NASA/CR-1998-208954
ICASE Interim Report No. 34

Construction of Three Dimensional Solutions for the Maxwell Equations

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Operated by Universities Space Research Association

National Aeronautics and
Space Administration

Langley Research Center
Hampton, Virginia 23681-2199

Prepared for Langley Research Center
under Contract NAS1-97046

December 1998

Available from the following:

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CONSTRUCTION OF THREE DIMENSIONAL SOLUTIONS FOR THE MAXWELL EQUATIONS*

A. YEFET† AND E. TURKEL‡

Abstract. We consider numerical solutions for the three dimensional time dependent Maxwell equations. We construct a fourth order accurate compact implicit scheme and compare it to the Yee scheme for free space in a box.

Subject classification. Applied and Numerical Mathematics

Key words. Maxwell equations, the Yee scheme, the Ty(2,4) scheme

1. Maxwell Equations in a Box. Let $\tau = ct = t/\sqrt{\mu\epsilon}$ and $Z = \sqrt{\frac{\mu}{\epsilon}}$. For the rest of this paper we replace τ by t. The three dimensional time dependent Maxwell equations then are:

$$
\begin{aligned}
\frac{\partial E_x}{\partial t} &= Z\left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right)\\
\frac{\partial E_y}{\partial t} &= Z\left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right)\\
\frac{\partial E_z}{\partial t} &= Z\left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right)\\
\frac{\partial H_x}{\partial t} &= \frac{1}{Z}\left(\frac{\partial E_y}{\partial z} - \frac{\partial E_z}{\partial y}\right)\\
\frac{\partial H_y}{\partial t} &= \frac{1}{Z}\left(\frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z}\right)\\
\frac{\partial H_z}{\partial t} &= \frac{1}{Z}\left(\frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x}\right).
\end{aligned}
\tag{1.1}
$$

We set $Z = 1$ in this paper.

A plane wave solution is given by

$$
\begin{aligned}
H_x &= H_x^0 \sin(\omega t)\sin(Ax + By + Cz)\\
H_y &= H_y^0 \sin(\omega t)\sin(Ax + By + Cz)\\
H_z &= H_z^0 \sin(\omega t)\sin(Ax + By + Cz)\\
E_x &= E_x^0 \cos(\omega t)\cos(Ax + By + Cz)\\
E_y &= E_y^0 \cos(\omega t)\cos(Ax + By + Cz)\\
E_z &= E_z^0 \cos(\omega t)\cos(Ax + By + Cz)
\end{aligned}
$$

Substituting into the Maxwell equations this is a solution if

$$\omega^2 = A^2 + B^2 + C^2 \tag{1.2}$$

$$0 = AH_x^0 + BH_y^0 + CH_z^0 \tag{1.3}$$

*This research was supported by the National Aeronautics and Space Administration under NASA Contract No. NAS1-97046 while the authors were in residence at the Institute for Computer Applications for Science and Engineering (ICASE), NASA Langley Research Center, Hampton, VA 23681-2199.

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We also demand that

$$\omega E_x^0 = H_y^0 C - H_z^0 B$$
$$\omega E_y^0 = H_z^0 A - H_x^0 C$$
$$\omega E_z^0 = H_x^0 B - H_y^0 A$$

2. Numerical Tests. We consider a case where $H_x^0 = H_y^0 = H_z^0 = 1$ and

$$A = \pi$$
$$B = -2\pi$$
$$C = \pi$$
$$\omega = \sqrt{6}\pi$$

We use this exact solution as a basis for comparison in the box $[0, 1/2] \times [0, 1/4] \times [0, 1/2]$. We shall compare two numerical methods: the Yee scheme [1] which is second order accurate in space and time and the Ty(2,4) scheme [2, 3] which is second order accurate in time but fourth order accurate in space. In order for the total error to be fourth order we must choose the time step small enough so that the temporal error does not swamp the spatial error. This requires $\Delta t \sim (\Delta x)^2$. If the error requirements are too severe then this is inefficient and the leapfrog in time should be replaced by a fourth order Runge-Kutta method. However, for the experiments in this paper we shall use the same leapfrog method for both schemes. Hence, both the Yee scheme and the Ty(2,4) have the electric and magnetic variables at the same staggered locations both in space and in time. The Yee scheme approximates the derivatives via the following approximation.

$$\frac{\partial}{\partial y}\begin{bmatrix} U^{1/2} \\ U^{3/2} \\ . \\ . \\ U^{(2p-1)/2} \end{bmatrix} = \frac{1}{\Delta y}\left(\begin{bmatrix} U^1 \\ U^2 \\ . \\ U^{p-1} \\ U^p \end{bmatrix} - \begin{bmatrix} U^0 \\ U^1 \\ . \\ U^{p-2} \\ U^{p-1} \end{bmatrix}\right).$$

A similar formula holds for the other variables shifted to other locations in each direction. The Ty(2,4) scheme is an implicit compact scheme given by

$$\frac{\partial}{\partial y}\begin{bmatrix} U^{1/2} \\ U^{3/2} \\ . \\ . \\ U^{(2p-1)/2} \end{bmatrix} = \mathbf{A}^{-1}\frac{\mathbf{1}}{\mathbf{\Delta y}}\left(\begin{bmatrix} U^1 \\ U^2 \\ . \\ U^{p-1} \\ U^p \end{bmatrix} - \begin{bmatrix} U^0 \\ U^1 \\ . \\ U^{p-2} \\ U^{p-1} \end{bmatrix}\right).$$

where A is defined the following way:

$$\mathbf{A} = \frac{\mathbf{1}}{\mathbf{24}}\begin{pmatrix} 26 & -5 & 4 & -1 & . & . & 0 \\ 1 & 22 & 1 & 0 & . & . & 0 \\ 0 & 1 & 22 & 1 & 0 & . & 0 \\ . & . & . & . & & & \\ 0 & . & . & 0 & 1 & 22 & 1 \\ 0 & . & . & -1 & 4 & -5 & 26 \end{pmatrix}.$$

For the Yee scheme we choose $\Delta t = \frac{4h}{7}$ while for the Ty(2,4) scheme we choose $\Delta t \sim h^2$ where $h = \Delta x = \Delta y$.

We measure the error in the L_2 norm between the approximate and exact electric field in the $\hat{z}$-direction. The Ty(2,4) behaves better than expected and gives almost fifth order accuracy. The Yee scheme gives a second order accuracy as expected.

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[1] K.S. YEE, *Numerical Solution of Initial Boundary Value Problems Involving Maxwell's Equation in Isotropic Media,* IEEE Transactions on Antennas and Propagation **AP-14** (May 1996), pp. 302-307.

[2] E. TURKEL AND A. YEFET, *Fourth Order Accurate Compact Implicit Method for the Maxwell Equations,* submitted to IEEE Transactions on Antennas and Propagation.

[3] A. YEFET AND E. TURKEL, *Fourth Order Compact Implicit Method for the Maxwell Equations with Discontinuous Coefficients,* submitted to Applied Numerical Mathematics.

scheme	h	Δt	t=10	reduction	rate
$Ty(2,4)$	$\frac{1}{20}$	$\frac{1}{400}$	3.62×10^{-4}		
$Ty(2,4)$	$\frac{1}{40}$	$\frac{1}{1600}$	1.1443×10^{-5}	31.6423	4.98
$Ty(2,4)$	$\frac{1}{80}$	$\frac{1}{6400}$	3.5621×10^{-7}	32.1255	5.0056
Yee	$\frac{1}{20}$	$\frac{1}{35}$	0.027		
Yee	$\frac{1}{40}$	$\frac{1}{70}$	7.3×10^{-4}	3.694	1.9028
Yee	$\frac{1}{80}$	$\frac{1}{140}$	1.82×10^{-4}	4.0042	2.0015

TABLE 2.1
Comparison of the maximum errors in L_2 norm

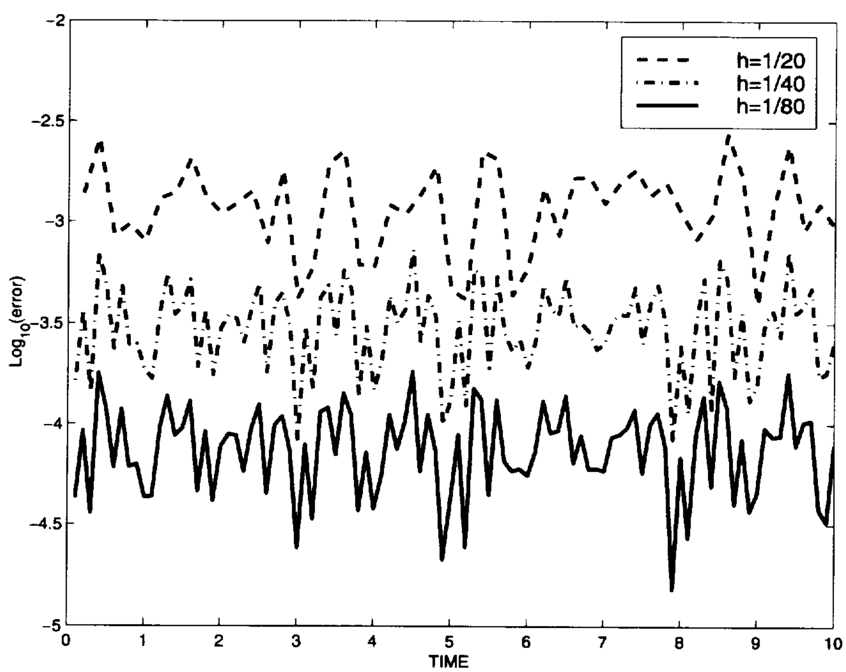

FIG. 2.1. $\log_{10}(error)$ *For the Yee scheme.*

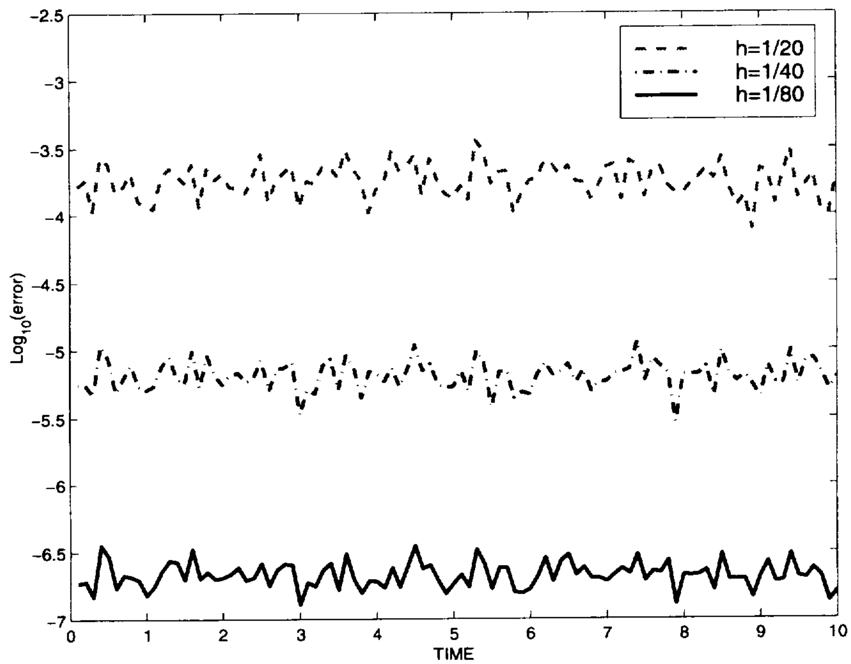

FIG. 2.2. $\log_{10}(errors)$ *For the Ty(2,4) scheme.*

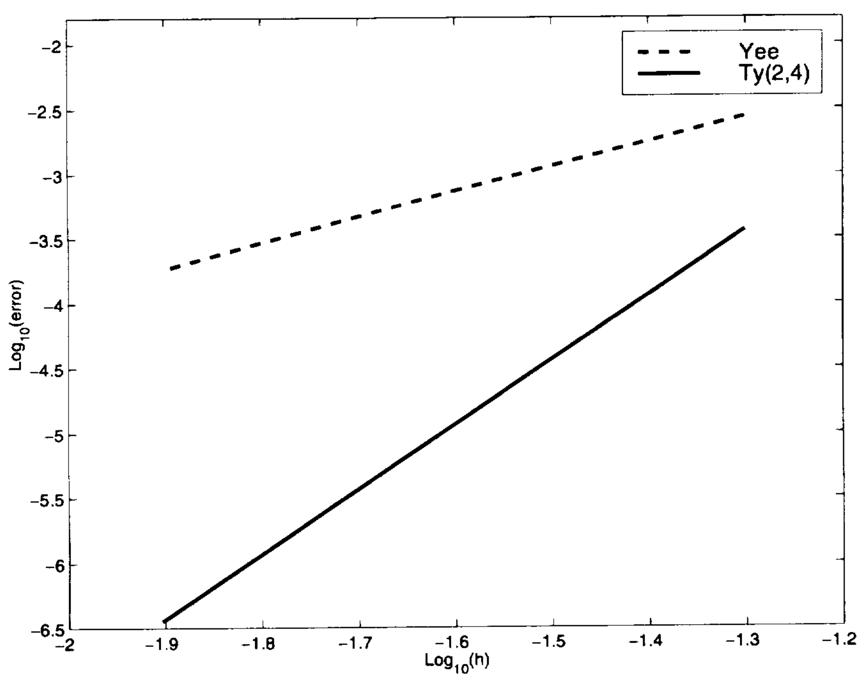

FIG. 2.3. *$Log_{10}(error)$ as a function of $Log_{10}(h)$ For the Yee and the Ty(2,4) schemes.*

REPORT DOCUMENTATION PAGE

Form Approved
OMB No. 0704-0188

1. AGENCY USE ONLY *(Leave blank)*

2. REPORT DATE
December 1998

3. REPORT TYPE AND DATES COVERED
Contractor Report

4. TITLE AND SUBTITLE
Construction of Three Dimensional Solutions for the Maxwell Equations

5. FUNDING NUMBERS
C NAS1-97046
WU 505-90-52-01

6. AUTHOR(S)
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7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)
Institute for Computer Applications in Science and Engineering
Mail Stop 403, NASA Langley Research Center
Hampton, VA 23681-2199

8. PERFORMING ORGANIZATION REPORT NUMBER
ICASE Interim Report No. 34

9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)
National Aeronautics and Space Administration
Langley Research Center
Hampton, VA 23681-2199

10. SPONSORING/MONITORING AGENCY REPORT NUMBER
NASA/CR-1998-208954
ICASE Interim Report No. 34

11. SUPPLEMENTARY NOTES
Langley Technical Monitor: Dennis M. Bushnell
Final Report
To be submitted to IEEE Transactions on Antennas and Propagation and Applied Numerical Mathematics

12a. DISTRIBUTION/AVAILABILITY STATEMENT
Unclassified Unlimited
Subject Category 64
Distribution: Nonstandard
Availability: NASA-CASI (301)621-0390

12b. DISTRIBUTION CODE

13. ABSTRACT *(Maximum 200 words)*
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14. SUBJECT TERMS
Maxwell equations; the Yee scheme; the Ty(2,4) scheme

15. NUMBER OF PAGES
10

16. PRICE CODE
A03

17. SECURITY CLASSIFICATION OF REPORT	18. SECURITY CLASSIFICATION OF THIS PAGE	19. SECURITY CLASSIFICATION OF ABSTRACT	20. LIMITATION OF ABSTRACT
Unclassified	Unclassified		