A Game Theoretic Fault Detection Filter

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Abstract. The fault detection process is modelled as a disturbance attenuation problem. The solution to this problem is found via differential game theory, leading to an H_∞ filter which bounds the transmission of all exogenous signals save the fault to be detected. For a general class of linear systems which includes some time-varying systems, it is shown that this trannission bound can be taken to zero by simultaneously bringing the sensor noise weighting to zero. Thus, in the limit, a complete trannission block can be achieved, making the game filter into a fault detection filter. When we specialize this result to time-invariant systems, it is found that the detection filter attained in the limit is identical the well known Beard-Jones Fault Detection Filter. That is, all fault inputs other than the one to be detected (the "nuisance faults") are restricted to an invariant subspace which is unobservable to a projection on the output. For time-invariant systems, it is also shown that in the limit, the order of the state-space and the game filter can be reduced by factoring out the invariant subspace. The result is a lower dimensional filter which can observe only the fault to be detected. A reduced-order filter can also be generated for time-varying systems, though the computational overhead may be intensive. An example given at the end of the paper demonstrates the effectiveness of the filter as a tool for fault detection and identification.

1. Introduction

The need for high reliability and low maintenance in complex systems such as the proposed intelligent vehicle highway system will require self-monitoring and fault tolerance. Schemes to carry out these tasks generally fall under the heading of health monitoring systems and without exception require a processor which uses sensor data to determine the presence and origin of failures within the system. One concept of such a processor is the fault detection filter. Often called the Beard-Jones Fault Detection Filter after its originators, the fault detection filter is a specially designed observer which isolates the influence of each fault upon the state trajectory, making the simultaneously detection and identification of failures possible.

Since its initial formulation, the fault detection filter has undergone many reinterpretations and refinements. White [1] derived a detection filter design algorithm based upon eigenstructure assignment. Massoumnia [2] gave a geometric interpretation of the filter and also derived a reduced-order fault detector based on geometric arguments in [3]. Most recently, Douglas robustified the filter and derived a new version of the filter which bounds disturbance transmission [4], [5], and [6].

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Common to all of these incarnations of the fault detection filter is an underlying structure of independent, invariant subspaces which are matched one-to-one with a particular fault and which wholly contain the state trajectory when the system is driven solely by that fault. Design algorithms have, thus, relied heavily on spectral methods - i.e. specifying eigenvalues or eigenvectors. Spectral methods lead directly to the needed filter structure, but they limit the applicability of the fault detection filter to linear, time-invariant systems and, due to their need for an accurate plant model, their robustness to parameter variations can be poor [7].

For these reasons, we have taken a different route to fault detection filter design. We look at the fault detection process as a disturbance attenuation problem. The filter design, as a result, comes out of a differential game in which the player is a state estimate and the adversaries are all of the exogenous signals, save the fault to be detected. To approximate the invariant subspace structure of the fault detection filter, the game is one in which the player attempts to exclude the adversaries from a specified portion of the state-space, assuming all the while that the adversaries play their best strategy. The solution to this game results in an H_∞-type filter which bounds disturbance transmission.

Since fault detection filters block transmission, it would seem reasonable to expect that in the limiting case when the H_∞ transmission bound is brought to zero, the game filter no longer approximates, but actually becomes a fault detection filter. We will prove that this is indeed the case. In fact, for linear time-invariant (LTI) systems, we will show that the game filter becomes a Beard-Jones Fault Detector in the sense of [4]: faults other than the one to be detected are restricted to a subspace which is invariant and unobservable.

The filter that we propose here, however, is not merely an alternative algorithm for detection filter design. The method we develop has wider applicability than existing design techniques. Time invariance is not assumed in the game solution and so it can be shown that, for a class of time-varying systems, results analogous to the LTI case exist in the limit as disturbance bounds are taken to zero. With this method, it is also possible to deal with model uncertainty by treating such uncertainty as another element in the differential game [8], [9]. Finally, we will show that with this approach the designer has the freedom to choose the extent to which the game filter behaves as an H^∞ filter and the extent to which it behaves like a detection filter. This flexibility is unique to the game theoretic approach to fault detection filter design.

Section 2 covers the basics of fault detection filter theory. We define a general fault detection filter problem and show how the existing theory matches our definition for LTI systems. In Section 3, we pose a disturbance attenuation problem which models the fault detection process for a large class of systems which includes some time-varying systems. The solution to this problems leads to a game theoretic filter which bounds the transmission of all exogenous inputs aside from the fault be detected. In Section 4, we analyze sufficient conditions for our game cost to be non-positive. This will enable us to show the existence of the filter in the limit and analyze its structure. In Section 5, we return to the LTI case and prove that the detection filter that we find in the limiting case is equivalent to the fault detection filter structures already existing in the literature. In Section 6, we use the limiting form of the game theoretic filter to derive a reduced-order estimator for fault detection. These results apply to the larger class of systems. Finally, in Section 7 we go through an example which shows that the filter is an effective fault detector for finite values of the disturbance attenuation bound and in the limit.

2. Fault Detection Filter Theory

In this section, we introduce and define the fault detection filter problem. The existing theory for fault detection filters is strongly tied to time-invariant systems and many of the key results of this theory depend on concepts which do not carry over into time-varying systems. Hence, we make a distinction between the existing problem and a definition that we propose for the fault detection problem based upon disturbance attenuation. We will show that our definition matches the existing definition when applied to LTI systems.

2.1 Time-Invariant Fault Detection Filter Theory

Current fault detection filter theory is largely developed in [1], [2], and [4]. The geometric contol concepts upon which much of the theory is based are developed in [10].

The systems that we consider here are linear, time-invariant with (C, A) observable:

$$\dot{x}(t) = Ax(t) + Bu(t)$$

$$y(t) = Cx(t).$$

Linear maps are denoted by mathematical italics (e.g. M) and linear spaces by caligraphic text (e.g. $\mathcal{M}$). Hence, $\mathcal{X}$, is the n-dimensional state-space; $\mathcal{Y}$ is the m-dimenstional output space, and $\mathcal{U}$ is the k-dimensional input space. $A : \mathcal{X} \rightarrow \mathcal{X}$, $B : \mathcal{U} \rightarrow \mathcal{X}$, and $C : \mathcal{X} \rightarrow \mathcal{Y}$ are matrices of appropriate dimesions and are typically called the state matrix, the input matrix, and the output matrix respectively.

It has been shown in the cited references that faults in the sensors, the actuators, and even the plant can be modelled as additive signals in the state equation:

$$\dot{x} = Ax + Bu + \sum_{j=1}^{q} F_j \mu_j \tag{1}$$

The $F_j : \mathcal{M}_j \to \mathcal{X}$'s are called *failure maps* and are fixed, known maps which represent the directional behavior of the fault. For an actuator failure, F_j is the corresponding column of the B matrix. For sensor failures [1]. F_j is derived from the corresponding row of the C matrix via:

$$y = Cx + E_i\mu_i.$$

E_i is an $m \times 1$ unit vector corresponding to a failure in the ith sensor. To convert this failure into a form compatible with (1), a two-column failure map [1]:

$$F_i = [\ f_i \quad f_i^*\ ] \tag{2}$$

is used where f_i is any vector such that $E_i = Cf_i$ and $f_i^* = Af_i$ - for time-invariant systems. For time-varying systems, $f_i^* = Af_i - \dot{f}_i$. Why this is so will be explained later in this section.

The μ_j's are *failure signals* and represent the time history of the failure signal amplitude. In nominal, no-failure conditions, the μ_j's are zero. Failure maps are generally assumed to be monic so that non-zero μ_j's result in non-zero vectors, $F_j\mu_j$. In addition to (C, A)-observability, it will also be assumed that the system is ***output separable*** [1]:

$$\text{rank}\,[\ CA^{\beta_1}V_1, \quad \ldots, \quad CA^{\beta_q}V_q\ ] = q \tag{3}$$

where:

$$V_i = \begin{cases} F_i & \text{if } i \text{ is an actuator fault,} \\ f_i & \text{if } i \text{ is a sensor fault.} \end{cases}$$

Output separability is a check that the problem is well-posed in the sense that the failures chosen in the design set are linearly independent. F_i is the failure map for actuator faults, f_i is the vector defined by Equation 2, and β_i is the smallest integer such that $CA^{\beta_i}V_i \neq 0$. We make the distinction between actuator and sensor faults. because F_i has only one column for the former and two columns for the latter. If (3) is not satisfied, then the designer needs to decide which failures he needs to discard from his design set. Note, that (3) is a time-invariant result. The analog for the time-varying case would look like a grammian matrix.

Definition 2.1. The *Fault Detection Filter Problem* as defined in [4] is to find subspaces $\mathcal{W}_j \subset \mathcal{X}$ such that:

1. Each subspace contains the image of one and only one of the failure maps ($\mathcal{F}_j := \text{Image}F_j$):

$$\mathcal{F}_j \subset \mathcal{W}_j \tag{4}$$

2. The subspaces are *output separable*:
$$CW_j \cap \sum_{i \neq j} CW_i = \emptyset. \tag{5}$$

3. The subspaces are (C, A)-invariant, i.e. there exists a map $L : \mathcal{Y} \rightarrow \mathcal{X}$ such that:
$$(A + LC)\mathcal{W}_j \subset \mathcal{W}_j \tag{6}$$

∎

Note that the output separability requirement in (5) refers to the solution of the fault detection filter problem, whereas the output separability specified by (3) refers to the well-posedness of the detection filter problem itself.

In [4] ***unobservability subspaces*** are used as the invariant subspaces for detection filter construction. Unobservability subspaces are the dual of controllablity subspaces [10] and are the unobservable subspaces, $\mathcal{T}_j$, of the pairs $(\hat{H}_jC, A + LC)$, where L is the observer gain from (6) and $\hat{H}_j$ is a natural projection such that $\ker \hat{H}_jC = \sum_{i \neq j} \mathcal{T}_i + \ker C$ ($\ker C$ refers to the Kernal or Null Space of C). $\hat{H}_j$ can be found via:

$$\hat{H}_j = I - (CA^{\beta_j}\hat{F}_j)[(CA^{\beta_j}\hat{F}_j)^T(CA^{\beta_j}\hat{F}_j)]^{-1}(CA^{\beta_j}\hat{F}_j)^T. \tag{7}$$

Again, β_j is the smallest integer such that $CA^{\beta_j}\hat{F}_j \neq 0$ and

$$\hat{F}_j = \begin{bmatrix} F_1 & \dots & F_{j-i} & F_{j+1} & \dots & F_q \end{bmatrix} \tag{8}$$

Roughly speaking, $\hat{H}_j$ blocks out the portion of the state-trajectory which lies in the other invariant subspaces.

Unobservability subspaces are crucial for spectral design methods because for any such subspace, $\mathcal{T}_i$, both the spectrum of $(A + LC|\mathcal{T}_i)$ - the restriction of $(A + LC)$ to $\mathcal{T}_i$ - and the spectrum of $(A + LC|\mathcal{X}/\mathcal{T}_i)$ - the restriction to the factor space - can be completely specified by L[1]. If an invariant subspace other than an unobservability subspace is used in a detection filter design, it is possible that some of the eigenvalues of $(A + LC)$ will be fixed by the invariant zeros of $(C, A, \hat{F}_1)$. This could be problematic if these zeros are in undesirable locations. We will briefly discuss invariant zeros and where they appear in the game filter in Section 5 so that our results can be directly compared to existing detection filter design methods.

When the three given conditions are realized, the result is a Luenberger Observer:

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x}).$$

[1] When there is only one unobservability subspace to consider, this means that the entire spectrum of closed-loop state matrix $(A + LC)$ can be specified. When more than one unobservability involved, an extra condition called ***mutual detectability*** [1] is needed to specify all of $(A + LC)$

with error dynamics governed by ($e := x - \hat{x}$):

$$\dot{e} = (A + LC)e + \sum_{j=1}^{q} F_i \mu_i \tag{9}$$

Equation 5 implies that μ_j is restricted to an invariant subspace $\mathcal{W}_j$ whose image under C is disjoint from all other such subspaces. Hence, by the definition of $\hat{H}_j$ and equations (4) - (9), the signal:

$$z_j = \hat{H}_j(y - C\hat{x}) = \hat{H}_j C e. \tag{10}$$

will be driven solely by the fault μ_j. A non-zero value of z_j should simultaneuosly detect and identify the fault μ_j. Given (10, we can derive the sensor failure map for the time-varying case. A sensor fault leads to $z = \hat{H}_1 C \bar{\epsilon}$ where $\bar{\epsilon} = e + f_i \mu_i$. Differentiating $\bar{\epsilon}$, gives us an equation in the form of (9), leading to the failure map:

$$\dot{\bar{\epsilon}} = (A + LC)\bar{\epsilon} - A f_i \mu_i + \dot{f}_i \mu_i + f_i \dot{\mu}_i = (A + LC)\bar{\epsilon} + \left[\begin{array}{cc} f_i & f_i^* \end{array}\right] \left\{\begin{array}{c} \dot{\mu}_i \\ -\mu_i \end{array}\right\}$$

2.2 A Definition of the Fault Detection Filter Problem based upon Disturbance Attenuation

Let us now consider an alternative and more encompassing definition of the fault detection filter problem. Consider a linear system in which q possible faults have been modelled:

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t) + F_1(t)\mu_1(t) + \sum_{i=2}^{q} F_i(t)\mu_i(t) \\ y(t) &= C(t)x(t) + v(t). \end{aligned}$$

It is desired to detect the appearance of μ_1 (the "target fault") in the presence of sensor noise, v, and the possible presence of other faults $\mu_i, i \neq 1$ (the "nuisance faults"). Following the standard assumptions of section 2.1, we will assume that each of the F_i's are monic and that (C, A) is an observable pair. Also, since u is a known function of $t \in [t_0, t_1]$, we will drop the Bu term for convenience. We will also neglect to explicitly show the possible time dependence of the system matrices, though the reader should keep this possibility in mind. For convenience, we define:

$$\hat{\mu}_2 = \left\{\begin{array}{c} \mu_2 \\ \vdots \\ \mu_q \end{array}\right\},$$

and use the definition of $\hat{F}_i$ (8) so that the state equation becomes:

$$\dot{x} = Ax + F_1\mu_1 + \hat{F}_1\hat{\mu}_2$$

The definition that we propose is based upon disturbance attenuation. We use (10) and define the corresponding residual signal z_1 associated with μ_1 as the output signal. A disturbance attenuation problem would be to limit the transmission of the nuisance faults and the sensor noise to this output. For a fault detection filter problem we want to block this transmission entirely.

Definition 2.2. The *Fault Detection Filter Problem* is to find an estimator such that:

$$\frac{\|z_1\|^2}{\|\hat{\mu}_2\|^2} = 0$$

and:

$$\frac{\|z_1\|^2}{\|\mu_1\|^2} \neq 0.$$

∎

Clearly, in the time-invariant case, the solution to the fault detection filter problem as defined by Definition 2.1 solves the general fault detection filter problem that we have defined above. Later on, we will show that these definitions are equivalent in the time-invariant case by showing that the solution to Definition 2.2 solves the problem defined by Definition 2.1.

The definition that we propose is needed when we consider time-varying systems. In such cases, we cannot talk about invariant subspaces and also observability becomes a trickier concept. Thus instead of defining the structure of the final filter, we must content ourselves with merely describing its action.

Now, because we are dealing with time-varying systems, we must define our projector $\hat{H}_1$ appropriately:

$$\hat{H}_1 = I - CB_i[(CB_i)^T(CB_i)]^{-1}(CB_i)^T \tag{11}$$

where:

$$B_1 = F_1 \tag{12}$$

$$B_j = AB_{j-1} + \dot{B}_{j-1} \tag{13}$$

and $i \in \mathbb{Z}_+$ is the smallest integer such that:

$$CB_i \neq 0 \tag{14}$$

For time-invariant systems, (11) reduces to (7). Equations (12) to (14) come from the Goh Transformation [11]. We will use the Goh Transformation again in later sections. It will be assumed that the system matrices $A(t), C(t), \hat{F}_1(t)$

the number of iterations of (13) needed so that (14) is satisfied is constant over the entire interval, $[t_0, t_1]$. That is, the time-variations of the system do not change the dimensionality of the detection problem. This restricts the applicability of this analysis to a subclass of time-varying systems, but it allows us to avoid expending the effort which would be needed to deal with exceptional cases. Calling (11) a projector is a slight misnomer since it will be a time-varying matrix, but by it's construction it will be such that Ker $\hat{H}(t)_1 =$ Ker $C(t) +$ Im B_i $\forall t \in [t_0, t_1]$ which is our main concern.

3. A Game Theoretic Filter for Fault Detection in a General Class of Systems

3.1 The Disturbance Attenuation Problem

We arrive at a solution to the fault detection filter problem as defined by Definition 2.2 by first solving the disturbance attenuation problem. The solution to the fault detection filter problem then comes when we take the limit of the disturbance attenuation solution. The results that we find here, however, are valuable in their own right. As we will see, the game filter that we get from the disturbance attenuation problem is itself a useful filter for fault detection.

We begin by quantifying the problem objective with a disturbance attenuation function, the ratio of the norm of the output to the norms of the inputs. For this problem, the function is:

$$D_{af} = \frac{\int_{t_0}^{t_1} \|\hat{H}_1 C(x - \hat{x})\|_{Q_1}^2 dt}{\int_{t_1}^{t_2} [\|\hat{\mu}_2\|_{M_2^{-1}}^2 + \|v\|_{V^{-1}}^2 + \|N_1 C(x - \hat{x})\|_{R_1}^2] dt + \|x(t_0) - \hat{x}_0\|_{P_0^{-1}}^2} \tag{15}$$

where $N_1 := I - \hat{H}_1$ and M_2, V, R_1, P_0 are weighting matrices. The disturbance attenuation problem is to find an estimator so that for all adversaries $\hat{\mu}_2$, $v \in L_2[t_1, t_2]$, $x(0) \in \mathbb{R}^n$:

$$D_{af} \leq \gamma. \tag{16}$$

We will refer to γ as the disturbance attenuation bound. Once again, the assumptions that we will make are: 1) (C, A) is a an observable pair 2) F_i, $i = 1 \ldots q$ is monic 3) i, the number of iterations of (13) needed so that (14) is satisfied, is a constant.

To solve (16) we convert it into a differential game with cost function:

$$J = \int_{t_0}^{t_1} [\|\hat{H}_1 C(x - \hat{x})\|_{Q_1}^2 - \gamma(\|\hat{\mu}_2\|_{M_2^{-1}}^2 + \|v\|_{V^{-1}}^2 + \|N_1 C(x - \hat{x})\|_{R_1}^2)] dt - \|x(t_0) - \hat{x}_0\|_{\Pi_0}^2 \tag{17}$$

Note that $\Pi_0 := \gamma P_0^{-1}$. We want to find:

$$\min_{\hat{x}} \max_{v} \max_{\hat{\mu}_2} \max_{x(t_0)} J \leq 0 \tag{18}$$

subject to:

$$\dot{x} = Ax + \hat{F}_1\hat{\mu}_2. \tag{19}$$

An element that is missing in our problem statement (17), (18), (19) is the target fault, μ_1. This is not an oversight. It would seem logical to include enhancing the transmission of μ_1 as part of the game, but there is no obvious way to include such an objective in the game cost. Moreover, extremizing the cost with respect to μ_1 leads to assumptions upon the temporal behavior of the target fault. This can be quite detrimental to filter performance if these assumptions are wrong (which is why fault detection filters are designed without any such assumptions). Thus, since μ_1 is not part of the differential game, we set it to zero for convenience when we work through the solution. This places the burden on the designer to make sure the set of faults that he chooses for the filter design leads to a well-posed problem. Well-posedness is discussed in Section 2 and for LTI systems is easily checked by Equation 3.

3.2 Maximization with Respect to $x(t_0)$ and $\hat{\mu}_2$

We will solve our problem in two steps beginning with the subproblem:

$$\max_{\hat{\mu}_2}\max_{x(t_0)} J \leq 0.$$

The reasoning for this particular order of the extremizations is given in [12].

We begin by converting the sensor noise term $\|v\|^2_{V^{-1}}$ to the equivalent $\|y - Cx\|^2_{V^{-1}}$ and by appending the dynamics of the system to the cost with a Lagrange multiplier, λ^T:

$$J = \int_{t_0}^{t_1} [\|\hat{H}_1C(x-\hat{x})\|^2_{Q_1} - \gamma(\|\hat{\mu}_2\|^2_{M_2^{-1}} + \|y-Cx\|^2_{V^{-1}} + \|N_1C(x-\hat{x})\|^2_{R_1}) + \lambda^T(Ax+\hat{F}_1\hat{\mu}_2-\dot{x})]dt - \|x(t_0)-\hat{x}_0\|^2_{\Pi_0}. \tag{20}$$

Integrate $\lambda\dot{x}$ by parts:

$$\begin{aligned} J = \int_{t_0}^{t_1} & [\|\hat{H}_1C(x-\hat{x})\|^2_{Q_1} - \gamma(\|\hat{\mu}_2\|^2_{M_2^{-1}} + \|y - Cx\|^2_{V^{-1}} + \|N_1C(x-\hat{x})\|^2_{R_1}) \\ & + \lambda^T(Ax + \hat{F}_1\hat{\mu}_2) + \dot{\lambda}^T x]dt - \|x(t_0) - \hat{x}_0\|^2_{\Pi_0} + \lambda(t_0)^T x(t_0) - \lambda(t_1)^T x(t_1) \end{aligned} \tag{21}$$

and then take the variation of (21) with respect to $\hat{\mu}_2$ and $x(t_0)$:

$$\begin{aligned} \delta J = \int_{t_0}^{t_1} & \{[(x-\hat{x})^T C^T \hat{H}_1^T Q_1 \hat{H}_1 C + \gamma(y - Cx)^T V^{-1} C - \gamma(x-\hat{x})^T C^T N_1^T R_1 N_1 C + \dot{\lambda}^T + \lambda^T A]\delta x \\ & + [-\gamma\hat{\mu}_2^T M_2^{-1} + \lambda^T\hat{F}_1]\delta\hat{\mu}_2\}dt - [(x(t_0) - \hat{x}_0)^T\Pi_0 + \lambda(t_0)^T]\delta x(t_0) - \lambda(t_1)^T\delta x(t_1). \end{aligned} \tag{22}$$

The above implies that first-order necessary conditions for J to be maximized are:

$$
\begin{aligned}
\hat{\mu}_2 &= \frac{1}{\gamma} M_2 \hat{F}_1^T \lambda && (23)\\
-\dot{\lambda} &= A^T\lambda + C^T(\hat{H}_1^T Q_1 \hat{H}_1^T - \gamma N_1^T R_1 N_1)C(x - \hat{x}) + \gamma C^T V^{-1}(y - Cx) && (24)\\
\lambda(t_1) &= 0 && (25)\\
\lambda(t_0) &= \Pi_0[x(t_0) - \hat{x}_0] && (26)
\end{aligned}
$$

Substituting (23) into our dynamics (19) and using (24), we obtain a two point boundary value problem:

$$
\left\{ \begin{array}{c} \dot{x} \\ \dot{\lambda} \end{array} \right\} = \left[\begin{array}{cc} A & \frac{1}{\gamma}\hat{F}_1 M_2 \hat{F}_1^T \\ -C^T(\hat{H}_1^T Q_1 \hat{H}_1^T - \gamma N_1^T R_1 N_1 - \gamma V^{-1})C & -A^T \end{array} \right] \left\{ \begin{array}{c} x \\ \lambda \end{array} \right\} +
$$
$$
\left\{ \begin{array}{c} 0 \\ C^T(\hat{H}_1^T Q_1 \hat{H}_1^T - \gamma N_1^T R_1 N_1)C\hat{x} - \gamma C^T V^{-1} y \end{array} \right\}. \qquad (27)
$$

If we assume solutions x^* and λ^* to (27) and a quadratic form of the optimal return function, then:

$$
\lambda^* = \Pi(x^* - x_p) \qquad (28)
$$

where x_p is a measurement dependent variable which will be shown to reduce to the estimate of the optimal state. Using (28) and the first equation of (27), the second equation of (27) becomes:

$$
\begin{aligned}
0 = [\dot{\Pi} + A^T\Pi + \Pi A + \frac{1}{\gamma}\Pi \hat{F}_1 M_2 \hat{F}_1^T \Pi + C^T(\hat{H}_1^T Q_1 \hat{H}_1^T - \gamma N_1^T R_1 N_1 - \gamma V^{-1})C]x^* \\
-\dot{\Pi} x_p - \Pi \dot{x_p} - A^T \Pi x_p - C^T(\hat{H}_1^T Q_1 \hat{H}_1^T - \gamma N_1^T R_1 N_1)C\hat{x} + \gamma C^T V^{-1} y. \qquad (29)
\end{aligned}
$$

Now, add and subtract $\gamma C^T V^{-1} C\hat{x}$ and $[\Pi A + C^T(\hat{H}_1^T Q_1 \hat{H}_1^T - \gamma N_1^T R_1 N_1 - \gamma V^{-1})C]x_p$ to (29):

$$
\begin{aligned}
0 = [\dot{\Pi} + A^T\Pi + \Pi A + \frac{1}{\gamma}\Pi \hat{F}_1 M_2 \hat{F}_1^T \Pi + C^T(\hat{H}_1^T Q_1 \hat{H}_1^T - \gamma N_1^T R_1 N_1 - \gamma V^{-1})C](x^* - x_p) \\
-\Pi \dot{x}_p - \Pi A x_p - [C^T(\hat{H}_1^T Q_1 \hat{H}_1^T - \gamma N_1^T R_1 N_1 - \gamma C^T V^{-1})C](\hat{x} - x_p) + \gamma C^T V^{-1}(y - C\hat{x}). \qquad (30)
\end{aligned}
$$

Thus, if we set:

$$
\begin{aligned}
-\dot{\Pi} &= A^T\Pi + \Pi A + \frac{1}{\gamma}\Pi \hat{F}_1 M_2 \hat{F}_1^T \Pi + C^T(\hat{H}_1^T Q_1 \hat{H}_1^T - \gamma N_1^T R_1 N_1 - \gamma V^{-1})C && (31)\\
\Pi \dot{x}_p &= \Pi A x_p - C^T(\hat{H}_1^T Q_1 \hat{H}_1^T - \gamma N_1^T R_1 N_1 - \gamma V^{-1})C(\hat{x} - x_p) + \gamma C^T V^{-1}(y - C\hat{x}). && (32)
\end{aligned}
$$

(30) is satisfied identically. (31) is an estimator Riccati Equation. If we set:

$$\Pi = \gamma P^{-1}, \tag{33}$$

we can convert (31) into:

$$\dot{P} = PA^T + PA - PC^T(V^{-1} + N_1^T R_1 N_1 - \frac{1}{\gamma}\hat{H}_1^T Q_1 \hat{H}_1^T)CP + \hat{F}_1 M_2 \hat{F}_1^T \tag{34}$$

which is a Riccati Equation of the sort seen in [12],[13], and [14]. (32) looks like an estimator, but its final form will not become apparent until we solve the second half of the game problem.

3.3 Minimization with Respect to $\hat{x}$ and Maximization with Respect to v

The first part of our game solution led to optimal values for μ and $x(t_0)$:

$$\begin{aligned} \mu^* &= \frac{1}{\gamma}\hat{F}_1 M_2 \hat{F}_1^T \lambda & (35) \\ x(t_0)^* &= \Pi_0^{-1}\lambda(t_0) + \hat{x}_0 & (36) \end{aligned}$$

If we substitute these optimal values into the cost function (17) we obtain a new cost, $\bar{J}$, which is written as:

$$\bar{J} = \int_{t_0}^{t_1} [||(x - \hat{x})||^2_{C^T(\hat{H}_1^T Q_1 \hat{H}_1 - \gamma N_1^T R_1 N_1)C} - ||\lambda||^2_{\frac{1}{\gamma}\hat{F}_1 M_2 \hat{F}_1^T} - \gamma||y - Cx||^2_{V^{-1}}]dt - ||\lambda(t_0)||^2_{\Pi_0^{-1}}. \tag{37}$$

Maximization of the cost with respect to v is unnecessary at this point. Since $v = y - Cx$ and y is a given measurement vector, v^* is determined once x^* has been. Hence, the second part of the solution reduces to a single minimization:

$$\min_{\hat{x}} \bar{J} \leq 0$$

To begin, we add the identically zero term:

$$||\lambda(t_0)||^2_{\Pi(t_0)^{-1}} - ||\lambda(t_1)||^2_{\Pi(t_1)^{-1}} + \int_{t_0}^{t_1} \frac{d}{dt}||\lambda(t)||^2_{\Pi^{-1}} dt = 0$$

to (37). After applying the boundary condition for λ at t_1 (25) and carrying out the differentiation of the $||\lambda||^2_{\Pi^{-1}}$ term, we get:

$$\begin{aligned} \bar{J} = \int_{t_0}^{t_1} [||(x - \hat{x})||^2_{C^T(\hat{H}_1^T Q_1 \hat{H}_1 - \gamma N_1^T R_1 N_1)C} - ||\lambda||^2_{\frac{1}{\gamma}\hat{F}_1 M_2 \hat{F}_1^T} - \gamma||y - Cx||^2_{V^{-1}} \\ + \dot{\lambda}^T \Pi^{-1} \lambda^T + \lambda^T \dot{\Pi}^{-1} \lambda + \lambda^T \Pi^{-1} \dot{\lambda}]dt + ||\lambda(t_0)||^2_{\Pi^{-1}(t_0) - \Pi_0^{-1}}. \end{aligned} \tag{38}$$

Note that (38) gives us a boundary condition for (31):

$$\Pi(t_0) = \Pi_0. \tag{39}$$

Applying this boundary condition and substituting the differential equation for λ, (24), into (38) leads to:

$$\begin{aligned}\bar{J} = \int_{t_0}^{t_1} [\lambda^T(-A\Pi^{-1} - \Pi^{-1}A^T - \hat{F}_1 M_2 \hat{F}_1^T + \dot{\Pi}^{-1})\lambda + (x - \hat{x})^T C^T (\hat{H}_1^T Q_1 \hat{H}_1 - \gamma N_1^T R_1 N_1) C (x - \hat{x}) \\ -(x - \hat{x})^T C^T (\hat{H}_1^T Q_1 \hat{H}_1 - \gamma N_1^T R_1 N_1) C \Pi^{-1} \lambda - \lambda^T \Pi^{-1} C^T (\hat{H}_1^T Q_1 \hat{H}_1 - \gamma N_1^T R_1 N_1) C (x - \hat{x}) - \\ \gamma[(y - Cx)^T V^{-1} (y - Cx) + (y - Cx)^T V^{-1} C \Pi^{-1} \lambda + \lambda^T \Pi^{-1} C^T V^{-1} (y - Cx)] dt.\end{aligned} \tag{40}$$

From (31) the differential equation for Π^{-1} is:

$$\dot{\Pi}^{-1} = -\Pi^{-1}\dot{\Pi}\Pi^{-1} = \Pi^{-1}A^T + A\Pi^{-1} + \frac{1}{\gamma}\hat{F}_1 M_2 \hat{F}_1^T + \Pi^{-1} C^T (\hat{H}_1^T Q_1 \hat{H}_1 - \gamma N_1^T R_1 N_1 - \gamma V^{-1}) C \Pi^{-1}. \tag{41}$$

After we insert (41) into (40) and cancel terms, we are left with what turns out to be a pair of quadratic terms:

$$\begin{aligned}\bar{J} = \int_{t_0}^{t_1} \{[\Pi^{-1}\lambda - (x - \hat{x})]^T C^T (\hat{H}_1^T Q_1 \hat{H}_1 - \gamma N_1^T R_1 N_1) C [\Pi^{-1}\lambda - (x - \hat{x})] - \\ \gamma[C\Pi^{-1}\lambda + (y - Cx)]^T V^{-1} [C\Pi^{-1}\lambda + (y - Cx)]\} dt\end{aligned} \tag{42}$$

Now, use the solution for the optimal value of λ (28) and substitute into (42) to get:

$$\bar{J} = \int_{t_0}^{t_1} [(\hat{x} - x_p)^T C^T (\hat{H}_1^T Q_1 \hat{H}_1 - \gamma N_1^T R_1 N_1) C (\hat{x} - x_p) - \gamma (y - Cx_p)^T V^{-1} (y - Cx_p)] dt. \tag{43}$$

Because of the projectors $\hat{H}_1$ and N_1, the minimizing value of $\hat{x}$ is:

$$\hat{x}^* = x_p \bmod \ker \hat{H}_1 C \tag{44}$$

Using (44) as a guide, we rewrite (32) as:

$$\Pi \dot{\hat{x}}^* = \Pi A \hat{x}^* + \gamma C^T V^{-1} (y - C\hat{x}^*) \tag{45}$$

In (45) we have used $\hat{x}^* = x_p$ instead of (44). However, this choice is justified in that an adjustment of (32) in the manner suggested by (44) would build in a bias to our estimate which would be better in the sense of the game but not in our ultimate objective which is to design a fault detection filter. Since Π is positive-definite for $\gamma > 0$, we can rewrite (45) as:

$$\dot{\hat{x}}^* = A\hat{x}^* + \gamma \Pi^{-1} C^T V^{-1} (y - C\hat{x}^*) \tag{46}$$

Alternatively, the analyst could use (34) and:

$$\dot{\hat{x}}^* = A\hat{x}^* + PC^TV^{-1}(y - C\hat{x}^*). \tag{47}$$

This form of the filter is equivalent to (46), however; experience has shown that numerical problems are more likely to be seen when trying to find a solution to (34) than (31) when γ is brought to extremely small values. For convenience, we will use $\hat{x}$ instead of $\hat{x}^*$ when referring to the optimal state estimate with the understanding that it is the estimate that comes from the game solution which is being used.

From (44) we can see how this game filter mimics a fault detection filter. $\hat{x}$ does better by inducing estimation errors in the space defined by the projector $\hat{H}_1$ than by trying to estimate the state exactly. This implies that knowledge of the time history of $\hat{\mu}_2$, either by measurement or estimation, is not needed.

The solution of the fault detection filter problem exists at the limit of the game solution when γ is taken to zero. Finding the solution or even showing that it exists is not a straightforward matter. In both versions of the game Riccati Equation (31) and (34), there are terms which go to infinity as γ goes to zero. Kwakernaak and Sivan [15], studied a similar situation for the linear quadratic regulator. The linear-quadratic cost function in their problem, however, is always non-negative while our game cost can be both positive and negative. Hence, their results are not directly applicable here. Also, it is well known [14] that for game Riccati Equations, there is a minimum value of γ for which the equation will yield a positive-definite solution. Below this critical value of γ any number of different phenomena could occur (e.g. eigenvalues on the imaginary axis) which make positive-definite solutions impossible.

It turns out that we can resolve the second difficulty fairly quickly. By decreasing the noise weighting V to zero along with γ - i.e. $V \to 0$ as $\gamma \to 0$ - we can find solutions to (31) and (34) for smaller and smaller γ. What we get in the limit turns out to be a singular filtering problem. This insight, in fact, is the key to resolving the first difficulty. By using a pair of techniques from singular optimal control theory in the next section, we will be able to show the conditions for the existence of the limiting solution and the nature of this solution when it exists.

4. The Limiting Case Solution via Singular Optimal Control Techniques

4.1 Conditions for the Non-Positivity of the Game Cost: An LMI for the Game

In this section, we will find sufficient conditions for the non-positivity of the game cost. These conditions fall out when we manipulate the cost slightly and then set $\hat{x}$ to its optimal strategy found in Section 2. The game cost then becomes a single quadratic form:

$$J(\hat{x}, x(t_0), \hat{\mu}_2, v) = \int_{t_0}^{t_1} \xi^T \overline{W} \xi dt \tag{48}$$

where ξ is some vector consisting of linear combinations of the game players. The non-negativity of the cost then hinges on the sign definiteness of $\overline{W}$, giving rise to a linear matrix inequality. Historically, this technique was first seen in the singular optimal control theory [16] and [17] and the derivation seen here follows in that vein.

We begin with the cost function as given by (20). Note that the $(x - \hat{x})$ terms have been combined:

$$J = \int_{t_0}^{t_1} [||(x - \hat{x})||^2_{C^T(\hat{H}_1^T Q_1 \hat{H}_1 - \gamma N_1^T R_1 N_1)C} - \gamma||\hat{\mu}_2||^2_{M_2^{-1}} - \gamma||y - Cx||^2_{V^{-1}}]dt - ||x(t_0) - \hat{x}_0||^2_{\Pi_0}. \tag{49}$$

We now append the dynamics of the system to (49) through the Lagrange Multiplier $(x - \hat{x})^T \Pi$:

$$\begin{aligned} J = \int_{t_0}^{t_1} [||(x - \hat{x})||^2_{C^T(\hat{H}_1^T Q_1 \hat{H}_1 - \gamma N_1^T R_1 N_1)C} - \gamma||\hat{\mu}_2||^2_{M_2^{-1}} - \gamma||y - Cx||^2_{V^{-1}} + \\ (x - \hat{x})^T \Pi(Ax + \hat{F}_1 \hat{\mu}_2 - \dot{x})]dt - ||x(t_0) - \hat{x}_0||^2_{\Pi_0}. \end{aligned} \tag{50}$$

Add and subtract to (21) the terms $(x - \hat{x})^T \Pi A \hat{x}$ and $(x - \hat{x})^T \Pi \dot{\hat{x}}$. Collect terms to get:

$$\begin{aligned} J = \int_{t_0}^{t_1} \{||(x - \hat{x})||^2_{\Pi A + C^T(\hat{H}_1^T Q_1 \hat{H}_1 - \gamma N_1^T R_1 N_1)C} - \gamma||\hat{\mu}_2||^2_{M_2^{-1}} - \gamma||y - Cx||^2_{V^{-1}} \\ +(x - \hat{x})^T \Pi \hat{F}_1 \hat{\mu}_2 - (x - \hat{x})^T \Pi(\dot{x} - \dot{\hat{x}}) + (x - \hat{x})^T [\Pi A \hat{x} - \Pi \dot{\hat{x}}]\}dt - ||x(t_0) - \hat{x}_0||^2_{\Pi_0} \end{aligned} \tag{51}$$

Note, we have moved ΠA into the weighting $||(x - \hat{x})||^2$. As we continue along, new terms will be appear in the weighting of $||(x - \hat{x})||^2$ as we manipulate the cost function. Integrate $(x - \hat{x})^T \Pi(\dot{x} - \dot{\hat{x}})$ by parts:

$$\begin{aligned} J = \int_{t_0}^{t_1} \{||(x - \hat{x})||^2_{\dot{\Pi} + \Pi A + C^T(\hat{H}_1^T Q_1 \hat{H}_1 - \gamma N_1^T R_1 N_1)C} - \gamma||\hat{\mu}_2||^2_{M_2^{-1}} - \gamma||y - Cx||^2_{V^{-1}} + (x - \hat{x})^T \Pi \hat{F}_1 \hat{\mu}_2 \\ +(x - \hat{x})^T [\Pi A \hat{x} - \Pi \dot{\hat{x}}] + (\dot{x} - \dot{\hat{x}})^T \Pi(x - \hat{x})\}dt - ||x(t_0) - \hat{x}_0||^2_{\Pi_0 - \Pi(t_0)} - ||x(t_1) - \hat{x}(t_1)||^2_{\Pi(t_1)} \end{aligned} \tag{52}$$

Substitute the state equation for $\dot{x}$ (19) and add and subtract $\hat{x}^T A^T \Pi(x - \hat{x})$:

$$\begin{aligned} J = \int_{t_0}^{t_1} \{||(x - \hat{x})||^2_{\dot{\Pi} + \Pi A + A^T \Pi + C^T(\hat{H}_1^T Q_1 \hat{H}_1 - \gamma N_1^T R_1 N_1)C} - \gamma||\hat{\mu}_2||^2_{M_2^{-1}} + (x - \hat{x})^T \Pi \hat{F}_1 \hat{\mu}_2 \\ +\hat{\mu}_2^T \hat{F}_1^T \Pi(x - \hat{x})^T - \gamma||y - Cx||^2_{V^{-1}} + (x - \hat{x})^T [-\Pi \dot{\hat{x}} + \Pi A \hat{x}] + [-\Pi \dot{\hat{x}} \\ +\Pi A \hat{x}]^T (x - \hat{x})\}dt - ||x(t_0) - \hat{x}_0||^2_{\Pi_0 - \Pi(t_0)} - ||x(t_1) - x(\hat{t}_1)||^2_{\Pi(t_1)}. \end{aligned} \tag{53}$$

We are now going to manipulate the $||y - Cx||^2_{V^{-1}}$ term by adding and subtracting $C\hat{x}$ inside of the term so that it reads $||(y - C\hat{x}) - C(x - \hat{x})||^2_{V^{-1}}$. Expand this quadratic term out and collect terms so that we end up with:

$$J = \int_{t_0}^{t_1} \{ \|(x-\hat{x})\|^2_{\dot{\Pi}+\Pi A+A^T\Pi+C^T(\hat{H}_1^T Q_1 \hat{H}_1 - \gamma N_1^T R_1 N_1 - \gamma V^{-1})C} - \gamma \|\hat{\mu}_2\|^2_{M_2^{-1}} + (x-\hat{x})^T \Pi \hat{F}_1 \hat{\mu}_2$$

$$+\hat{\mu}_2^T \hat{F}_1^T \Pi (x-\hat{x})^T - \gamma \|y - C\hat{x}\|^2_{V^{-1}} + (x-\hat{x})^T[-\Pi\dot{\hat{x}} + \Pi A\hat{x} + \gamma C^T V^{-1}(y - C\hat{x})]$$

$$-[\Pi\dot{\hat{x}} + \Pi A\hat{x} + \gamma C^T V^{-1}(y-C\hat{x})]^T(x-\hat{x})\}dt - \|x(t_0) - \hat{x}_0\|^2_{\Pi_0 - \Pi(t_0)} - \|x(t_1) - \hat{x}(t_1)\|^2_{\Pi(t_1)}. \quad (54)$$

Using (45) we can eliminate a pair of terms in (54). We are then left with a quadratric in the form:

$$J = \int_{t_0}^{t_1} \xi^T \overline{W} \xi dt - \|x(t_0) - \hat{x}_0\|^2_{\Pi_0 - \Pi(t_0)} - \|x(t_1) - x(\hat{t}_1)\|^2_{\Pi(t_1)},$$

where:

$$\xi = \left\{ \begin{array}{c} (x-\hat{x}) \\ \hat{\mu}_2 \\ (y - C\hat{x}) \end{array} \right\}$$

and:

$$\overline{W} := \begin{bmatrix} C^T(\hat{H}_1^T Q_1 \hat{H}_1 - \gamma V^{-1} - \gamma N_1^T R_1 N_1)C + A^T\Pi + \Pi A + \dot{\Pi} & \Pi\hat{F}_1 & 0 \\ \hat{F}_1^T \Pi & -\gamma M_2^{-1} & 0 \\ 0 & 0 & -\gamma V^{-1} \end{bmatrix}. \quad (55)$$

Define the upper 2×2 block of $\overline{W}$ as:

$$W(\Pi) := \begin{bmatrix} C^T(\hat{H}_1^T Q_1 \hat{H}_1 - \gamma V^{-1} - \gamma N_1^T R_1 N_1)C + A^T\Pi + \Pi A + \dot{\Pi} & \Pi\hat{F}_1 \\ \hat{F}_1^T \Pi & -\gamma M_2^{-1} \end{bmatrix}. \quad (56)$$

For matrices $\Pi \geq 0$ such that:

$$W(\Pi) \leq 0 \quad (57)$$

$$\Pi_0 - \Pi(t_0) \geq 0 \quad (58)$$

$$\Pi(t_1) \geq 0 \quad (59)$$

$\overline{W}$ is clearly negative semi-definite. Hence, we need only pay attention to the smaller LMI, $W(\Pi)$.

For $\gamma > 0$, it is easy to see that the Riccati Equation (31) of the previous section is embedded in (56). In fact, the solution of (31) is the solution of $W(\Pi)$ which minimizes its rank [18]. Thus with (56) and (45), we retain the results of the previous section, but in a form which can be easily analyzed in the limit $\gamma \to 0$. If we define $\overline{V} = \lim_{\gamma \to 0} \gamma V$, sufficient conditions for $J \leq 0$ in the limit as $\gamma \to 0$ are:

$$\Pi \dot{F}_1 = 0 \quad (60)$$

$$\dot{\Pi} + A^T\Pi + \Pi A + C^T(\hat{H}_1^T Q_1 \hat{H}_1 - \overline{V}^{-1})C \leq 0. \quad (61)$$

along with the boundary conditions (58) and (59).

(60) clearly shows that in the limit, the Riccati Matrix, Π, obtains a non-trivial null space which contains the image of the nuisance failure map, $\hat{F}_1$. Moreover, those familiar with singular optimal control theory will recognize (60) and (61) as conditions seen previously for the singular LQ regulator (see e.g. [16]). This tells us, first of all, that the limiting form of this game filter is a singular filter. It is likely that similar results hold for game theoretic (H_∞) filters and controllers in general. Secondly, singular optimal control provides a wealth of results and insights which we can apply to the analysis of this filter. This is, in fact, what we will do next.

4.2 A Riccati Equation for the Limiting Form of the Game Theoretic Filter

Goh (see [11]) used a transformation on the control input space to derive a full-order Riccati Equation for the totally singular linear quadratic regulator problem. By applying the same type of transformation on the nuisance fault input space, we can obtain a similar equation for the limiting game filter. The existence of the solution to this equation gives the condition for the existence of the game solution in the limit. We have already seen this transformation applied to the construction of the time-varying projector, $\hat{H}_1$, (11). Because this Riccati Matrix must also have a non-trivial null space, we will not be able to use the solution to this Riccati Equation directly in a game filter, but this matrix will also prove to be useful when we look at reduced-order filters.

We start with the game cost for the limiting case:

$$J^* = \lim_{\gamma\to 0} J = \int_{t_0}^{t_1} \|x - \hat{x}\|^2_{C^T \hat{H}_1^T Q_1 \hat{H}_1 C} - \|y - Cx\|^2_{\overline{V}^{-1}} dt,$$

where $\overline{V}^{-1} := \lim_{\gamma\to 0}(\gamma V)^{-1}$. Now, define a new nuisance fault vector, ρ_1 and a new state vector, α_1:

$$\rho_1 := \int_{t_0}^{t} \hat{\mu}_2 \, dt \tag{62}$$

$$\alpha_1 := x - \hat{F}_1 \rho_1 \equiv x - B_1 \rho_1 \tag{63}$$

Note that we have defined a matrix $B_1 := \hat{F}_1$. The reason for the numbered subscripts will become apparent later. Differentiating (63), we get:

$$\dot{\alpha}_1 = A\alpha_1 + (AB_1 - \dot{B}_1)\rho_1. \tag{64}$$

as a new state equation and:

$$J^{*} = \int_{t_0}^{t_1} [\|\alpha_1 - \hat{x}\|^2_{C^T \hat{H}_1^T Q_1 \hat{H}_1 C} + (\alpha_1 - \hat{x})^T C^T \hat{H}_1^T Q_1 \hat{H}_1 C B_1 \rho_1 + \rho_1^T B_1^T C^T \hat{H}_1^T Q_1 \hat{H}_1 C B_1 \rho_1 - \|y - C\alpha_1\|^2_{\overline{V}^{-1}} - (y - C\alpha_1)^T \overline{V}^{-1} C B_1 \rho_1 - \rho_1^T B_1^T C^T \overline{V}^{-1} (y - C\alpha_1) - \|\rho_1\|^2_{B_1^T C^T \overline{V}^{-1} C B_1}]\, dt \tag{65}$$

as the new game cost. Because $\hat{H}_1$ is a projector constructed so that $\hat{H}_1 C \hat{F}_1 = 0$, we can simplify (65) to:

$$J^{*} = \int_{t_0}^{t_1} [\|\alpha_1 - \hat{x}\|^2_{C^T \hat{H}_1^T Q_1 \hat{H}_1 C} - \|y - C\alpha_1\|^2_{\overline{V}^{-1}} - (y - C\alpha_1)^T \overline{V}^{-1} C B_1 \rho_1 - \rho_1^T B_1^T C^T \overline{V}^{-1} (y - C\alpha_1) - \|\rho_1\|^2_{B_1^T C^T \overline{V}^{-1} C B_1}]\, dt. \tag{66}$$

Now, if $B_1^T C^T \overline{V}^{-1} C B_1 > 0$, we can solve the following differential game:

$$\min_{\hat{x}} \max_{\rho_1} J^{*} \leq 0$$

subject to (64). Because of its similarity to the derivation given in section 3, we do not provide the solution here. But if the interested reader follows the steps in section 3[2], he will find that the solution leads to a Riccati equation:

$$-\dot{S} = SA + A^T S + C^T(\hat{H}_1^T Q_1 \hat{H}_1 - \overline{V}^{-1})C + [S(AB_1 - \dot{B}_1) - C^T \overline{V}^{-1} C B_1] \times (B_1^T C^T \overline{V}^{-1} C B_1)^{-1} [(AB_1 - \dot{B}_1)^T S - B_1^T C^T \overline{V}^{-1} C] \tag{67}$$

with the boundary condition:

$$S(t_0) = 0. \tag{68}$$

It may happen, however, that $CB_1 = 0$, which would make $B_1^T C^T \overline{V}^{-1} C B_1 = 0$ and which would invalidate our Riccati Equation (67). The remedy to this situation is to perform the same transformation as before but on the ρ_1 input space via the recursion equations:

$$\rho_i = \int_{t_0}^{t} \rho_{i-1} dt \tag{69}$$

$$B_i = AB_{i-1} - \dot{B}_{i-1} \tag{70}$$

$$\alpha_i = x - B_i \rho_i. \tag{71}$$

The process stops once a B_i is found such that $CB_i \neq 0$. The game is then:

[2] Hint: the first step involves converting $y - C\alpha$ into $(y - C\hat{x}) + C(\alpha - \hat{x})$

$$\min_{\hat{x}} \max_{\rho_i} J^* = \int_{t_0}^{t_1} [\|\alpha_i - \hat{x}\|^2_{C^T \hat{H}_1^T Q_1 H_1 C} - \|y - C\alpha_i\|^2_{\overline{V}^{-1}} - (y - C\alpha_i)^T \overline{V}^{-1} C B_i \rho_i$$
$$-\rho_i^T B_i^T C^T \overline{V}^{-1}(y - C\alpha_i) - \|\rho_i\|^2_{B_i^T C^T \overline{V}^{-1} C B_i}]\, dt \tag{72}$$

subject to:

$$\dot{\alpha}_i = A\alpha_i + (AB_i - \dot{B}_i)\rho_i. \tag{73}$$

The general form of the Goh Riccati Equation is then:

$$-\dot{S} = SA + A^T S + C^T(\hat{H}_1^T Q_1 \hat{H}_1 - \overline{V}^{-1})C + [S(AB_i - \dot{B}_i) - C^T \overline{V}^{-1} C B_i]$$
$$\times(B_i^T C^T \overline{V}^{-1} C B_i)^{-1}[(AB_i - \dot{B}_i)^T S - B_i^T C^T \overline{V}^{-1} C] \tag{74}$$

The following theorem shows that (74) is a Riccati Equation for the limiting form of the game theoretic filter.

Theorem 4.1. The solution S to (74) satisfies the sufficient conditions for the non-positivity of the game cost: Equations 60 and 61.

Proof. (The proof given here follows Bell and Jacobson [16] (pg. 121). Due to its importance, we list it here.) Clearly, (74) implies that:

$$\dot{S} + SA + A^T S + C^T(\hat{H}_1^T Q_1 \hat{H}_1 - \overline{V}^{-1})C \leq 0,\ \forall t \in [t_0, t_1]. \tag{75}$$

which is (60). Now, pre-multiply (74) by B_i and add $-\dot{B}_i^T S$ to both sides of the resulting equation to get:

$$-B_i^T \dot{S} - \dot{B}_i^T S = B_i^T SA - \dot{B}_i S + B_i^T A^T S - B_i^T C^T \overline{V}^{-1} C + [B_i^T S(AB_i - \dot{B}_i) - B_i^T C^T \overline{V}^{-1} C B_i]$$
$$\times(B_i^T C^T \overline{V}^{-1} C B_i)^{-1}[(AB_i - \dot{B}_i)^T S - C^T \overline{V}^{-1} C B_i] \tag{76}$$

Rearranging terms leads to a differential equation in $B_i^T S$ with (68) as the boundary condition:

$$-\frac{d}{dt}[B_i^T S] = B_i^T S[A + B_i^T S(AB_i - \dot{B}_i)(B_i^T C^T \overline{V}^{-1} C B_i)^{-1}[(AB_i - \dot{B}_i)^T S - C^T \overline{V}^{-1} C B_i]. \tag{77}$$

The solution to (77) given (68) is:

$$B_i^T(t)S(t) = 0,\ \forall t \in [t_0, t_1]$$

The necessary condition (60) actually requires that $\hat{F}_1^T S(t) = 0$. However, since $B_1 = \hat{F}_1$ and with the following proposition, (60) is satisfied. ∎

Proposition 4.2. Let $i \in \mathbb{N}$ be the smallest number such that $CB_i \neq 0$. Then, the solution, S, to (110) is such that

$$SB_j = 0, \; \forall j \leq i, \; \forall t \in [t_0, t_1] \tag{78}$$

Proof. That $SB_i = 0$ is a result of the solution to the game and is analogous to the $S\hat{F}_1 = 0$ condition found in the solution to the original game. The proof that $SB_j = 0$ for all $j < i$ can be found in [19]. The proof used there is an inductive argument which applies the same steps used above in Theorem 6.1 to prove $SB_i = 0 \forall t \in [t_0, t_1]$ to show that the same holds for all $j \leq i$. ∎

5. The Equivalence of the Game Theoretic Filter to the Beard-Jones Filter in the Limit

In this section, we return to time-invariant case and show that for these systems the solution to the fault detection filter problem as stated in Definition 2.2 also solves the problem as stated by Definition 2.1 - i.e. limiting form of the game theoretic filter is a Beard-Jones Fault Detection Filter. The required invariant subspace is the kernal of Π. To prove to this assertion, we first need a minor lemma:

Lemma 5.1. $(\ker\Pi \cap \ker C) \subset \ker\dot{\Pi}$

Proof. Let $x \in \ker\Pi \cap \ker C$. Pre-multiply (61) by x^T and post-multiply by x. This leaves:

$$x^T \dot{\Pi} x \leq 0.$$

Since $\dot{\Pi}$ can be negative or positive-definite, this can be true in general if and only if:

$$x^T \dot{\Pi} x = 0,$$

which implies our proposition. ∎

Theorem 5.2. $\ker\Pi$ is a subspace which solves the fault detection filter problem

Proof. The three conditions listed by Definition 2.1 are subspace inclusion (Equation 4), output separability (Equation 5), and (C, A)-invariance (Equation 6). (60) clearly implies subspace inclusion. Since we are trying to detect only one fault, output separability is satisfied trivially. Thus, all that remains is to show (C, A)-invariance.

From Wonham [10], a necessary and sufficient condition for $\ker\Pi$ to be (C, A)-invariant is that:

$$A(\ker\Pi \cap \ker C) \subset \ker\Pi$$

Therefore, let $x \in A(\ker\Pi \cap \ker C)$. That is, there exists a vector ς such that:

$$x = A\varsigma \quad \text{and} \quad \Pi\varsigma = C\varsigma = 0.$$

Now consider (61). If we post-multiply (61) by ς and use our lemma, we get:

$$\Pi A\varsigma \leq 0 \;\Rightarrow\; \varsigma^T A^T \Pi A\varsigma \leq 0.$$

Since $\Pi \geq 0$, this means that:

$$\varsigma^T A^T \Pi A\varsigma = 0.$$

which implies that:

$$\Pi A\varsigma = \Pi x = 0 \;\Rightarrow\; x \in \ker\Pi$$

From this we conclude that $A(\ker\Pi \cap \ker C) \subset \ker\Pi$ and so $\ker\Pi$ is (C, A)-invariant. ∎

Remark 1. In actual practice, it is not necessary to use the limiting form of the filter. In many H_∞ designs, γ is not taken to its smallest possible value, but left at one which results in an acceptable compromise between all of the (usually competing) design objectives. The virtue of a game theoretic approach to fault detection filter design is that it gives the designer a "knob" with which he can make the filter more like a Beard-Jones filter (small γ and small V) or more like an H_∞ filter with more emphasis on sensor noise attenuation (larger γ and V). ∎

Remark 2. It should be noted that a Beard-Jones fault detection filter can detect all of the μ_j's. The filter that we propose here can detect only one fault at a time. ∎

Remark 3. Lee and Gibson derive a filter for fault detection via linear quadratic methods in [7]. They do not discuss their work in relation to fault detection filters, but in many respects their final result is similar to the game theoretic filter described here. Moreover, they also apply their work to the same example that we examine in the next section with similar results. ∎

In Section 2 we noted that unobservability subspaces are used in current fault detection filter design methods because they allow the designer to specify (within complex conjugate symmetry) all of the eigenvalues of the filter. Such design freedom exists with these subspaces because they include any invariant zero directions which arise out of the triple $(C, A, \hat{F}_1)$.

Invariant zeros [20] are those complex numbers λ such that the system matrix:

$$P(\lambda) = \begin{bmatrix} A - \lambda I & \hat{F}_1 \\ C & 0 \end{bmatrix}$$

loses rank. Invariant zero directions x are those complex vectors such that:

$$\begin{bmatrix} A - \lambda I & \hat{F}_1 \\ C & 0 \end{bmatrix} \left\{ \begin{matrix} x \\ w \end{matrix} \right\} = 0$$

Invariant zeros are a concern in fault detection filter design because they cannot be eliminated by output injection feedback, i.e. an estimator. Moreover, Douglas (Proposition 2.9 [4]) has shown that if we do not include invariant zero directions in a (C, A)-invariant subspace, the corresponding invariant zero will show up as an eigenvalue in the factor space when we try to design an estimator for the system.

In the next theorem, we will show that the game theoretic filter places invariant zeros in the kernal of Π. Since we have already shown that kerΠ is a (C, A)-invariant subspace, this theorem then implies that kerΠ is an unobservability subspace. For our purposes, this result is not as important as it is for spectral design methods since pole locations fall out from the game solution. It does, however, connect this filter to existing detection filters by showing that the invariant subspace formed by the game theoretic approach is the same kind chosen in standard detection filter design algorithms.

The proof to the upcoming theorem is based on the fact that positive semi-definite, symmetric matrices such as Π always have non-singular, transformations - say Γ - that are orthonormal ($\Gamma^T \Gamma = I$) and that convert the matrix into the form:

$$\Gamma \Pi \Gamma^T = \begin{bmatrix} \overline{\Pi} & 0 \\ 0 & 0 \end{bmatrix} \tag{79}$$

,

where $\overline{\Pi}$ is positive definite. Define:

$$C\Gamma^T = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \qquad \Gamma A \Gamma^T = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \qquad \Gamma \hat{F}_1 = \begin{bmatrix} F_{11} \\ F_{12} \end{bmatrix}$$

Because $\Pi \hat{F}_1 = 0$ implies $\Gamma \Pi \hat{F}_1 = 0$, we can immediately conclude that:

$$\Gamma\Pi\Gamma^T\Gamma\hat{F}_1 = \begin{bmatrix} \overline{\Pi} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} F_{11} \\ F_{22} \end{bmatrix} = \overline{\Pi}F_{11} = 0.$$

Which, since $\overline{\Pi}$ is positive-definite, implies:

$$F_{11} = 0.$$

Theorem 5.3. The invariant zeros of $(C, A, \hat{F}_1)$ lie in $\ker\Pi$

Proof.

Transform the system matrix $P(\lambda)$ using the transformation matrix, Γ described by (79):

$$\begin{bmatrix} \Gamma & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A - \lambda I & \hat{F}_1 \\ C & 0 \end{bmatrix} \begin{bmatrix} \Gamma^T & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} A_{11} - \lambda I & A_{12} & 0 \\ A_{21} & A_{22} - \lambda I & F_{12} \\ C_1 & C_2 & 0 \end{bmatrix} \tag{80}$$

(80) clearly shows that if invariant zeros exist they can only arise from the triple (C_2, A_{22}, F_{12}) which corresponds to the subspace determined by $\ker\Pi$. In fact, (80) shows that the only kind of zeros which could occur in the range of Π are output decoupling zeros [20] arising from values of λ which make:

$$\begin{bmatrix} A_{11} - \lambda I \\ A_{21} \\ C_1 \end{bmatrix}$$

lose rank. However, since we have assumed that (C, A) is observable, these zeros cannot occur.

∎

6. Fault Detection with the Limiting Form of the Game Theoretic Filter

In this section, we will show that a reduced-order fault detector can be derived from the limiting form of the game theoretic filter. The results from this section are more easily applied to time-invariant systems, but we will give an overview of how to apply these results to time-varying systems. For the moment, let us restrict ourselves the LTI systems. Using the transformation, Γ, defined by (79), set:

$$\hat{\eta} = \begin{Bmatrix} \hat{\eta}_1 \\ \hat{\eta}_2 \end{Bmatrix} = \Gamma\hat{x}.$$

Pre-multiply (45) by Γ and make use of the identity $\Gamma^T\Gamma = I$ to get:

$$(\Gamma\Pi\Gamma^T)\dot{\hat{\eta}} = (\Gamma\Pi\Gamma^T)(\Gamma A\Gamma^T)\hat{\eta} + \Gamma C^T\overline{V}^{-1}(y - C\Gamma^T\hat{\eta})) \tag{81}$$

The transformed filter equation (81) is seen to be:

$$\begin{bmatrix} \overline{\Pi} & 0 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} \dot{\hat{\eta}}_1 \\ \dot{\hat{\eta}}_2 \end{Bmatrix} = \begin{bmatrix} \overline{\Pi} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{Bmatrix} \hat{\eta}_1 \\ \hat{\eta}_2 \end{Bmatrix} + \begin{Bmatrix} C_1^T \\ C_2^T \end{Bmatrix} \overline{V}^{-1} (y - \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{Bmatrix} \hat{\eta}_1 \\ \hat{\eta}_2 \end{Bmatrix}) \tag{82}$$

From (82) we get a dynamic equation for $\hat{\eta}_1$:

$$\overline{\Pi}\dot{\hat{\eta}}_1 = \overline{\Pi}A_{11}\hat{\eta}_1 + \overline{\Pi}A_{12}\hat{\eta}_2 + C_1^T\overline{V}^{-1}(y - C_1\hat{\eta}_1 - C_2\hat{\eta}_2) \tag{83}$$

and a static equation for $\hat{\eta}_2$:

$$\hat{\eta}_2 = (C_2^T\overline{V}^{-1}C_2)^{-1}C_2^T\overline{V}^{-1}(y - C_1\hat{\eta}_1). \tag{84}$$

Define

$$K := (C_2^T\overline{V}^{-1}C_2)^{-1}C_2^T\overline{V}^{-1} \tag{85}$$

so that the substitution of (85) and (84) into (83) gives us an estimator for $\hat{\eta}_1$:

$$\dot{\hat{\eta}}_1 = A_{11}\hat{\eta}_1 + [\overline{\Pi}^{-1}C_1^T\overline{V}^{-1}(I - C_2K) + A_{12}K](y - C_1\hat{\eta}_1). \tag{86}$$

To see that the reduced-order estimator (86) is unaffected by the nuisance fault $\hat{\mu}_2$, we will derive the error equation for the reduced-order filter. Define:

$$\eta = \begin{Bmatrix} \eta_1 \\ \eta_2 \end{Bmatrix} := \Gamma x \qquad e_1 := \hat{\eta}_1 - \eta_1 \qquad e_2 := \hat{\eta}_2 - \eta_2$$

We begin by premultiplying the dynamical equation (19) by the Riccati Matrix, Π. Since $\Pi\hat{F}_1 = 0$, we get:

$$\Pi\dot{x} = \Pi A x.$$

This can be pre-multiplied by Γ and manipulated into:

$$\begin{bmatrix} \overline{\Pi} & 0 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} \dot{\eta}_1 \\ \dot{\eta}_2 \end{Bmatrix} = \begin{bmatrix} \overline{\Pi} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{Bmatrix} \eta_1 \\ \eta_2 \end{Bmatrix}. \tag{87}$$

As with the estimator equation, (87) shows that only a portion of the state-space possesses dynamics:

$$\overline{\Pi}\dot{\eta}_1 = \overline{\Pi}A_{11}\eta_1 + \overline{\Pi}A_{12}\eta_2 \tag{88}$$

Using (88) to get an error equation would leave terms in η_2 or e_2. In anticipation of this, we transform the measurement equation:

$$y = Cx + v = C\Gamma^T\Gamma x + v = C_1\eta_1 + C_2\eta_2 + v \tag{89}$$

and use (89) and (84) to solve for ϵ_2:

$$\epsilon_2 = (C_2^T\overline{V}^{-1}C_2)^{-1}(C_2^T\overline{V}^{-1}C_1\epsilon_1 + C_2^T\overline{V}^{-1}v) = K(C_1\epsilon_1 - v) \tag{90}$$

Subtract (88) from (83) and substitute (89) for y:

$$\overline{\Pi}\dot{\epsilon}_1 = \overline{\Pi}A_{11}\epsilon_1 + \overline{\Pi}A_{12}\epsilon_2 + C_1^T\overline{V}^{-1}C_1e_1 + C_1^T\overline{V}^{-1}C_2e_2 + C^T\overline{V}^{-1}v$$

Using (90) and collecting terms, we can turn the previous equation into:

$$\dot{e}_1 = [A_{11} - \overline{\Pi}^{-1}C_1^T\overline{V}^{-1}(I - C_2K)C_1 - A_{12}KC_1]e_1 + [\overline{\Pi}^{-1}C_1^T\overline{V}^{-1}(I - C_2K) + A_{12}K]v. \tag{91}$$

Note that nuisance fault, $\hat{\mu}_2$, appears nowhere in the estimator (86) nor in the error equation (91). Thus, in the limit, we get a reduced-order estimator completely uninfluenced by the nuisance faults. The term $(C_2^T\overline{V}^{-1}C_2)^{-1}$ appears in various places in the reduced-order estimator. This inverse will always exist since $\overline{V}$ is positive definite and since the assumption of (C, A) observability guarantees that C_2 will have full column rank.

Remark 4. The reduced-order filter derived here is similar to the residual generator derived by Massoumnia, et al. in [3]. An important difference, however, is that Massoumnia begins his design process by factoring out the reachable space of the nuisance faults. As a result, he has the freedom to use any kind of filter design technique for the lower dimensional state-space. The trade-off, however, is that the system reduction in Massoumnia's filter is sensitive to the inexactness of the plant model. Variations in the plant will change the reachable subspace and may, as a result, degrade the performance of the reduced-order detector. In the game filter, the order reduction comes at the end of the design process. Thus, there is no design freedom left to tune the reduced-order filter, but the game formulation used to obtain the filter makes it possible to account for model uncertainties. Thus, it can be argued that the order reduction used here is more robust. ■

The Goh transformation and corresponding Riccati equation greatly extend our ability to analyze the reduced-order estimator. In fact with the Goh Riccati equation we can show that there always exist a stabilizing solution for the reduced order estimator. Applying the transformation Γ to (74), we get:

$$\begin{aligned} -\Gamma\dot{S}\Gamma^T = {} & \Gamma S\Gamma^T\Gamma A\Gamma^T + \Gamma A^T\Gamma^T\Gamma S\Gamma^T + \Gamma C^T(\hat{H}_1^TQ_1\hat{H}_1 - \overline{V}^{-1})C\Gamma^T \\ & + [\Gamma S\Gamma^T(\Gamma A\Gamma^T\Gamma B_i - \Gamma\dot{B}_i) - \Gamma C^T\overline{V}^{-1}C\Gamma^T\Gamma B_i](B_i^TC^T\overline{V}^{-1}CB_i)^{-1} \\ & \times[(\Gamma A\Gamma^T\Gamma B_i - \Gamma\dot{B}_i)^T\Gamma S\Gamma^T - \Gamma C^T\overline{V}^{-1}C\Gamma^T\Gamma B_i] \end{aligned} \tag{92}$$

Define:

$$\Gamma B_i = \begin{bmatrix} B_{11} \\ B_{12} \end{bmatrix}.$$

As in section 5, the necessary condition $SB_i = 0$ will lead to $B_{11} = 0$ since $\Gamma S \Gamma^T \Gamma B_i = 0 \Rightarrow \overline{S} B_{11} = 0$ and $\overline{S}$ is positive-definite. Also, if we carry the transformation through, a number of terms fall out because the projector $\hat{H}_1$ has been constructed so that:

$$\hat{H}_1 C B_i = 0 \implies \hat{H}_1 C \Gamma^T \Gamma B_i = 0 \implies \begin{bmatrix} \hat{H}_1 C_1 & \hat{H}_1 C_2 \end{bmatrix} \begin{bmatrix} 0 \\ B_{12} \end{bmatrix} = 0 \implies \hat{H}_1 C_2 B_{12} = 0 \tag{93}$$

We we show later that B_i can always be augmented so that B_{12} is an invertible square matrix. Hence (93) implies:

$$\hat{H}_1 C_2 = 0. \tag{94}$$

Using (94) and working through all of the transformations leads to:

$$\begin{aligned} \begin{bmatrix} -\dot{\overline{S}} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \overline{S}A_{11} & \overline{S}A_{12} \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} A_{11}^T\overline{S} & 0 \\ A_{12}^T\overline{S} & 0 \end{bmatrix} + \left(\begin{bmatrix} \overline{S}A_{12}B_{12} - \overline{S}\dot{B}_{11} \\ 0 \end{bmatrix} - \begin{bmatrix} C_1^T\overline{V}^{-1}C_2B_{12} \\ C_2^T\overline{V}^{-1}C_2B_{12} \end{bmatrix} \right) \\ \times (B_{12}^TC_2^T\overline{V}^{-1}C_2B_{12})^{-1} \left(\begin{bmatrix} B_{12}^TA_{12}\overline{S} - \dot{B}_{11}\overline{S} & 0 \end{bmatrix} - \begin{bmatrix} B_{12}^TC_2^T\overline{V}^{-1}C_1^T & B_{12}^TC_2^T\overline{V}^{-1}C_2^T \end{bmatrix} \right) \\ + \begin{bmatrix} C_1^T(\hat{H}_1^TQ_1\hat{H}_1 - \overline{V}^{-1})C_1 & -C_1^T\overline{V}^{-1}C_2 \\ -C_2^T\overline{V}^{-1}C_1 & -C_2^T\overline{V}^{-1}C_2 \end{bmatrix} \end{aligned} \tag{95}$$

From (95) we get three equations:

$$\begin{aligned} -\dot{\overline{S}} = C_1^T(\hat{H}_1^TQ_1\hat{H}_1 - \overline{V}^{-1})C_1 + \overline{S}A_{11} + A_{11}^T\overline{S} + (\overline{S}A_{12}B_{12} - \overline{S}\dot{B}_{11} - C_1^T\overline{V}^{-1}C_2B_{12})(B_{12}^TC_2^T\overline{V}^{-1}C_2B_{12})^{-1} \\ \times(\overline{S}A_{12}B_{12} - \overline{S}\dot{B}_{11} - C_1^T\overline{V}^{-1}C_2B_{12})^T \end{aligned} \tag{96}$$

$$0 = -C_1^T\overline{V}^{-1}C_2 + \overline{S}A_{12} - (\overline{S}A_{12}B_{12} - \overline{S}\dot{B}_{11} - C_1^T\overline{V}^{-1}C_2^TB_{12})(B_{12}^TC_2^T\overline{V}^{-1}C_2B_{12})^{-1}B_{12}^TC_2^T\overline{V}^{-1}C_2 \tag{97}$$

$$0 = -C_2^T\overline{V}^{-1}C_2 + C_2^T\overline{V}^{-1}C_2B_{12}(B_{12}^TC_2^T\overline{V}^{-1}C_2B_{12})^{-1}B_{12}^TC_2^T\overline{V}^{-1}C_2. \tag{98}$$

However, if we post-multiply (98) by B_{12} and cancel terms we obtain the identity $0 = 0$. If we post-multiply (97) by B_{12} we obtain:

$$0 = \overline{S}\dot{B}_{11} \Rightarrow \dot{B}_{11} = 0. \tag{99}$$

Thus, we need only (96), which thanks to (99) can be simplified to:

$$-\dot{\overline{S}} = C_1^T(\hat{H}_1^T Q_1 H_1 - \overline{V}^{-1})C_1 + \overline{S}A_{11} + A_{11}^T\overline{S} + (\overline{S}A_{12}B_{12} - C_1^T\overline{V}^{-1}C_2B_{12})$$
$$\times(B_{12}^T C_2^T\overline{V}^{-1}C_2B_{12})^{-1}(\overline{S}A_{12}B_{12} - C_1^T\overline{V}^{-1}C_2B_{12})^T. \qquad (100)$$

Now if i=1, then $B_i = \hat{F}_1$ and the rank of $\hat{F}_1$ equals the dimension of the kernal of S. $B_{12} = F_{12}$ will then be square and, moreover, it will be invertible since $\hat{F}_1$ was assumed monic. Given this, we can simplify (100) to:

$$-\dot{\overline{S}} = C_1^T(\hat{H}_1^T Q_1 \hat{H}_1 - \overline{V}^{-1})C_1 + \overline{S}A_{11} + A_{11}^T\overline{S} + (\overline{S}A_{12} - C_1^T\overline{V}^{-1}C_2)(C_2^T\overline{V}^{-1}C_2)^{-1}(\overline{S}A_{12} - C_1^T\overline{V}^{-1}C_2)^T \qquad (101)$$

$$\overline{S}(t_0) = 0 \qquad (102)$$

where the boundary condition comes from (68). This leads us to the key result of this section.

Theorem 6.1. The solution $\overline{S}$ to (101) gives a stabilizing solution for the reduced-order estimator (86).

Proof. Using the same transformation to derive both (101) and (86) will ensure that $\overline{S}$ is of proper dimension for (86). Substitute $\overline{S}$ into (86) directly for $\overline{\Pi}$. The resulting estimator is:

$$\dot{\hat{\eta}}_1 = (A_{11} - [\overline{S}^{-1}C_1^T\overline{V}^{-1}(I - C_2K) + A_{12}K]C_1)\hat{\eta}_1 + [\overline{S}^{-1}C_1^T\overline{V}^{-1}(I - C_2K) + A_{12}K]y. \qquad (103)$$

where $K := (C_2^T\overline{V}^{-1}C_2)^{-1}C_2^T\overline{V}^{-1}$. Clearly, the stability of the estimator depends upon the closed-loop state matrix, $(A_{11} - [\overline{S}^{-1}C_1^T\overline{V}^{-1}(I - C_2K) + A_{12}K]C_1)$. Now, if we go back to (101), multiply out the quadratic, and use the definition for K, we get:

$$-\dot{\overline{S}} = \overline{S}(A_{11} - A_{12}KC_1) + (A_{11} - A_{12}KC_1)^T\overline{S} + C_1^T[\hat{H}_1^T Q_1 \hat{H}_1 - \overline{V}^{-1}(I - C_2K)]C_1 + \overline{S}A_{12}(C_2^T\overline{V}^{-1}C_2)^{-1}A_{12}^T\overline{S}. \qquad (104)$$

If we add and subtract $C_1^T\overline{V}^{-1}(I - C_2K)C_1$ to (104) and rearrange terms we get:

$$-\dot{\overline{S}} = \overline{S}(A_{11} - A_{12}KC_1 - \overline{S}^{-1}C_1^T\overline{V}^{-1}(I - C_2K)C_1) + (A_{11} - A_{12}KC_1 - \overline{S}^{-1}C_1^T\overline{V}^{-1}(I - C_2K)C_1)^T\overline{S}$$
$$+C_1^T[\hat{H}_1^T Q_1 \hat{H}_1 + \overline{V}^{-1}(I - C_2K)]C_1 + \overline{S}A_{12}(C_2^T\overline{V}^{-1}C_2)^{-1}A_{12}^T\overline{S}]C_1. \qquad (105)$$

Note that $C_1^T\overline{V}^{-1}(I - C_2K)C_1$ is symmetric. (105) implies:

$$\dot{\overline{S}} + \overline{S}(A_{11} - A_{12}KC_1 - \overline{S}^{-1}C_1^T\overline{V}^{-1}(I - C_2K)C_1) + (A_{11} - A_{12}KC_1 - \overline{S}^{-1}C_1^T\overline{V}^{-1}(I - C_2K)C_1)^T\overline{S} \leq 0, \qquad (106)$$

which by Lyapunov's Direct Method [21] implies that $(A_{11} - A_{12}KC_1 - \overline{S}^{-1}C_1^T\overline{V}^{-1}(I - C_2K)C_1)$ is stable. For time-invariant systems, this implies that the closed-loop eigenvalues lie in the open left-half plane. ∎

What happens, however, when $i > 1$ and $\dim(\ker S) > \text{rank } B_i$? The matrix B_{12} will no longer be square and the reduced-order Riccati Equation will be stuck in the form of Equation 100 which is not the same as what is needed in the proof for stability (Equation 101). It would thus seem that we cannot guarantee stability in the general case.

It turns out, however, that by augmenting the failure map in the original problem statement, we can always convert the reduced-order Riccati equation into the desired form (101). The necessary augmentation turns out to be:

$$\overline{F}_1 = \begin{bmatrix} B_i & B_{i-1} & \cdots & B_1 \end{bmatrix} \tag{107}$$

The new game problem for the limiting case is:

$$\begin{aligned} \min_{\hat{x}} \max_{\overline{\mu}_2} J^* = \int_{t_0}^{t_1} [\|x - \hat{x}\|^2_{C^T \hat{H}_1^T Q_1 \hat{H}_1 C} + (x - \hat{x})^T C^T \hat{H}_1^T Q_1 \hat{H}_1 C \overline{F}_1 \overline{\mu}_2 + \|\overline{\mu}_2\|^2_{\overline{F}_1^T C^T \hat{H}_1^T Q_1 \hat{H}_1 C \overline{F}_1} \\ - \|y - Cx\|^2_{\overline{V}^{-1}} - (y - Cx)^T \overline{V}^{-1} C \overline{F}_1 \overline{\mu}_2 - \overline{\mu}_2^T \overline{F}_1^T C^T \overline{V}^{-1} (y - Cx) - \|\overline{\mu}_2\|^2_{\overline{F}_1^T C^T \overline{V}^{-1} C \overline{F}_1}] dt \end{aligned} \tag{108}$$

subject to:

$$\dot{x} = Ax + \overline{F}_1 \overline{\mu}_2 \tag{109}$$

where $\overline{\mu}_2$ is the augmented failure signal which has many inputs as there are columns in $\overline{F}_1$. Note, that here we have gone back to the pre-transformed problem (where the state is x, not α_i). We will show that this new problem leads to a Riccati equation which is equivalent to (74). In this equation, however, the reduced-order version is easily seen to reduce to the desired form (101). The equivalence of the two equations then implies that the same reduced form holds for both.

The augmented failure map, $\overline{F}_1$ is such that $C\overline{F}_1 \neq 0$, thereby stopping the transformation process after one iteration. The solution to this game leads to a Goh Riccati Equation:

$$\begin{aligned} -\dot{S} = SA + A^T S + C^T (\hat{H}_1^T Q_1 \hat{H}_1 - \overline{V}^{-1}) C + [S(A\overline{F}_1 - \dot{\overline{F}}_1) - C^T \overline{V}^{-1} C \overline{F}_1] \\ \times (\overline{F}_1^T C^T \overline{V}^{-1} C \overline{F}_1)^{-1} [(A\overline{F}_1 - \dot{\overline{F}}_1)^T S - \overline{F}_1^T C^T \overline{V}^{-1} C] \end{aligned} \tag{110}$$

with a boundary condition given by (68). The solution, S, to (110) is such that $\dim(\ker \overline{S}) = \text{rank} \overline{F}_1$. Hence, after the transformation and defining:

$$\begin{bmatrix} \overline{F}_{11} \\ \overline{F}_{12} \end{bmatrix} = \Gamma \overline{F}_1,$$

the reduced-order Riccati Equation:

$$-\dot{\overline{S}} = C_1^T(\hat{H}_1^T Q_1 \hat{H}_1 - \overline{V}^{-1})C_1 + \overline{S}A_{11} + A_{11}^T\overline{S} + (\overline{S}A_{12}\overline{F}_{12} - C_1^T\overline{V}^{-1}C_2\overline{F}_{12})$$
$$\times(\overline{F}_{12}^T C_2^T \overline{V}^{-1} C_2\overline{F}_{12})^{-1}(\overline{S}A_{12}\overline{F}_{12} - C_1^T\overline{V}^{-1}C_2\overline{F}_{12})^T. \tag{111}$$

can be simplified to (101) because $\overline{F}_{12}$ is square and invertible. We know that $\overline{F}_{12}$ is square and invertible because the construction of $\overline{F}_1$ ensures that $\overline{F}_1$ has full column rank and that the size of ker S, which determines the order reduction, is equal to this column rank.

Proposition 6.2. The Goh Riccati Equation of the augmented system (110) is equivalent to the Goh Riccati Equation of the original system (74).

Proof. It is immediate that

$$C\overline{F}_1 = C\begin{bmatrix} B_i & B_{i-1} & \dots & B_1 \end{bmatrix} = CB_i \tag{112}$$

If we examine the term $SA\overline{F}_i - \dot{\overline{F}}_1$ in (**110**):

$$\begin{aligned} S(A\overline{F}_1 - \dot{\overline{F}}_1) &= SA\begin{bmatrix} B_i & B_{i-1} & \dots & B_1 \end{bmatrix} + S\begin{bmatrix} \dot{B}_i & \dot{B}_{i-1} & \dots & \dot{B}_1 \end{bmatrix} \\ &= \begin{bmatrix} SAB_i - S\dot{B}_i, SAB_{i-1} - S\dot{B}_i, \dots, SAB_1 - S\dot{B}_1 \end{bmatrix} \\ &= \begin{bmatrix} SAB_i - S\dot{B}_i, SB_i, SB_{i-1}, \dots, SB_2 \end{bmatrix}. \end{aligned}$$

Because of Proposition 4.2, this simplifies to

$$S(A\overline{F}_1 - \dot{\overline{F}}_1) = S(AB_i - \dot{B}_i). \tag{113}$$

Given, (112) and (113), the Goh Riccati Equation for the augmented system (110) simplifies to (74). ▮

Reduced-order filters for the time-varying are much harder to come by since the transformation matrix, Γ, will now be a function of time. In this case, the only likely option left to the analyst is to use the results of [22] which give differential equations for the eigenvectors and eigenvalues of the solution to a time-varying Riccati Equation. From here the reduced-order Riccati Matrix, the transformed system equation, and finally the reduced order filter can be formed through a transformation matrix based upon the eigenvectors. Needless to say, the computation required here will be quite intensive. The state and measurement matrices will also have to be transformed at each time step and only then can the filter be formed and propagated. The point here is that it is possible to find a reduced filter for the time-varying case, though the effort may outweigh the benefits. Since the full-order filter is always available, this is not a serious problem.

The analyst has many options when designing a game theoretic filter. In the case of the full-order filter he has the freedom to choose the different weighting matrices and γ. For reduced-order filters, he can use either the solution to the Goh Riccati Equation (74) or the solution of linear matrix inequality (56) with $\gamma = 0$ to find the needed transformation matrix and reduced-order filter gain. He also has the reduced-order Riccati Equation (101). Moreover, he can mix the two approaches (e.g. using the LMI to find the transformation matrix and using the reduced-order Goh Riccati Equation to find the gain). This flexibility is important, because the solution to the Goh equations may be ill-conditioned when several iterations of the Goh Transformation are needed to generate the Riccati Equation. The appearance of powers of A in the resulting equation may cause problems with the numerical solution. Thus, though the Riccati solution represents the saddle-point solution, it may not be the best choice from numerical considerations.

7. Example: Accelerometer Fault Detection for the F16XL

7.1 Problem Statement

To demonstrate the effectiveness of the game theoretic filter, the F16XL example of [5] is re-examined. The aircraft dynamics are linearized about trimmed level flight at 10,000 ft altitude and Mach 0.9. For simplicity, a reduced-order, five-state model of the longitudinal dynamics (including a first-order wind gust model) is considered:

$$\dot{x} = Ax + Bw_{wg}$$

$$y = Cx + v.$$

The five components of the states are:

$$x = \left\{ \begin{array}{c} u \\ w \\ q \\ \theta \\ w_g \end{array} \right\} \begin{array}{l} \textit{long. velocity} \text{ (ft/sec)} \\ \textit{normal velocity} \text{ (ft/sec)} \\ \textit{pitch rate} \text{ (deg/sec)} \\ \textit{pitch} \text{ (deg)} \\ \textit{wind gust} \text{ (ft/sec)}. \end{array} \tag{114}$$

The measurements are:

$$y = \left\{ \begin{array}{c} q \\ \theta \\ A_z \\ A_x \end{array} \right\} \begin{array}{l} \textit{pitch rate} \text{ (deg/sec)} \\ \textit{pitch} \text{ (deg)} \\ \textit{long. acceleration} \text{ (ft/sec}^2\text{)} \\ \textit{normal acceleration} \text{ (ft/sec}^2\text{)}. \end{array} \tag{115}$$

The input, w_{wg}, is windgust and v is the sensor noise. v is also assumed to be weighted by $V = \nu I_4$. The resulting system matrices are:

$$A = \begin{bmatrix} -.0674 & .0430 & -.8886 & -.5587 & .0430 \\ .0205 & -1.4666 & 16.5800 & -.0299 & -1.4666 \\ .1377 & -1.6788 & -.6819 & 0 & -1.6788 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1.1948 \end{bmatrix} \tag{116}$$

$$B^T = \begin{bmatrix} 0 & 0 & 0 & 0 & 2.0156 \end{bmatrix}, \tag{117}$$

$$C = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0.0139 & 1.0517 & 0.1485 & -0.0299 & 0 \\ -0.0677 & 0.0431 & 0.0171 & 0 & 0 \end{bmatrix} \tag{118}$$

It is desired to detect a normal accelerometer fault, A_z, in the presence of the wind gust disturbance and the sensor noise. Following the modelling techniques described in section 2, we incorporate the accelerometer faults into the system by modifying our state equations to:

$$\dot{x} = Ax + Bw_{wg} \tag{119}$$

$$y = Cx + E_{A_z}\mu_{A_z} + v. \tag{120}$$

where $E_{A_z} = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}^T$ and μ_{A_z} is the acceleromter failure signal. We then convert (119) and (120) into:

$$\dot{x} = Ax + F_{A_z}\mu_{A_z} + F_{wg}\mu_{wg} \tag{121}$$

$$y = Cx + v, \tag{122}$$

F_{wg} is simply B_{wg}; μ_{wg} is the wind gust, w_{wg}. The failure map, F_{A_z} and the failure signal μ_{A_z} are:

$$F_{A_z} = \begin{bmatrix} 0.6003 & 0 \\ 0.9429 & -1.3706 \\ 0 & -1.5003 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \mu_{A_z} = \left\{ \begin{matrix} \mu_1 \\ \mu_2 \end{matrix} \right\}$$

Finally, the natural projections on the filter residual are:

$$\hat{H}_1 = I - (CAF_{wg})[(CAF_{wg})^T(CAF_{wg})]^{-1}(CAF_{wg})^T = \begin{bmatrix} 0.5330 & 0 & -0.4982 & -0.0264 \\ 0 & 1 & 0 & 0 \\ -0.4982 & 0 & 0.4685 & -0.0281 \\ -0.0264 & 0 & -0.0281 & 0.9985 \end{bmatrix} \tag{123}$$

$$N_1 = I - \hat{H}_1 = (CAF_{wg})[(CAF_{wg})^T(CAF_{wg})]^{-1}(CAF_{wg})^T = \begin{bmatrix} 0.4670 & 0 & 0.4982 & 0.0264 \\ 0 & 0 & 0 & 0 \\ 0.4982 & 0 & 0.5315 & 0.0281 \\ 0.0264 & 0 & 0.0281 & 0.0015 \end{bmatrix}. \tag{124}$$

7.2 Full-Order Filter Design

Equation 34, the Riccati Equation in terms of P, was used for this example, though we could have just as easily used (31), the Riccati Equation in terms of Π. To bring sensor noise weighting, V $(= \nu I)$, to zero with the disturbance bound, it is assumed that ν is some multiple of γ. By trial and error, it was found that:

$$\nu = 5 \times 10^{-8}, \quad \frac{\nu}{\gamma} = 2, \quad Q_1 = R_1 = M_2 = I$$

gave the results seen in Figure 1. For the given parameters above, the solution of (34) is:

$$P = 10^{-6} \times \begin{bmatrix} .019161 & -.25947 & -.20981 & -.00448 & 11.069 \\ -.25947 & 7.5971 & 7.1863 & .076754 & -376.33 \\ -.20981 & 7.1863 & 7.9932 & .011879 & -428.83 \\ -.00448 & .076754 & .011879 & .00397 & -.098976 \\ 11.069 & -376.33 & -428.83 & -.098976 & 45255 \end{bmatrix}, \tag{125}$$

resulting in a gain:

$$L = \begin{bmatrix} -2.0981 & -0.0448 & -3.0364 & -0.16068 \\ 71.863 & 0.7675 & 90.511 & 4.679 \\ 79.932 & 0.1188 & 87.416 & 4.6062 \\ 0.1188 & 0.0398 & 0.8230 & 0.0381 \\ -4288.3 & -0.9897 & -4593.1 & -243.02 \end{bmatrix}$$

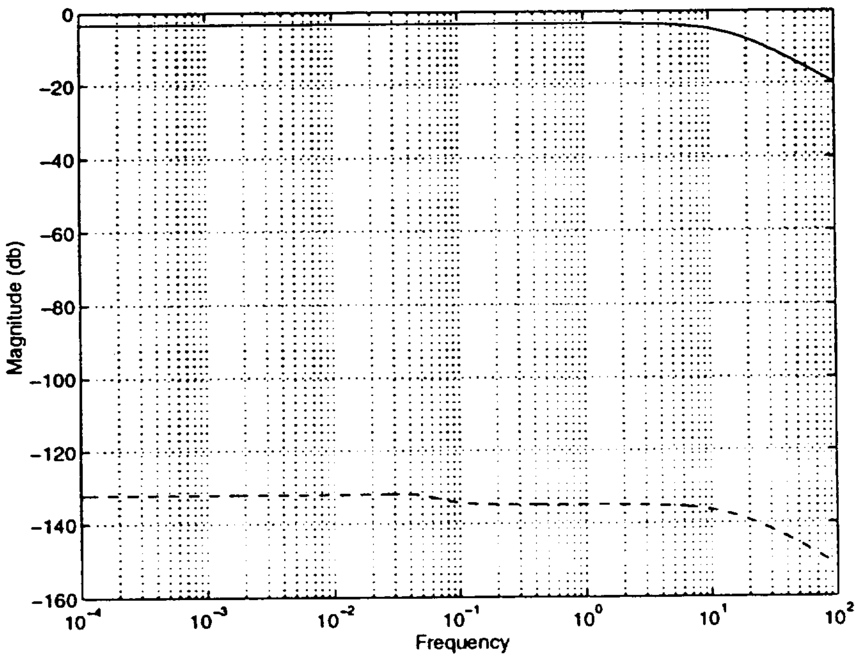

Figure 1: Frequency Response of Accerometer Fault vs. Frequency Response of Wind Gust (solid line - output due to μ_{A_z}; dashed line - output due to μ_{w_g})

Figure 1 is a plot of the Euclidean Norms of the signal $z_1 := \hat{H}_1 C(y - C\hat{x})$ due to the μ_{A_z} and μ_{w_g}. The pertinent transfer functions are:

$$\frac{z_1(s)}{\mu_{A_z}(s)} = \hat{H}_1 C(sI - A + LC)^{-1} L E_{a_z}$$

$$\frac{z_1(s)}{\mu_{wg}(s)} = \hat{H}_1 C(sI - A + LC)^{-1} \hat{F}_1$$

It can been seen that at least 130 dB of separation exists between the output due to the wind gust and the output due to the accelerometer fault. This should be satisfactory.

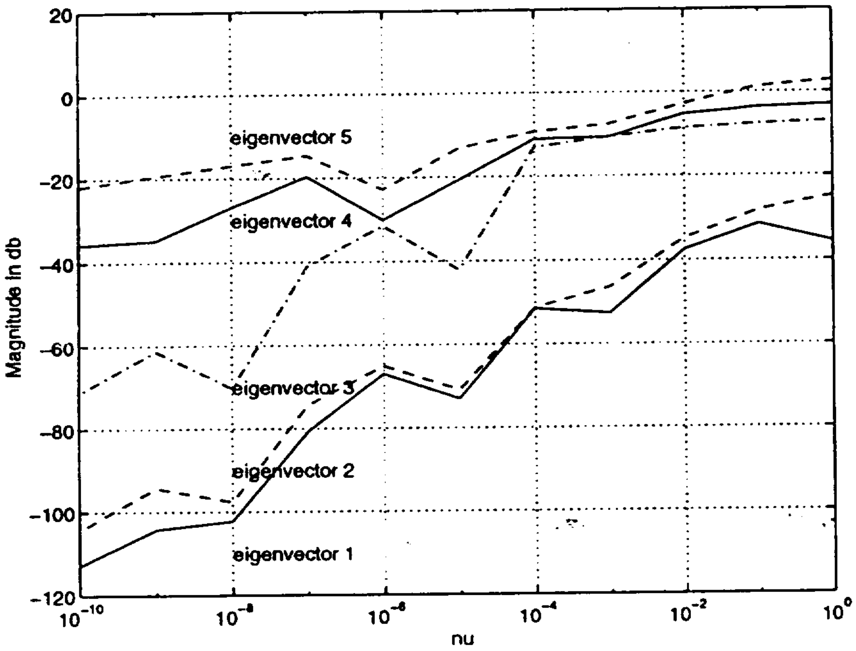

Figure 2: Inner Product of the Closed-Loop Eigenvectors and Fault Direction 1

The asymptotic convergence of the game theoretic filter to a fault detection filter structure can be demonstrated with the analysis of [4]. In that filter, the closed-loop eigenvectors are placed in such a way that for each fault j its complementary detection space, $\hat{\mathcal{F}}_i = \sum_{j \neq i} \mathcal{F}_j$ is decimated. The complementary detection space is spanned by the faults which are not meant to be seen. This condition can be represented by the linear equation:

$$\begin{bmatrix} A^T - \lambda_i I & C^T \\ \hat{F}^T & 0 \end{bmatrix} \begin{bmatrix} V_i \\ W_i \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{126}$$

From (126) we conclude that:

$$(A^T - \lambda_i I)V_i + C^T W_i = 0 \tag{127}$$

$$\hat{F}_i^T V_i = 0 \tag{128}$$

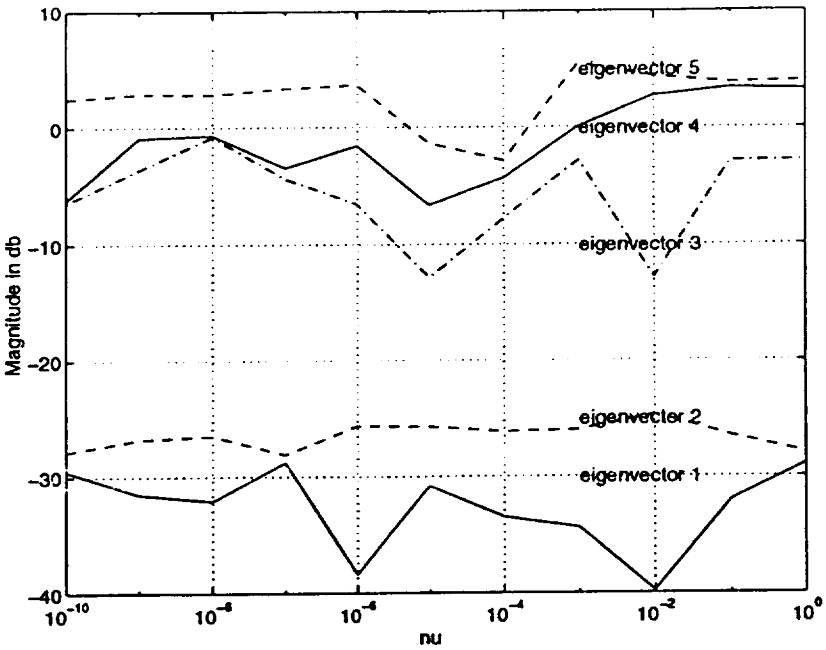

Figure 3: Inner Product of the Closed-Loop Eigenvectors and Fault Direction 2

The robust fault detection filter as described in [5] uses a result from [23] that W_i and V_i are related through a gain matrix:

$$W_i = L^T V_i$$

leading to the following conditions:

$$\begin{aligned} (A^T + C^T L^T - \lambda_i I)V_i &= 0 \qquad (129) \\ \hat{F}_i^T V_i &= 0. \qquad (130) \end{aligned}$$

We can conclude from (129) and (130), therefore, that in a fault detection filter the complementary fault directions are perpendicular to the closed-loop eigenvectors.

In Figures 2 and 3 we employ Equation 130 and examine the eigenstructure of the game theoretic filter as the parameters ν and γ are taken to zero ($\gamma = \nu/2$). Figure 2 shows a plot of $F_{wg}^T V_i$ ($i = 1 \ldots 5$) where V_i are the eigenvectors of $(A + PC^T V^{-1} C)$. Figure 2 shows that the closed-loop eigenvectors (two in particular) become perpendicular to the directions of F_{wg}. Since we have designed the filter to block out the transmission of μ_{wg} to a portion of the state-space, this is the result we would expect.

Figure 3 shows that the closed-loop eigenvectors do not align in directions normal to the space spanned by F_{A_1}. Decreasing values of γ seem to have no effect on alignment, in fact. Since this is the fault that we want to detect, this is the result that we want to see.

7.3 Reduced-Order Filter Design via the Goh Riccati Equations

We now repeat the example, but now we will design a lower order detection filter using the Goh Riccati Equations. The first step is to derive the transformation matrix, Γ. Since the transformation is determined via the null space of the full-order Riccati Matrix, the design process begins by finding the solution to the full-order Goh Riccati Equation (74). Because $C\hat{F}_1 = CB_1 = 0$, we need to go to a second iteration ($i = 2$), in which we find that:

$$B_2 = AB_1 = A\hat{F}_1 \tag{131}$$

Using (131) and the same weightings as in the full-order design, we find that the solution to the Goh Riccati Equation (74) is:

$$S = \begin{bmatrix} -4.6032 & 7.0275 & -6.0662 & -123.5565 & -0.2697 \\ -5.2351 & 2.5842 & -2.2990 & -48.2081 & -0.1353 \\ 4.4622 & -2.0830 & 1.8578 & 39.0486 & 0.1115 \\ 85.2718 & -36.2789 & 32.4847 & 698.8667 & 2.0423 \\ -0.0098 & 0.0079 & -0.0069 & -0.1432 & -0.0003 \end{bmatrix} \tag{132}$$

Using the QR decompostion we find obtain a transformation matrix:

$$\Gamma^T = \begin{bmatrix} -0.0537 & -0.9954 & -0.0763 & -0.0175 & 0.0091 \\ -0.0611 & -0.0667 & 0.7407 & 0.5738 & -0.3374 \\ 0.0521 & 0.0332 & -0.6631 & 0.6384 & -0.3860 \\ 0.9953 & -0.0596 & 0.0761 & 0.0009 & 0.0001 \\ -0.0001 & -0.0007 & -0.0062 & 0.5128 & 0.8585 \end{bmatrix} \tag{133}$$

Using this transformation, we reduce our state-space to a third-order system (i.e. we find the matrices A_{11}, C_1 etc.). From here we employ the reduced-order system matrices in the reduced order Goh Riccati Equation, (110). The soltution to (110) using (133) is:

$$\overline{S} = \begin{bmatrix} 219.8389 & -14.6664 & -0.2469 \\ -14.6664 & 5.2528 & 0.1179 \\ -0.2469 & 0.1179 & 0.0035 \end{bmatrix} \tag{134}$$

with a corresponding gain:

$$L = \begin{bmatrix} -0.0066 & 0.0061 & 0.0005 & 0.0033 \\ -0.3211 & 0.1887 & 0.0068 & 0.1483 \\ 8.4710 & -7.2336 & 4.3827 & -5.6715 \end{bmatrix} \tag{135}$$

The closed-loop eigenvalues are $-0.0375 \pm 0.0128j, -5.1698$. Compare this to the eigenvalues of the full-order filter which are: $-0.0344 \pm 0.0448j, -88.40 \pm 88.57j, -14.9142$. To demonstrate the effectiveness of the reduced-order filter a linear simulation of the system was run for two cases: one with a accelerometer fault input (modelled as a step) the other with a windgust input (also a step). Figures 5 and 4, shows that the reduced-order filter responds to the accelerometer fault input and is relatively insensitive to the wind gust input.

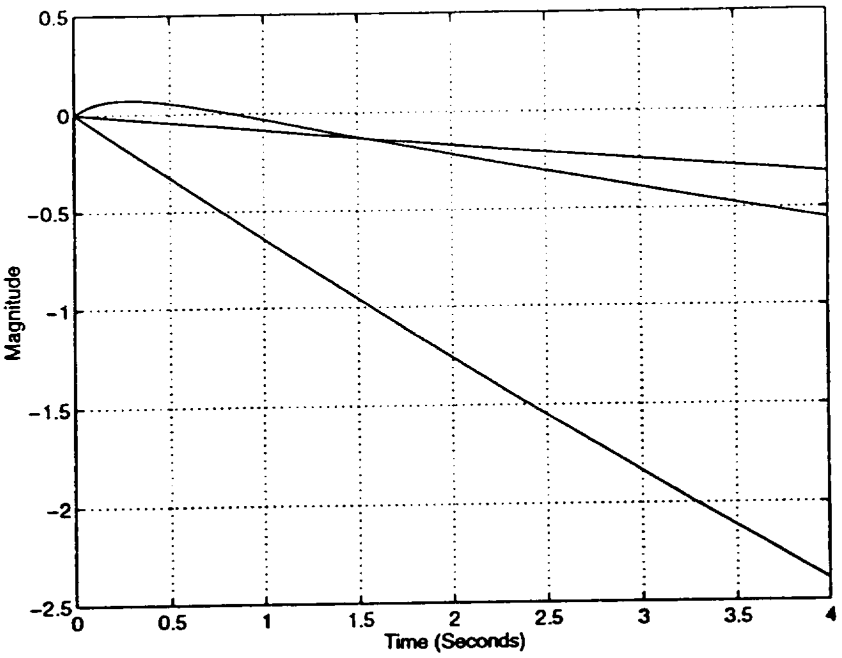

Figure 4: Reduced-Order Goh Filter Residual due to step in μ_A, (fault to be detected)

7.4 Reduced-Order Filter Design via the Linear Matrix Inequality

We will now demonstrate yet another design algorithm. Compared to the previous design, this approach requires more work and judgement from the engineer, but the final filter design is very effective and in some cases the approach may be numerically more reliable.

The linear matrix inequality (56) was solved in the steady-state case using software for semi-definite programming software[3]. For $\gamma = 0$ and the same weighting matrices as the full-order design, it was found that:

$$\Pi = \begin{bmatrix} 3.4282 & -0.1842 & 0.2482 & 6.4511 & -0.0000 \\ -0.1842 & 1.0263 & -0.9014 & -17.1934 & -0.0000 \\ 0.2482 & -0.9014 & 0.7978 & 15.1839 & 0.0000 \\ 6.4511 & -17.1934 & 15.1839 & 301.6193 & -0.0000 \\ -0.0000 & -0.0000 & 0.0000 & -0.0000 & 0.0000 \end{bmatrix} \tag{136}$$

Using the eigenvectors of the Riccati matrix (136), we obtain the following as a transformation matrix:

$$T = \begin{bmatrix} 0.0211 & 0.0002 & 0.0575 & 0.9979 & -0.0215 \\ -0.6823 & -0.0058 & -0.7266 & 0.0575 & 0.0568 \\ -0.7307 & -0.0066 & 0.6804 & -0.0249 & -0.0502 \\ -0.0026 & 0.0000 & -0.0769 & -0.0170 & -0.9969 \\ -0.0088 & 1.0000 & 0.0003 & -0.0000 & 0.0000 \end{bmatrix}$$

Following the results of the previous section, we choose the reduced-order estimator to be 3rd order, with the corresponding reduced Riccati Matrix:

[3] "SP" developed at Stanford University by Stephen Boyd and his students

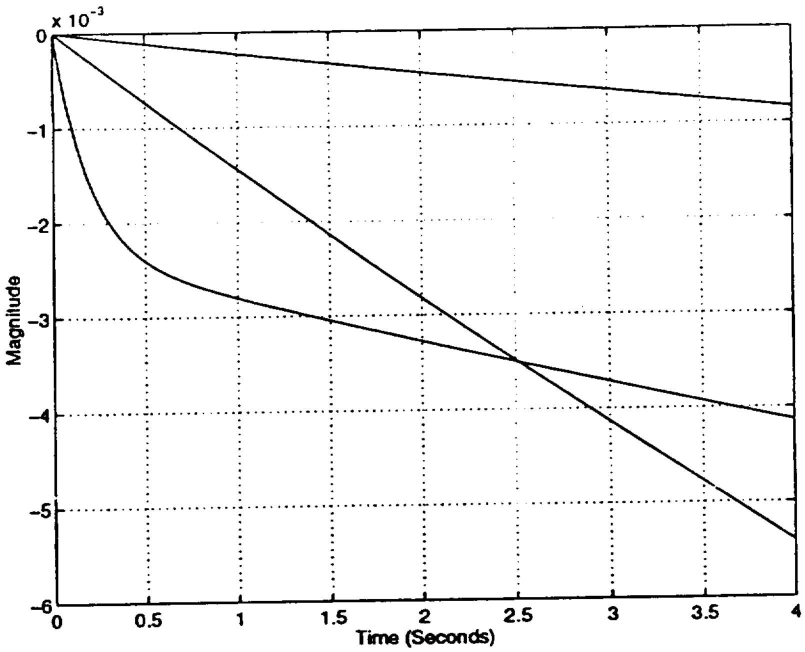

Figure 5: Reduced-Order Goh Filter Residual due to step in μ_{wg} (nuisance fault)

$$\overline{\Pi} = \begin{bmatrix} 3.3129 & -0.0000 & 0.0000 \\ -0.0000 & 305.3302 & -0.0000 \\ 0.0000 & -0.0000 & 0.0633 \end{bmatrix}$$

The resulting gain is:

$$L = \begin{bmatrix} 0.0162 & -0.0035 & -0.0153 & -0.0005 \\ -0.6843 & 0.0740 & 0.6642 & 0.0130 \\ 178.7185 & -19.9610 & -169.0275 & -5.8971 \end{bmatrix}. \tag{137}$$

The closed-loop poles are $-0.0347 \pm 0.0448j, -243.4$ (Note that the dominant second-order pair of the full-order filter has been retained).

Again by simulating the reduced-order system, we find in Figures 7 and 6 that the reduced-order filter works effectively as a fault detector for an accelerometer fault in the presence of a windgust fault. Figure 6 shows that the reduced-order estimator is sensitive to μ_{A_z}, the accelerometer fault, whereas Figure 7 shows quite clearly that the reduced-order filter is unaffected by the presence of a windgust fault (the residuals are on the order of 10^{-6}).

8. Conclusions

By solving the fault detection problem via disturbance attenuation, we obtain a game theoretic filter that bounds the transmission of disturbances and nuisance faults. By going to the limit of this solution, we get a fault detection filter which in the time-invariant case is equivalent to the Beard-Jones Fault Detection Filter. That is, the presence

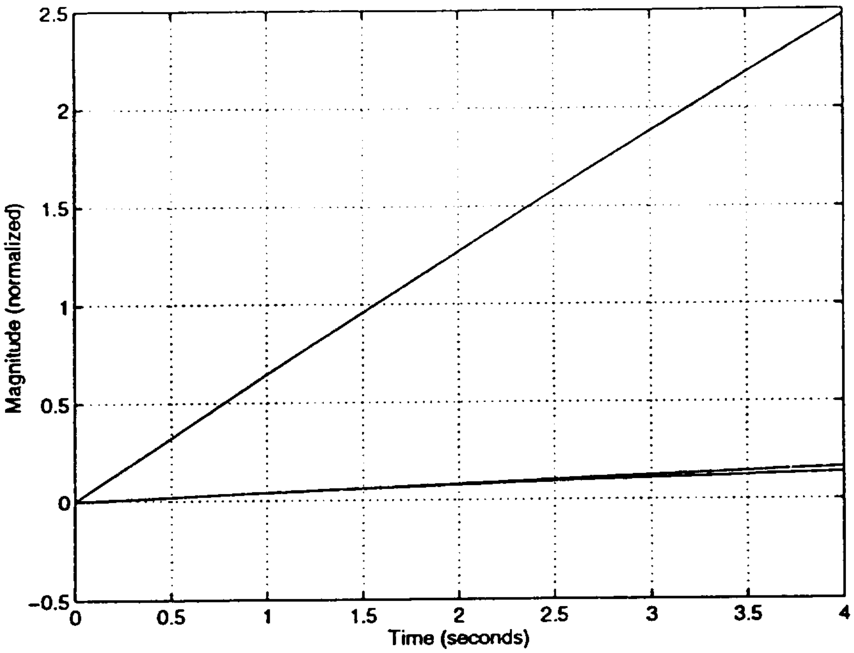

Figure 6: Reduced-Order Filter Residual due to step in μ_{A_t} (fault to be detected)

of the nuisance faults is restricted to an invariant subspace that can be made unobservable through a projection. This unobservable subspace can be factored out of total space to get a lower-order system which is uninfluenced by the nuisance faults. The same factoring process can then be applied to the game filter to get a reduced-order fault detector for the newly reduced state-space. Extensions of this latter result exist for the time-varying case, though the computation involved may be intensive.

The game theoretic approach to fault detection filter design is more flexible and applicable than current design methods. The designer can choose the degree to which the game filter possesses the structure of the Beard-Jones filter. This allows him to make tradeoffs between nuisance fault blocking and sensor noise rejection. The linear quadratic game used to solve the disturbance attenuation problem admits time-varying systems and can be used to incorporate parameter uncertainty into the filter design. Recent extensions of robust control such as designs which constrain pole-placement and designs with multiple objectives (e.g. the so-called mixed H_2/H_∞ problems) suggest that the same can be done here. The latter is of particular interest since it appears to be a logical way to detect and identify multiple faults with a single game theoretic filter.

Finally, we have shown that the limiting form of the game filter is a singular filter. Since any disturbance attenuation problem can be solved in the same manner as this one, it is likely that this result applies to all such problems. That is, the limiting form of a disturbance attenuation problem is a singular optimization problem. This

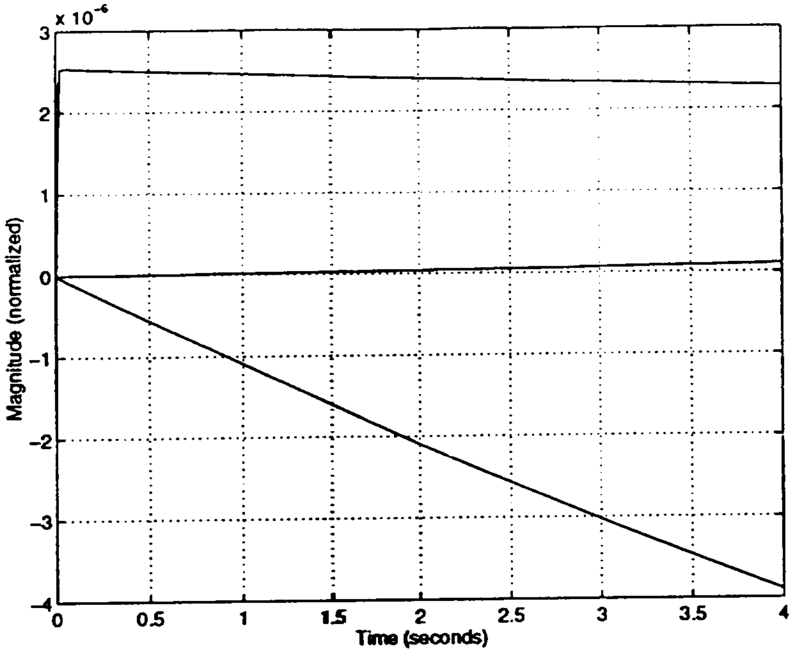

Figure 7: Reduced-Order Filter Residual due to step in μ_{w_g} (nuisance fault)

makes applicable a wealth of results from singular control and it provides a new way to understand H_∞ problems by looking at them as "almost" singular optimal control problems.

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