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A {3,2}-order bending theory for laminated composite and sandwich beams

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A higher-order bending theory is derived for laminated composite and sandwich beams thus extending the recent {1,2}-order theory to include third-order axial effects without introducing additional kinematic variables. The present theory is of order {3,2} and includes both transverse shear and transverse normal deformations. A closed-form solution to the cylindrical bending problem is derived and compared with the corresponding exact elasticity solution. The numerical comparisons are focused on the most challenging material systems and beam aspect ratios which include moderate-to-thick unsymmetric composite and sandwich laminates. Advantages and limitations of the theory are discussed. © 1998 Published by Elsevier Science Ltd. All rights reserved

INTRODUCTION

Higher performance and lower cost requirements for the next generation of aerospace vehicles often necessitate the use of advanced polymer-matrix composite materials. Composite materials can be tailored into highly efficient structures that combine high stiffness and strength, light weight, and improved fatigue and thermal performance. From the design perspective, accurate strain and stress predictions are required to avoid higher factors of safety that inevitably lead to an over design, reduced performance, and higher cost.

The structural modeling of composite and sandwich laminates with the use of approximate beam, plate, and shell theories is known to be the most efficient. Many significant developments in this area can be found, for example, in review papers by Reissner[1], Reddy[2], and Noor and Burton[3]. Numerous finite elements used in commercial and research codes have also been developed for composite structures. The most commonly used finite element models are those based on the first-order shear-deformation theory. The following brief discussion reviews the most pertinent aspects of composite beam theories, and also makes comparisons to similar plate and shell theories.

The classical Bernoulli–Euler beam theory, neglecting transverse shear and transverse normal deformations, is appropriate for thin, homogeneous beams and is known to be inadequate for composite and relatively thick beams. Timoshenko beam theory includes transverse shear deformation and provides more accurate response predictions for thin and moderately thick homogeneous beams. Reissner[4] stress-based and Mindlin[5] displacement-based first-order shear-deformation plate theories were originally developed for the analysis of homogeneous elastic plates. Many subsequent shear-deformation theories, utilizing the displacement-based approximation approach, focused on the analysis of laminated composites, e.g. refer to Stavsky[6], Yang *et al.*[7], Whitney and Pagano[8]. Such theories, commonly referred to as *single-layer* theories, treat a laminate as an equivalent single layer, with the displacement assumption representing a weighted-average distribution through the thickness.

Reddy and Liu[9] formulated a *layer-wise* theory which assumes piece-wise smooth displacement components through the thickness, i.e., while the displacement function is continuous through the thickness, the slope of the function at the ply interfaces may not be continuous. This type of theory produces a large number of unknowns and is computationally expensive, especially when a laminate consists of many layers which is usually the case in load-carrying structures.

Higher-order theories, which account for transverse shear and transverse normal stresses, generally provide a reasonable compromise between accuracy and simplicity; however, they are usually associated with higher-order boundary conditions that are difficult to interpret in practical engineering applications, e.g., refer to Essenburg[10], Whitney and Sun[11], Lo *et al.*[12], Reddy[13], and Phan and Reddy[14].

Recently, Tessler and coworkers[15–19] developed a higher-order theory for application to laminate composite beam, plate, and shell analyses. The theory maintains the simplicity and computational advantages of the first-order shear-deformation theory. It accounts for transverse shear

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and transverse normal deformations by assuming a special form of the {1,2}-order displacement assumption (The notation $\{m,n\}$ implies that the axial displacement is expanded with a polynomial of degree, m, whereas the transverse displacement may be of a different degree, n.) Additionally, the average shear strains are assumed to be parabolic, thus satisfying zero shear tractions on the top/bottom surfaces; and an average transverse normal strain is assumed in the form of a cubic polynomial satisfying one of the equilibrium equations of elasticity theory exactly. The approach requires that the transverse strains need only be least-squares compatible, through the laminate thickness, with the strains derived from strain-displacement relations. The resulting thickness distributions for the transverse stresses and strains produce adequate correlation with results given by elasticity theory, an improvement over previous higher-order theories. Tessler[16] improved the theory further for application to composites by introducing an independent polynomial assumption for the transverse normal stress to replace the cubic transverse normal strain assumption. The improved theory results in a more accurate representation of transverse normal stresses and strains, and is further substantiated by solutions given by Schleicher[20]. The {1,2} theory retains the simplicity of the first-order shear deformation theory in so far as the engineering boundary conditions are concerned. Furthermore, the theory gives rise to finite element formulations that are fully compatible with the first-order shear deformation elements.

Application of the {1,2} theory generally results in excellent predictions for thin and moderately thick homogeneous and laminated composites. Nevertheless, the theory has some limitations, particularly with respect to the modeling of relatively thick sandwich laminates. This is because in such laminates the distribution of the inplane displacement and strain can be highly non-linear. In thick laminates, this generally results in underestimation of the axial stress, typically the largest stress that governs the design of the structure. Another deficiency, which is only manifested in sandwich laminates, is the violation of the traction conditions on the top and bottom surfaces associated with the transverse normal stress.

In a recent NASA publication, Cook[21] explored a {3,2}-order beam theory which expands upon Tessler's {1,2} theory by including cubic axial effects. A special hierarchical form for the axial displacement is developed such that the theory employs the same five kinematic variables as its {1,2}-order counterpart, without introducing any additional kinematic variables. The hierarchical form of the displacement field ensures the exact fulfillment of traction-free shear stress boundary conditions and permits a straightforward reduction to several lower-order beam theories. As in Tessler[16], in addition to the assumed displacements, an independent polynomial expansion is employed for the transverse normal stress. The concepts of transverse shear and transverse normal correction factors are effectively incorporated using strain energy and traction equilibrium considerations. The theory enables more accurate predictions for the axial, transverse shear, and transverse normal stresses and strains, particularly for thick laminated composite and sandwich beams. Accurate piecewise smooth transverse shear stresses are determined by integrating two-dimensional equilibrium equation of elasticity theory. Cook[21] also developed a straightforward correction procedure that improves the accuracy of this approach for unsymmetric and sandwich laminates.

In this paper, the theoretical foundation and predictive characteristics of the {3,2}-order theory are closely examined. The theory, which begins with an assumed {3,2}-order displacement field and assumed cubic transverse normal stress, employs the virtual work principle from which the beam equilibrium equations and associated boundary conditions are derived. These field equations are solved in closed form for the problem of cylindrical bending of laminated composite and sandwich beams. Appropriate transverse shear and transverse normal correction factors are employed. Numerical results are presented for moderately thick and truly thick beams, and comparisons are made to the {1,2}-order theory and three-dimensional elasticity solutions.

{3,2}-ORDER BEAM THEORY

Consider a straight, linearly elastic beam laminated with N

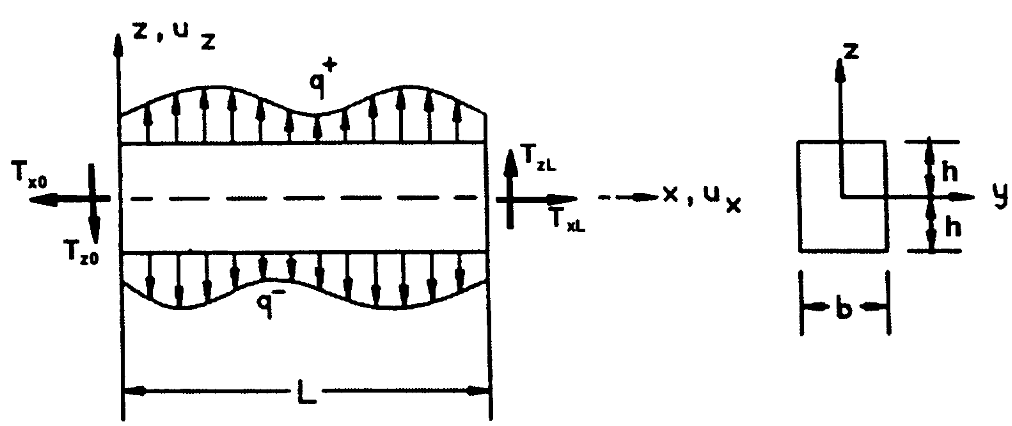

Figure 1 Beam notation

orthotropic plies subject to the loading shown in *Figure 1*. The beam has a span L and a rectangular cross-section thickness of $2h$ and width b. The orthotropic plies are stacked from the bottom ($z = -h$) such that the material properties, in general, are functions of the z coordinate. The tractions q^+ and q^- are applied normal to the top and bottom faces of the beam. T_{i0} and T_{iL} ($i = x,z$) are tractions prescribed at the ends of the beam.

Displacement assumptions

From the viewpoint of exact elasticity theory, the displacement components are piece-wise smooth and, in thick laminated composite and sandwich beams, they are non-linear through the thickness. This contrasts with the predominantly linear displacement distributions for thin beams. To represent both linear and non-linear deformation effects within the realm of a relatively simple, single-layer structural theory, the axial and transverse displacement components u_x and u_z are assumed to vary through the thickness as the cubic and quadratic polynomials

$$u_x(x,z) = \sum_{i=0}^{3} u_i(x)\zeta^i, \quad u_z(x,z) = w(x) + w_1(x)\zeta + w_2(x)(\zeta^2 + C) \tag{1}$$

where $\zeta = z/h \in [-1,1]$ is a dimensionless thickness coordinate such that $\zeta = 0$ defines the midplane of the beam. The four u_i coefficients in the axial displacement expression are yet to be defined, the w_i coefficients in the transverse displacement represent the same kinematic variables as those defined by Tessler[15]. The constant C is included in the transverse displacement equation to allow $w(x)$ to represent a weighted-average transverse displacement yet to be defined.

Three weighted-average kinematic variables are defined, as in Reissner[4], such that

$$u(x) = \frac{1}{2h}\int_{-h}^{h} u_x(x,z)\,\mathrm{d}z, \quad \theta(x) = \frac{3}{2h^3}\int_{-h}^{h} u_x(x,z)z\,\mathrm{d}z,$$

$$w(x) = \frac{3}{4h}\int_{-h}^{h} u_z(x,z)(1-\zeta^2)\,\mathrm{d}z \tag{2}$$

where $u(x)$ is the midplane displacement along the x axis, $\theta(x)$ is the rotation of the normal about the y axis, and $w(x)$ is the weighted-average of the transverse displacement. The displacement field in eqn (1) is substituted into eqn (2) resulting in $C = -{}^1/_5$, and the expressions for two u_i coefficients in terms of the $u(x)$ and $\theta(x)$ variables. The remaining two u_i coefficients are determined by enforcing zero shear traction conditions at the top and bottom beam faces. Since, from Hooke's law (eqn (5)), the shear stress is proportional to the shear strain, the shear strain at the top and bottom faces must also vanish:

$$\gamma_{xz} = \left(u_{x,z} + u_{z,x}\right)\Big|_{z=\pm h} = 0 \tag{3}$$

Enforcing the conditions in eqn (3) gives rise to the displacement components of the form:

$$u_x(x,z) = u + h\zeta\theta - \tfrac{1}{6}(3\zeta^2 - 1)hw_{1,x} - h\zeta\left(\tfrac{1}{3}\zeta^2 - \tfrac{1}{5}\right)\gamma,$$

$$u_z(x,z) = w + \zeta w_1 + \left(\zeta^2 - \tfrac{1}{5}\right)w_2, \tag{4}$$

where

$$\gamma = \tfrac{5}{4}\left[\theta(x) + w_{,x}(x)\right] + w_{2,x}(x)$$

Note that the resulting displacements in eqn (4) are in terms of the same five kinematic variables as in Tessler's {1,2}-order theory, and the quadratic u_z is the same for both theories. The first three variables, $u(x)$, $\theta(x)$, and $w(x)$, are the Reissner weighted-average displacements, defined in eqn (2), whereas $w_1(x)$ and $w_2(x)$ represent the higher-order terms that account for the stretching of the beam through the thickness. The cubic u_x has an hierarchical form such that if the higher-order terms $w_{1,x}$ and γ are eliminated, the displacement field is reduced to the {1,2}-order theory with a linear axial displacement distribution.

The displacements in eqn (4) represent the beam analog of the {3,2}-order plate displacement approximations explored by Tessler[17] in the context of a hierarchical recovery of the {1,2} results using the {3,2}-order displacement, strain, and stress expansions.

Stress–strain relations

Either plane strain or plane stress constitutive relations can be developed for a laminated beam resulting in the stress–strain relations in the form:

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{zz} \\ \tau_{xz} \end{Bmatrix}_g^{(k)} = \begin{bmatrix} \bar{C}_{11} & \bar{C}_{13} & 0 \\ \bar{C}_{13} & \bar{C}_{33} & 0 \\ 0 & 0 & \bar{C}_{55} \end{bmatrix}_g^{(k)} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{zz} \\ \gamma_{xz} \end{Bmatrix}_g^{(k)} \tag{5}$$

The complete derivation of eqn (5) can be found, for example, in Cook[21].

Strain-displacement relations

In this beam theory, two distinct approaches are used to express the strains in terms of the kinematic variables. The axial ε_{xx} and transverse shear γ_{xz} strains are determined directly from the strain-displacement relations of elasticity theory, where the γ_{xz} strain is further augmented with a shear correction factor. The strains derived in this manner will be represented by continuous and differentiable polynomial functions through the thickness whose distributions are independent of the individual laminate properties; hence the superscript (k) will be dropped for these strains. Clearly, such strains can be regarded as some average representations of the "true" strains (i.e., those strains that satisfy the requisite equations of elasticity theory). The derivation of the transverse normal strain $\varepsilon_{zz}^{(k)}$, however, begins with an average stress assumption for σ_{zz}, which is assumed to have a cubic polynomial distribution through the thickness. These strain developments are summarized as follows.

The axial, *average strain* ε_{xx} is obtained from the linear strain-displacement relations as

$$\varepsilon_{xx} = u_{x,x} = \varepsilon_{x0} + \kappa_{x0}\phi_1 + \varepsilon_H\phi_2 + \kappa_H\phi_3 \tag{6}$$

where the strain measures, curvatures, and the thickness distribution functions ϕ_i are defined as

$$\begin{aligned}(\varepsilon_{x0}, \varepsilon_H) &= (u_{,x}, hw_{1,xx}),\\ (\kappa_{x0}, \kappa_H) &= (\theta_{,x}, \tfrac{5}{4}(w_{,xx} + \theta_{,x}) + w_{2,xx}),\\ (\phi_1, \phi_2, \phi_3) &= (h\zeta, 1/6 - \zeta^2/2,\ h(\zeta/5 - \zeta^3/3))\end{aligned} \tag{7}$$

The transverse shear average strain is obtained from the linear strain-displacement relations of elasticity and is augmented with a shear correction factor k, i.e.,

$$\begin{aligned}\gamma_{xz}^{\text{corr}} &= k\gamma_{xz} = k(u_{x,z} + u_{z,x}) = k\gamma_{xz0}\phi_{xz},\\ (\gamma_{xz0}, \phi_{xz}) &= (\theta + w_{,x}, 5(1-\zeta^2)/4)\end{aligned} \tag{8}$$

The shear correction factor is introduced in eqn (8) in anticipation that for certain material systems and lay-ups, a correction in the value of the transverse shear energy may be necessary; the shear correction factor provides a simple and effective mechanism for implementing such a correction. The motivation for circumventing the determination of the transverse normal strain directly from the strain-displacement relations is as follows. The strain-displacement relations which employ the displacement assumptions eqn (4) give rise to a continuous through the thickness ε_{zz} strain which would represent only an average distribution of this strain through the thickness. This in turn would result in a σ_{zz} which for laminated beams may exhibit discontinuity along ply interfaces. However, according to elasticity theory, σ_{zz} must be continuous through the thickness and $\varepsilon_{zz}^{(k)}$ may be discontinuous along ply interfaces. The approach introduced by Tessler[16] enables the derivation of an improved $\varepsilon_{zz}^{(k)}$ that will be discontinuous at ply interfaces. Importantly, the desired simplicity of the theory is retained. For mechanical loading, σ_{zz} is closely approximated by a cubic expansion through the thickness as

$$\sigma_{zz} = \sum_{n=0}^{3} \sigma_{zn}\zeta^n \tag{9}$$

in which the four σ_{zn} coefficients need to be determined. Two of the coefficients are found from the equilibrium equation of elasticity theory, i.e.,

$$\tau_{xz,x} + \sigma_{zz,z} = 0 \tag{10}$$

Since the transverse shear stress satisfies traction-free boundary conditions on the top and bottom surfaces of the beam, i.e. $\tau_{xz}(x, \pm h) = 0$, the derivatives of the shear stress $\tau_{xz,x}$ at the top and bottom faces must vanish. To satisfy the equilibrium equation, the derivatives of the transverse normal stress must also vanish on the top and bottom surfaces:

$$\sigma_{zz,z}(x, \pm h) = 0 \tag{11}$$

These exact equilibrium traction conditions reduce σ_{zz} to the form

$$\sigma_{zz} = \sigma_{z0} + \sigma_{z1}\phi_5,\ \phi_5 = (\zeta - \zeta^3/3) \tag{12}$$

The remaining two coefficients are found by forcing the $\varepsilon_{zz}^{(k)}$ strain to be least-squares compatible with the *corrected* average strain derived from the strain-displacement relation (the notation $\varepsilon_{zz}^{(k)}$ with the superscript (k) implies that the strain is piece-wise (ply-level) continuous):

$$\text{minimize} \int_{-h}^{h} (\varepsilon_{zz}^{(k)} - u_{z,z}^{\text{corr}})^2\, dz \tag{13}$$

where the *corrected* average strain is determined from the strain-displacement relations as

$$u_{z,z}^{\text{corr}} = k_{z0}\varepsilon_{z0} + 2k_{z1}\kappa_{z0}\phi_1,\ (\varepsilon_{z0}, \kappa_{z0}) = (w_1/h, w_2/h^2) \tag{14}$$

where k_{z0} and k_{z1} are the transverse normal correction factors, and ε_{z0} and κ_{z0} denote the transverse strain measure and curvature, respectively. Obtaining $\varepsilon_{zz}^{(k)}$ from the constitutive relations, eqn (5), results in

$$\varepsilon_{zz}^{(k)} = 1/\bar{C}_{33}^{(k)}(\sigma_{zz} - \bar{C}_{13}^{(k)}\varepsilon_{xx}) \tag{15}$$

Introducing eqns (14) and (15) into eqn (13), where the minimization is performed with respect to the undetermined coefficients, σ_{z0} and σ_{z1}, results in two algebraic equations from which these coefficients are readily determined. Eqn (15) is then simplified to yield the transverse normal strain of the form

$$\begin{aligned}\varepsilon_{zz}^{(k)} &= \varepsilon_{x0}\psi_1^{(k)} + k_{z0}\varepsilon_{z0}\psi_2^{(k)} + \varepsilon_H\psi_3^{(k)} + \kappa_{x0}\psi_4^{(k)}\\ &\quad + k_{z1}\kappa_{z0}\psi_5^{(k)} + \kappa_H\psi_6^{(k)}\end{aligned} \tag{16}$$

where $\psi_i^{(k)}$ depend on the thickness coordinate, ζ, and the elastic stiffness coefficients, $\bar{C}_{mn}^{(k)}$. For their explicit form, refer to Cook[21].

In contrast to the linear distribution of $u_{z,z}^{\text{corr}}$ in eqn (14), $\varepsilon_{zz}^{(k)}$ can be discontinuous at the ply interfaces and is piece-wise cubic. As will be demonstrated by numerical comparisons with exact elasticity solutions, this form of $\varepsilon_{zz}^{(k)}$ ensures superior through-the-thickness predictions and improves the overall beam response.

Variational principle

The principle of virtual work is employed to construct the beam equilibrium equations and associated boundary conditions. Neglecting body forces, the virtual work principle can be stated as

$$\begin{aligned}&\int_V (\sigma_{xx}^{(k)}\delta\varepsilon_{xx} + \sigma_{zz}\delta\varepsilon_{zz}^{(k)} + \tau_{xz}^{(k)}\delta\gamma_{xz})\, dA\, dx\\ &- \int_{S^+} q^+\delta u_z(x,h)\, dx\, dy + \int_{S^-} q^-\delta u_z(x,-h)\, dx\, dy\\ &+ \int_A [T_{x0}\delta u_x(0,z) + T_{z0}\delta u_z(0,z)]\, dA\\ &- \int_A [T_{xL}\delta u_x(L,z) + T_{zL}\delta u_z(L,z)]\, dA = 0\end{aligned} \tag{17}$$

where δ is the variational operator, A is the cross-sectional area of the beam, and S^+ and S^- denote the top and bottom surfaces of the beam, which, respectively, are subject to the normal pressure loads q^+ and q^-. The first term in eqn (17) is the volume integral representing the virtual work done by the stresses. The surface integrals denote the virtual work done by the external surface tractions.

Introducing the beam displacement assumptions and strain-displacement relations into eqn (17), then integrating over the beam cross-section and performing integration by parts results in the one-dimensional form of the virtual work principle

$$
\begin{aligned}
&\int_0^L \Big[N_{x,x}\delta u+\left(Q_x-M_{x,x}-\tfrac{5}{4}M_{H,x}\right)\delta\theta \\
&\quad+\left(\tfrac{5}{4}M_{H,xx}-Q_{x,x}-\bar{q}_1\right)\delta w+\left(N_z/h+hN_{H,xx}-\bar{q}_2\right)\delta w_1 \\
&\quad+\left(M_z/h^2+M_{H,xx}-\tfrac{4}{5}\bar{q}_1\right)\delta w_2\Big]\,\mathrm{d}x \\
&\quad+\sum_{\alpha=0,L}(1-2\alpha/L)\Big\langle\left[\bar{N}_{x\alpha}-N_x(\alpha)\right]\delta u(\alpha) \\
&\quad+\left[\bar{M}_{x\alpha}-M_x(\alpha)\right]\delta\theta(\alpha)+\left[\bar{M}_{1\alpha}-hN_H(\alpha)\right]\delta w_{1,x}(\alpha) \\
&\quad+\left[\bar{M}_{2\alpha}-M_H(\alpha)\right]\delta\left\{\tfrac{5}{4}\left[\theta(\alpha)+w_{,x}(\alpha)\right]+w_{2,x}(\alpha)\right\} \\
&\quad+\left[\bar{Q}_{x\alpha}-Q_x(\alpha)\right]\delta w(\alpha)+\left[\bar{Q}_{1\alpha}-hN_{H,x}(\alpha)\right]\delta w_1(\alpha) \\
&\quad+\left[\bar{Q}_{2\alpha}-M_{H,x}(\alpha)\right]\delta w_2(\alpha)\Big\rangle=0 \qquad (18)
\end{aligned}
$$

where the beam reactive and prescribed (superscribed with a bar) stress resultants are defined as

$$
\begin{aligned}
(N_x,N_z,N_H)&=\int_A\Big(\sigma_{xx}^{(k)}+\sigma_{zz}\psi_1^{(k)},\ k_{z0}\sigma_{zz}\psi_2^{(k)},\ \sigma_{xx}^{(k)}\phi_2 \\
&\qquad+\sigma_{zz}\psi_3^{(k)}\Big)\,\mathrm{d}A, \\
Q_x&=\int_A\tau_{xz}^{(k)}\phi_{xz}\,\mathrm{d}z,\quad (M_x,M_z,M_H) \\
&=\int_A\Big(\sigma_{xx}^{(k)}\phi_1+\sigma_{zz}\psi_4^{(k)},\ k_{z1}\sigma_{zz}\psi_5^{(k)},\ \sigma_{xx}^{(k)}\phi_3 \\
&\qquad+\sigma_{zz}\psi_6^{(k)}\Big)\,\mathrm{d}A, \\
(\bar{q}_1,\bar{q}_2)&=b(q^+-q^-,q^++q^-), \\
\bar{N}_{x\alpha}&=\int_A T_{x\alpha}\,\mathrm{d}z,\quad \bar{M}_{x\alpha}=\int_A T_{x\alpha}z\,\mathrm{d}z, \\
\bar{M}_{1\alpha}&=h\int_A T_{x\alpha}\phi_2\,\mathrm{d}z,\quad \bar{M}_{2\alpha}=\int_A T_{x\alpha}\phi_3\,\mathrm{d}z, \\
\bar{Q}_{x\alpha}&=\int_A T_{z\alpha}\,\mathrm{d}z,\quad \bar{Q}_{1\alpha}=\int_A T_{z\alpha}\zeta\,\mathrm{d}z, \\
\bar{Q}_{2\alpha}&=\int_A T_{z\alpha}\left(\zeta^2-\tfrac{1}{5}\right)\mathrm{d}z,\quad (\alpha=0,L) \qquad (19)
\end{aligned}
$$

Equilibrium equations and boundary conditions

The equilibrium equations and boundary conditions are obtained from the principle of virtual work, eqn (18). The expressions associated with the arbitrary kinematic variations must vanish independently, resulting in the following equilibrium equations:

$$
\begin{aligned}
&N_{x,x}=0,\ \ N_z/h+hN_{H,xx}-\bar{q}_2=0,\ \ \tfrac{5}{4}M_{H,xx}-Q_{x,x}-\bar{q}_1=0, \\
&Q_x-M_{x,x}-\tfrac{5}{4}M_{H,x}=0,\ M_z/h^2+M_{H,xx}-\tfrac{4}{5}\bar{q}_1=0. \qquad (20)
\end{aligned}
$$

The remaining terms in eqn (18) must also vanish independently, thus giving rise to the boundary conditions for the theory. Evidently, either tractions or displacements can be prescribed at $x=0$ and L, such that

$$
\begin{aligned}
&\bar{N}_{x\alpha}=N_x(\alpha)\ \text{or}\ \delta u(\alpha)=0,\ \ \bar{M}_{x\alpha}=M_x(\alpha)\ \text{or}\ \delta\theta(\alpha)=0, \\
&\bar{M}_{1\alpha}=hN_H(\alpha)\ \text{or}\ \delta w_{1,x}(\alpha)=0,\ \ \bar{M}_{2\alpha}=M_H(\alpha)\ \text{or}\ \delta\gamma(\alpha)=0, \\
&\bar{Q}_{x\alpha}=Q_x(\alpha)\ \text{or}\ \delta w(\alpha)=0,\ \ \bar{Q}_{1\alpha}=hN_{H,x}(\alpha)\ \text{or}\ \delta w_1(\alpha)=0, \\
&\bar{Q}_{2\alpha}=M_{H,x}(\alpha)\ \text{or}\ \delta w_2(\alpha)=0,\ \ (\alpha=0,L). \qquad (21)
\end{aligned}
$$

Beam constitutive relations

Introducing the strains eqns (6), (8) and (16) into the stress-strain relations eqn (5), and integrating the stress resultants eqn (19), yields the beam constitutive relations of the form

$$
\begin{Bmatrix} N_x \\ N_z \\ N_H \\ M_x \\ M_z \\ M_H \\ Q_x \end{Bmatrix}
=\begin{bmatrix}
A_{11} & k_{z0}A_{12} & A_{13} & B_{11} & k_{z1}B_{12} & B_{13} & 0 \\
k_{z0}A_{12} & k_{z0}^2A_{22} & k_{z0}A_{23} & k_{z0}B_{21} & k_{z0}k_{z1}B_{22} & k_{z0}B_{23} & 0 \\
A_{13} & k_{z0}A_{23} & A_{33} & B_{31} & k_{z1}B_{32} & B_{33} & 0 \\
B_{11} & k_{z0}B_{21} & B_{31} & D_{11} & k_{z1}D_{12} & D_{13} & 0 \\
k_{z1}B_{12} & k_{z1}k_{z0}B_{22} & k_{z1}B_{32} & k_{z1}D_{12} & k_{z1}^2D_{22} & k_{z1}D_{23} & 0 \\
B_{13} & k_{z0}B_{23} & B_{33} & D_{13} & k_{z1}D_{23} & D_{33} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & k^2G
\end{bmatrix}
\times\begin{Bmatrix} \varepsilon_{x0} \\ \varepsilon_{z0} \\ \varepsilon_H \\ \kappa_{x0} \\ \kappa_{z0} \\ \kappa_H \\ \gamma_{xz0} \end{Bmatrix} \qquad (22)
$$

where the A_{ij}, B_{ij}, D_{ij} and G coefficients represent the membrane, membrane-bending coupling, bending, and shear rigidities. For their explicit form, refer to Cook[21].

Equilibrium equations in terms of displacements

To facilitate a closed-form solution to the equilibrium eqn (20), subject to the appropriate boundary conditions eqn

(21), it is convenient to express eqn (20) in terms of the five kinematic variables of the theory. First, the strain measures and curvatures can be expressed in terms of the kinematic variables in matrix form as

$$\begin{Bmatrix} \varepsilon_{x0} \\ \varepsilon_{z0} \\ \varepsilon_H \\ \kappa_{x0} \\ \kappa_{z0} \\ \kappa_H \\ \gamma_{xz0} \end{Bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{h} & 0 & 0 & 0 \\ 0 & h\frac{\partial^2}{\partial x^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\partial}{\partial x} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{h^2} \\ 0 & 0 & \frac{5}{4}\frac{\partial^2}{\partial x^2} & \frac{5}{4}\frac{\partial}{\partial x} & \frac{\partial^2}{\partial x^2} \\ 0 & 0 & \frac{\partial}{\partial x} & 1 & 0 \end{bmatrix} \begin{Bmatrix} u \\ w_1 \\ w \\ \theta \\ w_2 \end{Bmatrix} \tag{23}$$

Substituting eqns (22) and (23) into eqn (20), the equilibrium equations in terms of the kinematic variables take the form

$$A_{11}u_{,xx} + \frac{k_{z0}A_{12}}{h}w_{1,x} + hA_{13}w_{1,xxx} + B_{11}\theta_{,xx} + \frac{k_{z1}B_{12}}{h^2}w_{2,x} + B_{13}\left[\frac{5}{4}(\theta_{,xx} + w_{,xxx}) + w_{2,xxx}\right] = 0 \tag{24}$$

$$\begin{aligned} h\Big\{ & A_{13}u_{,xxx} + \frac{k_{z0}A_{23}}{h}w_{1,xx} + hA_{33}w_{1,4x} + B_{31}\theta_{,xxx} \\ & + \frac{k_{z1}B_{32}}{h^2}w_{2,xx} + B_{33}\left[\tfrac{5}{4}(\theta_{,xxx} + w_{,4x}) + w_{2,4x}\right]\Big\} \\ & + \frac{1}{h}\Big\{k_{z0}A_{12}u_{,x} + \frac{k_{z0}^2A_{22}}{h}w_1 + hK_{z0}A_{23}w_{1,xx} + k_{z0}B_{21}\theta_{,x} \\ & + \frac{k_{z0}k_{z1}B_{22}}{h^2}w_2 + k_{z0}B_{23}\left[\tfrac{5}{4}(\theta_{,x} + w_{,xx}) + w_{2,xx}\right]\Big\} \\ & - \bar{q}_2 = 0 \end{aligned} \tag{25}$$

$$\begin{aligned} \tfrac{5}{4}\Big\{ & B_{13}u_{,xxx} + \frac{k_{z0}B_{23}}{h}w_{1,xx} + hB_{33}w_{1,4x} + D_{13}\theta_{,xxx} \\ & + \frac{k_{z1}D_{23}}{h^2}w_{2,xx} + D_{33}\left[\tfrac{5}{4}(\theta_{,xxx} + w_{,4x}) + w_{2,4x}\right]\Big\} \\ & - k^2G(\theta_{,x} + w_{,xx}) - \bar{q}_1 = 0 \end{aligned} \tag{26}$$

$$\begin{aligned} & B_{11}u_{,xx} + \frac{k_{z0}B_{21}}{h}w_{1,x} + hB_{31}w_{1,xxx} + D_{11}\theta_{,xx} \\ & + \frac{k_{z1}D_{12}}{h^2}w_{2,x} + D_{13}\left[\tfrac{5}{4}(\theta_{,xx} + w_{,xxx}) + w_{2,xxx}\right] \\ & + \tfrac{5}{4}\Big\{B_{13}u_{,xx} + \frac{k_{z0}B_{23}}{h}w_{1,x} + hB_{33}w_{1,xxx} + D_{13}\theta_{,xx} \\ & + \frac{k_{z1}D_{23}}{h^2}w_{2,x} + D_{33}\left[\tfrac{5}{4}(\theta_{,xx} + w_{,xxx}) + w_{2,xxx}\right]\Big\} \\ & - k^2G(\theta + w_{,x}) = 0 \end{aligned} \tag{27}$$

$$\begin{aligned} & B_{13}u_{,xxx} + \frac{k_{z0}B_{23}}{h}w_{1,xx} + hB_{33}w_{1,4x} + D_{13}\theta_{,xxx} \\ & - \frac{k_{z1}D_{23}}{h^2}w_{2,xx} + D_{33}\left[\tfrac{5}{4}(\theta_{,xxx} + w_{,4x}) + w_{2,4x}\right] \\ & - \frac{1}{h^2}\Big\{k_{z1}B_{12}u_{,x} + \frac{k_{z1}k_{z0}B_{22}}{h}w_1 + k_{z1}hB_{32}w_{1,xx} \\ & - k_{z1}D_{12}\theta_{,x} + \frac{k_{z1}k_{z0}D_{22}}{h^2}w_2 \\ & - k_{z1}D_{23}\left[\tfrac{5}{4}(\theta_{,x} + w_{,xx}) + w_{2,xx}\right]\Big\} - \tfrac{4}{5}\bar{q}_1 = 0 \end{aligned} \tag{28}$$

where $w_{,4x} = w_{,xxxx}$.

Similarly, the boundary conditions eqn (21) can also be readily expressed in terms of the five kinematic variables if necessary. Eqns (24)–(28), subject to the boundary conditions eqn (21), can then be solved simultaneously to determine the five kinematic variables and subsequent displacement, strain, and stress distributions in the beam.

Reduction to lower-order theories

The hierarchical displacement approximation of the {3,2} order theory permits a straightforward reduction to several lower-order theories. By eliminating the higher-order displacement terms $w_{1,x}(x)$ and $\gamma(x)$ from eqn (4), the displacement field reduces to the {1,2} form given by Tessler[15]:

$$u_x(x,z) = u + h\zeta\theta, \quad u_z(x,z) = w + \zeta w_1 + \left(\zeta^2 - \tfrac{1}{5}\right)w_2 \tag{29}$$

Consequently, the higher-order strain and curvature terms ε_H and κ_H are eliminated from the theory. This results in the simplification of the equilibrium equations, boundary conditions and stress resultants, respectively, such that all of the terms with a subscript H are eliminated.

The {1,2} displacement theory can further be reduced to Timoshenko theory by neglecting the Poisson effect (i.e., by setting $\nu_{13} = 0$), thus ignoring the coupling between the axial and transverse stretching of the beam. Furthermore, the weighting function associated with the computation of the transverse shear stiffness, which is parabolic, needs to be set to unity to simulate the constant shear distribution according to Timoshenko theory. While this yields the Timoshenko theory equilibrium equations, the boundary conditions for both {1,2}-order and Timoshenko theories are the same. The results of Timoshenko theory can further be reduced to those of classical beam theory by setting the transverse shear rigidity to be infinite, i.e., $G = \infty$.

CYLINDRICAL BENDING PROBLEM

The problem of cylindrical bending is considered for the beam in a state of plane-strain. The beam is simply-supported at the ends $x = 0$ and $x = L$ and is subjected to a transverse load in the form of a half-sine wave applied at the top surface, i.e.,

$$q^+(x) = q_o \sin(\pi x/L), \quad q^-(x) = 0 \tag{30}$$

where q_o is the amplitude of the loading.

A closed-form solution is derived by first assuming appropriate trigonometric distributions of the kinematic variables

$$u = U\cos(\pi x/L),\ \theta = \Theta\cos(\pi x/L),$$
$$w = W\sin(\pi x/L),\ w_1 = W_1\sin(\pi x/L),\ w_2 = W_2\sin(\pi x/L) \tag{31}$$

which satisfy the simply-supported end conditions exactly:

$$\begin{aligned} \text{At } x=0: N_x(0) &= M_x(0) = N_H(0) = M_H(0) \\ &= w(0) = w_1(0) = w_2(0) = 0 \\ \text{At } x=L: N_x(L) &= M_x(L) = N_H(L) = M_H(L) \\ &= w(L) = w_1(L) = w_2(L) = 0 \end{aligned} \tag{32}$$

Introducing eqns (30) and (31) into the equilibrium eqns (24)–(28) results in five algebraic equations in which the trigonometric functions are factored out, leaving only the amplitudes U, Θ, W, W_1, and W_2 as unknowns. Once the displacement amplitudes are determined, the kinematic variables are completely defined, giving rise to the strain measures and curvatures. The displacements, strains and stresses are then computed in a straightforward manner and are subsequently compared with the corresponding exact elasticity solutions, e.g., refer to Pagano[22] and Burton and Noor[23].

Since, in composite and sandwich laminates, the actual shear strain distribution is generally discontinuous at the ply interfaces and the shear stress is only piecewise continuous, the two-dimensional equilibrium equation of elasticity theory needs to be integrated to obtain an improved approximation for the transverse shear stress. This well-established procedure has been modified by Cook[21] to ensure accurate shear stress computations for unsymmetric and sandwich laminates—the type of laminates for which the integration approach results in rather inaccurate shear stresses.

Numerical results

Cook[21] assessed the {3,2}-order theory by examining a wide range of laminates and material systems. As expected, the best performance is achieved for homogeneous beams, where the displacement, strains and stresses, both due to the {3,2} and {1,2} theories, correlate exceptionally well with exact elasticity solutions even for the thick beams with $L/2h = 4$. For homogeneous beams, all correction factors take on the value of unity ($k^2 = k_{z0} = k_{z1} = 1.0$), i.e. no corrections are required.

Presently, the numerical assessment is focused on the material systems and aspect ratios which expose the highest degree of modeling difficulty for the theory. In particular, the results for two types of moderately thick and thick composite beams ($L/2h = 10$ and 4) are presented: (a) graphite/epoxy (GR/EP) unsymmetric laminated beams with a lay-up of $[0_4/90_4/0_4/90_4]_T$ and (b) GR/EP, PVC-core symmetric sandwich beams with a lay-up of $[0_4/90_2/0_4/90_2/0_4/\text{PVC Core}]_S$. The material properties and geometric data are summarized in *Table 1*, and the transverse shear and transverse normal correction factors are given in *Table 2*. For details on the determination of the correction factors, the reader is referred to Cook[21].

Laminated composite beams

In *Figures 2* and *3*, the displacement, strain, and stress through-thickness distributions for the moderately thick ($L/2h = 10$) and thick ($L/2h = 4$) unsymmetric, GR/EP laminated beams $[0_4/90_4/0_4/90_4]_T$ are shown. For comparison purposes, the {1,2}-theory results are included for the thick case only where the differences in results are most pronounced. Due to the lack of symmetry in the lay-up, the midplane is in tension with respect to the ε_{xx} strain and is under a compressive ε_{zz} strain. The transverse displacement is non-linear through the thickness, and is within 0.1% of the exact solution for the $L/2h = 10$ case and within 2% for the $L/2h = 4$ beam. The exact σ_{zz} stress is seen to be more complex through the thickness than its cubic approximation within the present (and {1,2}) theory. Nonetheless, the qualitative comparison is quite adequate. Also, the cubic distribution of the axial strain, ε_{xx}, is quite accurate, underestimating the maximum value only slightly for $L/2h = 10$. For $L/2h = 4$, however, the results are significantly less accurate, with the present theory results being consistently superior to the {1,2} theory results. As

Table 1 Material properties and lamina geometric data

Graphite/Epoxy (GR/EP)	$E_L = 22.9$ Msi	$\nu_{LT} = 0.32$	$G_{LT} = 0.86$ Msi
	$E_T = 1.39$ Msi	$\nu_{TT} = 0.49$	$G_{TT} = 0.468$ Msi
Polyvinyl chloride (PVC)	$E = 15.08$ ksi	$\nu = 0.3$	$G = 5.80$ ksi
Ply thickness		0.00625 in	
Beam width		1.0 in	
Sandwich core thickness		0.8 × Total thickness	

Notation: L = Longitudinal direction, T = Transverse direction

Table 2 {3,2} and {1,2} theory correction factors

Material system	k^2		k_{z0}		k_{z1}	
	{3,2}	{1,2}	{3,2}	{1,2}	{3,2}	{1,2}
GR/EP laminate	0.76187	0.73262	1.21668	1.0	1.01975	1.0
GR/EP–PVC sandwich	0.30666	0.37301	1.24326	1.24326	1.59569	1.76405

expected, the results are in excellent agreement with the exact solutions for the moderately thick beam and are somewhat less accurate for the thick beam.

Sandwich beams

Sandwich laminates present a unique challenge to any approximate bending theory owing to the drastic change in the material properties through the thickness. The face sheets of a sandwich are stiff while the core material is lightweight and, generally, is several orders of magnitude more compliant.

Figures 4 and *5* show the displacement, strain and stress variations, through-thickness, for the moderately thick ($L/2h = 10$) and thick ($L/2h = 4$) symmetric sandwich beams. Characteristically for a sandwich laminate, the axial stress σ_{xx} is carried by the stiff GR/EP face sheets whereas the transverse shear stress τ_{xz} is almost exclusively carried

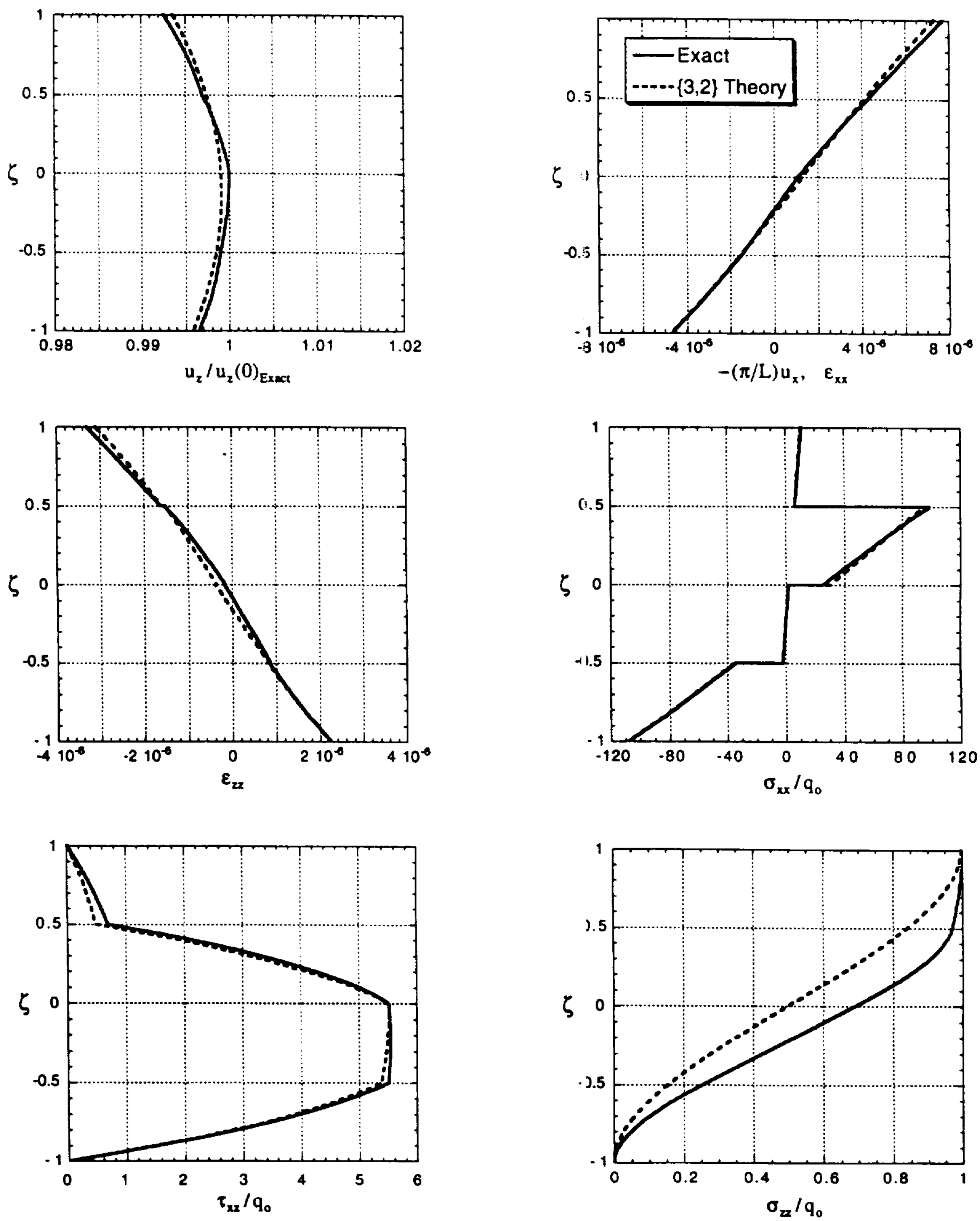

Figure 2 Unsymmetric GR/EP laminate, $L/2h = 10$

by the PVC core. Note that for the moderately thick beam, the deflection is over estimated by about 2%. For the thick beam, the {3,2}- and {1,2}-order theories over estimate the deflection by 12% and 37%, respectively. Such large discrepancies could have been avoided by way of correction factors appropriate for the thick regime. The axial displacement and strain have a pronounced zigzag distribution through the thickness according to the exact solution. For these quantities, the cubic variations of the {3,2}-order theory predict the response adequately in the face sheets and at the midplane. Consequently, the stresses and strains on the top and bottom faces, where these quantities are usually the largest, are accurately predicted by the theory. Notice that the linear approximation for the axial displacement in the {1,2} theory underestimates the axial strain, resulting in a significant underestimation of the axial stress. The {3,2}-order theory captures σ_{xx} at the top and bottom surfaces adequately, while the same stress for the {1,2} theory is

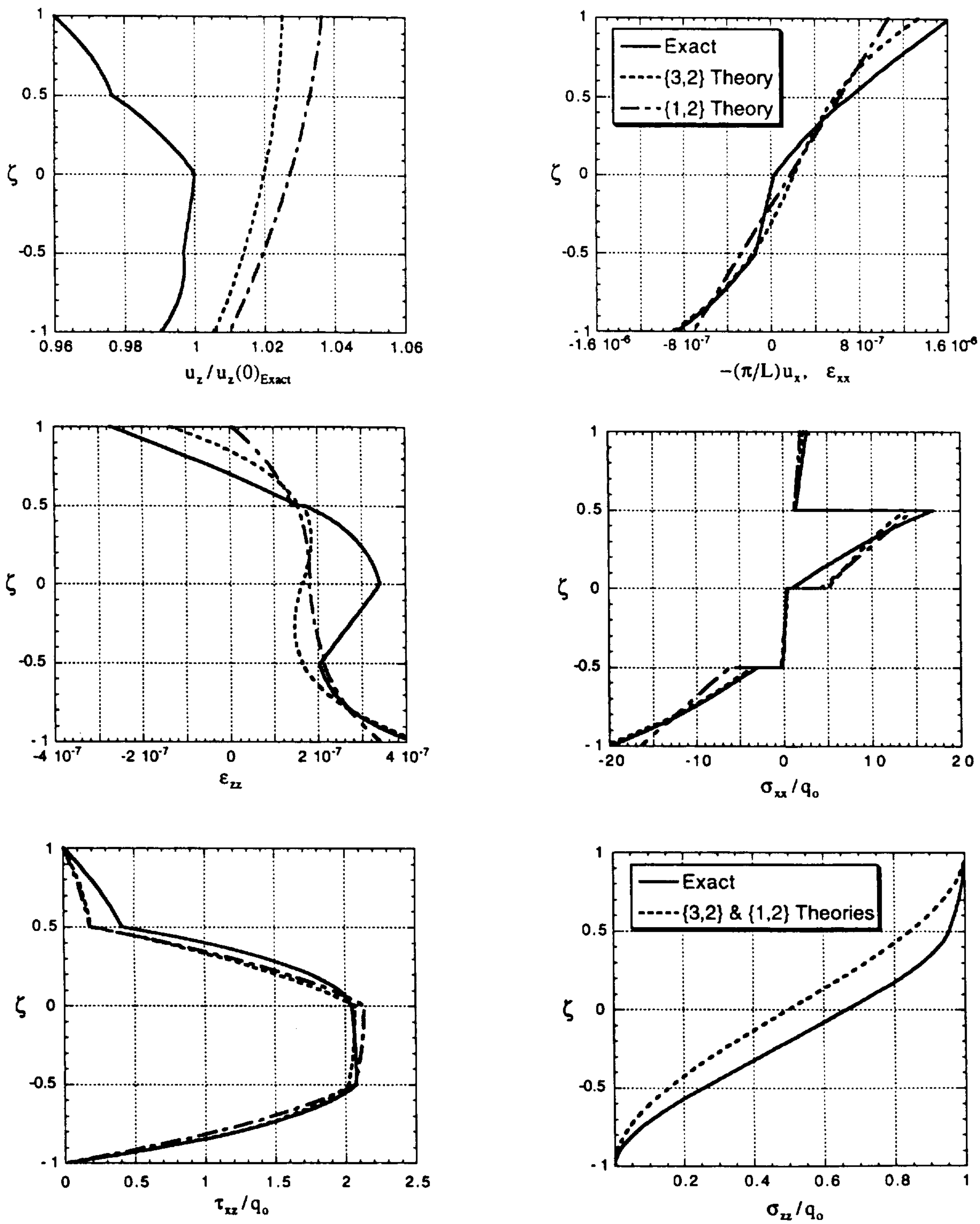

Figure 3 Unsymmetric GR/EP laminate, $L/2h = 4$

85% in error. The quality of these results suggests that the span to thickness ratio of $L/2h = 4$ may constitute the practical limit for application of this theory to sandwich beams.

CONCLUSIONS

A {3,2}-order bending theory for laminated composite and sandwich beams has been developed. The theory employs a hierarchical form of a third-order axial displacement and a quadratic transverse normal displacement, and possesses the same kinematic variables as the {1,2}-order theory. The assumed kinematic field results in an *average* parabolic shear strain such that zero shear-stress boundary conditions on the top and bottom beam surfaces are fulfilled exactly. An independent expansion for the transverse normal stress is also introduced, thus enabling accurate transverse normal

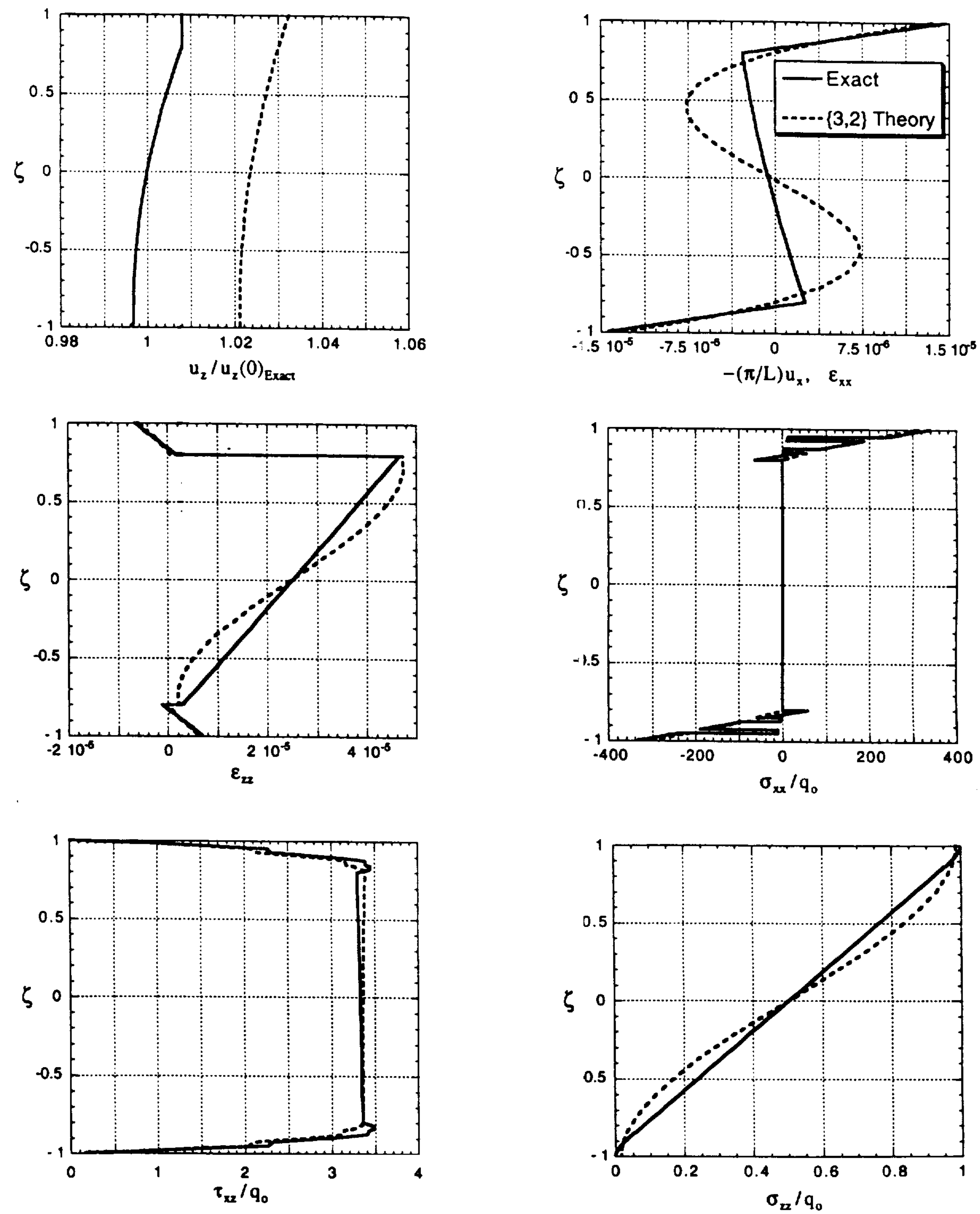

Figure 4 GR/EP–PVC symmetric sandwich, $L/2h = 10$

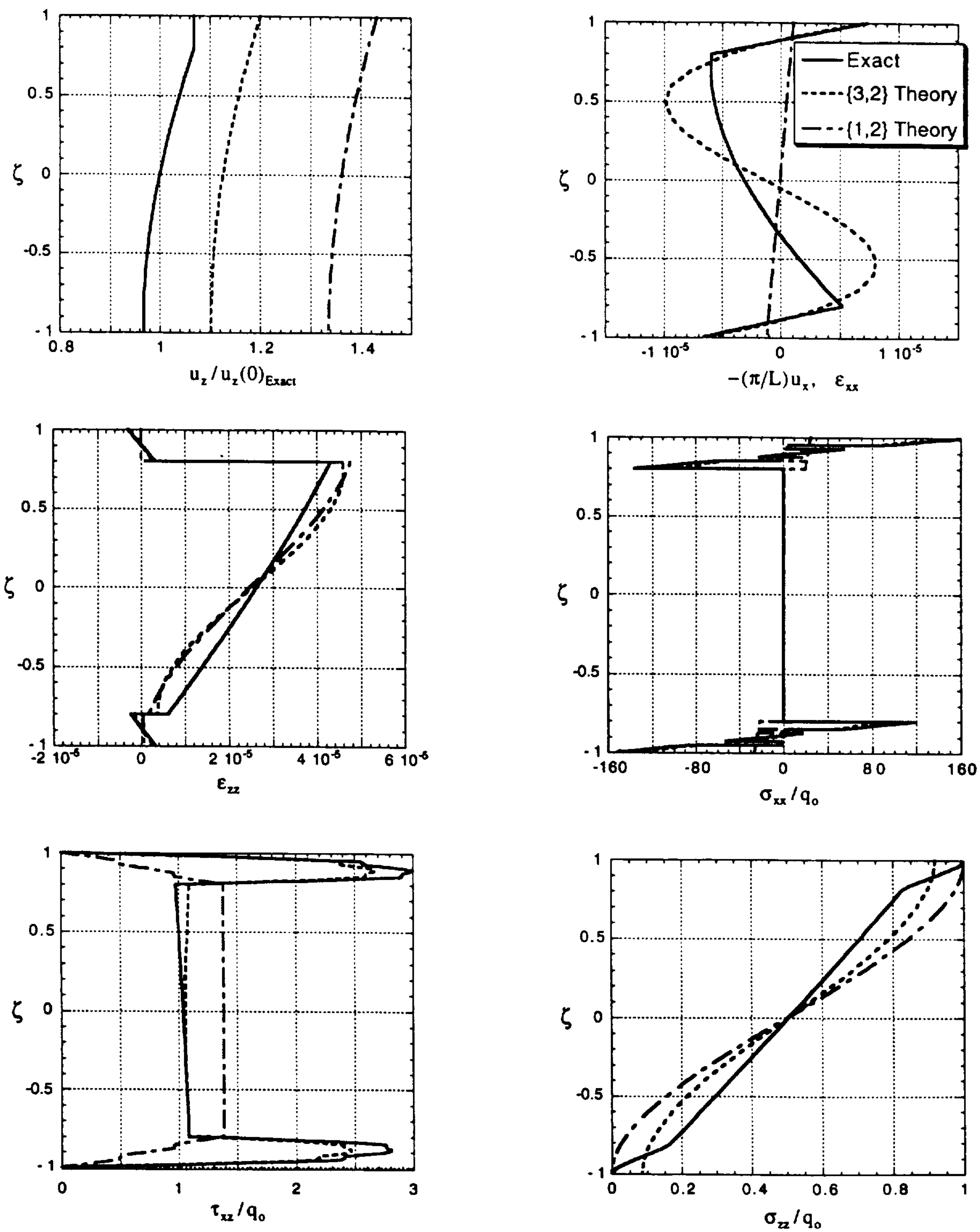

Figure 5 GR/EP–PVC symmetric sandwich, $L/2h = 4$

strain and stress predictions. Appropriate transverse shear and transverse normal correction factors are used to adjust the shear and thickness-stretch response of the beam. A closed-form solution to the cylindrical bending of moderately thick and thick unsymmetric laminated composite and symmetric sandwich beams has been developed. The numerical results show that the {3,2}-order theory has some advantages over the {1,2}-order theory, particularly in predicting the axial response in thick sandwich laminates.

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