} (\overline{\mathbf{P}}_{12_{\delta_i}})^\dagger]^2 = (\overline{\mathbf{P}}_{21_{\delta_i}})^\dagger \overline{\mathbf{S}}_{\Delta_{1_{\delta_i}}} (\overline{\mathbf{P}}_{12_{\delta_i}})^\dagger (\overline{\mathbf{P}}_{21_{\delta_i}})^\dagger \overline{\mathbf{S}}_{\Delta_{1_{\delta_i}}} (\overline{\mathbf{P}}_{12_{\delta_i}})^\dagger$$

$$= (\overline{\mathbf{P}}_{21_{\delta_i}})^\dagger \mathbf{M}_{\delta_i} \overline{\mathbf{S}}_{\Delta_{0_{\delta_i}}} (\overline{\mathbf{P}}_{12_{\delta_i}})^\dagger (\overline{\mathbf{P}}_{21_{\delta_i}})^\dagger \mathbf{M}_{\delta_i} \overline{\mathbf{S}}_{\Delta_{0_{\delta_i}}} (\overline{\mathbf{P}}_{12_{\delta_i}})^\dagger$$

$$= (\overline{\mathbf{P}}_{21_{\delta_i}})^\dagger \mathbf{M}_{\delta_i} \overline{\mathbf{P}}_{21_{\delta_i}} \overline{\mathbf{P}}_{12_{\delta_i}} (\overline{\mathbf{P}}_{12_{\delta_i}})^\dagger (\overline{\mathbf{P}}_{21_{\delta_i}})^\dagger \mathbf{M}_{\delta_i} \overline{\mathbf{S}}_{\Delta_{0_{\delta_i}}} (\overline{\mathbf{P}}_{12_{\delta_i}})^\dagger$$

$$= (\overline{\mathbf{P}}_{21_{\delta_i}})^\dagger \mathbf{M}_{\delta_i}{}^2 \overline{\mathbf{S}}_{\Delta_{0_{\delta_i}}} (\overline{\mathbf{P}}_{12_{\delta_i}})^\dagger$$

$$\Rightarrow \qquad (\mathbf{P}_{11_{\delta_i\delta_i}})^n = (\overline{\mathbf{P}}_{21_{\delta_i}})^\dagger \mathbf{M}_{\delta_i}{}^n \overline{\mathbf{S}}_{\Delta_{0_{\delta_i}}} (\overline{\mathbf{P}}_{12_{\delta_i}})^\dagger \qquad (3.34)$$

Now, the constant coefficient matrices of equation (3.26) can be rewritten as follows:

$$\mathbf{S}_{\Delta_{n-1_{\delta_i n}}} = \begin{bmatrix} \mathbf{I}_{n_{rows}} & \mathbf{0} \end{bmatrix} \mathbf{M}_{\delta_i}{}^{n-1} \overline{\mathbf{S}}_{\Delta_{0_{\delta_i}}} \begin{bmatrix} \mathbf{I}_{n_{cols}} \\ \mathbf{0} \end{bmatrix} \qquad (3.35)$$

Substituting equation (3.29) into this expression yields:

$$\Rightarrow \quad \mathbf{S}_{\Delta_{n-1_{\delta_i n}}} = \begin{bmatrix} \mathbf{I}_{n_{rows}} & \mathbf{0} \end{bmatrix} \mathbf{M}_{\delta_i}{}^{n-1} \overline{\mathbf{S}}_{\Delta_{0_{\delta_i}}} (\overline{\mathbf{S}}_{\Delta_{0_{\delta_i}}})^\dagger \overline{\mathbf{S}}_{\Delta_{0_{\delta_i}}} \begin{bmatrix} \mathbf{I}_{n_{cols}} \\ \mathbf{0} \end{bmatrix}$$

Substitution of equation (3.31) into this equation yields the following:

$$\Rightarrow \quad S_{\Delta_{n-1}}{}_{\delta_i n} = \begin{bmatrix} \mathbf{I}_{n_{rows}} & \mathbf{0} \end{bmatrix} \overline{\mathbf{S}}_{\Delta_0}{}_{\delta_i} \mathbf{N}_{\delta_i}{}^{n-1} (\overline{\mathbf{S}}_{\Delta_0}{}_{\delta_i})^{\dagger} \overline{\mathbf{S}}_{\Delta_0}{}_{\delta_i} \begin{bmatrix} \mathbf{I}_{n_{cols}} \\ \mathbf{0} \end{bmatrix}$$

Substituting equation (3.29) into this expression yields:

$$\Rightarrow \quad S_{\Delta_{n-1}}{}_{\delta_i n} = \begin{bmatrix} \mathbf{I}_{n_{rows}} & \mathbf{0} \end{bmatrix} \overline{\mathbf{S}}_{\Delta_0}{}_{\delta_i} (\overline{\mathbf{S}}_{\Delta_0}{}_{\delta_i})^{\dagger} \overline{\mathbf{S}}_{\Delta_0}{}_{\delta_i} \mathbf{N}_{\delta_i}{}^{n-1} (\overline{\mathbf{S}}_{\Delta_0}{}_{\delta_i})^{\dagger} \overline{\mathbf{S}}_{\Delta_0}{}_{\delta_i} \begin{bmatrix} \mathbf{I}_{n_{cols}} \\ \mathbf{0} \end{bmatrix}$$

Substitution of equation (3.31) into this equation yields the following:

$$\Rightarrow \quad S_{\Delta_{n-1}}{}_{\delta_i n} = \begin{bmatrix} \mathbf{I}_{n_{rows}} & \mathbf{0} \end{bmatrix} \overline{\mathbf{S}}_{\Delta_0}{}_{\delta_i} (\overline{\mathbf{S}}_{\Delta_0}{}_{\delta_i})^{\dagger} \mathbf{M}_{\delta_i}{}^{n-1} \overline{\mathbf{S}}_{\Delta_0}{}_{\delta_i} (\overline{\mathbf{S}}_{\Delta_0}{}_{\delta_i})^{\dagger} \overline{\mathbf{S}}_{\Delta_0}{}_{\delta_i} \begin{bmatrix} \mathbf{I}_{n_{cols}} \\ \mathbf{0} \end{bmatrix}$$

Substituting equations (3.24) and (3.28) into this equation yields the following result:

$$S_{\Delta_{n-1}}{}_{\delta_i n} = \begin{bmatrix} \mathbf{I}_{n_{rows}} & \mathbf{0} \end{bmatrix} \overline{\mathbf{P}}_{21}{}_{\delta_i} \overline{\mathbf{P}}_{12}{}_{\delta_i} (\overline{\mathbf{P}}_{12}{}_{\delta_i})^{\dagger} (\overline{\mathbf{P}}_{21}{}_{\delta_i})^{\dagger} \mathbf{M}_{\delta_i}{}^{n-1} \overline{\mathbf{S}}_{\Delta_0}{}_{\delta_i} (\overline{\mathbf{P}}_{12}{}_{\delta_i})^{\dagger} (\overline{\mathbf{P}}_{21}{}_{\delta_i})^{\dagger} \overline{\mathbf{P}}_{21}{}_{\delta_i} \overline{\mathbf{P}}_{12}{}_{\delta_i} \begin{bmatrix} \mathbf{I}_{n_{cols}} \\ \mathbf{0} \end{bmatrix}$$

$$\Rightarrow \quad S_{\Delta_{n-1}}{}_{\delta_i n} = \begin{bmatrix} \mathbf{I}_{n_{rows}} & \mathbf{0} \end{bmatrix} \overline{\mathbf{P}}_{21}{}_{\delta_i} \{ (\overline{\mathbf{P}}_{21}{}_{\delta_i})^{\dagger} \mathbf{M}_{\delta_i}{}^{n-1} \overline{\mathbf{S}}_{\Delta_0}{}_{\delta_i} (\overline{\mathbf{P}}_{12}{}_{\delta_i})^{\dagger} \} \overline{\mathbf{P}}_{12}{}_{\delta_i} \begin{bmatrix} \mathbf{I}_{n_{cols}} \\ \mathbf{0} \end{bmatrix}$$

Then, using equations (3.25) - (3.27) and (3.34) yields the following result:

$$S_{\Delta_{n-1}}{}_{\delta_i n} = P_{21}{}_{\delta_i} (\mathbf{P}_{11}{}_{\delta_i \delta_i})^{n-1} \mathbf{P}_{12}{}_{\delta_i} \quad ; \quad n = 1, 2, \ldots, \eta_i \tag{3.36}$$

Recalling equations (3.18) and (3.19), equation (3.36) shows that equations (3.25) - (3.27) are a realization of $S_{\Delta_{L,\zeta}}(\delta)$, as definedby equation (3.21). To show irreducibility, consider the following:

$$n_i = \dim(\mathbf{P}_{11}{}_{\delta_i \delta_i}) = \mathrm{rank}(\overline{\mathbf{S}}_{\Delta_0}{}_{\delta_i}) \leq \min\{\mathrm{rank}(\overline{\mathbf{P}}_{21}{}_{\delta_i}), \mathrm{rank}(\overline{\mathbf{P}}_{12}{}_{\delta_i})\} \tag{3.37}$$

Using equations (3.18) and (3.19), the following matrices can be defined to be consistent with the Hankel matrix given by equation (3.22) and its svd given by (3.24).

$$\overline{\mathbf{P}}_{21_{\delta_i}} = \begin{bmatrix} \mathbf{P}_{21_{\delta_i}} \\ \mathbf{P}_{21_{\delta_i}}(\mathbf{P}_{11_{\delta_i\delta_i}}) \\ \mathbf{P}_{21_{\delta_i}}(\mathbf{P}_{11_{\delta_i\delta_i}})^2 \\ \vdots \\ \mathbf{P}_{21_{\delta_i}}(\mathbf{P}_{11_{\delta_i\delta_i}})^{\eta_i-1} \end{bmatrix}, \quad \overline{\mathbf{P}}_{21_{\delta_i}} \in \mathbf{R}^{(\eta_i n_{rows})\times n_i} \tag{3.38}$$

$$\overline{\mathbf{P}}_{12_{\delta_i}} = \begin{bmatrix} \mathbf{P}_{12_{\delta_i}} & (\mathbf{P}_{11_{\delta_i\delta_i}})\mathbf{P}_{12_{\delta_i}} & (\mathbf{P}_{11_{\delta_i\delta_i}})^2\mathbf{P}_{12_{\delta_i}} & \cdots & (\mathbf{P}_{11_{\delta_i\delta_i}})^{\eta_i-1}\mathbf{P}_{12_{\delta_i}} \end{bmatrix},$$

$$\overline{\mathbf{P}}_{12_{\delta_i}} \in \mathbf{R}^{n_i\times(\eta_i n_{cols})} \tag{3.39}$$

Since $\overline{\mathbf{P}}_{21_{\delta_i}} \in \mathbf{R}^{(\eta_i n_{rows})\times n_i}$ and $\overline{\mathbf{P}}_{12_{\delta_i}} \in \mathbf{R}^{n_i\times(\eta_i n_{cols})}$ are tall and wide matrices (respectively) that result from the svd computation of equation (3.24), the rank of each equals n_i and equation (3.37) can be evaluated as follows.

$$n_i = \dim(\mathbf{P}_{11_{\delta_i\delta_i}}) = \operatorname{rank}(\overline{\mathbf{S}}_{\Delta_{0_{\delta_i}}}) = \operatorname{rank}(\overline{\mathbf{P}}_{21_{\delta_i}}) = \operatorname{rank}(\overline{\mathbf{P}}_{12_{\delta_i}}) \tag{3.40}$$

Hence, the realization given by equations (3.25) - (3.27) is irreducible. QED

Note that as stated in equation (3.17), each main diagonal block of P_{11} must be nilpotent of index η_i, i.e.:

$$(\mathbf{P}_{11_{\delta_i\delta_i}})^{\eta_i} = 0$$

The following theorem establishes the nilpotency of $P_{11_{\delta_i\delta_i}}$.

<u>Theorem 3.2</u>

The $P_{11_{\delta_i\delta_i}}$ matrix computed using the result of Theorem 3.1 is nilpotent with index η_i.

<u>Proof</u>:

Consider the following equation:

$$(\mathbf{P}_{11_{\delta_i\delta_i}})^{\eta_i} = [(\overline{\mathbf{P}}_{21_{\delta_i}})^{\dagger}\overline{\mathbf{S}}_{\Delta_{1_{\delta_i}}}(\overline{\mathbf{P}}_{12_{\delta_i}})^{\dagger}]^{\eta_i}$$

Substituting from equations (3.24) and (3.30), and using the fact that $\mathbf{U}_{\delta_i}$ and $\mathbf{V}_{\delta_i}$ are unitary matrices yields the following result.

$$(\mathbf{P}_{11_{\delta_i\delta_i}})^{\eta_i} = [(\mathbf{U}_{\delta_i}\Sigma_{\delta_i}{}^{1/2})^{\dagger}\mathbf{M}_{\delta_i}\overline{\mathbf{S}}_{\Delta_{0_{\delta_i}}}(\Sigma_{\delta_i}{}^{1/2}\mathbf{V}_{\delta_i})^{\dagger}]^{\eta_i}$$

$$\Rightarrow \quad (\mathbf{P}_{11_{\delta_i\delta_i}})^{\eta_i} = [\Sigma_{\delta_i}{}^{-1/2}\mathbf{U}_{\delta_i}{}^{T}\mathbf{M}_{\delta_i}\mathbf{U}_{\delta_i}\Sigma_{\delta_i}\mathbf{V}_{\delta_i}\mathbf{V}_{\delta_i}{}^{T}\Sigma_{\delta_i}{}^{-1/2}]^{\eta_i}$$

$$\Rightarrow \quad (\mathbf{P}_{11_{\delta_i\delta_i}})^{\eta_i} = [\Sigma_{\delta_i}{}^{-1/2}\mathbf{U}_{\delta_i}{}^{T}\mathbf{M}_{\delta_i}\mathbf{U}_{\delta_i}\Sigma_{\delta_i}\Sigma_{\delta_i}{}^{-1/2}]^{\eta_i}$$

$$\Rightarrow \quad (\mathbf{P}_{11_{\delta_i\delta_i}})^{\eta_i} = [\Sigma_{\delta_i}{}^{-1/2}\mathbf{U}_{\delta_i}{}^{T}\mathbf{M}_{\delta_i}\mathbf{U}_{\delta_i}\Sigma_{\delta_i}{}^{1/2}]^{\eta_i}$$

Then, the right-hand side of the equation can be separated into the product of matrix components as follows.

$$(\mathbf{P}_{11_{\delta_i\delta_i}})^{\eta_i} = [\Sigma_{\delta_i}{}^{-1/2}\mathbf{U}_{\delta_i}{}^{T}\mathbf{M}_{\delta_i}\mathbf{U}_{\delta_i}\Sigma_{\delta_i}{}^{1/2}]^{2}[\Sigma_{\delta_i}{}^{-1/2}\mathbf{U}_{\delta_i}{}^{T}\mathbf{M}_{\delta_i}\mathbf{U}_{\delta_i}\Sigma_{\delta_i}{}^{1/2}]^{\eta_i-2}$$

Squaring the first term yields the following.

$$(\mathbf{P}_{11_{\delta_i\delta_i}})^{\eta_i} = [(\Sigma_{\delta_i}{}^{-1/2}\mathbf{U}_{\delta_i}{}^{T}\mathbf{M}_{\delta_i}\mathbf{U}_{\delta_i}\Sigma_{\delta_i}{}^{1/2})(\Sigma_{\delta_i}{}^{-1/2}\mathbf{U}_{\delta_i}{}^{T}\mathbf{M}_{\delta_i}\mathbf{U}_{\delta_i}\Sigma_{\delta_i}{}^{1/2})][\Sigma_{\delta_i}{}^{-1/2}\mathbf{U}_{\delta_i}{}^{T}\mathbf{M}_{\delta_i}\mathbf{U}_{\delta_i}\Sigma_{\delta_i}{}^{1/2}]^{\eta_i-2}$$

$$\Rightarrow \quad (\mathbf{P}_{11_{\delta_i\delta_i}})^{\eta_i} = [\Sigma_{\delta_i}{}^{-1/2}\mathbf{U}_{\delta_i}{}^{T}\mathbf{M}_{\delta_i}{}^{2}\mathbf{U}_{\delta_i}\Sigma_{\delta_i}{}^{1/2}][\Sigma_{\delta_i}{}^{-1/2}\mathbf{U}_{\delta_i}{}^{T}\mathbf{M}_{\delta_i}\mathbf{U}_{\delta_i}\Sigma_{\delta_i}{}^{1/2}]^{\eta_i-2}$$

Continuing this process yields the following result (which is consistent with equation (3.34) for n = η_i).

$$(\mathbf{P}_{11_{\delta_i\delta_i}})^{\eta_i} = \Sigma_{\delta_i}{}^{-1/2}\mathbf{U}_{\delta_i}{}^{T}\mathbf{M}_{\delta_i}{}^{\eta_i}\mathbf{U}_{\delta_i}\Sigma_{\delta_i}{}^{1/2}$$

Since $\mathbf{M}_{\delta_i}$ has η_i block rows and columns and is defined by equation (3.32), it is a nilpotent matrix with index η_i (see Reference [10]). Therefore, the deired result is obtained, i.e.:

$$(\mathbf{P}_{11_{\delta_i\delta_i}})^{\eta_i} = 0$$

QED

In summary, this section has presented a simple numerical technique for computing $P_{21_{\delta_i}}$, $P_{12_{\delta_i}}$, and $P_{11_{\delta_i\delta_i}}$ for each uncertain parameter. The result is irreducible, and each main-diagonal block is guaranteed to be nilpotent of index η_i, where η_i is the highest degree of δ_i appearing in $S_{\Delta}(\delta)$.

3.2.2 Solution of P_{11} Off-Diagonal Blocks

The P_{11} off-diagonal blocks are each solved using the appropriate crossterms of $S_\Delta(\delta)$, as defined by equation (3.20). The number of off-diagonal blocks to be solved is given by the following equation.

$$n_{ODB} = \sum_{i=1}^{m-1}(m-i) \tag{3.41}$$

The equation to be solved for each off-diagonal block of P_{11} is a generalized linear matrix equation. The general equation is given below for computing the off-diagonal block $P_{11_{\delta_n\delta_j}}$, where n = 1, 2, ... , m–1 and j = n+1, n+2, ... , m.

$$(\overline{\mathbf{P}}_{21_{\delta_n}}{}^{[n]}\overline{\mathbf{P}}_{11_{\delta_n}}{}^{[n]})\mathbf{P}_{11_{\delta_n\delta_j}}(\overline{\mathbf{P}}_{12_{\delta_j}}) = \overline{\mathbf{S}}_{\Delta_{\delta_n}}{}^{[n]} \tag{3.42}$$

The matrices $\overline{\mathbf{P}}_{21_{\delta_n}}{}^{[n]}$, $\overline{\mathbf{P}}_{11_{\delta_n}}{}^{[n]}$, $\overline{\mathbf{P}}_{12_{\delta_j}}$, and $\overline{\mathbf{S}}_{\Delta_{\delta_n}}{}^{[n]}$ in equation (3.42) are comprised of known matrices as well as matrices that have already been computed at this point in the solution process. Their explicit general definition is given in the following pages.

The matrix $\overline{\mathbf{P}}_{21_{\delta_n}}{}^{[n]}$ in equation (3.58) is a block-diagonal matrix with n partitions along the main-diagonal, which is comprised of known matrices (i.e., matrices that have already been computed at this point). This matrix can be defined as follows.

$$\overline{\mathbf{P}}_{21_{\delta_n}}{}^{[n]} = \mathrm{diag}[\overline{\mathbf{P}}_{21_{\delta_n}}, \overline{\mathbf{P}}_{21_{\delta_{i_1}\delta_n}}{}^{[2]}, \overline{\mathbf{P}}_{21_{\delta_{i_1}\delta_{i_2}\delta_n}}{}^{[2]}, \overline{\mathbf{P}}_{21_{\delta_{i_1}\delta_{i_2}\delta_{i_3}\delta_n}}{}^{[2]}, \cdots,$$

$$\overline{\mathbf{P}}_{21_{\delta_{i_1}\delta_{i_2}\cdots\delta_{i_{k-1}}\delta_n}}{}^{[2]}, \cdots, \mathbf{P}_{21_{\delta_1\delta_2\cdots\delta_{i_{n-1}}\delta_n}}{}^{[2]}]$$

where:

Partition 1:
$$\overline{\mathbf{P}}_{21_{\delta_n}} = \begin{bmatrix} \mathbf{P}_{21_{\delta_n}} \\ \mathbf{P}_{21_{\delta_n}}(\mathbf{P}_{11_{\delta_n\delta_n}}) \\ \mathbf{P}_{21_{\delta_n}}(\mathbf{P}_{11_{\delta_n\delta_n}})^2 \\ \vdots \\ \mathbf{P}_{21_{\delta_n}}(\mathbf{P}_{11_{\delta_n\delta_n}})^{\eta_n - 1} \end{bmatrix}, \quad \binom{n-1}{0} = 1 \text{ Block}$$

Partition 2:
$$\overline{\mathbf{P}}_{21_{\delta_{i_1}\delta_n}}{}^{[2]} = \mathrm{diag}[\mathbf{P}_{21_{\delta_1\delta_n}}{}^{[2]}, \mathbf{P}_{21_{\delta_2\delta_n}}{}^{[2]}, \cdots, \mathbf{P}_{21_{\delta_{n-1}\delta_n}}{}^{[2]}] \;,$$

$$\binom{n-1}{1} = n-1 \text{ Blocks}$$

$$\mathbf{P}_{21_{\delta_{i_1}\delta_n}}{}^{[2]} = \overline{\mathbf{P}}_{21_{\delta_{i_1}}} \otimes \mathbf{I}_{\eta_n} \qquad ; \qquad i_1 = 1, 2, \ldots, n-1$$

Partition 3: $\overline{\mathbf{P}}_{21_{\delta_{i_1}\delta_{i_2}\delta_n}}{}^{[2]} = \text{diag}[\mathbf{P}_{21_{\delta_1\delta_2\delta_n}}{}^{[2]}, \mathbf{P}_{21_{\delta_1\delta_3\delta_n}}{}^{[2]}, \cdots, \mathbf{P}_{21_{\delta_1\delta_{n-1}\delta_n}}{}^{[2]},$

$$\mathbf{P}_{21_{\delta_2\delta_3\delta_n}}{}^{[2]}, \mathbf{P}_{21_{\delta_2\delta_4\delta_n}}{}^{[2]}, \cdots, \mathbf{P}_{21_{\delta_2\delta_{n-1}\delta_n}}{}^{[2]}, \cdots, \mathbf{P}_{21_{\delta_{n-2}\delta_{n-1}\delta_n}}{}^{[2]}] \, ;$$

$$\binom{n-1}{2} = \frac{(n-1)(n-2)}{2!} \text{ Blocks}$$

$$\mathbf{P}_{21_{\delta_{i_1}\delta_{i_2}\delta_n}}{}^{[2]} = \overline{\mathbf{P}}_{21_{\delta_{i_1}}} \otimes \mathbf{I}_{\eta_{i_2} \cdot \eta_n} \qquad ;$$

$$i_1 = 1, 2, \ldots, n-2 \quad , \quad i_2 = i_1 + 1, i_1 + 2, \ldots, n-1$$

Partition 4:

$$\overline{\mathbf{P}}_{21_{\delta_{i_1}\delta_{i_2}\delta_{i_3}\delta_n}}{}^{[2]} = \text{diag}[\mathbf{P}_{21_{\delta_1\delta_2\delta_3\delta_n}}{}^{[2]}, \mathbf{P}_{21_{\delta_1\delta_2\delta_4\delta_n}}{}^{[2]}, \cdots, \mathbf{P}_{21_{\delta_1\delta_2\delta_{n-1}\delta_n}}{}^{[2]},$$

$$\mathbf{P}_{21_{\delta_1\delta_3\delta_4\delta_n}}{}^{[2]}, \mathbf{P}_{21_{\delta_1\delta_3\delta_5\delta_n}}{}^{[2]}, \cdots, \mathbf{P}_{21_{\delta_1\delta_3\delta_{n-1}\delta_n}}{}^{[2]}, \cdots, \mathbf{P}_{21_{\delta_{n-3}\delta_{n-2}\delta_{n-1}\delta_n}}{}^{[2]}] \, ;$$

$$\binom{n-1}{3} = \frac{(n-1)(n-2)(n-3)}{3!} \text{ Blocks}$$

$$\mathbf{P}_{21_{\delta_{i_1}\delta_{i_2}\delta_{i_3}\delta_n}}{}^{[2]} = \overline{\mathbf{P}}_{21_{\delta_{i_1}}} \otimes \mathbf{I}_{\eta_{i_2} \cdot \eta_{i_3} \cdot \eta_n} \qquad ;$$

$$i_1 = 1, 2, \ldots, n-3 \;\; ; \;\; i_2 = i_1 + 1, i_1 + 2, \ldots, n-2 \;\; ; \;\; i_3 = i_1 + 2, i_1 + 3, \ldots, n-1$$

Partition k:

$$\overline{\mathbf{P}}_{21_{\delta_{i_1}\delta_{i_2}\cdots\delta_{i_{k-1}}\delta_n}}{}^{[2]} =$$

$$\mathrm{diag}[\mathbf{P}_{21_{\delta_1\delta_2\cdots\delta_{k-2}\delta_{k-1}\delta_n}}{}^{[2]}, \mathbf{P}_{21_{\delta_1\delta_2\cdots\delta_{k-2}\delta_k\delta_n}}{}^{[2]}, \cdots, \mathbf{P}_{21_{\delta_1\delta_2\cdots\delta_{k-2}\delta_{n-1}\delta_n}}{}^{[2]},$$

$$\mathbf{P}_{21_{\delta_1\delta_3\cdots\delta_{k-1}\delta_k\delta_n}}{}^{[2]}, \mathbf{P}_{21_{\delta_1\delta_3\cdots\delta_{k-1}\delta_{k+1}\delta_n}}{}^{[2]}, \cdots, \mathbf{P}_{21_{\delta_1\delta_3\cdots\delta_{k-1}\delta_{n-1}\delta_n}}{}^{[2]},$$

$$\cdots, \mathbf{P}_{21_{\delta_{n-3}\delta_{n-2}\delta_{n-1}\delta_n}}{}^{[2]}] \quad ;$$

$$\binom{n-1}{k-1} = \frac{(n-1)!}{(k-1)!(n-k)!} = \frac{(n-1)(n-2)\cdots(n-k+1)}{(k-1)!} \text{ Blocks}$$

$$\mathbf{P}_{21_{\delta_{i_1}\delta_{i_2}\cdots\delta_{i_{k-1}}\delta_n}}{}^{[2]} = \overline{\mathbf{P}}_{21_{\delta_{i_1}}} \otimes \mathbf{I}_{\eta_{i_2}\cdot\cdots\cdot\eta_{i_{k-1}}\cdot\eta_n} \quad ;$$

$$i_1 = 1, 2, \ldots, n-k+1 \;;\quad i_2 = i_1+1, i_1+2, \ldots, n-k+2 \;;$$

$$i_{k-2} = i_1+k-3, \ldots, n-2 \;;\quad i_{k-1} = i_1+k-2, \ldots, n-1$$

Partition n:

$$\mathbf{P}_{21_{\delta_1\delta_2\cdots\delta_{n-1}\delta_n}}{}^{[2]} = \overline{\mathbf{P}}_{21_{\delta_1}} \otimes \mathbf{I}_{\eta_2\cdot\eta_3\cdot\cdots\cdot\eta_{n-1}\cdot\eta_n} \quad ; \quad \binom{n-1}{n-1} = 1 \text{ Block}$$

Note that all $\overline{\mathbf{P}}_{21_{\delta_i}}$ terms in the above equations are defined by the $\overline{\mathbf{P}}_{21_{\delta_n}}$ equation given for Partition 1.

The matrix $\overline{\mathbf{P}}_{11_{\delta_n}}{}^{[n]}$ in equation (3.42) is a block-column matrix with n partitions, and is comprised of known matrices (i.e., matrices that have already been computed at this point). This matrix is defined as follows.

$$\overline{\mathbf{P}}_{11_{\delta_n}}{}^{[n]} = \begin{bmatrix} \mathbf{I}_{n_n} \\ \overline{\mathbf{P}}_{11_{\delta_{i_1}\delta_n}}{}^{[2]} \\ \overline{\mathbf{P}}_{11_{\delta_{i_1}\delta_{i_2}\delta_n}}{}^{[3]} \\ \overline{\mathbf{P}}_{11_{\delta_{i_1}\delta_{i_2}\delta_{i_3}\delta_n}}{}^{[4]} \\ \vdots \\ \overline{\mathbf{P}}_{11_{\delta_{i_1}\delta_{i_2}\cdots\delta_{i_{k-1}}\delta_n}}{}^{[k]} \\ \vdots \\ \mathbf{P}_{11_{\delta_1\delta_2\cdots\delta_{n-1}\delta_n}}{}^{[n]} \end{bmatrix}$$

Partition 1: $\quad I_{n_n}$ = Identity Matrix of Dimension determined by δ_n

Partition 2: $$\overline{\mathbf{P}}_{11_{\delta_{i_1}\delta_n}}{}^{[2]} = \begin{bmatrix} \mathbf{P}_{11_{\delta_1\delta_n}}{}^{[2]} \\ \mathbf{P}_{11_{\delta_2\delta_n}}{}^{[2]} \\ \vdots \\ \mathbf{P}_{11_{\delta_{n-1}\delta_n}}{}^{[2]} \end{bmatrix} \quad ; \quad \binom{n-1}{1} = n-1 \text{ Blocks}$$

$$\mathbf{P}_{11_{\delta_{i_1}\delta_n}}{}^{[2]} = \mathbf{P}_{11_{\delta_{i_1}\delta_n}}{}^{[1]}\overline{\mathbf{P}}_{11_{\delta_n}}$$

$$\mathbf{P}_{11_{\delta_{i_1}\delta_n}}{}^{[1]} = \mathbf{P}_{11_{\delta_{i_1}\delta_n}} \otimes \mathbf{I}_{\eta_n} \quad ; \quad i_1 = 1, 2, \ldots, n-1$$

$$\overline{\mathbf{P}}_{11_{\delta_n}} = \begin{bmatrix} \mathbf{I}_{n_n} \\ (\mathbf{P}_{11_{\delta_n\delta_n}}) \\ \vdots \\ (\mathbf{P}_{11_{\delta_n\delta_n}})^{\eta_n - 1} \end{bmatrix}$$

Partition 3:

$$\overline{\mathbf{P}}_{11_{\delta_{i_1}\delta_{i_2}\delta_n}}{}^{[3]} = \begin{bmatrix} \mathbf{P}_{11_{\delta_1\delta_2\delta_n}}{}^{[3]} \\ \mathbf{P}_{11_{\delta_1\delta_3\delta_n}}{}^{[3]} \\ \vdots \\ \mathbf{P}_{11_{\delta_1\delta_{n-1}\delta_n}}{}^{[3]} \\ \mathbf{P}_{11_{\delta_2\delta_3\delta_n}}{}^{[3]} \\ \mathbf{P}_{11_{\delta_2\delta_4\delta_n}}{}^{[3]} \\ \vdots \\ \mathbf{P}_{11_{\delta_2\delta_{n-1}\delta_n}}{}^{[3]} \\ \vdots \\ \mathbf{P}_{11_{\delta_{n-2}\delta_{n-1}\delta_n}}{}^{[3]} \end{bmatrix} \quad ; \quad \binom{n-1}{2} = \frac{(n-1)(n-2)}{2!} \text{ Blocks}$$

$$\mathbf{P}_{11_{\delta_{i_1}\delta_{i_2}\delta_n}}{}^{[3]} = \mathbf{P}_{11_{\delta_{i_1}\delta_{i_2}\delta_n}}{}^{[2]}\mathbf{P}_{11_{\delta_{i_2}\delta_n}}{}^{[1]}\overline{\mathbf{P}}_{11_{\delta_n}}$$

$$\mathbf{P}_{11_{\delta_{i_1}\delta_{i_2}\delta_n}}{}^{[2]} = \mathbf{P}_{11_{\delta_{i_1}\delta_{i_2}}}{}^{[2]} \otimes \mathbf{I}_{\eta_n}$$

$$\mathbf{P}_{11_{\delta_{i_1}\delta_{i_2}}}{}^{[2]} = \mathbf{P}_{11_{\delta_{i_1}\delta_{i_2}}}{}^{[1]}\overline{\mathbf{P}}_{11_{\delta_{i_2}}}$$

$$\mathbf{P}_{11_{\delta_{i_1}\delta_{i_2}}}{}^{[1]} = \mathbf{P}_{11_{\delta_{i_1}\delta_{i_2}}} \otimes \mathbf{I}_{\eta_{i_2}}$$

$$\mathbf{P}_{11_{\delta_{i_2}\delta_n}}{}^{[1]} = \mathbf{P}_{11_{\delta_{i_2}\delta_n}} \otimes \mathbf{I}_{\eta_n}$$

$$i_1 = 1, 2, \ldots, n-2 \quad ; \quad i_2 = i_1 + 1, i_1 + 2, \ldots, n-1$$

Partition 4:

$$\overline{\mathbf{P}}_{11_{\delta_{i_1}\delta_{i_2}\delta_{i_3}\delta_n}}{}^{[4]} = \begin{bmatrix} \mathbf{P}_{11_{\delta_1\delta_2\delta_3\delta_n}}{}^{[4]} \\ \mathbf{P}_{11_{\delta_1\delta_2\delta_4\delta_n}}{}^{[4]} \\ \vdots \\ \mathbf{P}_{11_{\delta_1\delta_2\delta_{n-1}\delta_n}}{}^{[4]} \\ \mathbf{P}_{11_{\delta_1\delta_3\delta_4\delta_n}}{}^{[4]} \\ \mathbf{P}_{11_{\delta_1\delta_3\delta_5\delta_n}}{}^{[4]} \\ \vdots \\ \mathbf{P}_{11_{\delta_1\delta_3\delta_{n-1}\delta_n}}{}^{[4]} \\ \vdots \\ \mathbf{P}_{11_{\delta_{n-3}\delta_{n-2}\delta_{n-1}\delta_n}}{}^{[4]} \end{bmatrix} \; ; \; \binom{n-1}{3} = \frac{(n-1)(n-2)(n-3)}{3!} \text{ Blocks}$$

$$\mathbf{P}_{11_{\delta_{i_1}\delta_{i_2}\delta_{i_3}\delta_n}}{}^{[4]} = \mathbf{P}_{11_{\delta_{i_1}\delta_{i_2}\delta_{i_3}\delta_n}}{}^{[3]}\mathbf{P}_{11_{\delta_{i_3}\delta_n}}{}^{[1]}\overline{\mathbf{P}}_{11_{\delta_n}}$$

$$\mathbf{P}_{11_{\delta_{i_1}\delta_{i_2}\delta_{i_3}\delta_n}}{}^{[3]} = \mathbf{P}_{11_{\delta_{i_1}\delta_{i_2}\delta_{i_3}}}{}^{[3]} \otimes \mathbf{I}_{\eta_n}$$

$$\mathbf{P}_{11_{\delta_{i_1}\delta_{i_2}\delta_{i_3}}}{}^{[3]} = \mathbf{P}_{11_{\delta_{i_1}\delta_{i_2}\delta_{i_3}}}{}^{[2]}\mathbf{P}_{11_{\delta_{i_2}\delta_{i_3}}}{}^{[1]}\overline{\mathbf{P}}_{11_{\delta_{i_3}}}$$

etc. (see above)

$i_1 = 1, 2, \ldots, n-3 \quad ; \quad i_2 = i_1+1, i_1+2, \ldots, n-2 \quad ; \quad i_3 = i_1+2, i_1+3, \ldots, n-1$

Partition k:

$$\overline{\mathbf{P}}_{11_{\delta_{i_1}\delta_{i_2}\cdots\delta_{i_{k-1}}\delta_n}}^{[k]} = \begin{bmatrix} \mathbf{P}_{11_{\delta_1\delta_2\cdots\delta_{k-2}\delta_{k-1}\delta_n}}^{[k]} \\ \mathbf{P}_{11_{\delta_1\delta_2\cdots\delta_{k-2}\delta_k\delta_n}}^{[k]} \\ \vdots \\ \mathbf{P}_{11_{\delta_1\delta_2\cdots\delta_{k-2}\delta_{n-1}\delta_n}}^{[k]} \\ \mathbf{P}_{11_{\delta_1\delta_3\cdots\delta_{k-1}\delta_k\delta_n}}^{[k]} \\ \mathbf{P}_{11_{\delta_1\delta_3\cdots\delta_{k-1}\delta_{k+1}\delta_n}}^{[k]} \\ \vdots \\ \mathbf{P}_{11_{\delta_1\delta_3\cdots\delta_{k-1}\delta_{n-1}\delta_n}}^{[k]} \\ \vdots \\ \mathbf{P}_{11_{\delta_{n-k+1}\delta_{n-k+2}\cdots\delta_{n-1}\delta_n}}^{[k]} \end{bmatrix};$$

$$\binom{n-1}{k-1} = \frac{(n-1)!}{(k-1)!(n-k)!} = \frac{(n-1)(n-2)\cdots(n-k+1)}{(k-1)!} \quad \text{Blocks}$$

$$\mathbf{P}_{11_{\delta_{i_1}\delta_{i_2}\cdots\delta_{i_{k-1}}\delta_n}}^{[k]} = \mathbf{P}_{11_{\delta_{i_1}\delta_{i_2}\cdots\delta_{i_{k-1}}\delta_n}}^{[k-1]}\mathbf{P}_{11_{\delta_{i_{k-1}}\delta_n}}^{[1]}\overline{\mathbf{P}}_{11_{\delta_n}}$$

$$\mathbf{P}_{11_{\delta_{i_1}\delta_{i_2}\cdots\delta_{i_{k-1}}\delta_n}}^{[k-1]} = \mathbf{P}_{11_{\delta_{i_1}\delta_{i_2}\cdots\delta_{i_{k-1}}}}^{[k-1]} \otimes \mathbf{I}_{\eta_n}$$

$$\mathbf{P}_{11_{\delta_{i_1}\delta_{i_2}\cdots\delta_{i_{k-1}}}}^{[k-1]} = \mathbf{P}_{11_{\delta_{i_1}\delta_{i_2}\cdots\delta_{i_{k-1}}}}^{[k-2]}\mathbf{P}_{11_{\delta_{i_{k-2}}\delta_{i_{k-1}}}}^{[1]}\overline{\mathbf{P}}_{11_{\delta_{i_{k-1}}}}$$

$$\mathbf{P}_{11_{\delta_{i_1}\delta_{i_2}\cdots\delta_{i_{k-1}}}}^{[k-2]} = \mathbf{P}_{11_{\delta_{i_1}\delta_{i_2}\cdots\delta_{i_{k-2}}}}^{[k-2]} \otimes \mathbf{I}_{\eta_{k-1}}$$

$$\vdots$$

$$\mathbf{P}_{11_{\delta_{i_1}\delta_{i_2}\delta_{i_3}}}^{[2]} = \mathbf{P}_{11_{\delta_{i_1}\delta_{i_2}}}^{[2]} \otimes \mathbf{I}_{\eta_{i_3}}$$

$$\mathbf{P}_{11_{\delta_{i_1}\delta_{i_2}}}^{[2]} = \mathbf{P}_{11_{\delta_{i_1}\delta_{i_2}}}^{[1]}\overline{\mathbf{P}}_{11_{\delta_{i_2}}}$$

$$\mathbf{P}_{11_{\delta_{i_1}\delta_{i_2}}}{}^{[1]} = \mathbf{P}_{11_{\delta_{i_1}\delta_{i_2}}} \otimes \mathbf{I}_{\eta_{i_2}}$$

$$i_1 = 1, 2, \ldots, n-k+1 \quad ; \quad i_2 = i_1+1, i_1+2, \ldots, n-k+2$$

$$\vdots$$

$$i_{k-2} = i_1+k-3, \ldots, n-2 \quad ; \quad i_{k-1} = i_1+k-2, \ldots, n-1$$

Partition n: $\mathbf{P}_{11_{\delta_1\delta_2\cdots\delta_{n-1}\delta_n}}{}^{[n]} = \mathbf{P}_{11_{\delta_1\delta_2\cdots\delta_{n-1}\delta_n}}{}^{[n-1]}\mathbf{P}_{11_{\delta_{n-1}\delta_n}}{}^{[1]}\overline{\mathbf{P}}_{11_{\delta_n}} \quad ; \quad \binom{n-1}{n-1} = 1$ Block

The first two matrices on the right side of the above equation for Partition n are defined by the preceeding equations for Partition k. Also, all $\overline{\mathbf{P}}_{11_{\delta_i}}$ terms in the above equations are defined by the $\overline{\mathbf{P}}_{11_{\delta_n}}$ equation given for Partition 2.

The matrix $\overline{\mathbf{P}}_{12_{\delta_j}}$ in equation (3.42) is a block-row matrix with j partitions, and is comprised of known matrices (i.e., matrices that have already been computed at this point). This matrix can be defined as follows.

$$\overline{\mathbf{P}}_{12_{\delta_j}} = [\mathbf{P}_{12_{\delta_j}}, \mathbf{P}_{11_{\delta_j\delta_j}}\mathbf{P}_{12_{\delta_j}}, (\mathbf{P}_{11_{\delta_j\delta_j}})^2\mathbf{P}_{12_{\delta_j}}, \cdots, (\mathbf{P}_{11_{\delta_j\delta_j}})^{\eta_j-1}\mathbf{P}_{12_{\delta_j}}]$$

The matrix $\overline{\mathbf{S}}_{\Delta_{\delta_n}}{}^{[n]}$ on the right side of equation (3.42) is a block-column matrix with n partitions, and is comprised of known coefficeint matrices from the expansion of $S_\Delta(\delta)$. This matrix can be defined as follows.

$$\bar{\mathbf{S}}_{\Delta_{\delta_n}}{}^{[n]} = \begin{bmatrix} \bar{\mathbf{S}}_{\Delta_{\delta_n\delta_j}} \\ \bar{\mathbf{S}}_{\Delta_{\delta_{i_1}\delta_n\delta_j}}{}^{[2]} \\ \bar{\mathbf{S}}_{\Delta_{\delta_{i_1}\delta_{i_2}\delta_n\delta_j}}{}^{[2]} \\ \bar{\mathbf{S}}_{\Delta_{\delta_{i_1}\delta_{i_2}\delta_{i_3}\delta_n\delta_j}}{}^{[2]} \\ \vdots \\ \bar{\mathbf{S}}_{\Delta_{\delta_{i_1}\delta_{i_2}\cdots\delta_{i_{k-1}}\delta_n\delta_j}}{}^{[2]} \\ \vdots \\ \bar{\mathbf{S}}_{\Delta_{\delta_1\delta_2\cdots\delta_{n-1}\delta_n\delta_j}}{}^{[2]} \end{bmatrix}$$

Partition 1:

$$\bar{\mathbf{S}}_{\Delta_{\delta_n\delta_j}} = \begin{bmatrix} \mathbf{S}_{\Delta_1}{}_{\delta_n\delta_j} & \mathbf{S}_{\Delta_2}{}_{\delta_n\delta_j{}^2} & \cdots & \mathbf{S}_{\Delta_{\eta_j}}{}_{\delta_n\delta_j{}^{\eta_j}} \\ \mathbf{S}_{\Delta_2}{}_{\delta_n{}^2\delta_j} & \mathbf{S}_{\Delta_3}{}_{\delta_n{}^2\delta_j{}^2} & \cdots & \mathbf{S}_{\Delta_{(\eta_j+1)}}{}_{\delta_n{}^2\delta_j{}^{\eta_j}} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{S}_{\Delta_{\eta_n}}{}_{\delta_n{}^{\eta_n}\delta_j} & \mathbf{S}_{\Delta_{(\eta_n+1)}}{}_{\delta_n{}^{\eta_n}\delta_j{}^2} & \cdots & \mathbf{S}_{\Delta_{(\eta_n+\eta_j-1)}}{}_{\delta_n{}^{\eta_n}\delta_j{}^{\eta_j}} \end{bmatrix} ;$$

$$j = n+1, n+2, \dots, m$$

Partition 2:

$$\bar{\mathbf{S}}_{\Delta_{\delta_{i_1}\delta_n\delta_j}}{}^{[2]} = \begin{bmatrix} \bar{\mathbf{S}}_{\Delta_{\delta_{i_1}\delta_n\delta_j}} \\ \bar{\mathbf{S}}_{\Delta_{\delta_{i_1}\delta_n{}^2\delta_j}} \\ \vdots \\ \bar{\mathbf{S}}_{\Delta_{\delta_{i_1}\delta_n{}^{\eta_n}\delta_j}} \end{bmatrix} ; \quad i_1 = 1, 2, \dots, n-1 \quad ; \quad j = n+1, n+2, \dots, m$$

$$\overline{\mathbf{S}}_{\Delta_{\delta_{i_1}\delta_n{}^{\ell}\delta_j}} = \begin{bmatrix} \mathbf{S}_{\Delta_{(1+\ell)}}{}_{\delta_{i_1}\delta_n{}^{\ell}\delta_j} & \mathbf{S}_{\Delta_{(2+\ell)}}{}_{\delta_{i_1}\delta_n{}^{\ell}\delta_j{}^2} & \cdots & \cdots & \mathbf{S}_{\Delta_{(\eta_j+\ell)}}{}_{\delta_{i_1}\delta_n{}^{\ell}\delta_j{}^{\eta_j}} \\ \mathbf{S}_{\Delta_{(2+\ell)}}{}_{\delta_{i_1}{}^2\delta_n{}^{\ell}\delta_j} & \mathbf{S}_{\Delta_{(3+\ell)}}{}_{\delta_{i_1}{}^2\delta_n{}^{\ell}\delta_j{}^2} & \cdots & \cdots & \mathbf{S}_{\Delta_{(\eta_j+\ell+1)}}{}_{\delta_{i_1}{}^2\delta_n{}^{\ell}\delta_j{}^{\eta_j}} \\ \vdots & \vdots & \cdots & \cdots & \vdots \\ \mathbf{S}_{\Delta_{(\eta_{i_1}+\ell)}}{}_{\delta_{i_1}{}^{\eta_{i_1}}\delta_n{}^{\ell}\delta_j} & \mathbf{S}_{\Delta_{(\eta_{i_1}+\ell+1)}}{}_{\delta_{i_1}{}^{\eta_{i_1}}\delta_n{}^{\ell}\delta_j{}^2} & \cdots & \cdots & \mathbf{S}_{\Delta_{(\eta_{i_1}+\eta_j+\ell-1)}}{}_{\delta_{i_1}{}^{\eta_{i_1}}\delta_n{}^{\ell}\delta_j{}^{\eta_j}} \end{bmatrix} \quad ; \quad \ell = 1, 2, \ldots, \eta_n$$

Partition 3:

$$\overline{\mathbf{S}}_{\Delta_{\delta_{i_1}\delta_{i_2}\delta_n\delta_j}}{}^{[2]} = \begin{bmatrix} \overline{\mathbf{S}}_{\Delta_{\delta_{i_1}\delta_{i_2}{}^{\ell_{i_2}}\delta_n\delta_j}} \\ \overline{\mathbf{S}}_{\Delta_{\delta_{i_1}\delta_{i_2}{}^{\ell_{i_2}}\delta_n{}^2\delta_j}} \\ \vdots \\ \overline{\mathbf{S}}_{\Delta_{\delta_{i_1}\delta_{i_2}{}^{\ell_{i_2}}\delta_n{}^{\eta_n}\delta_j}} \end{bmatrix}$$

$$i_1 = 1, 2, \ldots, n-2 \quad ; \quad i_2 = i_1 + 1, i_1 + 2, \ldots, n-1$$

$$\ell_{i_2} = 1, 2, \ldots, \eta_{i_2} \quad ; \quad j = n+1, n+2, \ldots, m$$

$$
\overline{\mathbf{S}}_{\Delta_{\delta_{i_1}\bar{\delta}\delta_j}} = \begin{bmatrix}
\mathbf{S}_{\Delta_{(1+\bar{\ell})}{}_{\delta_{i_1}\bar{\delta}\delta_j}} & \mathbf{S}_{\Delta_{(2+\bar{\ell})}{}_{\delta_{i_1}\bar{\delta}\delta_j{}^2}} & \cdots & \mathbf{S}_{\Delta_{(\eta_j+\bar{\ell})}{}_{\delta_{i_1}\bar{\delta}\delta_j{}^{\eta_j}}} \\
\mathbf{S}_{\Delta_{(2+\bar{\ell})}{}_{\delta_{i_1}{}^2\bar{\delta}\delta_j}} & \mathbf{S}_{\Delta_{(3+\bar{\ell})}{}_{\delta_{i_1}{}^2\bar{\delta}\delta_j{}^2}} & \cdots & \mathbf{S}_{\Delta_{(\eta_j+\bar{\ell}+1)}{}_{\delta_{i_1}{}^2\bar{\delta}\delta_j{}^{\eta_j}}} \\
\vdots & \vdots & \ddots & \vdots \\
\mathbf{S}_{\Delta_{(\eta_{i_1}+\bar{\ell})}{}_{\delta_{i_1}{}^{\eta_{i_1}}\bar{\delta}\delta_j}} & \mathbf{S}_{\Delta_{(\eta_{i_1}+\bar{\ell}+1)}{}_{\delta_{i_1}{}^{\eta_{i_1}}\bar{\delta}\delta_j{}^2}} & \cdots & \mathbf{S}_{\Delta_{(\eta_{i_1}+\eta_j+\bar{\ell}-1)}{}_{\delta_{i_1}{}^{\eta_{i_1}}\bar{\delta}\delta_j{}^{\eta_j}}}
\end{bmatrix}
$$

$$
\bar{\delta} = (\delta_{i_2})^{\ell_{i_2}}(\delta_n)^{\ell_n} \quad ; \quad \ell_n = 1, 2, \ldots, \eta_n \quad ; \quad \bar{\ell} = \ell_{i_2} + \ell_n
$$

Note: ℓ_{i_2} is updated before ℓ_n

Partition 4:

$$
\overline{\mathbf{S}}_{\Delta_{\delta_{i_1}\delta_{i_2}\delta_{i_3}\delta_n\delta_j}}{}^{[2]} = \begin{bmatrix}
\overline{\mathbf{S}}_{\Delta_{\delta_{i_1}\delta_{i_2}{}^{\ell_{i_2}}\delta_{i_3}{}^{\ell_{i_3}}\delta_n\delta_j}} \\
\overline{\mathbf{S}}_{\Delta_{\delta_{i_1}\delta_{i_2}{}^{\ell_{i_2}}\delta_{i_3}{}^{\ell_{i_3}}\delta_n{}^2\delta_j}} \\
\vdots \\
\overline{\mathbf{S}}_{\Delta_{\delta_{i_1}\delta_{i_2}{}^{\ell_{i_2}}\delta_{i_3}{}^{\ell_{i_3}}\delta_n{}^{\eta_n}\delta_j}}
\end{bmatrix}
$$

$$
i_1 = 1, 2, \ldots, n-3 \quad ; \quad i_2 = i_1+1, i_1+2, \ldots, n-2
$$

$$
i_3 = i_1+2, i_1+3, \ldots, n-1 \quad ; \quad j = n+1, n+2, \ldots, m
$$

$$
\ell_{i_2} = 1, 2, \ldots, \eta_{i_2} \quad ; \quad \ell_{i_3} = 1, 2, \ldots, \eta_{i_3}
$$

$$\overline{\mathbf{S}}_{\Delta_{\delta_{i_1}\bar{\delta}\delta_j}} = \begin{bmatrix} \mathbf{S}_{\Delta_{(1+\bar{\ell})}\,\delta_{i_1}\bar{\delta}\delta_j} & \mathbf{S}_{\Delta_{(2+\bar{\ell})}\,\delta_{i_1}\bar{\delta}\delta_j{}^2} & \cdots & \mathbf{S}_{\Delta_{(\eta_j+\bar{\ell})}\,\delta_{i_1}\bar{\delta}\delta_j{}^{\eta_j}} \\ \mathbf{S}_{\Delta_{(2+\bar{\ell})}\,\delta_{i_1}{}^2\bar{\delta}\delta_j} & \mathbf{S}_{\Delta_{(3+\bar{\ell})}\,\delta_{i_1}{}^2\bar{\delta}\delta_j{}^2} & \cdots & \mathbf{S}_{\Delta_{(\eta_j+\bar{\ell}+1)}\,\delta_{i_1}{}^2\bar{\delta}\delta_j{}^{\eta_j}} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{S}_{\Delta_{(\eta_{i_1}+\bar{\ell})}\,\delta_{i_1}{}^{\eta_{i_1}}\bar{\delta}\delta_j} & \mathbf{S}_{\Delta_{(\eta_{i_1}+\bar{\ell}+1)}\,\delta_{i_1}{}^{\eta_{i_1}}\bar{\delta}\delta_j{}^2} & \cdots & \mathbf{S}_{\Delta_{(\eta_{i_1}+\eta_j+\bar{\ell}-1)}\,\delta_{i_1}{}^{\eta_{i_1}}\bar{\delta}\delta_j{}^{\eta_j}} \end{bmatrix}$$

$$\bar{\delta} = (\delta_{i_2})^{\ell_{i_2}} (\delta_{i_3})^{\ell_{i_3}} (\delta_n)^{\ell_n} \quad ; \quad \ell_n = 1, 2, \ldots, \eta_n \quad ; \quad \bar{\ell} = \ell_{i_2} + \ell_{i_3} + \ell_n$$

<u>Note</u>: ℓ_{i_2} is updated before ℓ_{i_3} ; ℓ_{i_3} is updated before ℓ_n

Partition k:

$$\overline{\mathbf{S}}_{\Delta_{\delta_{i_1}\delta_{i_2}\cdots\delta_{i_{k-1}}\delta_n\delta_j}}{}^{[2]} = \begin{bmatrix} \overline{\mathbf{S}}_{\Delta_{\delta_{i_1}\delta_{i_2}{}^{\ell_{i_2}}\cdots\delta_{i_{k-1}}{}^{\ell_{i_{k-1}}}\delta_n\delta_j}} \\ \overline{\mathbf{S}}_{\Delta_{\delta_{i_1}\delta_{i_2}{}^{\ell_{i_2}}\cdots\delta_{i_{k-1}}{}^{\ell_{i_{k-1}}}\delta_n{}^2\delta_j}} \\ \vdots \\ \overline{\mathbf{S}}_{\Delta_{\delta_{i_1}\delta_{i_2}{}^{\ell_{i_2}}\cdots\delta_{i_{k-1}}{}^{\ell_{i_{k-1}}}\delta_n{}^{\eta_n}\delta_j}} \end{bmatrix}$$

$$i_1 = 1, 2, \ldots, n - \ell + 1 \quad ; \quad i_2 = i_1 + 1, i_1 + 2, \ldots, n - \ell + 2$$

$$\vdots$$

$$i_{\ell-2} = i_1 + \ell - 3, i_1 + \ell - 2, \ldots, n - 2 \quad ; \quad i_{\ell-1} = i_1 + \ell - 2, i_1 + \ell - 1, \ldots, n - 1$$

$$j = n + 1, n + 2, \ldots, m$$

$$\ell_{i_2} = 1, 2, \ldots, \eta_{i_2} \quad ; \quad \ldots \quad ; \quad \ell_{i_{k-1}} = 1, 2, \ldots, \eta_{i_{k-1}}$$

$$
\bar{S}_{\Delta_{\delta_{i_1}\bar{\delta}\delta_j}} = \begin{bmatrix}
S_{\Delta_{(1+\bar{\ell})}\,\delta_{i_1}\bar{\delta}\delta_j} & S_{\Delta_{(2+\bar{\ell})}\,\delta_{i_1}\bar{\delta}\delta_j{}^2} & \cdots & S_{\Delta_{(\eta_j+\bar{\ell})}\,\delta_{i_1}\bar{\delta}\delta_j{}^{\eta_j}} \\
S_{\Delta_{(2+\bar{\ell})}\,\delta_{i_1}{}^2\bar{\delta}\delta_j} & S_{\Delta_{(3+\bar{\ell})}\,\delta_{i_1}{}^2\bar{\delta}\delta_j{}^2} & \cdots & S_{\Delta_{(\eta_j+\bar{\ell}+1)}\,\delta_{i_1}{}^2\bar{\delta}\delta_j{}^{\eta_j}} \\
\vdots & \vdots & \ddots & \vdots \\
S_{\Delta_{(\eta_{i_1}+\bar{\ell})}\,\delta_{i_1}{}^{\eta_{i_1}}\bar{\delta}\delta_j} & S_{\Delta_{(\eta_{i_1}+\bar{\ell}+1)}\,\delta_{i_1}{}^{\eta_{i_1}}\bar{\delta}\delta_j{}^2} & \cdots & S_{\Delta_{(\eta_{i_1}+\eta_j+\bar{\ell}-1)}\,\delta_{i_1}{}^{\eta_{i_1}}\bar{\delta}\delta_j{}^{\eta_j}}
\end{bmatrix}
$$

$$
\bar{\delta} = (\delta_{i_2})^{\ell_{i_2}} \cdots (\delta_{i_{k-1}})^{\ell_{i_{k-1}}} (\delta_n)^{\ell_n} \quad ; \quad \ell_n = 1, 2, \ldots, \eta_n
$$

$$
\bar{\ell} = \ell_{i_2} + \ell_{i_3} + \cdots + \ell_{i_{k-1}} + \ell_n
$$

<u>Note</u>: ℓ_{i_2} is updated before ℓ_{i_3} ; ℓ_{i_3} is updated before ℓ_{i_4} ; ... ; $\ell_{i_{k-1}}$ is updated before ℓ_n

Partition n:

$$
\bar{S}_{\Delta_{\delta_1\delta_2\cdots\delta_{n-1}\delta_n\delta_j}}{}^{[2]} = \begin{bmatrix}
\bar{S}_{\Delta_{\delta_1\delta_2{}^{\ell_2}\cdots\delta_{n-1}{}^{\ell_{n-1}}\delta_n\delta_j}} \\
\bar{S}_{\Delta_{\delta_1\delta_2{}^{\ell_2}\cdots\delta_{n-1}{}^{\ell_{n-1}}\delta_n{}^2\delta_j}} \\
\vdots \\
\bar{S}_{\Delta_{\delta_1\delta_2{}^{\ell_2}\cdots\delta_{n-1}{}^{\ell_{n-1}}\delta_n{}^{\eta_n}\delta_j}}
\end{bmatrix}
$$

$$
j = n+1, n+2, \ldots, m
$$

$$
\ell_2 = 1, 2, \ldots, \eta_2 \quad ; \quad \ldots \quad ; \quad \ell_{n-1} = 1, 2, \ldots, \eta_{n-1}
$$

$$\bar{\mathbf{S}}_{\Delta_{\delta_1\bar{\delta}\delta_j}} = \begin{bmatrix} \mathbf{S}_{\Delta_{(1+\bar{\ell})}{}_{\delta_1\bar{\delta}\delta_j}} & \mathbf{S}_{\Delta_{(2+\bar{\ell})}{}_{\delta_1\bar{\delta}\delta_j{}^2}} & \cdots & \mathbf{S}_{\Delta_{(\eta_j+\bar{\ell})}{}_{\delta_1\bar{\delta}\delta_j{}^{\eta_j}}} \\ \mathbf{S}_{\Delta_{(2+\bar{\ell})}{}_{\delta_1{}^2\bar{\delta}\delta_j}} & \mathbf{S}_{\Delta_{(3+\bar{\ell})}{}_{\delta_1{}^2\bar{\delta}\delta_j{}^2}} & \cdots & \mathbf{S}_{\Delta_{(\eta_j+\bar{\ell}+1)}{}_{\delta_1{}^2\bar{\delta}\delta_j{}^{\eta_j}}} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{S}_{\Delta_{(\eta_1+\bar{\ell})}{}_{\delta_1{}^{\eta_1}\bar{\delta}\delta_j}} & \mathbf{S}_{\Delta_{(\eta_1+\bar{\ell}+1)}{}_{\delta_1{}^{\eta_1}\bar{\delta}\delta_j{}^2}} & \cdots & \mathbf{S}_{\Delta_{(\eta_1+\eta_j+\bar{\ell}-1)}{}_{\delta_1{}^{\eta_1}\bar{\delta}\delta_j{}^{\eta_j}}} \end{bmatrix}$$

$$\bar{\delta} = (\delta_2)^{\ell_2} \cdots (\delta_{n-1})^{\ell_{n-1}} (\delta_n)^{\ell_n} \quad ; \quad \ell_n = 1, 2, \ldots, \eta_n$$

$$\bar{\ell} = \ell_2 + \ell_3 + \cdots + \ell_{n-1} + \ell_n$$

Note: ℓ_2 is updated before ℓ_3 ; ℓ_3 is updated before ℓ_4 ; ... ; ℓ_{n-1} is updated before ℓ_n

The above general equations, which define the matrices given in equations (3.42) for generating the off-diagonal block equations, are complicated due to the large number of cross-product terms that can arise in solving the general problem and due to the notation required to generate the associated equations. As an illustration of generating these equations based on equations (3.42) and the above defining equations, the off-diagonal block equations for the case of three parameters (m = 3) with maximum degree of 2 for each δ_i parameter ($\eta_1 = \eta_2 = \eta_3 = 2$) are shown below.

Off-Diagonal Block Equations for m = 3 ($\eta_1 = \eta_2 = \eta_3 = 2$)

$$\begin{bmatrix} \mathbf{P}_{21_{\delta_1}} \\ \mathbf{P}_{21_{\delta_1}} \mathbf{P}_{11_{\delta_1\delta_1}} \end{bmatrix} \mathbf{P}_{11_{\delta_1\delta_j}} \begin{bmatrix} \mathbf{P}_{12_{\delta_j}} & \mathbf{P}_{11_{\delta_j\delta_j}} \mathbf{P}_{12_{\delta_j}} \end{bmatrix} = \begin{bmatrix} \mathbf{S}_{\Delta_{1_{\delta_1\delta_j}}} & \mathbf{S}_{\Delta_{2_{\delta_1\delta_j{}^2}}} \\ \mathbf{S}_{\Delta_{2_{\delta_1{}^2\delta_j}}} & \mathbf{S}_{\Delta_{3_{\delta_1{}^2\delta_j{}^2}}} \end{bmatrix}, \quad j = 1, 2$$

$$\begin{bmatrix} \mathbf{P}_{21_{\delta_2}} \\ \mathbf{P}_{21_{\delta_2}} \mathbf{P}_{11_{\delta_2\delta_2}} \\ \mathbf{P}_{21_{\delta_1}} \mathbf{P}_{11_{\delta_1\delta_2}} \\ \mathbf{P}_{21_{\delta_1}} \mathbf{P}_{11_{\delta_1\delta_1}} \mathbf{P}_{11_{\delta_1\delta_2}} \\ \mathbf{P}_{21_{\delta_1}} \mathbf{P}_{11_{\delta_1\delta_2}} \mathbf{P}_{11_{\delta_2\delta_2}} \\ \mathbf{P}_{21_{\delta_1}} \mathbf{P}_{11_{\delta_1\delta_1}} \mathbf{P}_{11_{\delta_1\delta_2}} \mathbf{P}_{11_{\delta_2\delta_2}} \end{bmatrix} \mathbf{P}_{11_{\delta_2\delta_j}} \begin{bmatrix} \mathbf{P}_{12_{\delta_j}} & \mathbf{P}_{11_{\delta_j\delta_j}} \mathbf{P}_{12_{\delta_j}} \end{bmatrix} = \begin{bmatrix} \mathbf{S}_{\Delta_{1_{\delta_2\delta_j}}} & \mathbf{S}_{\Delta_{2_{\delta_2\delta_j{}^2}}} \\ \mathbf{S}_{\Delta_{2_{\delta_2{}^2\delta_j}}} & \mathbf{S}_{\Delta_{3_{\delta_2{}^2\delta_j{}^2}}} \\ \mathbf{S}_{\Delta_{2_{\delta_1\delta_2\delta_j}}} & \mathbf{S}_{\Delta_{3_{\delta_1\delta_2\delta_j{}^2}}} \\ \mathbf{S}_{\Delta_{3_{\delta_1{}^2\delta_2\delta_j}}} & \mathbf{S}_{\Delta_{4_{\delta_1{}^2\delta_2\delta_j{}^2}}} \\ \mathbf{S}_{\Delta_{3_{\delta_1\delta_2{}^2\delta_j}}} & \mathbf{S}_{\Delta_{4_{\delta_1\delta_2{}^2\delta_j{}^2}}} \\ \mathbf{S}_{\Delta_{4_{\delta_1{}^2\delta_2{}^2\delta_j}}} & \mathbf{S}_{\Delta_{5_{\delta_1{}^2\delta_2{}^2\delta_j{}^2}}} \end{bmatrix}, \quad j = 3$$

The general equation for computing each P_{11} off-diagonal block, $P_{11_{\delta_n \delta_j}}$, given by equation (3.42), can be written as a generalized linear matrix equation of the following form:

$$\mathbf{AXB} = \mathbf{C} \tag{3.43}$$

where A, B, and C are known constant matrices. The following Lemma is stated without proof as an extension of Lemma 2.2 given in Reference [11].

Lemma 3.2

Consider the generalized linear matrix equation given by equation (3.43), where $\mathbf{A} \in \mathbf{R}^{nxm}$, $\mathbf{B} \in \mathbf{R}^{rxp}$, and $\mathbf{C} \in \mathbf{R}^{nxp}$ are given matrices. Then the following statements are equivalent:

(1) there exists a solution $\mathbf{X} \in \mathbf{R}^{mxr}$;
(2) the columns of $\mathbf{C} \in$ Im $[\mathbf{A}]$ and the rows of $C \in$ Im $[\mathbf{B}^T]$;
(3) rank $[\,\mathbf{A}\;\; \mathbf{C}\,]$ = rank $[\,\mathbf{A}\,]$ and rank $[\,\mathbf{B}^T\;\; \mathbf{C}^T\,]^T$ = rank $[\,\mathbf{B}\,]$;
(4) Ker $(\mathbf{A}^*) \subset$ Ker $(\mathbf{C}^*)$ and Ker $(\mathbf{B}) \subset$ Ker $(\mathbf{C})$.

Furthermore, the solution, if it exists, is unique if and only if **A** has full column rank and **B** has full row rank.

Equation (3.43) and Lemma 3.2 can be used in computing a solution for each off-diagonal block of P_{11}, based on equation (3.42). This solution has the following form.

$$\mathbf{X}^{\downarrow} = \mathbf{M} \setminus \mathbf{N} \tag{3.44}$$

where: $$\mathbf{M} = \mathbf{B}^T \otimes \mathbf{A} \quad ; \quad \mathbf{N} = \mathbf{C}^{\downarrow} \tag{3.45}$$

Note: $\mathbf{C}^{\downarrow}$ is the column-form vector of matrix C obtained by stacking the columns of C into one column vector

$$\Rightarrow \qquad \mathbf{X} = [\,\mathbf{X}_1^{\downarrow}\;\; \mathbf{X}_2^{\downarrow}\;\; \ldots\;\; \mathbf{X}_r^{\downarrow}\,] \quad ; \; \mathbf{X}_i^{\downarrow} \in \mathbf{R}^{mx1} \quad ; \quad i = 1, 2, \ldots, r \tag{3.46}$$

Then the following theorem is stated.

Theorem 3.3

Given a general linear matrix equation of the form given by equation (3.42) for each off-diagonal block of P_{11}, i.e.:

$$(\overline{\mathbf{P}}_{21_{\delta_n}}{}^{[n]}\overline{\mathbf{P}}_{11_{\delta_n}}{}^{[n]})\mathbf{P}_{11_{\delta_n\delta_j}}(\overline{\mathbf{P}}_{12_{\delta_j}}) = \overline{\mathbf{S}}_{\Delta_{\delta_n}}{}^{[n]}$$

where: $n = 1, 2, \ldots, m-1$ and $j = n+1, n+2, \ldots, m$

then a solution for $P_{11_{\delta_n \delta_j}}$ of the form given by equations (3.43) - (3.46) and which satisfies rank test (3) of Lemma 3.2 always exists and is irreducible.

Proof: (Sketch)

The rank test (3) of Lemma 3.2 can be used to determine whether a solution for $P_{11_{\delta_n \delta_j}}$ exists, based on the $P_{21_{\delta_i}}$, $P_{12_{\delta_i}}$, and $P_{11_{\delta_i \delta_i}}$ matrices computed as described in Section 3.2.1. If not, these matrices can be augmented using the appropriate columns and/or rows of the matrix $\overline{S}_{\Delta_{\delta_n}}^{[n]}$ given on the right side of equation (3.42). Thus, a solution can always be found. The resulting solution is irreducible, because satisfaction of rank condition (3) in obtaining a solution prevents unnecessary redundancy from being built into the solution process.

QED

To summarize, this section has presented a simple numerical technique for computing the off-diagonal blocks of P_{11}, i.e. $P_{11_{\delta_n \delta_j}}$, for each block-row, n, and each block-column, j, (as defined by equation (3.16)), where n = 1, 2, ... , m–1 and j = n+1, n+2, ... , m. The numerical computation involves the solution of a generalized linear matrix equation, and such a solution can always be found by augmenting the previously computed $P_{21_{\delta_i}}$, $P_{12_{\delta_i}}$, $P_{11_{\delta_i \delta_i}}$, and $P_{11_{\delta_i \delta_j}}$ matrices as required to obtain a solution for equation (3.42) based on equations (3.43) - (3.46). The result is irreducible, because a solution for each off-diagonal block is computed to just meet the rank conditions (3) given by Lemma 3.2.

3.2.3 Full P-Δ Model Solution, Nilpotency and 1-D Irreducibility

Once the $P_{21_{\delta_i}}$, $P_{12_{\delta_i}}$, $P_{11_{\delta_i\delta_i}}$, and $P_{11_{\delta_i\delta_j}}$ partitions for each parameter have been determined as described in Sections 3.2.1 and 3.2.2, the full solution is determined using equations (3.9) - (3.12). This is a simple matter of collecting the matrix partitions together into a single matrix for P_{21}, P_{12}, and P_{11}. The Δ matrix is also known and given by equation (3.13), where the number of repetitions for each parameter, n_i, was determined in solving the $P_{21_{\delta_i}}$, $P_{12_{\delta_i}}$, $P_{11_{\delta_i\delta_i}}$, and $P_{11_{\delta_i\delta_j}}$ matrices.

The following theorem is given regarding the satisfaction of the nilpotency condition of equation (3.3) for the full P-Δ model solution.

Theorem 3.4

The P_{11} matrix defined by equation (3.9) and computed using Theorems 3.1 and 3.3 as described in Sections 3.2.1 and 3.2.2 satisfies the nilpotency condition of equation (3.3), as defined below.

$$[\,\Delta\, P_{11}\,]^{r+1} = 0 \quad ; \quad r+1 \le \eta_1 + \eta_2 + \dots + \eta_m$$

Proof: (Sketch)

For $r+1 = \eta_1 + \eta_2 + \dots + \eta_m$, nilpotency is satisfied by Lemma 3.1. For this case, solution of the off-diagonal blocks does not enter into satisfying the nilpotency condition. That is, the nilpotency of the main-diagonal blocks is sufficient to satisfy the nilpotency of the full solution.

For $r+1 < \eta_1 + \eta_2 + \ldots + \eta_m$, nilpotency is satisfied by the solution of the off-diagonal blocks. That is, this case arises when there are zero crossterm coefficient matrices that are factored into the solution of the off-diagonal blocks. Thus, inclusion of these zero matrix coefficients in the solution of the off-diagonal blocks automatically satisfies the nilpotency of the full solution.

QED

An objective of the P-Δ modeling process was to determine a model which is low-order. The following theorem is therefore given regarding the reducibility of the full P-Δ model solution.

Theorem 3.5

The P-Δ model matrices defined by equations (3.9) - (3.13) and solved using Theorems 3.1 and 3.3 is 1-D Irreducible.

Proof: (Sketch)

The $P_{21_{\delta_i}}$, $P_{12_{\delta_i}}$, and $P_{11_{\delta_i\delta_i}}$ matrices determined using Theorem 3.1 represent an irreducible realization of the linear and n^{th}-degree terms of $S_\Delta(\delta)$ associated with the δ_i parameter. Solving equation (3.42) using Theorem 3.3 results in an irreducible solution of the off-diagonal blocks of P_{11} based on the solution obtained previously for $P_{21_{\delta_i}}$, $P_{12_{\delta_i}}$, and $P_{11_{\delta_i\delta_i}}$. Thus, putting the full solution together results in a 1-D irreducible LFT model of the given system.

QED

4. Example: Multivariate Quadratic Problem (See Reference [4])

Consider the following compound inertia matrix problem presented in [4], and first posed in [13].

$$\mathbf{J} = \begin{bmatrix} 0 & -2yz & 2y^2 & 4(y^2 - z^2) & -3xy & xz \\ 2yz & 0 & -2xy & -4xy & 3(x^2 - z^2) & yz \\ -2y^2 & 2xy & 0 & 4xz & -3yz & y^2 - x^2 \end{bmatrix} \tag{4.1}$$

The x, y, and z terms represent displacement parameters from some reference (zero) point for the system. Thus, the parameters x, y, and z are the uncertain parameters, δ, of the system. The results obtained using the above computational solution (in Matlab) are shown below. However, the details of obtaining this solution are omitted for brevity.

$$\mathbf{P}_{21} = [\, \mathbf{P}_{21_{\delta x}} \; \mathbf{P}_{21_{\delta y}} \; \mathbf{P}_{21_{\delta z}} \,] \tag{4.2a}$$

$$\mathbf{P}_{21_{\delta x}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 1.7321 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}, \quad \mathbf{P}_{21_{\delta y}} = \begin{bmatrix} 0 & 2.1147 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1.4953 & 0 & 0 & 0 \end{bmatrix} \tag{4.2b}$$

$$\mathbf{P}_{21_{\delta z}} = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1.7321 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{4.2c}$$

$$\mathbf{P}_{12} = \begin{bmatrix} \mathbf{P}_{12_{\delta x}} \\ \mathbf{P}_{12_{\delta y}} \\ \mathbf{P}_{12_{\delta z}} \end{bmatrix} \tag{4.3a}$$

$$\mathbf{P}_{12_{\delta x}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.7321 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{P}_{12_{\delta y}} = \begin{bmatrix} 0 & 0 & .94574 & 1.8915 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1.3375 & 0 & 0 & 0 & 0 & -.66874 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{4.3b}$$

$$\mathbf{P}_{12_{\delta z}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.7321 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{4.3c}$$

$$\mathbf{P}_{11} = \begin{bmatrix} \mathbf{P}_{11_{\delta_x\delta_x}} & \mathbf{P}_{11_{\delta_x\delta_y}} & \mathbf{P}_{11_{\delta_x\delta_z}} \\ 0 & \mathbf{P}_{11_{\delta_y\delta_y}} & 0 \\ 0 & \mathbf{P}_{11_{\delta_z\delta_y}} & \mathbf{P}_{11_{\delta_z\delta_z}} \end{bmatrix} \tag{4.4a}$$

$$\mathbf{P}_{11_{\delta_x\delta_x}} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \ \mathbf{P}_{11_{\delta_y\delta_y}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \ \mathbf{P}_{11_{\delta_z\delta_z}} = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{4.4b}$$

$$P_{11_{\delta_x\delta_y}} = \begin{bmatrix} -1.2209 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -3 & 0 & 0 \end{bmatrix}, \quad P_{11_{\delta_x\delta_z}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \qquad (4.4c)$$

$$P_{11_{\delta_z\delta_y}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & .28778 & 0 & 0 & 0 & .7698 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -3 & 0 & 0 \end{bmatrix} \qquad (4.4d)$$

$$\Delta = \operatorname{diag}\,[\delta_x I_5, \delta_y I_7, \delta_z I_6] \qquad (\, n_\Delta = 18 \,) \qquad (4.5)$$

Note that the solution of this problem was not restricted to a quasi-upper-triangular P_{11} matrix. In particular, it was determined in solving this problem that the quasi-upper-triangular structure for P_{11} required an extra repetition in Δ to obtain a solution.

A comparison of this solution with those obtained in [4] and [13] is shown in Table 1.

Method	n_x	n_y	n_z	n_Δ	**Comments**
Belcastro & Chang	5	7	6	18	Direct Numerical Solution for Nonlinear Problem, No Decomposition, No Model Reduction
Cockburn & Morton [4]	9	10	9	28	Decomposition to Linear Components, Solution for Each Linear Component, Combination of Component Solutions
	7	8	5	20	Same As Above with Model Reduction
	7	5	7	19	Special Decomp. to Linear Components, Solution for Each Linear Component, Combination of Component Solutions
	6	5	6	17	Same As Above with Model Reduction
Doyle, Elgersma, et. al. [13]	9	9	9	27	Decomposition to Linear Components, Solution for Each Linear Component, Combination of Component Solutions
	4	5	4	13	Special Matrix Decomp. to Linear Products, Solution for Each Linear Component, Combination of Component Solutions

Table 1. Comparison of LFT Models Obtained Using Current Methods

The solution obtained using this LFT modeling approach required a total of 18 parameters in Δ, with 5 repetitions for δ_x ($n_x = 5$), 7 for δ_y ($n_y = 7$), and 6 for δ_z ($n_z = 6$). Note that the LFT modeling approach of this paper does not require matrix decompositions for a solution to this example, since it was already in a multivariate polynomial form. Moreover, this approach achieves a low-order model directly (without the use of model reduction), and can be readily implemented in Matlab. The result presented in [4] for a direct decomposition required 28 and 20 parameters in Δ before and after model resuction, respectively. The result obtained using a specialized decomposition approach developed in [4] to reduce the resulting LFT model dimension required 19 and 17 parameters in Δ before and after model reduction, respectively. Note that this approach decomposed the J matrix of equation (4.1) to linear matrix products and sums. Then an LFT model for each linear matrix was obtained separately, the individual LFT models combined to form the full LFT model, and reduction methods applied to remove unnecessary repetitions. The result presented in [13] required 27 parameters in Δ using a linear decomposition approach, and 13 parameters in Δ by recognizing that J can be factoed into the product of two matrices containing only linear x, y, and z terms. Although this yields the lowest-order LFT model, it is specific for this particular matrix structure and can therefore not be generally applied to other problems.

5. Concluding Remarks

A numerical approach was presented in this paper to directly compute low-order LFT models for multivariate polynomial problems. The LFT modeling approach does not require matrix decompositions for multivariate polynomial problems, and a low-order model is directly obtained without model reduction. The computations depend only on simple matrix computations, including the singular value decomposition (svd) and solving generalized linear matrix equations. A matrix svd is used to simultaneously compute a solution for the $P_{21_{\delta_i}}$, $P_{12_{\delta_i}}$, and $P_{11_{\delta_i \delta_i}}$ matrices for each δ_i parameter. Generalized linear matrix equations are used to solve for the $P_{11_{\delta_i \delta_j}}$ matrices. The full LFT model is constructed by simply collecting the partitioned solutions together into the P_{21}, P_{12}, and P_{11} matrices. The resulting LFT model is low-order, because matrix structure is exploited during the computations in satisfying the rank conditions required for a solution. Future work will include developing a Matlab implementation of this LFT modeling approach.

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13. ABSTRACT (Maximum 200 words)

Robust control system analysis and design is based on an uncertainty description, called a linear fractional transformation (LFT), which separates the uncertain (or varying) part of the system from the nominal system. These models are also useful in the design of gain-scheduled control systems based on Linear Parameter Varying (LPV) methods. Low-order LFT models are difficult to form for problems involving nonlinear parameter variations. This paper presents a numerical computational method for constructing and LFT model for a given LPV model. The method is developed for multivariate polynomial problems, and uses simple matrix computations to obtain an exact low-order LFT representation of the given LPV system without the use of model reduction. Although the method is developed for multivariate polynomial problems, multivariate rational problems can also be solved using this method by reformulating the rational problem into a polynomial form.

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