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F. G. Yuan
North Carolina State University, Raleigh, North Carolina

National Aeronautics and
Space Administration

Langley Research Center
Hampton, Virginia 23681-2199

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F. G. Yuan
Department of Mechanical and Aerospace Engineering
North Carolina State University
Raleigh, NC 27695

Abstract

Determination of all the coefficients in the crack tip field expansion for monoclinic materials under two-dimensional deformation is presented in this report. For monoclinic materials with a plane of material symmetry at $x_3 = 0$, the in-plane deformation is decoupled from the anti-plane deformation. In the case of in-plane deformation, utilizing conservation laws of elasticity and Betti's reciprocal theorem, together with selected auxiliary fields, T-stress and third-order stress coefficients near the crack tip are evaluated first from path-independent line integrals. To determine the T-stress terms using the J-integral and Betti's reciprocal work theorem, auxiliary fields under a concentrated force and moment acting at the crack tip are used respectively. Through the use of the Stroh formalism in anisotropic elasticity, analytical expressions for all the coefficients including the stress intensity factors are derived in a compact form that has surprisingly simple structure in terms of one of the Barnett-Lothe tensors, $\boldsymbol{L}$. The solution forms for degenerated materials, orthotropic, and isotropic materials are also presented.

Introduction

The use of fracture mechanics to assess the failure behavior in a flawed structure requires the identification of critical parameters which govern the severity of stress and deformation field in the vicinity of the flaw, and which can be evaluated using information obtained from the flaw geometry, loading, and material properties. In the linear elastic solids, stress intensity factors, k_i (i = I, II, III), represent the leading singular terms in the Williams eigenfunction expansion series near a crack tip. k_i are often assumed to be unique parameters associated with crack extension. The physical implications of the higher-order non-singular terms have been noted by Cotterell (1966). Especially, the so-called T-stress, second term of the crack tip stress field which represents the constant normal stress parallel to the crack surfaces, has been found as an additional parameter in characterizing the behavior of a crack (Larsson and Carlsson, 1973; Rice, 1974). Cotterell and Rice (1980) showed that T-stress substantially influences the fracture path stability of a mode-I crack. The stress biaxiality parameter (Leevers and Radon, 1982; Sham, 1991) has been tabulated as a function of relative crack lengths and overall geometry in many fracture test specimens for the isotropic solid using computational techniques (e.g., Kfouri, 1986 and Sham, 1989 and 1991). Kardomateas *et al.* (1993) examined the third-term of the Williams solution and concluded its significance in the center-cracked and single-edge specimens with short crack lengths.

In anisotropic linear elastic solids, Gao and Chiu (1992) examined the T-stress term of a crack in infinite orthotropic solids under mode-I loading. Because of the material anisotropy involved, the T-stress term is affected by the material properties. It is also expected that, in general, mixed-mode crack behavior and the biaxiality parameter are also dependent on the material anisotropy. Thus, it is essential to develop efficient computational techniques to determine T-stress term coefficients including the stress intensity factors in anisotropic cracked materials with finite geometry. In this report, two methods based on the J-integral and Betti's reciprocal theorem are proposed to obtain compact forms in calculating all the stress coefficient terms in the crack tip field expansion for monoclinic materials with a plane of material symmetry at $x_3 = 0$. To determine T-stress term using the J-integral, the method by Kfouri (1986) is extended to anisotropic solids. The closed form solution of the auxiliary field, a point force acting at the crack tip, is derived for this purpose. A path-independent integral based on the Betti's reciprocal work concept has been used for determining the stress intensity factors by Stern, Becker, and Dunham (1976), Hong and Stern (1978), Sinclair, Okajima, and Griffin (1984) for isotropic materials; Soni and Stern (1976) for orthotropic materials; and An (1987) for rectlinearly anisotropic materials. This path-independent line integral is also extended to determine all the stress coefficient terms with auxiliary fields.

Mathematical Formulation

In a fixed Cartesian coordinate system x_i, $(i = 1, 2, 3)$, consider a two-dimensional deformation of an anisotropic elastic body in which the deformation field is independent of the x_3 coordinate. In this report, attention focuses on the monoclinic material having three mutually perpendicular symmetry planes and one of the planes coinciding with the coordinate plane $x_3 = 0$. In this case, the in-plane and out-of-plane deformations are uncoupled. For in-plane deformation the strain and stress relations can be written as

$$\varepsilon = s'\sigma \tag{1}$$

where $\varepsilon = [\varepsilon_1, \varepsilon_2, \gamma_{12}]^T$, $\sigma = [\sigma_{11}, \sigma_{22}, \sigma_{12}]^T$

or

$$\varepsilon_i = s'_{ij}\,\sigma_j, \quad i, j = 1, 2, 6$$

where $s'_{ij} = s'_{ji}$ are reduced compliance coefficients defined by $s'_{ij} = s_{ij} - s_{i3}s_{j3} / s_{33}$.

Throughout the report, all indices range from 1 to 2 and the summation convention is applied to repeated Latin index unless otherwise noted. The bold-face letters are used to represent matrices or vectors. A comma stands for differentiation; overbar denotes complex conjugate. A symbol Re stands for real part; Im for imaginary part.

In the absence of body forces, general solutions of the displacement vector $\boldsymbol{u}$, the stress function ϕ, and stresses σ for in-plane deformation, according to Stroh formalism (Ting, 1996), can be represented by

$$u = \mathrm{Re}[\sum_{\alpha=1}^{2} a_\alpha d_\alpha f(z_\alpha)]$$
$$\phi = \mathrm{Re}[\sum_{\alpha=1}^{2} b_\alpha d_\alpha f(z_\alpha)] \tag{2}$$

or

$$u = \mathrm{Re}\left[A\langle f(z)\rangle d\right]$$
$$\phi = \mathrm{Re}\left[B\langle f(z)\rangle d\right] \tag{3}$$

$$\sigma_{i1} = -\phi_{i,2}, \quad \sigma_{i2} = \phi_{i,1} \tag{4}$$

where

$$\langle f(z)\rangle = diag[f(z_1), f(z_2)]$$

$$z_\alpha = x_1 + \mu_\alpha x_2, \ \mathrm{Im}[\mu_\alpha] > 0$$

f(z) is an arbitrary function, d is a unknown complex constant vector to be determined. μ_α, a_α, and b_α are the Stroh eigenvalues and corresponding eigenvectors determined by elastic constants only. For in-plane deformation, μ_α are given by the roots of the characteristics equation:

$$s'_{11}\mu^4 - 2s'_{16}\mu^3 + \left(2s'_{12} + s'_{66}\right)\mu^2 - 2s'_{26}\mu + s'_{22} = 0 \tag{5}$$

with positive imaginary parts. From energy consideration, Lekhnitskii (1963) showed that the roots are either complex or purely imaginary and cannot be real. A and B are Stroh matrices given by

$$A = \begin{bmatrix} a_1 & a_2 \end{bmatrix} = \begin{bmatrix} p_1 & p_2 \\ q_1 & q_2 \end{bmatrix} \tag{6}$$

$$B = \begin{bmatrix} b_1, & b_2 \end{bmatrix} = \begin{bmatrix} -\mu_1 & -\mu_2 \\ 1 & 1 \end{bmatrix}, \ B^{-1} = \frac{1}{\mu_1 - \mu_2}\begin{bmatrix} -1 & -\mu_2 \\ 1 & \mu_1 \end{bmatrix} \tag{7}$$

$$p_\alpha = s'_{11}\mu_\alpha^2 - s'_{16}\mu_\alpha + s'_{12}, \ q_\alpha = s'_{12}\mu_\alpha - s'_{26} + s'_{22} / \mu_\alpha . \tag{8}$$

The eigenvectors a_α and b_α are unique to an arbitrary multiplier. Introducing normalization factors k_α, we have

$$A = \begin{bmatrix} k_1 p_1 & k_2 p_2 \\ k_1 q_1 & k_2 q_2 \end{bmatrix}, \ B = \begin{bmatrix} -k_1\mu_1 & -k_2\mu_2 \\ k_1 & k_2 \end{bmatrix} \tag{9}$$

The values of k_α satisfy the conditions

$$2a_\alpha^T b_\beta = \delta_{\alpha\beta}$$

Based on the Stroh formalism, the matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ defined in eq. (9) satisfy orthogonality relations. For in-plane deformation, these relations can be expressed by

$$\boldsymbol{B}^T\boldsymbol{A} + \boldsymbol{A}^T\boldsymbol{B} = \mathbf{I} \tag{10}$$

$$\boldsymbol{B}^T\overline{\boldsymbol{A}} + \boldsymbol{A}^T\overline{\boldsymbol{B}} = 0 \tag{11}$$

where

$$2k_1^2(q_1 - p_1\mu_1) = 1, \quad 2k_2^2(q_2 - p_2\mu_2) = 1$$

It can be proved that

$$\boldsymbol{B}^{-T}\boldsymbol{B}^{-1} = -2i\boldsymbol{L}^{-1}, \quad \boldsymbol{L}^{-1} = -\mathrm{Im}[\boldsymbol{A}\boldsymbol{B}^{-1}] \tag{12}$$

where $\boldsymbol{L}$ is a real, symmetric, and positive definite matrix which will be used frequently in the sequel.

Note that the normalization factors cancel each other for the term $\boldsymbol{A}\boldsymbol{B}^{-1}$ in the second equation of (12). Therefore there is no need to introduce the normalization factors in computing $\boldsymbol{L}^{-1}$. From eq. $(12)_2$, it is easy to get

$$\boldsymbol{L}^{-1} = s'_{11}\begin{bmatrix} b & d \\ d & e \end{bmatrix}, \quad \boldsymbol{L} = \frac{1}{s'_{11}(be - d^2)}\begin{bmatrix} e & -d \\ -d & b \end{bmatrix} \tag{13}$$

$$\mu_1 + \mu_2 = a + ib, \quad \mu_1\mu_2 = c + id,$$

$$e = ad - bc = \mathrm{Im}[\mu_1\mu_2(\overline{\mu}_1 + \overline{\mu}_2)] \tag{14}$$

For crack problems, it may be convenient to introduce a complex potential function $\boldsymbol{\Phi}$ (Guo, 1991) such that

$$\boldsymbol{\Phi} = \boldsymbol{B}\langle f(z)\rangle\boldsymbol{B}^{-1}\boldsymbol{g} \tag{15}$$

where $\boldsymbol{g} = \boldsymbol{B}\boldsymbol{d}$, then the displacement expression in eq. (3) and stresses in eq. (4) can be rewritten in terms of the potential function $\boldsymbol{\Phi}$

$$\boldsymbol{u} = \mathrm{Re}\left[\boldsymbol{A}\boldsymbol{B}^{-1}\boldsymbol{\Phi}\right] \tag{16}$$

$$\sigma_{i1} = -\mathrm{Re}[\Phi_{i,2}], \quad \sigma_{i2} = \mathrm{Re}[\Phi_{i,1}] \tag{17}$$

The traction vector $\boldsymbol{t}$ at a point on a curve Γ with unit outward normal $\boldsymbol{n}$ is given by

$$t_i = -\mathrm{Re}\left[\frac{d\Phi_i}{ds}\right], \quad \boldsymbol{t} = -\mathrm{Re}\left[\frac{d\boldsymbol{\Phi}}{ds}\right] \tag{18}$$

where s is an arc length measured along Γ as shown in Fig. 1. Thus, without loss of generality, the traction free boundary conditions on a boundary may be written as

$$\mathrm{Re}[\boldsymbol{\Phi}] = 0 \text{ on } \Gamma \tag{19}$$

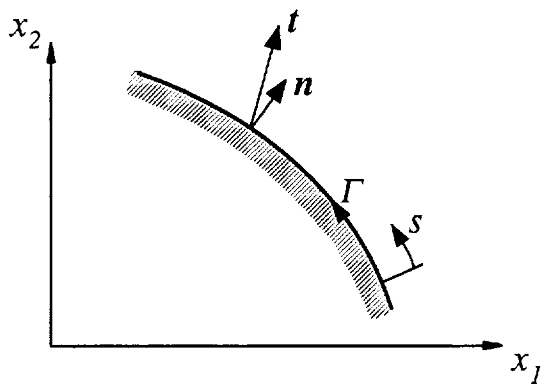

Fig. 1 The surface traction $\boldsymbol{t}$ on a curved boundary Γ with a unit outward normal vector $\boldsymbol{n}$.

The resultant force and moment about the x_3 axis due to the surface traction $\boldsymbol{t}$ acting on Γ between s_1 and s_2 ($s_2 > s_1$) are

$$\int_{s_1}^{s_2} \boldsymbol{t}(s)\,ds = \mathrm{Re}[\boldsymbol{\Phi}(s_1) - \boldsymbol{\Phi}(s_2)] \tag{20}$$

$$\int_{s_1}^{s_2} (x_1 t_2 - x_2 t_1)\,ds = -\mathrm{Re}[x_1\Phi_2 - x_2\Phi_1 - \chi]\Big|_{s_1}^{s_2} \tag{21}$$

where

$$\chi(z) = \int^{z} \Phi_2(\lambda)\,d\lambda$$

If Γ encloses a region and there are concentrated force f and moment M inside the region, then the equilibrium of the body demands that

$$-\int_{\Gamma} \boldsymbol{t}(s)\,ds = \boldsymbol{f} \tag{22}$$

$$-\int_{\Gamma} (x_1 t_2 - x_2 t_1)\,ds = M \tag{23}$$

Crack-tip fields

Consider a crack in the anisotropic body. Let a coordinate system be attached to the crack tip and crack plane lies on the x_1 - x_3 coordinate plane. The configuration is shown in Fig. 2. The crack faces are assumed to be traction-free. Note that the crack plane may not coincide with the symmetry plane of the material.

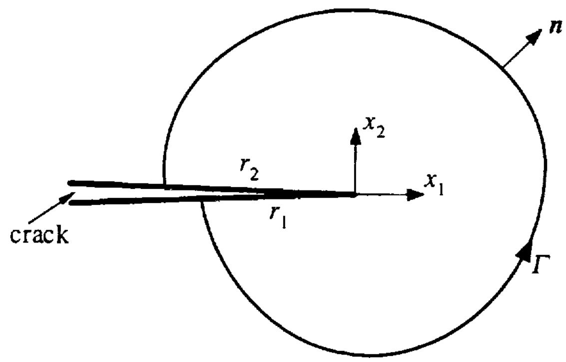

Fig. 2 A cracked body and a contour around a crack

To find the solution for the crack tip field, employing eq. (15) and (16) with $f_\alpha(z_\alpha) = z_\alpha^{\delta+1}$, we take the solutions $\boldsymbol{u}$ and $\boldsymbol{\Phi}$ in the form

$$\boldsymbol{u} = \mathrm{Re}[\boldsymbol{A}\langle z^{\delta+1}\rangle \boldsymbol{B}^{-1}\boldsymbol{g}] \tag{24}$$

$$\boldsymbol{\Phi} = \boldsymbol{B}\langle z^{\delta+1}\rangle \boldsymbol{B}^{-1}\boldsymbol{g} \tag{25}$$

where the complex variable z_α is defined by

$$z_\alpha = r(\cos\theta + \mu_\alpha \sin\theta), \quad -\pi \le \theta \le \pi$$

$\boldsymbol{g}$ is a complex constant vector and δ is the complex constant. We seek admissible values of δ subjected to a restriction in which the strain energy is bounded as $r \to 0$, that is

$$-1 < \mathrm{Re}(\delta)$$

To maintain a unique solution for $\boldsymbol{\Phi}$ in eq. (25), we introduce a branch cut along the negative x_1 axis with a convention, that is

$$z_\alpha^{\delta+1} = r^{\delta+1} e^{\pm i(\delta+1)\pi} \text{ at } \theta = \pm\pi$$

Insertion of eq. (25) into traction-free boundary conditions

$$\mathrm{Re}[\boldsymbol{\Phi}] = 0 \text{ at } \theta = \pm\pi$$

yields

$$r^{\delta-\bar{\delta}} e^{i\pi(\delta+\bar{\delta}+2)} \boldsymbol{g} + \bar{\boldsymbol{g}} = 0 \tag{26}$$

$$r^{\delta-\bar{\delta}} e^{-i\pi(\delta+\bar{\delta}+2)} \boldsymbol{g} + \bar{\boldsymbol{g}} = 0 \tag{27}$$

Subtracting of eq. (27) from eq. (26) leads to

$$[e^{i2(\delta+\bar{\delta}+2)\pi} - 1]\mathbf{I}\boldsymbol{g} = 0 \tag{28}$$

where $\mathbf{I}$ is a 2×2 identity matrix. For a nontrivial solution of $\boldsymbol{g}$, we must have

$$\det[(e^{i2(\delta+\bar{\delta}+2)\pi} - 1)\mathbf{I}] = \left|(e^{i2(\delta+\bar{\delta}+2)\pi} - 1)\mathbf{I}\right| = 0 \tag{29}$$

or

$$\mathrm{Re}(\delta) = (n-2)/2\,,\;\; n = 1, 2, 3, \cdots \tag{30}$$

With eq. (30), from eq. (26) or (27), we have

$$\bar{\boldsymbol{g}} = -r^{i2\,\mathrm{Im}(\delta)}\cos[2\pi\mathrm{Re}(\delta)]\,\boldsymbol{g} \tag{31}$$

Since $\bar{\boldsymbol{g}}$ is a constant, from eq. (31), (30),

$$\mathrm{Im}(\delta) = 0\,,\;\; \delta = (n-2)/2 \tag{32}$$

$$\bar{\boldsymbol{g}} = \begin{cases} \boldsymbol{g}, & \delta = -\dfrac{1}{2}, \dfrac{1}{2}, \dfrac{3}{2}, \cdots \\ -\boldsymbol{g}, & \delta = 0, 1, 2, \cdots \end{cases}$$

that is, δ is real, the vector $\boldsymbol{g}$ associated with δ is real if δ = -1/2, ½, 3/2, …; and $\boldsymbol{g}$ is pure imaginary if δ = 0, 1, 2, …. It is clear from the determinant given in eq. (29) that each of the eigenvalues δ is a root of multiplicity two. Since $\boldsymbol{g}$ has two arbitrary components, we have two independent eigenfunctions associated with the double eigenvalues δ. Therefore the assumed forms of $\boldsymbol{u}$ and $\boldsymbol{\Phi}$ given by eq. (24) and (25) are justified; no logarithmic type of solution form exists.

Superimposing all the solutions with different orders of r, the crack tip field can be constructed as

$$\begin{aligned} \boldsymbol{u} &= \sum_{n=1} \mathrm{Re}[\boldsymbol{A}\left\langle z^{\delta_n+1}\right\rangle \boldsymbol{B}^{-1}\boldsymbol{g}_n] \\ \boldsymbol{\Phi} &= \sum_{n=1} \boldsymbol{B}\left\langle z^{\delta_n+1}\right\rangle \boldsymbol{B}^{-1}\boldsymbol{g}_n \end{aligned} \tag{33}$$

where $\delta_n = (n-2)/2$. $\boldsymbol{g}_n$ is real for n = 1, 3, 5, ⋯, $\boldsymbol{g}_n$ is pure imaginary for n = 2, 4, 6,⋯, and $\boldsymbol{g}_n$ are dependent on the geometry of the cracked body, material properties, and loading conditions. Similarly, there is no need to introduce k_α in calculating σ_{ij} and u_i from eq. (33).

Performing algebraic calculation and defining $\boldsymbol{g}_n = [g_{n1}, g_{n2}]^{\mathrm{T}}$, the stress and displacement components can be written as

$$\begin{aligned} \sigma_{11} &= \sum_{n=1}(\delta_n+1)r^{\delta_n}\,\mathrm{Re}\left\{\frac{1}{\mu_1-\mu_2}\left[g_{n1}\left(\mu_2^2\varsigma_2^{\delta_n} - \mu_1^2\varsigma_1^{\delta_n}\right) + g_{n2}\mu_1\mu_2\left(\mu_2\varsigma_2^{\delta_n} - \mu_1\varsigma_1^{\delta_n}\right)\right]\right\} \\ \sigma_{22} &= \sum_{n=1}(\delta_n+1)r^{\delta_n}\,\mathrm{Re}\left\{\frac{1}{\mu_1-\mu_2}\left[g_{n1}\left(\varsigma_2^{\delta_n} - \varsigma_1^{\delta_n}\right) + g_{n2}\left(\mu_1\varsigma_2^{\delta_n} - \mu_2\varsigma_1^{\delta_n}\right)\right]\right\} \\ \sigma_{12} &= \sum_{n=1}(\delta_n+1)r^{\delta_n}\,\mathrm{Re}\left\{\frac{1}{\mu_1-\mu_2}\left[g_{n1}\left(\mu_1\varsigma_1^{\delta_n} - \mu_2\varsigma_2^{\delta_n}\right) + g_{n2}\mu_1\mu_2\left(\varsigma_1^{\delta_n} - \varsigma_2^{\delta_n}\right)\right]\right\} \end{aligned} \tag{34}$$

$$u_1 = \sum_{n=1} r^{\delta_n+1} \operatorname{Re}\left\{\frac{1}{\mu_1 - \mu_2}\left[g_{n1}\left(p_2\varsigma_2^{\delta_n+1} - p_1\varsigma_1^{\delta_n+1}\right) + g_{n2}\left(\mu_1 p_2 \varsigma_2^{\delta_n+1} - \mu_2 p_1 \varsigma_1^{\delta_n+1}\right)\right]\right\}$$

$$u_2 = \sum_{n=1} r^{\delta_n+1} \operatorname{Re}\left\{\frac{1}{\mu_1 - \mu_2}\left[g_{n1}\left(q_2\varsigma_2^{\delta_n+1} - q_1\varsigma_1^{\delta_n+1}\right) + g_{n2}\left(\mu_1 q_2 \varsigma_2^{\delta_n+1} - \mu_2 q_1 \varsigma_1^{\delta_n+1}\right)\right]\right\} \tag{35}$$

where $\varsigma_\alpha = \cos\theta + \mu_\alpha \sin\theta$.

For the leading-order term, $\delta_I = -1/2$, letting $\boldsymbol{g}_1 = \sqrt{2/\pi}\,\boldsymbol{k}$, the singular solution and T-stress term for the monoclinic solids can be written in the form:

$$\boldsymbol{\sigma} = \frac{1}{\sqrt{2\pi r}}\left(k_I \boldsymbol{\sigma}_I^{(1)} + k_{II}\boldsymbol{\sigma}_{II}^{(1)}\right) + T\begin{bmatrix}1\\0\\0\end{bmatrix} + O(r^{1/2}) \tag{36}$$

$$\boldsymbol{\varepsilon} = \frac{1}{\sqrt{2\pi r}}\left(k_I \boldsymbol{\varepsilon}_I^{(1)} + k_{II}\boldsymbol{\varepsilon}_{II}^{(1)}\right) + T\begin{bmatrix}s'_{11}\\s'_{12}\\s'_{16}\end{bmatrix} + O(r^{1/2}) \tag{37}$$

$$\boldsymbol{\varepsilon}_I^{(1)} = \boldsymbol{s}'\boldsymbol{\sigma}_I^{(1)}, \quad \boldsymbol{\varepsilon}_{II}^{(1)} = \boldsymbol{s}'\boldsymbol{\sigma}_{II}^{(1)}$$

$$\boldsymbol{u} = \sqrt{\frac{2r}{\pi}}\left(k_I \boldsymbol{u}_I^{(1)} + k_{II}\boldsymbol{u}_{II}^{(1)}\right) + Tr\begin{bmatrix}s'_{11}\cos\theta + s'_{16}\sin\theta\\ s'_{12}\sin\theta\end{bmatrix} + \omega r\begin{bmatrix}-\sin\theta\\ \cos\theta\end{bmatrix} + O(r^{3/2}) \tag{38}$$

where $\boldsymbol{\sigma}^{(n)}$, $\boldsymbol{u}^{(n)}$, $\cdots$, are the n-th order terms of the crack tip field; the subscripts I or II indicates the distribution associated with mode-I or mode-II. $\boldsymbol{k} = [k_{II},\, k_I]^T$; k_I and k_{II} are stress intensity factors for mode-I and mode-II respectively, ω is a constant representing rigid body rotation, and

$$(\sigma_{11})_I = \operatorname{Re}\left[\frac{\mu_1\mu_2}{\mu_1-\mu_2}\left(\frac{\mu_2}{\sqrt{\varsigma_2}} - \frac{\mu_1}{\sqrt{\varsigma_1}}\right)\right] \qquad (\sigma_{11})_{II} = \operatorname{Re}\left[\frac{1}{\mu_1-\mu_2}\left(\frac{\mu_2^2}{\sqrt{\varsigma_2}} - \frac{\mu_1^2}{\sqrt{\varsigma_1}}\right)\right]$$

$$(\sigma_{22})_I = \operatorname{Re}\left[\frac{1}{\mu_1-\mu_2}\left(\frac{\mu_1}{\sqrt{\varsigma_2}} - \frac{\mu_2}{\sqrt{\varsigma_1}}\right)\right], \qquad (\sigma_{22})_{II} = \operatorname{Re}\left[\frac{1}{\mu_1-\mu_2}\left(\frac{1}{\sqrt{\varsigma_2}} - \frac{1}{\sqrt{\varsigma_1}}\right)\right]$$

$$(\sigma_{12})_I = \operatorname{Re}\left[\frac{\mu_1\mu_2}{\mu_1-\mu_2}\left(\frac{1}{\sqrt{\varsigma_1}} - \frac{1}{\sqrt{\varsigma_2}}\right)\right] \qquad (\sigma_{12})_{II} = \operatorname{Re}\left[\frac{1}{\mu_1-\mu_2}\left(\frac{\mu_1}{\sqrt{\varsigma_1}} - \frac{\mu_2}{\sqrt{\varsigma_2}}\right)\right]$$

$$(u_1)_I = \mathrm{Re}\left[\frac{1}{\mu_1-\mu_2}\left(\mu_1 p_2\sqrt{\varsigma_2}-\mu_2 p_1\sqrt{\varsigma_1}\right)\right] \qquad (u_1)_{II} = \mathrm{Re}\left[\frac{1}{\mu_1-\mu_2}\left(p_2\sqrt{\varsigma_2}-p_1\sqrt{\varsigma_1}\right)\right]$$

$$(u_2)_I = \mathrm{Re}\left[\frac{1}{\mu_1-\mu_2}\left(\mu_1 q_2\sqrt{\varsigma_2}-\mu_2 q_1\sqrt{\varsigma_1}\right)\right], \qquad (u_2)_{II} = \mathrm{Re}\left[\frac{1}{\mu_1-\mu_2}\left(q_2\sqrt{\varsigma_2}-q_1\sqrt{\varsigma_1}\right)\right]$$

For the second-order term, $\delta_2 = 0$,

$$\boldsymbol{u}^{(2)} = Tr\begin{bmatrix} s'_{11}\cos\theta + s'_{16}\sin\theta \\ s'_{12}\sin\theta \end{bmatrix} + \omega r\begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix} \tag{39}$$

$$\sigma_{11}^{(2)} = T\ , \quad \sigma_{22}^{(2)} = \sigma_{12}^{(2)} = 0 \tag{40}$$

where ω represents the rigid body rotation, and

$$T = -i\,(g_{21}b + g_{22}d)\,, \quad \omega = -i\,s'_{11}(g_{21}d + g_{22}e) \tag{41}$$

Clearly the T-stress term is dependent on the material properties in anisotropic solids.

For orthotropic materials where x_1 and x_2 coincide with the material symmetry axes, the T-terms become

$$\boldsymbol{u}^{(2)} = Tr\begin{bmatrix} s'_{11}\cos\theta \\ s'_{12}\sin\theta \end{bmatrix} + \omega r\begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}, \quad \sigma_{11}^{(2)} = T$$

In isotropic solids, the second-order terms are

$$\boldsymbol{u}^{(2)} = \frac{Tr}{E}\begin{bmatrix} \cos\theta \\ -\nu\sin\theta \end{bmatrix} + \omega r\begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}, \quad \sigma_{11}^{(2)} = T \quad \text{for plane stress}$$

$$\boldsymbol{u}^{(2)} = \frac{T(1+\nu)r}{E}\begin{bmatrix} (1-\nu)\cos\theta \\ -\nu\sin\theta \end{bmatrix} + \omega r\begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}, \quad \sigma_{11}^{(2)} = T \quad \text{for plane strain}$$

For the degenerate materials, the crack tip fields are derived in Appendix A.

Auxiliary Fields associated with the Crack Tip Fields

Some auxiliary fields with higher-order singularities are needed in order to determine the coefficients in the expansion of the crack tip field by the use of conservation laws of elasticity and Betti's reciprocal theorem. Since the negative integers are also the eigenvalues of the crack problem which satisfy zero traction on the crack surfaces and satisfy the field governing equations of the anisotropic solids, the associated eigenfunctions can be conveniently used as auxiliary (pseduo) fields. Note that each eigenfunction with higher-order singularity has unbounded energy near the crack tip and thus corresponds to some concentrated source at the tip. This eigenfunction can be imagined as a self-equilibrated solution to the crack problem under some specified loads. These auxiliary fields may be obtained by choosing the values of n in eq. (33) as negative integers, that is,

$$
\begin{aligned}
\boldsymbol{u}^a &= \mathrm{Re}[\boldsymbol{A}\langle z^{\Delta_m+1}\rangle \boldsymbol{B}^{-1}\boldsymbol{h}_m] \\
\boldsymbol{\Phi}^a &= \boldsymbol{B}\langle z^{\Delta_m+1}\rangle \boldsymbol{B}^{-1}\boldsymbol{h}_m
\end{aligned}
\tag{42}
$$

$$
\sigma_{i1}^a = -\mathrm{Re}[\Phi_{i,2}], \quad \sigma_{i2}^a = \mathrm{Re}[\Phi_{i,1}] \tag{43}
$$

$$
\Delta_m = -m/2\,, m = 1, 3, 4, \cdots, \tag{44}
$$

$$
\boldsymbol{h}_m = [h_{m1}\,, h_{m2}]^T
$$

where $\boldsymbol{h}_m$ are arbitrary constant vectors. $\boldsymbol{h}_m$ is real for m = 1, 3, 5, ⋯; $\boldsymbol{h}_m$ is pure imaginary for m = 4, 6, 8,⋯, Utilizing eq. (22-23), the auxiliary fields, except $m = 4$, defined by eq. (42) yield zero resultant force on any contour Γ which encloses the crack tip shown in Fig. 2. The corresponding resultant moment about the x_3-axis, produced by the tractions acting on the contour Γ is also zero for the auxiliary fields in eq. (42). The special case of $\Delta_m = -2$ or $m = 4$ can be also directly explained from eq. (23). In this case, the function associated with $[0, h_{42}]^T$ corresponds to the particular solution for a crack under a concentrated moment about x_3-axis, $(-2\pi\; i\, h_{42})$, applied at the crack tip; the function associated with $[h_{41}\,, 0]^T$ corresponds to the homogeneous solution which satisfies zero concentrated force and moment at the crack tip.

From eq. (42)-(44), the stress and displacement components of the auxiliary fields are

$$
\begin{aligned}
\sigma_{11}^{(a)} &= (\Delta_m+1)r^{\Delta_m}\,\mathrm{Re}\left\{\frac{1}{\mu_1-\mu_2}\left[h_{m1}\left(\mu_2^2\varsigma_2^{\Delta_m}-\mu_1^2\varsigma_1^{\Delta_m}\right)+h_{m2}\mu_1\mu_2\left(\mu_2\varsigma_2^{\Delta_m}-\mu_1\varsigma_1^{\Delta_m}\right)\right]\right\} \\
\sigma_{22}^{(a)} &= (\Delta_m+1)r^{\Delta_m}\,\mathrm{Re}\left\{\frac{1}{\mu_1-\mu_2}\left[h_{m1}\left(\varsigma_2^{\Delta_m}-\varsigma_1^{\Delta_m}\right)+h_{m2}\left(\mu_1\varsigma_2^{\Delta_m}-\mu_2\varsigma_1^{\Delta_m}\right)\right]\right\} \\
\sigma_{12}^{(a)} &= (\Delta_m+1)r^{\Delta_m}\,\mathrm{Re}\left\{\frac{1}{\mu_1-\mu_2}\left[h_{m1}\left(\mu_1\varsigma_1^{\Delta_m}-\mu_2\varsigma_2^{\Delta_m}\right)+h_{m2}\mu_1\mu_2\left(\varsigma_1^{\Delta_m}-\varsigma_2^{\Delta_m}\right)\right]\right\}
\end{aligned}
\tag{45}
$$

$$
\begin{aligned}
u_1^a &= r^{\Delta_m+1}\,\mathrm{Re}\left\{\frac{1}{\mu_1-\mu_2}\left[h_{m1}\left(p_2\varsigma_2^{\Delta_m+1}-p_1\varsigma_1^{\Delta_m+1}\right)+h_{m2}\left(\mu_1p_2\varsigma_2^{\Delta_m+1}-\mu_2p_1\varsigma_1^{\Delta_m+1}\right)\right]\right\} \\
u_2^a &= r^{\Delta_m+1}\,\mathrm{Re}\left\{\frac{1}{\mu_1-\mu_2}\left[h_{m1}\left(q_2\varsigma_2^{\Delta_m+1}-q_1\varsigma_1^{\Delta_m+1}\right)+h_{m2}\left(\mu_1q_2\varsigma_2^{\Delta_m+1}-\mu_2q_1\varsigma_1^{\Delta_m+1}\right)\right]\right\}
\end{aligned}
\tag{46}
$$

In the following two sections, stress intensity factors, T-stress term, and coefficients of higher-order terms are determined using the J-integral and the Betti's reciprocal theorem including the use of above auxiliary fields.

J-Integral

(a) T-stress Term

J_i conservation laws (Knowles and Sternberg, 1972) for a plane anisotropic elasticity problem may be written as

$$J_k = \int_C \left(W n_k - t_i u_{i,k}\right) ds = 0, \quad k = 1, 2 \tag{47}$$

for an arbitrary closed contour C that encloses no defects, cracks, or material inhomogeneities. In the above equations, W is the strain energy density, $W = \sigma_{ij}\varepsilon_{ij}/2$, where σ_{ij} and ε_{ij} are the stresses and strains respectively; t_i are the traction components defined along the contour, $t_i = \sigma_{ij} n_j$; n_k are the unit outward vector normal to the contour path. Letting $k = 1$, the conservation law is reduced to the Rice's path-independent J-integral or the rate of energy release rate per unit of crack extension along the x_1 - axis, which is given by

$$J = \int_\Gamma \left(\sigma^T \varepsilon n_1 / 2 - t^T u_{,1}\right) ds \tag{48}$$

where Γ is an arbitrary path which starts on the straight lower face of the crack, encloses the crack tip and ends on the upper straight face with the positive direction in a counterclockwise direction shown in Fig. 2. Here, the crack surfaces are assumed to be traction-free.

Consider a cracked body under the two-dimensional deformation. The components of stress, strain, and displacement are represented by σ_{ij}, ε_{ij}, and u_i, respectively. As $r \to 0$, the asymptotic fields including the constant T-stress terms are given before. Now the coefficients of the T-stress terms and third-order terms are derived using conservation laws.

In general, for the purpose of determining the coefficients g_n of the term r^{δ_n} ($\delta_n \geq -1/2$) in the actual crack tip stress field, we may employ the J-integral method and follow the following procedure:

(i) find an auxiliary (pseudo) field that has singularity $\sigma_{ij}^a \sim r^{-\delta_n - 1}$ as $r \to 0$. It is convenient to select auxiliary stress field which gives zero traction on the crack surfaces and contains only the stress singular term $r^{-\delta_n - 1}$;

(ii) superimpose the actual field (the mixed-mode boundary value problem, in general) on the auxiliary field and represent the J-integral for the superimposed state as

$$J_s = J + J_a + J_M \tag{49}$$

where

$$J_s = \int_\Gamma [(\sigma + \sigma^a)^T (\varepsilon + \varepsilon^a) n_1 / 2 - (t + t^a)^T (u_{,1} + u_{,1}^a)] ds$$

$$J_a = \int_\Gamma [(\sigma^a)^T \varepsilon^a \, n_1 / 2 - (t^a)^T u_{,1}^a] \, ds$$

and

$$\begin{aligned} J_M &= J_s - J - J_a \\ &= \int_\Gamma \{[\sigma^T \varepsilon^a + (\sigma^a)^T \varepsilon] n_1 / 2 - t^T u_{,1}^a - (t^a)^T u_{,1}\} ds \end{aligned}$$

$$= \int_\Gamma [\sigma^T \varepsilon^a n_1 - t^T u^a_{,l} - (t^a)^T u_{,1}] ds \quad (50)$$

$$= \int_\Gamma (\sigma_{ij} u^a_{i,j} n_1 - t_i u^a_{i,1} - t^a_i u_{i,1}) ds$$

where the superscript or subscript "a" denote quantities referred to the auxiliary field; J_s is the J-integral for the superimposed state; J for the actual state; and J_a for the auxiliary field and J_M is the interaction integral. In the sequel, we assume that the J-integral is path-independent for both the actual field and the selected auxiliary fields, denoted by J and J_a. Then the integral J_s for the superimposed state, thus J_M, is also path-independent. If the auxiliary fields given by eq. (42) are used, it is readily proved that

$$\begin{aligned} J_a &\neq 0, \quad \text{for } \Delta_m = -1/2; \\ J_a &= 0, \quad \text{for } \Delta_m < -1/2 \end{aligned} \quad (51)$$

(iii) evaluate J_M as $\Gamma \to 0$. For simplicity, Γ may be taken as a circle with radius r, as $r \to 0$, the only terms in the integrand that contribute to J_M are the cross terms between r^{δ_n} in the actual stress field and the auxiliary stress term with order $r^{-\delta_n - 1}$;

(iv) carry out the routine manipulation, the exact expression for J_M can be obtained as $J_M = J_M(g_n)$ when $r \to 0$;

(v) evaluate J_M for a finite contour Γ using the computed actual field and the exact auxiliary solution; and determine the coefficients g_n from the value of J_M and the expression of $J_M = J_M(g_n)$ as $r \to 0$.

In extracting the T-stress in eq. (33) or g_2, we make use of another auxiliary field, that is the solution to a point force f (per unit thickness) applied at the crack tip. Under the load, $\sigma^a_{ij} \propto r^{-1}$. Note that the point force f must be resisted by traction t applied to some boundary C in achieving equilibrium. In the Stroh formalism (Ting, 1996), the real form solutions due to the point-force application can be written as

$$2u^a = -\left[\frac{\ln r}{\pi}\mathbf{I} + S(\theta)\right] h \quad (52)$$

$$2\phi^a = L(\theta) h \quad (53)$$

where $h = L^{-1} f$, $f = [f_1, f_2]^T$

$$S(\theta) = \frac{2}{\pi} \mathrm{Re}[A \langle \ln(\cos\theta + \mu \sin\theta) \rangle B^T]$$

$$L(\theta) = -\frac{2}{\pi} \mathrm{Re}[B \langle \ln(\cos\theta + \mu \sin\theta) \rangle B^T]$$

$x_1 = r\cos\theta$, $x_2 = r\sin\theta$.

It assumes the values

$$\ln(\cos\theta + \mu_\alpha \sin\theta) = \begin{cases} 0 & \theta = 0 \\ \pm i\pi & \theta = \pm\pi \end{cases}$$

In a cylindrical coordinate system (r, θ, x_3), let t_r and t_θ be the traction vectors on a cylindrical surface r = constant and on a radial plane θ = constant, then

$$t_r = -\frac{1}{r}\phi^a_{,\theta}\ ,\ \ t_\theta = \phi^a_{,r} \tag{54}$$

$$\sigma^a_r = n^T t_r,\ \ \sigma^a_{r\theta} = m^T t_r = n^T t_\theta,\ \ \sigma^a_\theta = m^T t_\theta$$

where $n^T = [\cos\theta, \sin\theta]$ and $m^T = [-\sin\theta, \cos\theta]$.

It follows from eq. (52) and (53) that

$$t_\theta = \phi_{,r} = 0,\ \ \sigma^a_\theta = \sigma^a_{r\theta} = 0$$

$$\sigma^a_r = -n^T\phi^a_{,\theta}/r$$

or

$$\begin{aligned} \sigma^a_r = &-\frac{f_1}{2\pi r}\mathrm{Im}\left\{\frac{[(\mu_1\mu_2-1)\sin\theta + (\mu_1+\mu_2)\cos\theta]\cos^2\theta - \sin\theta}{\varsigma_1\varsigma_2}\right\} \\ &-\frac{f_2}{2\pi r}\mathrm{Im}\left\{\frac{[(\mu_1\mu_2-1)\cos\theta - (\mu_1+\mu_2)\sin\theta]\sin^2\theta + \mu_1\mu_2\cos\theta}{\varsigma_1\varsigma_2}\right\} \end{aligned} \tag{55}$$

and

$$\begin{aligned} u^a = &-\frac{f_1}{2\pi}\left\{s'_{11}\ln r\begin{bmatrix} b \\ d \end{bmatrix} - \mathrm{Im}\left[\frac{1}{\mu_1-\mu_2}\begin{bmatrix} p_2\ln\varsigma_2 - p_1\ln\varsigma_1 \\ q_2\ln\varsigma_2 - q_1\ln\varsigma_1 \end{bmatrix}\right]\right\} \\ &-\frac{f_2}{2\pi}\left\{s'_{11}\ln r\begin{bmatrix} d \\ e \end{bmatrix} - \mathrm{Im}\left[\frac{1}{\mu_1-\mu_2}\begin{bmatrix} \mu_1 p_2\ln\varsigma_2 - \mu_2 p_1\ln\varsigma_1 \\ \mu_1 q_2\ln\varsigma_2 - \mu_2 q_1\ln\varsigma_1 \end{bmatrix}\right]\right\} \end{aligned} \tag{56}$$

For isotropic materials,

$$\sigma^a_{r\theta} = \sigma^a_\theta = 0,\quad \sigma^a_r = -\frac{1}{\pi r}(f_1\cos\theta + f_2\sin\theta) \tag{57}$$

$$u^a = -\frac{1}{8\pi G}\left\{\begin{bmatrix} (\kappa+1)\ln r & -(\kappa-1)\theta \\ (\kappa-1)\theta & (\kappa+1)\ln r \end{bmatrix} - \begin{bmatrix} -2\sin^2\theta & \sin 2\theta \\ \sin 2\theta & 2\sin^2\theta \end{bmatrix}\right\}\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \tag{58}$$

$$\kappa = 3 - 4\nu$$

By superimposing the actual field on the auxiliary field, and using the path-independent J-integral for the monoclinic elastic cracked body, it can be proved that

$$J_s = J + T s'_{11} f_1 + \omega f_2$$

$$J_M = T s'_{11} f_1 + \omega f_2 \tag{59}$$

Note that because $\sigma_{ij}^a \propto r^{-1}$, $u_{i,j}^a \propto r^{-1}$

$$J_a = \int_\Gamma \left(W^a n_1 - t_i^a u_{i,1}^a\right) ds = 0$$

Kfouri (1986) used the method to calculate the T-term for isotropic materials. Wang *et al.* (1980) and Wu (1989) applied the J-integral to determine the stress intensity factors for rectilinear anisotropic solids and general anisotropic materials respectively. In this report, the method is extended to determine all the coefficients in the crack tip field expansion for monoclinic materials. From eq. (59), it follows that

$$T = \frac{J_M\big|_{f_2=0}}{s'_{11} f_1} \quad \text{and} \quad \omega = \frac{J_M\big|_{f_1=0}}{f_2}$$

Detailed proof of eq. (59) is given below:

For general anisotropic linear elastic solids, we have the following relations

$$\sigma_{ij}\varepsilon_{ij}^a = c_{ijkl}\varepsilon_{kl}\varepsilon_{ij}^a = \sigma_{ij}^a\varepsilon_{ij} \quad \text{and} \quad \sigma_{ij}\varepsilon_{ij}^a = \sigma_{ij}u_{i,j}^a$$

where $c_{ijkl} = c_{klij}$. From eq. (50),

$$J_M = \int_\Gamma (\sigma_{ij}\, u_{i,j}^a\, n_1 - t_i\, u_{i,1}^a - t_i^a\, u_{i,1})\, ds = \int_\Gamma (\sigma_{i2} \frac{du_i^a}{ds} - t_i^a\, u_{i,1})\, ds \tag{60}$$

where du_i^a / ds is the tangential derivative of u_i^a. As $r \to 0$, we evaluate J_M and note that the only terms that contribute to J_M are the cross terms between T and f. After substituting these fields into the integral and performing the routine algebra, the integral J_M may be evaluated as

$$\begin{aligned} J_M &= \lim_{\Gamma \to 0} \int_\Gamma (\sigma_{i2} \frac{du_i^a}{ds} - t_i^a u_{i,1})\, ds \\ &= \int_\Gamma (\sigma_{i2}^{(2)} \frac{du_i^a}{ds} - t_i^a u_{i,1}^{(2)})\, ds \\ &= -\int_\Gamma u_{i,1}^{(2)}\, t_i^a\ ds \\ &= -\int_\Gamma (\boldsymbol{u}_{,1}^{(2)})^T\, \boldsymbol{t}^a\ ds \\ &= -(\boldsymbol{u}_{,1}^{(2)})^T \int_\Gamma \boldsymbol{t}^a\ ds \\ &= (\boldsymbol{u}_{,1}^{(2)})^T \boldsymbol{f} \end{aligned} \tag{61}$$

In the above derivation, eq. (22) has been used. From eq. (33) and (39) via $(12)_2$,

$$\boldsymbol{u}_{,1}^{(2)} = -i\,\boldsymbol{L}^{-1}\,\boldsymbol{g}_2 = \begin{bmatrix} Ts'_{11} \\ \omega \end{bmatrix} \tag{62}$$

Insertion eq. (62) into (61) leads to

$$J_M = -i\boldsymbol{g}_2^T\boldsymbol{L}^{-1}\boldsymbol{f} = -i\boldsymbol{f}^T\boldsymbol{L}^{-1}\boldsymbol{g}_2 \tag{63}$$

and

$$J_M = T\,s'_{11}f_1 + \omega f_2 \tag{64}$$

Eq. (63) and (64) can be used to calculate $\boldsymbol{g}_2$ and T. As $\boldsymbol{f}$ is arbitrary, it is convenient to choose $\boldsymbol{f}$ to be the following values

$$[1, 0]^T \equiv \boldsymbol{e}_1, \quad [0, 1]^T \equiv \boldsymbol{e}_2$$

respectively. Note that $\boldsymbol{e}_k$ has a dimension force/length. Correspondingly, eq. (63) yields two linear equations and they are, in matrix notation,

$$\tilde{\boldsymbol{J}}_M = -i\,\boldsymbol{L}^{-1}\boldsymbol{g}_2$$

where

$$\tilde{\boldsymbol{J}}_M = [J_M^{(1)}, J_M^{(2)}]^T$$

and $J_M^{(k)}$ is the value of J_M when $\boldsymbol{f} = \boldsymbol{e}_k$. Therefore,

$$\boldsymbol{g}_2 = i\,\boldsymbol{L}\tilde{\boldsymbol{J}}_M \tag{65}$$

Using eq. (64), The T-stress and ω can be obtained as

$$T = J_M^{(1)}/s'_{11}, \quad \omega = J_M^{(2)} \tag{66}$$

However, the choice of auxiliary field is not unique. For instance, we may choose the auxiliary fields as the sum of the field $(\boldsymbol{\sigma}', \boldsymbol{u}')$ for the point force $\boldsymbol{f}$ and a field $(\boldsymbol{\sigma}'', \boldsymbol{u}'')$ which is a known solution for the same cracked body under some loads on the outer boundary, then

$$\begin{aligned} \boldsymbol{\sigma}^a &= \boldsymbol{\sigma}' + \boldsymbol{\sigma}'' \\ \boldsymbol{u}^a &= \boldsymbol{u}' + \boldsymbol{u}'' \end{aligned},$$

Superimposing the actual field on the new auxiliary field, we have

$$J_s[\sigma + \sigma^a] = J[\sigma] + J_a[\sigma^a] + J_M[\sigma, \sigma^a] \tag{67}$$

From the definition of J_M,

$$J_M[\sigma, \sigma^a] = J_M[\sigma, \sigma' + \sigma''] = J_M[\sigma, \sigma'] + J_M[\sigma, \sigma''] \tag{68}$$

$$J_M[\sigma, \sigma''] = J_s[\sigma + \sigma''] - J[\sigma] - J[\sigma''] \tag{69}$$

In elastic materials, J is equal to the energy release rate G in the absence of body forces and dislocations and is related to the stress intensity factors (see Appendices B and C) through

$$G=\frac{1}{2}\boldsymbol{k}^T\boldsymbol{L}^{-1}\boldsymbol{k} \tag{70}$$

Therefore, eq. (69) leads to

$$J_M[\sigma,\sigma'']=\boldsymbol{k}^T\boldsymbol{L}^{-1}\boldsymbol{k}'' \tag{71}$$

where $\boldsymbol{k}$ is the stress intensity factor for the state $(\sigma,\boldsymbol{u})$, and $\boldsymbol{k}''$ for $(\sigma'',\boldsymbol{u}'')$.

Using eq. (71) and (64), eq. (68) can be expressed by

$$J_M[\sigma,\sigma^a]=T\,s'_{11}f_1+\omega f_2+\boldsymbol{k}^T\boldsymbol{L}^{-1}\boldsymbol{k}'' \tag{72}$$

Inserting eq. (72) into (67) and utilizing eq. (13) yield

$$\begin{aligned}J_s[\sigma+\sigma^a]=&\,J[\sigma]+J_a[\sigma^a]+T\,s'_{11}f_1+\omega f_2\\&+s'_{11}\,\mathrm{Im}\big[(\mu_1+\mu_2)k_{II}k''_{II}+\mu_1\mu_2(k_{II}k''_I+k''_{II}k_I)+\mu_1\mu_2(\bar{\mu}_1+\bar{\mu}_2)k_Ik''_I\big]\end{aligned} \tag{73}$$

For isotropic case,

$$J_s[\sigma+\sigma^a]=J[\sigma]+J_a[\sigma^a]+T\,s'_{11}f_1+\omega f_2++2s'_{11}(k_{II}k''_{II}+k_Ik''_I) \tag{74}$$

For plane strain under mode-I loading, if $f=(f_1,0)$, then eq. (64) and (74) reduce to the results given by Kfouri (1986) and $s'_{11}=(1-\nu^2)/E$

(b) The third term

The third-term coefficient in the asymptotic solution can be also obtained from the J-integral method. An auxiliary field with singularity $\sigma^a_{ij}\sim O(r^{-3/2})$ can be introduced by selecting $m=3$ in eq. (42). By superimposing the actual field (the mixed mode boundary value problem) on the auxiliary field, the interaction integral J_M may be evaluated as

$$J_M=-\frac{3}{2}\pi\boldsymbol{h}_3^T\boldsymbol{L}^{-1}\boldsymbol{g}_3 \tag{75}$$

Following in a similar manner,

$$\boldsymbol{g}_3=-\frac{2}{3\pi}\boldsymbol{L}\tilde{\boldsymbol{J}}_M \tag{76}$$

where $\tilde{\boldsymbol{J}}_M=[J_M^{(1)},J_M^{(2)}]^T$ and $J_M^{(k)}$ is the value of J_M when $\boldsymbol{h}_3=\boldsymbol{e}_k$, $(k=1,2)$. $\boldsymbol{e}_k$ has a dimension of force/(length)$^{1/2}$.

In general, superimposing of an auxiliary field with $\sigma^a_{ij}\propto r^{\Delta_n}$ on the actual field and applying the J-integral to this combined state with derivations proved in Appendix D for $n\neq 2$, we can get the interaction integral J_M denoted by J_{Mn}, that is

$$J_{Mn}=-2\pi\delta_n(\delta_n+1)\boldsymbol{h}_n^T\boldsymbol{L}^{-1}\boldsymbol{g}_n,\quad n=1,3,4,5,\cdots. \tag{77}$$

Then

$$g_n = \begin{cases} -\dfrac{L\tilde{J}_{Mn}}{2\pi\delta_n(\delta_n+1)}, & n = 1, 3, 5, \cdots \\ \dfrac{iL\tilde{J}_{Mn}}{2\pi\delta_n(\delta_n+1)}, & n = 4, 6, 8, \cdots \end{cases} \tag{78}$$

where

$$\tilde{J}_{Mn} = [J_{Mn}^{(1)}, J_{Mn}^{(2)}]^T$$

and $J_{Mn}^{(k)}$ is the value of J_{Mn} when

$$h_n = \begin{cases} e_k, & n = 1, 3, 5, \cdots \\ ie_k, & n = 4, 6, 8, \cdots \end{cases}$$

Here, e_k possesses dimension force / (length)$^{1-\delta_n}$. For the first singular term, introducing stress intensity factors, $k = [k_{II}, k_I]^T = \sqrt{\pi/2}\, g_1$ for the actual field and $k^a = [k_{II}^a, k_I^a]^T = \sqrt{\pi/2}\, h_I$ for the auxiliary field, J_{MI} and k can be rewritten from eq. (77), (78) as

$$J_{M1} = (k^a)^T L^{-1} k$$

$$k = L\hat{J}_{M1} \tag{79}$$

where $\hat{J}_{M1} = [J_{M1}^{(1)}, J_{M1}^{(2)}]^T$ and $J_{M1}^{(k)}$ is the value of J_{M1} when $k^a = e_k$.

Betti's Reciprocal Theorem

(a) T-stress Term

For a linear elastic plane problem, Betti's reciprocal theorem can be stated as

$$\int_C (t \cdot u^a - t^a \cdot u)\, ds = 0 \tag{80}$$

where C is an any closed contour enclosing a simple connected region in the elastic body; u is the displacement vector and t the traction on C corresponding to the solution of any particular elastic boundary value problem for the elastic body; u^a and t^a are corresponding quantities of the solution of any other problem for the body. Considering a crack in an anisotropic linear elastic material, and suppose the crack surfaces are free of tractions for both elastic states. If the closed contour C encloses the crack tip and extends along the crack surfaces, then it can be shown that the integral

$$I = \int_\Gamma (t \cdot u^a - t^a \cdot u)\, ds \tag{81}$$

is path independent where Γ is an any path which starts from the lower crack face and ends on the upper. Let (t, u) be an actual state for the crack under consideration, then eq. (81) provides sufficient information for determining the amplitude for each term in the asymptotic crack-tip fields if proper auxiliary solutions ($\boldsymbol{t}^a$, $\boldsymbol{u}^a$) are provided. In this section the Betti's reciprocal work contour integral is used for computing stress intensity factors, T-stress and other higher-order coefficients for monoclinic materials. The procedure can be evaluated from the analysis as follows.

For determining the coefficients $\boldsymbol{g}_n$ of the term r^{δ_n} ($\delta_n \geq -1/2$) in the actual crack tip stress field, an auxiliary (pseudo) field with $\sigma_{ij}^a \propto r^{-\delta_n - 2}$ or $u_i^a \propto r^{-\delta_n - 1}$ can be chosen. As $r \to 0$, take a Γ as a circle around the crack tip and evaluate integral I. When $r \to 0$, the only product between $\boldsymbol{g}_n$ and the auxiliary terms in the integrand given above can contribute to the integral I. Therefore, the expression for $I = I(\boldsymbol{g}_n)$ can be obtained as $r \to 0$. The value of I for a finite contour Γ shown in Fig. 2 is available from the numerical solutions for $\boldsymbol{t}$ and $\boldsymbol{u}$ of the boundary value problems and the exact auxiliary solution. The $\boldsymbol{g}_n$ can be computed from the expression for $I = I(\boldsymbol{g}_n)$ and the value of I.

To determine the T-stress or $\boldsymbol{g}_2$ for the crack-tip field from eq. (34), the auxiliary elastic field with stress singularity $\sigma_{ij}^a \propto r^{-2}$ as $r \to 0$ is used and can be obtained from the eq. (42) by choosing $m = 4$, that is, in Stroh formalism,

$$\begin{aligned} \boldsymbol{u}^a &= \mathrm{Re}[\boldsymbol{A}\langle z^{-1}\rangle \boldsymbol{B}^{-1}\boldsymbol{h}_4] \\ \Phi^a &= \boldsymbol{B}\langle z^{-1}\rangle \boldsymbol{B}^{-1}\boldsymbol{h}_4 \end{aligned} \tag{82}$$

The moment about x_3-axis applied at the crack tip, using eq. (23) and (82), is given by

$$M = -2\pi\, i\, h_{42}$$

When Γ shrinks to the crack tip, it is clear that only those parts of the integrand in eq. (81) which behaves like O($1/r$) as $r \to 0$ can contribute this portion of the integral. Substituting these fields of the two states into eq. (81), performing the integration for the circle surrounding the crack tip and evaluating the results in the limit of vanishing radius, the results may be derived, and

$$I = \lim_{r \to 0} \int_{\Gamma} \left(\boldsymbol{t} \cdot \boldsymbol{u}^a - \boldsymbol{t}^a \cdot \boldsymbol{u} \right) ds = -2\pi\, \boldsymbol{h}_4^T \boldsymbol{L}^{-1} \boldsymbol{g}_2 \tag{83}$$

$$\boldsymbol{g}_2 = \frac{i}{2\pi} \boldsymbol{L} \tilde{\boldsymbol{I}} \tag{84}$$

where $\tilde{I} = [I^{(1)}, I^{(2)}]$ and $I^{(k)}$ is the value of I when $\boldsymbol{h}_4 = i\,\boldsymbol{e}_k$, ($k$ = 1, 2). (Dimension of $\boldsymbol{e}_k$ is force).

From eq. (13), (41), and (84),

$$T = \frac{I^{(1)}}{2\pi\, s_{11}'}, \quad \omega = \frac{I^{(2)}}{2\pi} \tag{85}$$

For isotropic materials, the auxiliary displacement vector and stress functions can be modified as

$$\mathbf{\Phi}^a = \frac{\sin\theta}{r}\begin{bmatrix} 1+\cos 2\theta & \sin 2\theta \\ \sin 2\theta & 1-\cos 2\theta \end{bmatrix}\mathrm{Im}[\boldsymbol{h}_4] \tag{86}$$

$$\begin{bmatrix} \sigma_{11}^a \\ \sigma_{22}^a \\ \sigma_{12}^a \end{bmatrix} = \frac{1}{r^2}\begin{bmatrix} -2\cos 3\theta\cos\theta & -\sin 4\theta \\ -2\sin 3\theta\sin\theta & 2(\cos 2\theta - 1)\sin 2\theta \\ -\sin 4\theta & -2\sin 3\theta\sin\theta \end{bmatrix}\mathrm{Im}[\boldsymbol{h}_4] \tag{87}$$

$$\begin{bmatrix} u_1^a \\ u_2^a \end{bmatrix} = -\frac{1}{4Gr}\begin{bmatrix} -(\kappa+1)\cos\theta + 2\sin 2\theta\sin\theta & -(\kappa-1)\sin\vartheta - \cos 2\theta\sin\theta \\ (\kappa-1)\sin\theta - \cos 2\theta\sin\theta & -(\kappa+1)\cos\theta - 2\sin 2\theta\sin\theta \end{bmatrix}\mathrm{Im}[\boldsymbol{h}_4] \tag{88}$$

Then the path-independent integral I has the same form as eq. (83). Eqs. (84) and (85) are still valid.

(b) The Third Term

The coefficients of the third term in the eigenfunction expansion of the stress field can also be obtained using Betti's theorem. Selecting $m = 5$ in eq. (42), an auxiliary field with stress singularity $\sigma_{ij}^a \sim r^{-5/2}$ desired for this purpose can be obtained. Applying the Betti theorem of reciprocity to the actual field and the auxiliary field and evaluating the integral I as $\Gamma \rightarrow 0$ near the crack tip, we obtain

$$I = -3\pi\, \boldsymbol{h}_5^T \boldsymbol{L}^{-1} \boldsymbol{g}_3 \tag{89}$$

Eqs. (89) will be used to calculate $\boldsymbol{g}_3$ for mixed-mode problem when the two proper auxiliary field solutions are provided. g_3 can be expressed in the form

$$\boldsymbol{g}_3 = -\frac{1}{3\pi}\boldsymbol{L}\tilde{\boldsymbol{I}} \tag{90}$$

Applying Betti's reciprocal theorem to the actual fields and auxiliary fields with $\sigma_{ij}^a \propto r^{\Delta_{n+2}}$, the path independent I denoted by I_{n+2} can be evaluated by

$$I_{n+2} = -2\pi\,(\delta_n + 1)\, \boldsymbol{h}_{n+2}^T\, \boldsymbol{L}^{-1}\, \boldsymbol{g}_n \tag{91}$$

It follows from eq. (91)

$$\boldsymbol{g}_n = \begin{cases} -\dfrac{\boldsymbol{L}\tilde{\boldsymbol{I}}_{n+2}}{2\pi(\delta_n + 1)}, & n = 1, 3, 5, \cdots \\[2ex] \dfrac{i\boldsymbol{L}\tilde{\boldsymbol{I}}_{n+2}}{2\pi(\delta_n + 1)}, & n = 2, 4, 6, \cdots \end{cases} \tag{92}$$

where

$$\boldsymbol{I}_j = [I_j^{(1)},\ \ I_j^{(2)}]^T$$

and $I_j^{(k)}$ is the value of I_j when

$$h_n = \begin{cases} e_k, & n = 1, 3, 5, \cdots \\ i e_k, & n = 2, 4, 6, \cdots \end{cases}$$

Here, e_k possesses dimension $\text{force} \times (\text{length})^{\delta_n}$. The detailed proof is shown in Appendix E.

For the first singular term from eq. (91) and (92), I_3 and k can be written as

$$I_3 = -\pi\, h_3^T L^{-1} g_1 = -\sqrt{2\pi}\; h_3^T L^{-1}\, k \tag{93}$$

$$k = -\frac{1}{\sqrt{2\pi}} L\, \tilde{I}_3 \tag{94}$$

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Appendix A

Deformation Field for Degenerate Materials

The solutions for the nongenerate materials can be modified so that they can be applied for degenerate materials. We write the general solutions for nongenerate materials as

$$u = \mathrm{Re}\left[AB^{-1}X(z)g\right]$$
$$\Phi = X(z)g$$

where

$$X(z) = B\langle f(z)\rangle B^{-1}$$

In the limit $\mu_1 = \mu_2 = \mu$, it can be proved that the matrix reduces to

$$X(z) = f(z)I + x_2 f'(z)V$$

where $z = x_1 + \mu x_2$;

$$V = \begin{bmatrix} \mu & \mu^2 \\ -1 & -\mu \end{bmatrix}$$

Hence, for isotropic materials, we can obtain

$$u = \frac{1}{2G}\mathrm{Re}\left\{\left[f(z)E - ix_2 f'(z)V\right]g\right\}$$

$$\Phi = \left[f(z)I + x_2 f'(z)V\right]g$$

where

$$E = \frac{1}{s'_{11} - s'_{12}}\begin{bmatrix} -i2s'_{11} & s'_{11} + s'_{12} \\ -(s'_{11} + s'_{12}) & -i2s'_{11} \end{bmatrix}, \qquad V = \begin{bmatrix} i & -1 \\ -1 & -i \end{bmatrix}$$

For a crack in isotropic materials, choosing $f(z) = z^{\delta+1}$ and performing routine manipulations, the crack tip fields can be represented as

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \sum_n (\delta_n + 1) r^{\delta_n} \mathrm{Re}\left\{\begin{bmatrix} 2e^{i(\delta_n\theta - \pi/2)} + \delta_n e^{i(\delta_n - 1)\theta}\sin\theta & e^{i\delta_n\theta} + \delta_n e^{i[\pi/2 + (\delta_n - 1)\theta]}\sin\theta \\ -\delta_n e^{i(\delta_n - 1)\theta}\sin\theta & e^{i\delta_n\theta} - \delta_n e^{i[\pi/2 + (\delta_n - 1)\theta]}\sin\theta \\ e^{i\delta_n\theta} + \delta_n e^{i[\pi/2 + (\delta_n - 1)\theta]}\sin\theta & -\delta_n e^{i(\delta_n - 1)\theta}\sin\theta \end{bmatrix}\begin{bmatrix} g_{n1} \\ g_{n2} \end{bmatrix}\right\}$$

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \sum_n \frac{r^{\delta_n + 1}}{4G}\mathrm{Re}\left\{\begin{bmatrix} (\kappa + 1)e^{-i[\pi/2 - (\delta_n + 1)\theta]} + 2(\delta_n + 1)e^{i\delta_n\theta}\sin\theta & (\kappa - 1)e^{i(\delta_n + 1)\theta} + 2(\delta_n + 1)e^{i(\pi/2 + \delta_n\theta)}\sin\theta \\ -(\kappa - 1)e^{i(\delta_n + 1)\theta} + 2(\delta_n + 1)e^{i(\pi/2 + \delta_n\theta)}\sin\theta & (\kappa + 1)e^{-i[\pi/2 - (\delta_n + 1)\theta]} - 2(\delta_n + 1)e^{i\delta_n\theta}\sin\theta \end{bmatrix}\begin{bmatrix} g_{n1} \\ g_{n2} \end{bmatrix}\right\}$$

where $\kappa = 3 - 4\nu$ for plane strain; $\kappa = (3 - \nu)/(1 + \nu)$ for plane stress.

Appendix B

J-integral

The J-integral with a contour Γ around a crack shown in Fig. 2 can be written as

$$\begin{aligned} J &= \int_\Gamma (W n_1 - t_i u_{i,1})\, ds \\ &= \frac{1}{2}\int_\Gamma (\sigma_{i2} \frac{du_i}{ds} - t_i u_{i,1})\, ds \\ &= \frac{1}{2}\int_\Gamma (\boldsymbol{\tau}_2^T\, d\boldsymbol{u} - \boldsymbol{t}^T \boldsymbol{u}_{,1})\, ds \end{aligned} \tag{B1}$$

Substituting the expressions for $\boldsymbol{\tau}_2 = \boldsymbol{\phi}_{,1}$, $\boldsymbol{u}$ and $\boldsymbol{t}$ and using the orthogonality relations

$$\boldsymbol{B}^T\boldsymbol{A} + \boldsymbol{A}^T\boldsymbol{B} = \mathbf{I}, \quad \boldsymbol{B}^T\overline{\boldsymbol{A}} + \boldsymbol{A}^T\overline{\boldsymbol{B}} = \mathbf{0} \tag{B2}$$

J can be further expressed by

$$J = \frac{1}{4}\mathrm{Re}\left[\sum_m\sum_n \boldsymbol{g}_m^T\, \boldsymbol{B}^{-T}\left\langle(\delta_m + 1)(\delta_n + 1)\int_\Gamma z^{\delta_m+\delta_n} dz\right\rangle \boldsymbol{B}^{-1}\, \boldsymbol{g}_n\right] \tag{B3}$$

It can be readily shown that

$$\int_\Gamma z_\alpha^{\delta_m+\delta_n}\, dz_\alpha = \ln(r_2/r_1) + i\,2\pi \qquad \delta_m = \delta_n = -\frac{1}{2} \tag{B4}$$

The term contributing to J is the first term and

$$J = \frac{1}{16}\mathrm{Re}\left\{[\ln(r_2/r_1) + i\,2\pi]\boldsymbol{g}_1^T\, \boldsymbol{B}^{-T}\boldsymbol{B}^{-1}\, \boldsymbol{g}_1\right\} \tag{B5}$$

By the use of the identity

$$\boldsymbol{B}^{-T}\boldsymbol{B}^{-1} = -2i\,\boldsymbol{L}^{-1} \tag{B6}$$

eq. (B5) becomes

$$J = \frac{1}{8}\mathrm{Im}\left\{[\ln(r_2/r_1) + i\,2\pi]\boldsymbol{g}_1^T\, \boldsymbol{L}^{-1}\, \boldsymbol{g}_1\right\} = \frac{1}{4}\boldsymbol{g}_1^T\, \boldsymbol{L}^{-1}\, \boldsymbol{g}_1 \tag{B7}$$

The above derivation has used the real values of $\boldsymbol{g}_1$ and $\boldsymbol{L}^{-1}$. Since

$$\boldsymbol{g}_1 = \sqrt{\frac{2}{\pi}}\,\boldsymbol{k} \tag{B8}$$

we have

$$J = \frac{1}{2}\boldsymbol{k}^T\, \boldsymbol{L}^{-1}\, \boldsymbol{k} \tag{B9}$$

Appendix C

Energy Release Rate G

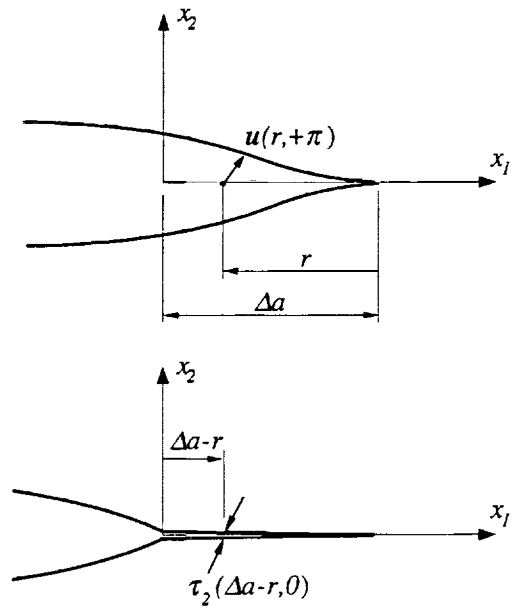

Fig. 3 Crack opening displacements for the extended crack and stress distribution ahead of the crack tip prior to crack extension.

Irwin (1957) realized that if a crack extends by a small amount Δa , the energy absorbed in the process is equal to the work required to close the crack to its original length. Using a polar coordinate system with the origin at the extended crack tip, the energy released for a unit area to extend is written as

$$\mathrm{G} = \lim_{\Delta a \to 0} \frac{1}{2\Delta a} \int_0^{\Delta a} \sigma_{i2}(\Delta a - r, 0)\,[u_i(r,\pi) - u_i(r,-\pi)]\,dr \qquad \text{(C1)}$$

where G is the energy release rate; Δa is the crack extension at the crack tip; $\sigma_{i2}(\Delta a - r,0)$ are the stresses prior to the crack extension; $u_i(r,\pm\pi)$ are the displacements due to extension. (C1) can be rewritten by

$$\mathrm{G} = \lim_{\Delta a \to 0} \frac{1}{2\Delta a} \int_0^{\Delta a} \tau_2^T(\Delta a - r, 0)\,[\boldsymbol{u}(r,\pi) - \boldsymbol{u}(r,-\pi)]\,dr \qquad \text{(C2)}$$

By Stroh formulation, the stress vector τ_2 is written as

$$\tau_2(r.\theta) = \phi_{,1} = \mathrm{Re} \sum_{m=1,2,3,\cdots} \boldsymbol{B}\left\langle(\delta_m + 1) z^{\delta_m}\right\rangle \boldsymbol{B}^{-1} \boldsymbol{g}_m$$

Along the crack plane, $\theta = 0$,

$$\begin{aligned}\tau_2(\Delta a - r, 0) &= \sum_{m=1,2,3,\cdots} (\delta_m + 1)(\Delta a - r)^{\delta_m} \mathrm{Re}[\boldsymbol{g}_m] \\ &= \sum_{m=1,3,5,\cdots} (\delta_m + 1)(\Delta a - r)^{\delta_m} \boldsymbol{g}_m \\ &= \sum_{j=0,1,2,\cdots} (j + \frac{1}{2})(\Delta a - r)^{j-1/2} \boldsymbol{g}_{2j+1}\end{aligned} \tag{C3}$$

Note that $\boldsymbol{g}_m$ is pure imaginary for $m = 2, 4, 6, \cdots$. For the displacement vector $\boldsymbol{u}$,

$$\boldsymbol{u} = \mathrm{Re}[\sum_{n=1,2,3,\cdots} \boldsymbol{A}\left\langle z^{\delta_n+1}\right\rangle \boldsymbol{B}^{-1} \boldsymbol{g}_n]$$

At the crack flanks, $\theta = \pm\pi$,

$$z = r\cos(\pm\pi) = re^{\pm i\pi}\ ,\ z^{\delta_n+1} = r^{\delta_n+1} e^{\pm i(\delta_n+1)\pi}$$

Thus

$$\begin{aligned}\Delta\boldsymbol{u} &= \boldsymbol{u}(r,\pi) - \boldsymbol{u}(r,-\pi) \\ &= \mathrm{Re}\{\sum_{n=1,2,3,\cdots} [\boldsymbol{A}\left\langle r^{\delta_n+1}e^{i(\delta_n+1)\pi} - r^{\delta_n+1}e^{-i(\delta_n+1)\pi}\right\rangle \boldsymbol{B}^{-1}\boldsymbol{g}_n]\} \\ &= \mathrm{Re}\{\sum_{n=1,2,3,\cdots} r^{\delta_n+1} 2i\sin[(\delta_n+1)\pi]\boldsymbol{A}\boldsymbol{B}^{-1}\boldsymbol{g}_n\} \\ &= 2\sum_{n=1,3,5,\cdots} r^{\delta_n+1}(-1)^{(n-1)/2}\,\mathrm{Re}[\boldsymbol{L}^{-1} - i\boldsymbol{S}\boldsymbol{L}^{-1}]\boldsymbol{g}_n \\ &= 2\sum_{n=1,3,5,\cdots} (-1)^{(n-1)/2}\boldsymbol{L}^{-1}\boldsymbol{g}_n r^{\delta_n+1} \\ &= 2\sum_{l=0,1,2,\cdots} (-1)^l \boldsymbol{L}^{-1}\boldsymbol{g}_{2l+1} r^{l+1/2}\end{aligned} \tag{C4}$$

where the identity $-\boldsymbol{A}\boldsymbol{B}^{-1} = \boldsymbol{S}\boldsymbol{L}^{-1} + i\boldsymbol{L}^{-1}$ has been used in deriving eq. (C4).
Substituting (C3), (C4) into (C2)

$$\mathrm{G} = \lim_{\Delta a\to 0}\frac{1}{2\Delta a}\sum_j\sum_l 2(j+\frac{1}{2})(-1)^l\,\boldsymbol{g}_{2j+1}^T\,\boldsymbol{L}^{-1}\,\boldsymbol{g}_{2l+1}\int_0^{\Delta a} r^l(\Delta a - r)^j\sqrt{\frac{r}{\Delta a - r}}\,dr \tag{C5}$$

Introducing a nondimensional variable, $x = r/\Delta a$

$$\mathrm{G} = \lim_{\Delta a\to 0}\frac{1}{2\Delta a}\sum_j\sum_l 2(j+\frac{1}{2})(-1)^l\,\boldsymbol{g}_{2j+1}^T\,\boldsymbol{L}^{-1}\,\boldsymbol{g}_{2l+1}(\Delta a)^{j+l+1}\int_0^1 x^l(1-x)^j\sqrt{\frac{x}{1-x}}\,dx$$

In the integrand, the linear term of Δa corresponds to the first term or $j = l = 0$

$$\begin{aligned}\mathrm{G} &= \lim_{\Delta a\to 0}\frac{1}{2\Delta a}[\boldsymbol{g}_1^T\boldsymbol{L}^{-1}\,\boldsymbol{g}_1\Delta a\frac{\pi}{2} + o(\Delta a)] \\ &= \frac{1}{2}\boldsymbol{k}^T\,\boldsymbol{L}^{-1}\,\boldsymbol{k}\end{aligned} \tag{C6}$$

For the monoclinic solid with plane of symmetry at $x_3 = 0$, $\boldsymbol{L}^{-1}$ under in-plane deformation can be expressed by

$$\boldsymbol{L}^{-1} = s'_{11}\begin{bmatrix} b & d \\ d & e \end{bmatrix} \tag{C7}$$

$$\mu_1 + \mu_2 = a + ib,\ \mu_1\mu_2 = c + id$$
$$e = ad - bc = \mathrm{Im}[\mu_1\mu_2(\bar{\mu}_1 + \bar{\mu}_2)]$$

$$\begin{aligned}
\boldsymbol{G} &= \frac{1}{2}s'_{11}[k_2, k_1]\begin{bmatrix} b & d \\ d & e \end{bmatrix}\begin{bmatrix} k_2 \\ k_1 \end{bmatrix} \\
&= \frac{1}{2}s'_{11}[ek_1^2 + 2dk_1k_2 + bk_2^2] \qquad \text{(C8)} \\
&= \frac{1}{2}s'_{11}\left\{\mathrm{Im}[\mu_1\mu_2(\bar{\mu}_1 + \bar{\mu}_2)]k_1^2 + 2\,\mathrm{Im}(\mu_1\mu_2)k_1k_2 + \mathrm{Im}(\mu_1 + \mu_2)k_2^2\right\}
\end{aligned}$$

This expression can be reduced to orthotropic solids without cross term of k_1 and k_2 in eq. (C8). This special case has been formulated from Lekhnitskii formulation and the results are shown in eq. (26) of ref. (16) by G. C. Sih, P. C. Paris, and G. R. Irwin.

Appendix D

Determining the Unknown Coefficients in the Crack Field Using the J-integral

The path-independent integral J_M from eq. (60) is

$$J_M = \int_\Gamma (\sigma_{i2} \frac{du_i^a}{ds} - t_i^a u_{i,1}) ds \tag{D1}$$

Let the actual and auxiliary crack fields

$$\Phi = \sum_n B\langle f_n(z)\rangle \boldsymbol{B}^{-1} \boldsymbol{g}_n \tag{D2}$$

$$\Phi^a = B\langle f^a(z)\rangle \boldsymbol{B}^{-1} \boldsymbol{h} \tag{D3}$$

where

$$\begin{aligned} f_n(z) &= z^{\delta_n+1}, \ \delta_n = (n-2)/2, \quad n = 1, 2, 3, \cdots. \\ f^a(z) &= z^{\Delta_m+1}, \quad \Delta_m = -m/2, \quad m = 1, 3, 4, \cdots. \end{aligned}$$

From the identity

$$\mathrm{Re}(\boldsymbol{C})\,\mathrm{Re}(\boldsymbol{D}) = \frac{1}{2}\mathrm{Re}[(\boldsymbol{C} + \overline{\boldsymbol{C}})\,\boldsymbol{D}] = \frac{1}{2}\mathrm{Re}[\boldsymbol{C}(\boldsymbol{D} + \overline{\boldsymbol{D}})] \tag{D4}$$

where $\boldsymbol{C}$ and $\boldsymbol{D}$ are complex matrices, we have

$$\begin{aligned} \sigma_{i2}\, du_i^a &= du_i^a\, \mathrm{Re}[\Phi_{i,1}] = d(\boldsymbol{u}^a)^T\, \mathrm{Re}[\Phi_{,1}] \\ &= d\,[\mathrm{Re}(\boldsymbol{AB}^{-1}\Phi^a)^T]\,\mathrm{Re}[\Phi_{,1}] \\ &= \frac{1}{2}\mathrm{Re}[(\boldsymbol{AB}^{-1} d\Phi^a)^T (\Phi_{,1} + c.c.)] \end{aligned} \tag{D5}$$

and

$$\begin{aligned} t_i^a u_{i,1}\, ds &= -\mathrm{Re}(\frac{d\Phi_i^a}{ds}) u_{i,1}\, ds \\ &= -\mathrm{Re}(d\Phi^a)^T\, \mathrm{Re}[\boldsymbol{AB}^{-1}\Phi_{,1}] \\ &= -\frac{1}{2}\mathrm{Re}[(d\Phi^a)^T (\boldsymbol{AB}^{-1}\Phi_{,1} + c.c.)] \end{aligned} \tag{D6}$$

where $c.c.$ denotes the complex conjugate of the preceding term, i.e.,

$$F + c.c. = F + \overline{F}$$

Therefore

$$
\begin{aligned}
J_M &= \frac{1}{2}\mathrm{Re}\left\{\int_\Gamma [(\boldsymbol{AB}^{-1}d\Phi^a)^T(\Phi_{,1}+c.c.)+(d\Phi^a)^T(\boldsymbol{AB}^{-1}\Phi_{,l}+c.c.)]\right\}_. \\
&= \frac{1}{2}\mathrm{Re}\Big[\sum_n\int_\Gamma (\boldsymbol{h}^T\boldsymbol{B}^{-T}\langle df^a\rangle\boldsymbol{A}^T\boldsymbol{B}\langle f_n'\rangle\boldsymbol{B}^{-1}\boldsymbol{g}_n+\boldsymbol{h}^T\boldsymbol{B}^{-T}\langle df^a\rangle\boldsymbol{A}^T\overline{\boldsymbol{B}}\langle\overline{f}_n'\rangle\overline{\boldsymbol{B}}^{-1}\overline{\boldsymbol{g}}_n \\
&\qquad +\boldsymbol{h}^T\boldsymbol{B}^{-T}\langle df^a\rangle\boldsymbol{B}^T\boldsymbol{A}\langle f_n'\rangle\boldsymbol{B}^{-1}\boldsymbol{g}_n+\boldsymbol{h}^T\boldsymbol{B}^{-T}\langle df^a\rangle\boldsymbol{B}^T\overline{\boldsymbol{A}}\langle\overline{f}_n'\rangle\overline{\boldsymbol{B}}^{-1}\overline{\boldsymbol{g}}_n)\Big] \\
&= \frac{1}{2}\mathrm{Re}\Big[\sum_n\int_\Gamma (\boldsymbol{h}^T\boldsymbol{B}^{-T}\langle df^a\rangle(\boldsymbol{A}^T\boldsymbol{B}+\boldsymbol{B}^T\boldsymbol{A})\langle f_n'\rangle\boldsymbol{B}^{-1}\boldsymbol{g}_n \\
&\qquad +\boldsymbol{h}^T\boldsymbol{B}^{-T}\langle df^a\rangle(\boldsymbol{A}^T\overline{\boldsymbol{B}}+\boldsymbol{B}^T\overline{\boldsymbol{A}})\langle\overline{f}_n'\rangle\overline{\boldsymbol{B}}^{-1}\overline{\boldsymbol{g}}_n)\Big]
\end{aligned}
$$

Using the orthogonality relations in eq. (10) and (11),

$$\boldsymbol{A}^T\boldsymbol{B}+\boldsymbol{B}^T\boldsymbol{A}=\mathbf{I} \quad \text{and} \quad \boldsymbol{A}^T\overline{\boldsymbol{B}}+\boldsymbol{B}^T\overline{\boldsymbol{A}}=0, \tag{D7}$$

J_M can be further rewritten as

$$
\begin{aligned}
J_M &= \frac{1}{2}\mathrm{Re}\left[\sum_n\int_\Gamma \boldsymbol{h}^T\boldsymbol{B}^{-T}\langle df^a\rangle\langle f_n'\rangle\boldsymbol{B}^{-1}\boldsymbol{g}_n\right] \\
&= \frac{1}{2}\mathrm{Re}\left[\sum_n \boldsymbol{h}^T\boldsymbol{B}^{-T}\left\langle\int_\Gamma f_n'(z)\,df^a(z)\right\rangle\boldsymbol{B}^{-1}\boldsymbol{g}_n\right]
\end{aligned}
\tag{D8}
$$

Defining

$$R_{mn}\equiv\int_\Gamma f_n'(z_\alpha)\,df^a(z_\alpha)=(\delta_n+1)(\Delta_m+1)\int_\Gamma z_\alpha^{\delta_n+\Delta_m}dz_\alpha$$

it is readily shown that

(a) $n=m$ or $\delta_n+\Delta_m+1=0$,

$$R_{mn}=(\delta_n+1)(\Delta_m+1)\int_\Gamma\frac{dz_\alpha}{z_\alpha}=-\delta_m(\delta_m+1)[\ln(r_2/r_1)+i2\pi];$$

(b) $n\neq m$ or $\delta_n+\Delta_m+1\neq0$,

$$R_{mn}=\frac{(\delta_n+1)(2-m)}{n-m}\{[r_2^{(n-m)/2}-r_1^{(n-m)/2}]\cos(n-m)\pi/2+i[r_2^{(n-m)/2}+r_1^{(n-m)/2}]\sin(n-m)\pi/2\}$$

i.e.

$$R_{mn}=\begin{cases}(-1)^j\dfrac{(\delta_n+1)(2-m)}{n-m}[r_2^{(n-m)/2}-r_1^{(n-m)/2}] & \text{for}\quad n-m=2j\\ i(-1)^j\dfrac{(\delta_n+1)(2-m)}{n-m}[r_2^{(n-m)/2}+r_1^{(n-m)/2}] & \text{for}\quad n-m=2j+1\end{cases}$$

Using R_{mn} and the identity

$$\boldsymbol{B}^{-T}\boldsymbol{B}^{-1}=-2i\boldsymbol{L}^{-1}$$

denoting J_M as J_{Mm}, $\boldsymbol{h}$ as $\boldsymbol{h}_m$, eq. (D8) becomes

$$
\begin{aligned}
J_{Mm} &= \frac{1}{2}\mathrm{Re}\left[\sum_n R_{mn}\boldsymbol{h}_m^T \boldsymbol{B}^{-T}\boldsymbol{B}^{-1}\boldsymbol{g}_n\right] \\
&= \frac{1}{2}\mathrm{Re}\left[\sum_n R_{mn}(-2i)\boldsymbol{h}_m^T \boldsymbol{L}^{-1}\boldsymbol{g}_n\right] = \mathrm{Im}\left[\sum_n R_{mn}\boldsymbol{h}_m^T \boldsymbol{L}^{-1}\boldsymbol{g}_n\right] \\
&= \mathrm{Im}\left[R_{mm}\boldsymbol{h}_m^T \boldsymbol{L}^{-1}\boldsymbol{g}_m\right] + \sum_{\substack{n \\ n=m+2j,\, n\neq m}} \mathrm{Im}(R_{mn})\boldsymbol{h}_m^T \boldsymbol{L}^{-1}\boldsymbol{g}_n + \sum_{\substack{n \\ n=m+2j+1}} \mathrm{Im}(R_{mn}\boldsymbol{h}_m^T \boldsymbol{L}^{-1}\boldsymbol{g}_n)
\end{aligned}
$$

The last two terms of the above equation are zero. Thus the term contributing to the J_{Mm} is the term between $\boldsymbol{g}_m$ and $\boldsymbol{h}_m$ only, and

$$
\begin{aligned}
J_{Mm} &= \mathrm{Im}\left[R_{mm}\boldsymbol{h}_m{}^T \boldsymbol{L}^{-1}\boldsymbol{g}_m\right] \\
&= \boldsymbol{h}_m{}^T \boldsymbol{L}^{-1}\,\boldsymbol{g}_m\,\mathrm{Im}\{-\delta_m(\delta_m+1)[\ln(r_2/r_1)+i2\pi]\}\,, \quad n\neq 0 \\
&= -2\pi\,\delta_m(\delta_m+1)\,\boldsymbol{h}_m{}^T \boldsymbol{L}^{-1}\,\boldsymbol{g}_m
\end{aligned}
$$

or

$$
J_{Mn} = -2\pi\,\delta_n(\delta_n+1)\,\boldsymbol{h}_n{}^T \boldsymbol{L}^{-1}\,\boldsymbol{g}_n \qquad \text{(D9)}
$$

As $\boldsymbol{h}_n$ is arbitrary, we choose

$$
\boldsymbol{h}_n = \begin{cases} \boldsymbol{e}_k\,, & n=1,3,5,\cdots \\ i\boldsymbol{e}_k\,, & n=4,6,8,\cdots \end{cases}
$$

Here, $\boldsymbol{e}_k$ (k = 1, 2) possesses dimension force / (length)$^{1-\delta_n}$. Therefore, for a given n, there will be two different values of J_{Mn} denoted by $J_{Mn}^{(1)}$ and $J_{Mn}^{(2)}$ correspondingly. For the two choices of $\boldsymbol{h}_n$, eq. (D9) leads to

$$
\boldsymbol{g}_n = \begin{cases} -\dfrac{\boldsymbol{L}\tilde{\boldsymbol{J}}_{Mn}}{2\pi\,\delta_n(\delta_n+1)}, & n=1,3,5,\cdots \\[2ex] \dfrac{i\boldsymbol{L}\tilde{\boldsymbol{J}}_{Mn}}{2\pi\,\delta_n(\delta_n+1)}, & n=4,6,8,\cdots \end{cases} \qquad \text{(D10)}
$$

where

$$
\tilde{\boldsymbol{J}}_{Mn} = [J_{Mn}^{(1)}\,,\, J_{Mn}^{(2)}]^T
$$

Appendix E

Determining the Unknown Coefficients in the Crack Field Using Betti's Theorem

Following arguments similar to those presented for the J_M integral in Appendix D, one can get the expression for the integral I. The path-independent integral from Betti's theorem is

$$I = \int_\Gamma [(\boldsymbol{u}^a)^T \boldsymbol{t} - (\boldsymbol{t}^a)^T \boldsymbol{u}] ds \tag{E1}$$

Using the complex potential functions, $\boldsymbol{\Phi}$ and $\boldsymbol{\Phi}^a$ and identity eq. (D4),

$$\begin{aligned}(\boldsymbol{u}^a)\boldsymbol{t}ds &= -\mathrm{Re}(\boldsymbol{AB}^{-1}\boldsymbol{\Phi}^a)^T]\mathrm{Re}[d\boldsymbol{\Phi}] \\ &= -\frac{1}{2}\mathrm{Re}[(\boldsymbol{AB}^{-1}d\boldsymbol{\Phi}^a + c.c.)^T d\boldsymbol{\Phi}]\end{aligned} \tag{E2}$$

$$\begin{aligned}(\boldsymbol{t}^a)^\mathrm{T}\boldsymbol{u}\, ds &= -\mathrm{Re}(d\boldsymbol{\Phi}^a)^T\, \mathrm{Re}[\boldsymbol{AB}^{-1}\boldsymbol{\Phi}] \\ &= -\frac{1}{2}\mathrm{Re}\{[d(\boldsymbol{\Phi}^a + c.c.)^T]\boldsymbol{AB}^{-1}\boldsymbol{\Phi}\} \\ &= -\frac{1}{2}\mathrm{Re}\{d[(\boldsymbol{\Phi}^a + c.c.)^T\boldsymbol{AB}^{-1}\boldsymbol{\Phi}] - (\boldsymbol{\Phi}^a + c.c.)^T\boldsymbol{AB}^{-1}d\boldsymbol{\Phi}\}\end{aligned} \tag{E3}$$

$$\int_\Gamma (\boldsymbol{t}^a)^\mathrm{T}\boldsymbol{u}\, ds = -\frac{1}{2}\mathrm{Re}\{[(\boldsymbol{\Phi}^a + c.c.)^T\boldsymbol{AB}^{-1}\boldsymbol{\Phi}]\Big|_{(r_1,-\pi)}^{(r_2,\pi)} - \int_\Gamma (\boldsymbol{\Phi}^a + c.c.)^T\boldsymbol{AB}^{-1}d\boldsymbol{\Phi}\} \tag{E4}$$

According to eq. (19), the traction free conditions on the crack faces for the auxiliary field may be written as

$$\mathrm{Re}[\boldsymbol{\Phi}^a(z)] = 0 \text{ or } \boldsymbol{\Phi}^a + c.c. = 0 \text{ at } \theta = \pm\pi .$$

Therefore the first term on the right hand side of (B4) vanishes and

$$\int_\Gamma (\boldsymbol{t}^a)^\mathrm{T}\boldsymbol{u}\, ds = \frac{1}{2}\mathrm{Re}[\int_\Gamma (\boldsymbol{\Phi}^a + c.c.)^T\boldsymbol{AB}^{-1}d\boldsymbol{\Phi}] \tag{E5}$$

Substituting (E2) and (E5) into (E1) and using the expressions for $\boldsymbol{\Phi}$ and $\boldsymbol{\Phi}^a$ (eq. (D2) and (D3)) yield

$$\begin{aligned}I &= -\frac{1}{2}\mathrm{Re}\left\{\int_\Gamma (\boldsymbol{AB}^{-1}\boldsymbol{\Phi}^a + c.c.)^T d\boldsymbol{\Phi} + (\boldsymbol{\Phi}^a + c.c.)^T\boldsymbol{AB}^{-1}d\boldsymbol{\Phi}\right\} \\ &= -\frac{1}{2}\mathrm{Re}\Big[\sum_n \int_\Gamma \boldsymbol{h}^T\boldsymbol{B}^{-T}\langle f^a\rangle\boldsymbol{A}^T\boldsymbol{B}\langle df_n\rangle\boldsymbol{B}^{-1}\boldsymbol{g}_n + \bar{\boldsymbol{h}}^T\bar{\boldsymbol{B}}^{-T}\langle \bar{f}^a\rangle\bar{\boldsymbol{A}}^T\boldsymbol{B}\langle df_n\rangle\boldsymbol{B}^{-1}\boldsymbol{g}_n \\ &\qquad + \boldsymbol{h}^T\boldsymbol{B}^{-T}\langle f^a\rangle\boldsymbol{B}^T\boldsymbol{A}\langle df_n\rangle\boldsymbol{B}^{-1}\boldsymbol{g}_n + \bar{\boldsymbol{h}}^T\bar{\boldsymbol{B}}^{-T}\langle \bar{f}^a\rangle\bar{\boldsymbol{B}}^T\boldsymbol{A}\langle df_n\rangle\boldsymbol{B}^{-1}\boldsymbol{g}_n\Big]\end{aligned}$$

$$\begin{aligned}I = -\frac{1}{2}\mathrm{Re}\Big[\sum_n \int_\Gamma &\boldsymbol{h}^T\boldsymbol{B}^{-T}\langle f^a\rangle(\boldsymbol{A}^T\boldsymbol{B} + \boldsymbol{B}^T\boldsymbol{A})\langle df_n\rangle\boldsymbol{B}^{-1}\boldsymbol{g}_n + \\ &\left[\bar{\boldsymbol{h}}^T\bar{\boldsymbol{B}}^{-T}\langle \bar{f}^a\rangle(\bar{\boldsymbol{A}}^T\boldsymbol{B} + \bar{\boldsymbol{B}}^T\boldsymbol{A})\langle df_n\rangle\boldsymbol{B}^{-1}\boldsymbol{g}_n\right]\Big]\end{aligned}$$

Using the orthogonality relations, eq. (D7),

$$I = -\frac{1}{2}\mathrm{Re}\left[\sum_n \int_\Gamma \boldsymbol{h}^T \boldsymbol{B}^{-T} \langle f^a \rangle \langle df_n \rangle \boldsymbol{B}^{-1} \boldsymbol{g}_n\right]$$

$$= -\frac{1}{2}\mathrm{Re}\left[\sum_n \boldsymbol{h}^T \boldsymbol{B}^{-T} \left\langle \int_\Gamma f^a(z)\, df_n(z) \right\rangle \boldsymbol{B}^{-1} \boldsymbol{g}_n\right] \qquad \text{(E6)}$$

where

$$f^a(z) = z^{\Delta_m+1}, \quad \Delta_m = -m/2, \quad m = 1, 3, 4, \cdots.$$

Defining

$$Q_{mn} \equiv \int_\Gamma f^a(z_\alpha)\, df_n(z_\alpha) = (\delta_n + 1)\int_\Gamma z_\alpha^{\delta_n+\Delta_m+1} dz_\alpha$$

it is readily shown that

(a) $n = m - 2$ or $\delta_n + \Delta_m + 2 = 0$

$$Q_{m,m-2} = (\delta_{m-2} + 1)\int_\Gamma \frac{dz_\alpha}{z_\alpha} = (\delta_{m-2} + 1)(\ln r_2 / r_1 + i2\pi);$$

(b) $n \neq m$-2 or $\delta_n + \Delta_m + 2 \neq 0$

$$Q_{mn} = -2\frac{\delta_n + 1}{n-m+2}[(r_2^{(n-m+2)/2} - r_1^{(n-m+2)/2})\cos[(n-m)\pi/2] + i(r_2^{(n-m+2)/2} + r_1^{(n-m+2)/2})\sin[(n-m)\pi/2]\},$$

i.e. $$Q_{mn} = \begin{cases} (-1)^{j+1} 2\dfrac{(\delta_n + 1)}{n-m+2}(r_2^{(n-m+2)/2} - r_1^{(n-m+2)/2}) & \text{for } n - m = 2j \\ 2i(-1)^{j+1}\dfrac{(\delta_n + 1)}{n-m+2}(r_2^{(n-m+2)/2} + r_1^{(n-m+2)/2}) & \text{for } n - m = 2j+1 \end{cases}$$

Using Q_{mn} and expression for $\boldsymbol{L}^{-1}$, writing I as I_m, and $\boldsymbol{h}$ as $\boldsymbol{h}_m$, the I-integral from eq. (E6), for a given Δ_m, becomes

$$I_m = -\frac{1}{2}\mathrm{Re}\{\sum_n Q_{mn}\boldsymbol{h}_m^T \boldsymbol{B}^{-T}\boldsymbol{B}^{-1}\boldsymbol{g}_n\}$$

$$= -\frac{1}{2}\mathrm{Re}\{\sum_n Q_{mn}(-2i)\boldsymbol{h}_m^T \boldsymbol{L}^{-1}\boldsymbol{g}_n\} = -\mathrm{Im}[\sum_n Q_{mn}\, \boldsymbol{h}_m^T \boldsymbol{L}^{-1}\boldsymbol{g}_n]$$

$$= -\mathrm{Im}\{Q_{m,\,m-2}\boldsymbol{h}_m^T \boldsymbol{L}^{-1}\boldsymbol{g}_{m-2}\} - \mathrm{Im}\sum_{\substack{n \\ n=m+2j,\, n\neq m-2}}\{Q_{mn}\, \boldsymbol{h}_m^T \boldsymbol{L}^{-1}\boldsymbol{g}_n\} - \mathrm{Im}\sum_{\substack{n \\ m=m+2j+1}}\{Q_{mn}\, \boldsymbol{h}_m^T \boldsymbol{L}^{-1}\boldsymbol{g}_n\}$$

Since the last two terms of the above equation are equal to zero, it can be clearly seen that the term contributing to the I is the cross terms between $\boldsymbol{g}_{m-2}$ and $\boldsymbol{h}_m$ only.

Thus

$$I_m = -\mathrm{Im}\left[Q_{m,m-2}\boldsymbol{h}_m{}^T \boldsymbol{L}^{-1}\boldsymbol{g}_{m-2}\right] = -\boldsymbol{h}_m{}^T \boldsymbol{L}^{-1}\boldsymbol{g}_{m-2}\,\mathrm{Im}\{(\delta_{m-2}+1)[\ln(r_2/r_1) + i2\pi]\}$$

$$= -2\pi(\delta_{m-2} + 1)\boldsymbol{h}_m{}^T \boldsymbol{L}^{-1}\boldsymbol{g}_{m-2}$$

or

$$I_{n+2} = -2\pi(\delta_n + 1)\,\boldsymbol{h}_{n+2}{}^T \boldsymbol{L}^{-1} \boldsymbol{g}_n \tag{E7}$$

From eq. (E7), following a similar procedure as before, $\boldsymbol{g}_n$ can be expressed by

$$\boldsymbol{g}_n = \begin{cases} -\dfrac{\boldsymbol{L}\tilde{\boldsymbol{I}}_{n+2}}{2\pi(\delta_n + 1)}, & n = 1, 3, 5, \cdots \\ \dfrac{i\,\boldsymbol{L}\tilde{\boldsymbol{I}}_{n+2}}{2\pi(\delta_n + 1)}, & n = 2, 4, 6, \cdots \end{cases} \tag{E8}$$

where

$$\tilde{\boldsymbol{I}}_{n+2} = [I_{n+2}^{(1)}, I_{n+2}^{(2)}]^T$$

$I_{n+2}^{(k)}$ is the value of I_{n+2} when

$$\boldsymbol{h}_n = \begin{cases} \boldsymbol{e}_k, & n = 1, 3, 5, \cdots \\ i\boldsymbol{e}_k, & n = 2, 4, 6, \cdots \end{cases}$$

Here, $\boldsymbol{e}_k$ possesses dimension $\text{force} \times (\text{length})^{\delta_n}$.

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F. G. Yuan

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Mars Mission Research Center
Department of Mechanical and Aerospace Engineering
North Carolina State University
Raleigh, NC 27695

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Determination of all the coefficients in the crack tip field expansion for monoclinic materials under two-dimensional deformation is presented in this report. For monoclinic materials with a plane of material symmetry at $x_3 = 0$, the in-plane deformation is decoupled from the anti-plane deformation. In the case of in-plane deformation, utilizing conservation laws of elasticity and Betti's reciprocal theorem, together with selected auxiliary fields, T-stress and third-order stress coefficients near the crack tip are evaluated first from path-independent line integrals. To determine the T-stress terms using the J-integral and Betti's reciprocal work theorem, auxiliary fields under a concentrated force and moment acting at the crack tip are used respectively. Through the use of Stroh formalism in anisotropic elasticity, analytical expressions for all the coefficients including the stress intensity factors are derived in a compact form that has surprisingly simple structure in terms of the Barnett-Lothe tensors, $\boldsymbol{L}$. The solution forms for degenerated materials, orthotropic, and isotropic materials are presented.

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