Design and Performance Evaluation of a UWB Communication and Tracking System for Mini-AERCam

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Abstract

NASA Johnson Space Center (JSC) is developing a low-volume, low-mass, robotic free-flying camera known as Mini-AERCam (Autonomous Extra-vehicular Robotic Camera) to assist the International Space Station (ISS) operations. Mini-AERCam is designed to provide astronauts and ground control real-time video for camera views of ISS. The system will assist ISS crewmembers and ground personnel to monitor ongoing operations and perform visual inspections of exterior ISS components without requiring extravehicular activity (EAV).

Mini-AERCam consists of a great number of subsystems. Many institutions and companies have been involved in the R&D for this project. A Mini-AERCam ground control system has been studied at Texas A&M University [3]. The path planning and control algorithms that direct the motions of Mini-AERCam have been developed through the joint effort of Carnegie Mellon University and the Texas Robotics and Automation Center [5]. NASA JSC has designed a layered control architecture that integrates all functions of Mini-AERCam [8]. The research described in this report is part of a larger effort focused on the communication and tracking subsystem that is designed to perform three major tasks:

1. To transmit commands from ISS to Mini-AERCam for control of robotic camera motions (downlink);
2. To transmit real-time video from Mini-AERCam to ISS for inspections (uplink);
3. To track the position of Mini-AERCam for precise motion control.

The ISS propagation environment is unique due to the nature of the ISS structure and multiple RF interference sources [9]. The ISS is composed of various truss segments, solar panels, thermal radiator panels, and modules for laboratories and crew accommodations. A tracking system supplemental to GPS is desirable both to improve accuracy and to eliminate the structural blockage due to the close proximity of the ISS which could at times limit the number of GPS satellites accessible to the Mini-AERCam. Ideally, the tracking system will be a passive component of the communication system which will need to operate in a time-varying multipath environment created as the robot camera moves over the ISS structure. In addition, due to many interference sources located on the ISS, SSO, LEO satellites and ground-based transmitters, selecting a frequency for the ISS and Mini-AERCam link which will coexist with all interferers poses a major design challenge. To meet all of these challenges, ultrawideband (UWB) radio technology is being studied for use in the Mini-AERCam communication and tracking subsystem. The research described in this report is focused on design and evaluation of passive tracking system algorithms based on UWB radio transmissions from mini-AERCam.

Introduction

UWB Technology

Ultrawideband radio, in particular impulse or carrier-free radio technology, is one promising new technology for low-power communications and tracking systems. It has been utilized for decades by the military and law enforcement agencies for fine-resolution ranging, covert communications and ground-penetrating radar applications. In February 2002, the Federal Communications Commission (FCC) approved the deployment of this technology in the commercial sector under Part 15 of its regulations [6]. UWB technology holds great potential to provide significant benefits in many applications such as precise positioning, short-range multimedia services and high-speed mobile wireless communications.

The DARPA study panel that coined the term *ultrawideband* in the 1990s defines it as a system with a fractional bandwidth greater than 25%. The basic concept of current UWB impulse radio technology is to transmit and receive an extremely short duration burst of RF energy – typically a few tens of picoseconds to a few nanoseconds in duration. Whereas conventional continuous-wave radio systems operate within a relatively narrow bandwidth, UWB operates across a wide range of frequency spectrum (a few GHz) by transmitting a series of low-power impulsive signals.

For the emerging technology of UWB radar and UWB wireless communications, the transmitted signal can be regarded as a uniform train of UWB pulses represented as

$$s(t) = \sum_{n=-\infty}^{+\infty} \omega(t - nT_r),$$

where T_r is the pulse repetition interval, and $\omega(t)$ is the pulse-shaping waveform which is often a Gaussian monocycle. In the time domain, the Gaussian monocycle is mathematically similar to the first derivative of Gaussian function. It has the form

$$\omega(t) = \frac{t}{\tau} e^{-(t/\tau)^2},$$

where τ is regarded as the duration of the monocycle. Figure 1 shows an ideal monocycle centered at 2 GHz in both the time and frequency domains [1].

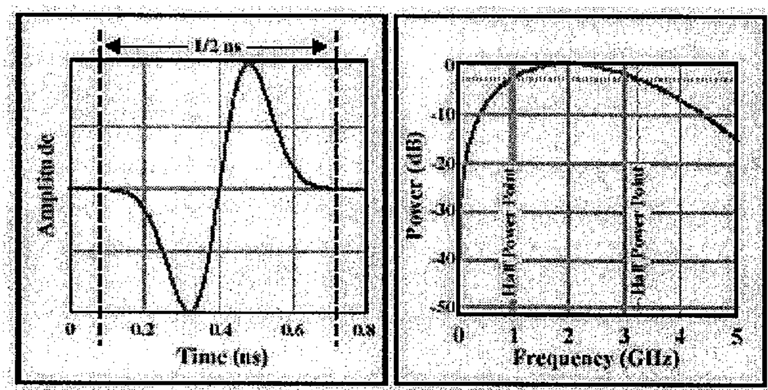

Figure 1. Gaussian monocycle in time and frequency domains

UWB impulse radio is characterized by several uniquely attractive features:

- Low-power, carrier-free, ultra-wide bandwidth signal transmissions
- Minimal interference on other RF systems due to extremely low power spectral density
- Immunity to interference from narrow-band RF systems due to extremely large bandwidth
- Immunity to multipath fading due to ample multipath diversity (RAKE receiver)
- Capable of precise positioning due to fine time resolution
- Capable of high data rate, multi-channel performance due to extremely large bandwidth
- Low-complexity, low-power baseband transceivers without intermediate frequency stage

Rapid technological advances have enabled the implementation of cost-effective UWB radar, communication, and tracking systems. Furthermore, antenna-array beamforming and space–time processing techniques promise further advancement in the capability of UWB technology to achieve long-range coverage, high capacity, and interference-free quality of reception [7].

Tracking Algorithm

To make Mini-AERCam coordinated maneuvers feasible, an accurate, robust, and self-contained tracking system that is small, low power, and low cost is required. Compared to GPS receivers, which can offer range resolution on the order of one meter, UWB radio can achieve sub-centimeter range resolution much faster and with less effort [1]. The experiment described in [2] demonstrates that UWB systems can provide range measurements accurate to the centimeter level over distances of kilometers, using only milliwatts of power from an omni-directional transceiver no bigger than a pager. In this research effort, the tracking subsystem will be designed to provide the precise positioning required for Mini-AERCam motion control as a byproduct of the UWB video communication system.

Many technologies have been applied to locating the source of radio signals, such as angle of arrival (AOA), time difference of arrival (TDOA) and relative signal strength (RSS). The extremely high fidelity of the UWB timing circuitry permits very precise measurement of propagation times for transmitted signals. This fine time resolution feature of UWB motivates us to apply a TDOA approach for tracking system design. We will utilize the header of the video data packets as the pilot signal to implement a passive tracking system in which the tracking rides on top of video communications [10].

Since electromagnetic waves travel with constant velocity in free space, the distance between the transmitter and the receiver is directly proportional to the propagation time. The TDOA approach determines the possible position of the transmitter by examining the difference in time at which the same signal arrives at multiple receivers. Each TDOA measurement determines a hyperboloid corresponding to the surface of constant range difference between the two receivers. At least three receivers are needed for a 2-D location estimate and four receivers for a 3-D location estimation. The intersection of the hyperboloids corresponding to all the TDOA measurements provides the location of the transmitter.

Suppose N receivers measure the time of arrival (TOA) of pilot signals from the transmitter in a 2-D case. The TOA estimates of the signal from receiver i and j are denoted t_i and t_j respectively. A range difference measurement r_k can be calculated from the TDOA measurement as follows:

$$r_k = c(t_i - t_j) = d_i - d_j = f_k(x,y), \tag{1}$$

where d_i and d_j are the distances from the transmitter to receivers i and j, respectively, and c is the propagation velocity of the signals, which is generally taken to be the speed of light in free space. Since the positions of all the receivers are known, this equation is a function $f_k(x,y)$ only of the unknown coordinate position of the transmitter (x,y).

In many cases, the transmitter location is determined by finding a least-squares solution to a linearized version of Equation (1). The linearization is given by

$$f_k(x,y) = f_k(x_0,y_0) + \frac{\partial f_k}{\partial x}(x - x_0) + \frac{\partial f_k}{\partial y}(y - y_0),$$

where the partial derivatives are evaluated at the *a priori* estimate for the transmitter position (x_0,y_0). This estimate is normally a previous solution for the transmitter position. The linearized version of Equation (1) can then be expressed as a matrix equation of the form $\mathbf{A}\mathbf{p}_T = \mathbf{b}$, where

$$
\mathbf{A} = \begin{bmatrix} \dfrac{\partial f_1}{\partial x} & \dfrac{\partial f_1}{\partial y} \\ \vdots & \vdots \\ \dfrac{\partial f_K}{\partial x} & \dfrac{\partial f_K}{\partial y} \end{bmatrix} \quad \mathbf{p}_T = \begin{bmatrix} (x - x_0) \\ (y - y_0) \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} r_1 - f_1(x_0, y_0) \\ \vdots \\ r_K - f_K(x_0, y_0) \end{bmatrix}.
$$

The least squares solution to this matrix equation is the estimated position of the transmitter, which is given by

$$
\hat{\mathbf{p}}_T = (\mathbf{A}^T \mathbf{A})^{-1} \mathbf{A}^T \mathbf{b}.
$$

The variance of the position estimate is related to the variance of the time estimate. Tracking requires that the direct path portion of the UWB signal be located and its arrival time inserted into the tracking algorithm. Hence, the accuracy of the TDOA estimates is very critical for position tracking. The conventional approach to estimating TDOA is to compute the cross-correlation of an identical transmitted signal arriving at different receivers. The TDOA estimate for each pair of receivers is given by the delay that maximizes the cross-correlation function. To complete the tracking algorithm, the sequence of position estimates is passed to a Kalman filter to update the current estimate of position.

Discussion of Results

The results of the research completed by the PI during the 2004 NASA Faculty Fellowship Program are discussed in the remainder of this report. The discussion is divided into two sections. The first presents a careful analysis of the statistical properties of a particular TDOA localization algorithm currently identified for use in the mini-AERCam communication and tracking subsystem. The second presents a proposed modification to this algorithm that should improve the localization accuracy.

Analysis of Current Algorithm

The TDOA localization algorithm currently proposed for use in the mini-AERCam communication and tracking subsystem was developed and analyzed by Chan and Ho in [4]. The statistical properties of the algorithm are analyzed to some extent in [4], and while the analysis presented there is essentially correct, it is also sketchy and incomplete, making a thorough performance evaluation of the algorithm problematic. To correct this deficiency, the PI performed a careful and complete analysis of the statistical properties of this algorithm in two dimensions. The results of that analysis are discussed in this section.

Assume that there is one transmitter located at an unknown location (x_0, y_0) in two-dimensional space and $M+1$ receivers located at positions $\{(0,0), (x_1, y_1), (x_2, y_2), \ldots, (x_M, y_M)\}$, which are assumed to be known precisely.

Further, assume that measurements of the relative time delays $\{d_1, d_2, \ldots, d_M\}$ between the arrival of the transmitted signal at receiver $(0,0)$ and each of the other locations $(x_1, y_1), \ldots, (x_M, y_M)$ are available. If the propagation velocity of the signals is given by the constant c, then it can be shown that the following system of linear equations is satisfied:

$$\mathbf{G}_0 \mathbf{u}_0 = \mathbf{h}_0, \tag{2}$$

where,

$$\mathbf{u}_0 = \begin{bmatrix} x_0 \\ y_0 \\ r_0 \end{bmatrix}, \quad r_0 = \sqrt{x_0^2 + y_0^2}, \quad \mathbf{G}_0 = -2 \cdot \begin{bmatrix} x_1 & y_1 & cd_1 \\ x_2 & y_2 & cd_2 \\ \vdots & \vdots & \vdots \\ x_M & y_M & cd_M \end{bmatrix}, \quad \mathbf{h}_0 = \begin{bmatrix} c^2 d_1^2 - x_1^2 - y_1^2 \\ c^2 d_2^2 - x_2^2 - y_2^2 \\ \vdots \\ c^2 d_M^2 - x_M^2 - y_M^2 \end{bmatrix}.$$

If the time delay measurements are not precisely correct, we have instead the system

$$\mathbf{G}_1 \mathbf{u}_0 = \mathbf{h}_1 - (\Delta \mathbf{h}_1 - \Delta \mathbf{G}_1 \mathbf{u}_0), \tag{3}$$

where

$$\mathbf{G}_1 = \mathbf{G}_0 + \Delta \mathbf{G}_1, \quad \mathbf{h}_1 = \mathbf{h}_0 + \Delta \mathbf{h}_1, \quad \Delta \mathbf{G}_1 = -2 \cdot \begin{bmatrix} 0 & 0 & c\delta_1 \\ 0 & 0 & c\delta_2 \\ \vdots & \vdots & \vdots \\ 0 & 0 & c\delta_M \end{bmatrix}, \quad \Delta \mathbf{h}_1 = \begin{bmatrix} c^2 \delta_1^2 + 2c^2 d_1 \delta_1 \\ c^2 \delta_2^2 + 2c^2 d_2 \delta_2 \\ \vdots \\ c^2 \delta_M^2 + 2c^2 d_M \delta_M \end{bmatrix},$$

and $\boldsymbol{\delta} = [\delta_1 \quad \delta_2 \quad \cdots \quad \delta_M]^T$ represents the vector of errors in the relative time delay measurements, which is assumed to be a zero-mean Gaussian random vector with covariance matrix $\mathbf{Q} = E\{\boldsymbol{\delta}\boldsymbol{\delta}^T\}$. For future reference, it is worthwhile to note that $\Delta \mathbf{h}_1$ can be rewritten as

$$\Delta \mathbf{h}_1 = 2c\mathbf{R}\boldsymbol{\delta} + c^2 \boldsymbol{\delta} \circ \boldsymbol{\delta},$$

where $\mathbf{R} = \mathrm{diag}(r_1, r_2, \ldots, r_M)$, $r_i = cd_i$, for $i = 1, 2, \ldots, M$, and $\boldsymbol{\delta} \circ \boldsymbol{\delta} = \begin{bmatrix} \delta_1^2 & \delta_2^2 & \cdots & \delta_M^2 \end{bmatrix}^T$ represents the Hadamard product of the vector $\boldsymbol{\delta}$ with itself.

The estimate $(\hat{x}, \hat{y})$ of (x_0, y_0) is computed recursively in three stages. In stage 1, a weighted least squares approach is used to find an estimate of $\mathbf{u}_0 = [x_0 \quad y_0 \quad r_0]^T$, which is given by

$$\hat{\mathbf{u}}_1 = \left(\mathbf{G}_1^T \mathbf{W}_1 \mathbf{G}_1\right)^{-1} \mathbf{G}_1^T \mathbf{W}_1 \mathbf{h}_1, \tag{4}$$

where the weighting matrix $\mathbf{W}_1$ is chosen as an approximation to the matrix $\mathbf{W}_0 = \left[E\left\{(\Delta\mathbf{h}_1 - \Delta\mathbf{G}_1\mathbf{u}_0)(\Delta\mathbf{h}_1 - \Delta\mathbf{G}_1\mathbf{u}_0)^T\right\}\right]^{-1}$. Since the statistical properties of the solution given by Equation (4) are relatively insensitive to small variations in the choice of the weighting matrix, we will make the simplifying assumption that $\mathbf{W}_1 = \mathbf{W}_0$ throughout the remainder of the statistical analysis. In practice, the matrix $\mathbf{W}_1$ is an estimate of $\mathbf{W}_0$ based on the observed data, but the error in the derived statistical properties introduced by this assumption is insignificant in comparison to the other simplifying assumptions made in the analysis.

Letting $\hat{\mathbf{u}}_1 = \mathbf{u}_0 + \Delta\mathbf{u}_1$, Equation (4) gives

$$(\mathbf{G}_0 + \Delta\mathbf{G})^T \mathbf{W}_1 (\mathbf{G}_0 + \Delta\mathbf{G})(\mathbf{u}_0 + \Delta\mathbf{u}_1) = (\mathbf{G}_0 + \Delta\mathbf{G})^T \mathbf{W}_1 (\mathbf{h}_0 + \Delta\mathbf{h}).$$

This expression can be solved for an approximation to $\Delta\mathbf{u}_1$ by expanding, simplifying, and dropping terms involving powers of the vector $\boldsymbol{\delta}$ greater than two. Proceeding in this manner and recalling that $\mathbf{G}_0\mathbf{u}_0 = \mathbf{h}_0$ and $\mathbf{W}_1 = \mathbf{W}_0$, we get

$$\begin{aligned}
\Delta\mathbf{u}_1 &= \left[\mathbf{G}_0^T\mathbf{W}_0\mathbf{G}_0 + \Delta\mathbf{G}_1^T\mathbf{W}_0\mathbf{G}_0 + \mathbf{G}_0^T\mathbf{W}_0\Delta\mathbf{G}_1 + \Delta\mathbf{G}_1^T\mathbf{W}_0\Delta\mathbf{G}_1\right]^{-1} \\
&\quad \cdot (\mathbf{G}_0 + \Delta\mathbf{G}_1)^T \mathbf{W}_0 (\Delta\mathbf{h}_1 - \Delta\mathbf{G}_1\mathbf{u}_0) \\
&\approx \left(\mathbf{G}_0^T\mathbf{W}_0\mathbf{G}_0\right)^{-1} (\mathbf{G}_0 + \Delta\mathbf{G}_1)^T \mathbf{W}_0 (\Delta\mathbf{h}_1 - \Delta\mathbf{G}_1\mathbf{u}_0) \\
&\quad - \left(\mathbf{G}_0^T\mathbf{W}_0\mathbf{G}_0\right)^{-1} \left(\Delta\mathbf{G}_1^T\mathbf{W}_0\mathbf{G}_0 + \mathbf{G}_0^T\mathbf{W}_0\Delta\mathbf{G}_1\right)\left(\mathbf{G}_0^T\mathbf{W}_0\mathbf{G}_0\right)^{-1} \mathbf{G}_0^T\mathbf{W}_0 (\Delta\mathbf{h}_1 - \Delta\mathbf{G}_1\mathbf{u}_0).
\end{aligned}$$

Using this result, and assuming that $E\{\boldsymbol{\delta}\} = 0$, we see that an approximation for the bias in Stage 1 of the algorithm is given by

$$\begin{aligned}
\boldsymbol{\mu}_1 &= E\{\Delta\mathbf{u}_1\} \\
&\approx c^2 \left(\mathbf{G}_0^T\mathbf{W}_0\mathbf{G}_0\right)^{-1} \mathbf{G}_0^T\mathbf{W}_0 \left(\begin{bmatrix} \sigma_1^2 \\ \sigma_2^2 \\ \vdots \\ \sigma_M^2 \end{bmatrix} + 4\mathbf{Q}(\mathbf{R} + r_0\mathbf{I})\mathbf{W}_0\mathbf{G}_0 \left(\mathbf{G}_0^T\mathbf{W}_0\mathbf{G}_0\right)^{-1} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right) \qquad (5) \\
&\quad - 4c^2 \mathrm{Tr}\left(\mathbf{W}_0 \left[\mathbf{G}_0 \left(\mathbf{G}_0^T\mathbf{W}_0\mathbf{G}_0\right)^{-1} \mathbf{G}_0^T\mathbf{W}_0 - \mathbf{I} \right] (\mathbf{R} + r_0\mathbf{I})\mathbf{Q} \right) \left(\mathbf{G}_0^T\mathbf{W}_0\mathbf{G}_0\right)^{-1} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},
\end{aligned}$$

where $\sigma_1^2 = E\{\delta_1^2\}$, $\sigma_2^2 = E\{\delta_2^2\}$, …, $\sigma_M^2 = E\{\delta_M^2\}$. Proceeding in a similar fashion, and recalling that $\mathbf{W}_0 = \left[E\left\{(\Delta\mathbf{h}_1 - \Delta\mathbf{G}_1\mathbf{u}_0)(\Delta\mathbf{h}_1 - \Delta\mathbf{G}_1\mathbf{u}_0)^T\right\}\right]^{-1}$, we can derive an approximation for the autocorrelation matrix of $\Delta\mathbf{u}_1$. In particular, we get

$$\begin{aligned} E\{\Delta\mathbf{u}_1\Delta\mathbf{u}_1^T\} &\approx \left(\mathbf{G}_0^T\mathbf{W}_0\mathbf{G}_0\right)^{-1}\mathbf{G}_0^T\mathbf{W}_0 E\left\{(\Delta\mathbf{h}_1 - \Delta\mathbf{G}_1\mathbf{u}_0)(\Delta\mathbf{h}_1 - \Delta\mathbf{G}_1\mathbf{u}_0)^T\right\} \\ &\quad \cdot \mathbf{W}_0\mathbf{G}_0\left(\mathbf{G}_0^T\mathbf{W}_0\mathbf{G}_0\right)^{-1} \\ &= \left(\mathbf{G}_0^T\mathbf{W}_0\mathbf{G}_0\right)^{-1}. \end{aligned} \tag{6}$$

Note that this implies that the approximation for the covariance matrix of $\Delta\mathbf{u}_1$ takes the form

$$\Sigma_1 \approx \left(\mathbf{G}_0^T\mathbf{W}_0\mathbf{G}_0\right)^{-1} - \mu_1\mu_1^T,$$

which is slightly different than the result presented in [4], where the bias of the estimate was ignored. Finally, it is straightforward to show that

$$\mathbf{W}_0 = \left[E\left\{(\Delta\mathbf{h}_1 - \Delta\mathbf{G}_1\mathbf{u}_0)(\Delta\mathbf{h}_1 - \Delta\mathbf{G}_1\mathbf{u}_0)^T\right\}\right]^{-1} = \frac{1}{4c^2}\left(\mathbf{B}_1\mathbf{Q}\mathbf{B}_1\right)^{-1}, \tag{7}$$

where $\mathbf{B}_1 = \mathbf{R} + r_0\mathbf{I}$. Equations (5) and (6) with $\mathbf{W}_0$ given by Equation (7) constitute the desired statistical properties for Stage 1 of the algorithm.

For the second stage of the algorithm, the possible inconsistency between the estimated values for (x_0, y_0) and $r_0 = \sqrt{x_0^2 + y_0^2}$ obtained in the vector $\hat{\mathbf{u}}_1$ is resolved by computing a new estimate of (x_0^2, y_0^2) based on $\hat{\mathbf{u}}_1$. The approach is again weighted least squares, and we begin by defining

$$\mathbf{G}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}, \quad \mathbf{h}_2 = \hat{\mathbf{u}}_1 \circ \hat{\mathbf{u}}_1 = \begin{bmatrix} (\hat{\mathbf{u}}_1(1))^2 \\ (\hat{\mathbf{u}}_1(2))^2 \\ (\hat{\mathbf{u}}_1(3))^2 \end{bmatrix}, \quad \Delta\mathbf{h}_2 = \mathbf{h}_2 - \mathbf{u}_0 \circ \mathbf{u}_0 = \mathbf{h}_2 - \mathbf{G}_2\begin{bmatrix} x_0^2 \\ y_0^2 \end{bmatrix}.$$

Clearly, the equation

$$\mathbf{G}_2\begin{bmatrix} x_0^2 \\ y_0^2 \end{bmatrix} = \mathbf{u}_0 \circ \mathbf{u}_0$$

is always satisfied, while the associated equation

$$\mathbf{G}_2 \begin{bmatrix} (\hat{\mathbf{u}}_1(1))^2 \\ (\hat{\mathbf{u}}_1(2))^2 \end{bmatrix} = \mathbf{h}_2 \tag{8}$$

will not generally be satisfied. The estimate $\hat{\mathbf{u}}_2$ of $\left(x_0^2, y_0^2\right)$ is computed as a weighted least squares solution to Equation (8) given by

$$\hat{\mathbf{u}}_2 = \left(\mathbf{G}_2^T \mathbf{W}_2 \mathbf{G}_2\right)^{-1} \mathbf{G}_2^T \mathbf{W}_2 \mathbf{h}_2,$$

where $\mathbf{W}_2$ is an estimate of the inverse of the autocorrelation matrix $E\left\{\Delta\mathbf{h}_2 \Delta\mathbf{h}_2^T\right\}$. Assuming that the error $\Delta\mathbf{u}_1$ in the Stage 1 estimate $\hat{\mathbf{u}}_1$ is small, we have

$$\Delta\mathbf{h}_2 \approx 2\mathbf{B}_2 \Delta\mathbf{u}_1,$$

where $\mathbf{B}_2 = \mathrm{diag}\left(x_0, y_0, r_0\right)$, and it follows from Equation (6) that

$$E\left\{\Delta\mathbf{h}_2 \Delta\mathbf{h}_2^T\right\} \approx 4\mathbf{B}_2 E\left\{\Delta\mathbf{u}_1 \Delta\mathbf{u}_1^T\right\}\mathbf{B}_2 \approx 4\mathbf{B}_2\left(\mathbf{G}_0^T \mathbf{W}_0 \mathbf{G}_0\right)^{-1}\mathbf{B}_2.$$

Again, the statistical properties of the estimate $\hat{\mathbf{u}}_2$ are insensitive to small variations in $\mathbf{W}_2$, so we simply assume for the remaining analysis that $\mathbf{W}_2^{-1} = 4\mathbf{B}_2\left(\mathbf{G}_0^T \mathbf{W}_0 \mathbf{G}_0\right)^{-1}\mathbf{B}_2$. Finally, letting

$$\hat{\mathbf{u}}_2 = \begin{bmatrix} x_0^2 \\ y_0^2 \end{bmatrix} + \Delta\mathbf{u}_2,$$

it follows that

$$\begin{aligned}
\Delta\mathbf{u}_2 = \hat{\mathbf{u}}_2 - \begin{bmatrix} x_0^2 \\ y_0^2 \end{bmatrix} &= \left(\mathbf{G}_2^T \mathbf{W}_2 \mathbf{G}_2\right)^{-1} \mathbf{G}_2^T \mathbf{W}_2 \mathbf{h}_2 - \begin{bmatrix} x_0^2 \\ y_0^2 \end{bmatrix} \\
&= \left(\mathbf{G}_2^T \mathbf{W}_2 \mathbf{G}_2\right)^{-1} \mathbf{G}_2^T \mathbf{W}_2 \mathbf{h}_2 - \left(\mathbf{G}_2^T \mathbf{W}_2 \mathbf{G}_2\right)^{-1} \mathbf{G}_2^T \mathbf{W}_2 \mathbf{G}_2 \begin{bmatrix} x_0^2 \\ y_0^2 \end{bmatrix} \\
&= \left(\mathbf{G}_2^T \mathbf{W}_2 \mathbf{G}_2\right)^{-1} \mathbf{G}_2^T \mathbf{W}_2 \left(\mathbf{h}_2 - \mathbf{G}_2 \begin{bmatrix} x_0^2 \\ y_0^2 \end{bmatrix}\right) = \left(\mathbf{G}_2^T \mathbf{W}_2 \mathbf{G}_2\right)^{-1} \mathbf{G}_2^T \mathbf{W}_2 \Delta\mathbf{h}_2 \\
&\approx 2\left(\mathbf{G}_2^T \mathbf{W}_2 \mathbf{G}_2\right)^{-1} \mathbf{G}_2^T \mathbf{W}_2 \mathbf{B}_2 \Delta\mathbf{u}_1 \\
&\approx 2\left(\mathbf{G}_2^T \mathbf{B}_2^{-1} \mathbf{G}_0^T \mathbf{W}_0 \mathbf{G}_0 \mathbf{B}_2^{-1} \mathbf{G}_2\right)^{-1} \mathbf{G}_2^T \mathbf{B}_2^{-1} \mathbf{G}_0^T \mathbf{W}_0 \mathbf{G}_0 \Delta\mathbf{u}_1.
\end{aligned}$$

Hence,

$$
\begin{aligned}
\mu_2 &= E\{\Delta\mathbf{u}_2\} \\
&\approx 2\left(\mathbf{G}_2^T\mathbf{B}_2^{-1}\mathbf{G}_0^T\mathbf{W}_0\mathbf{G}_0\mathbf{B}_2^{-1}\mathbf{G}_2\right)^{-1}\mathbf{G}_2^T\mathbf{B}_2^{-1}\mathbf{G}_0^T\mathbf{W}_0\mathbf{G}_0E\{\Delta\mathbf{u}_1\} \\
&\approx 2c^2\left(\mathbf{G}_2^T\mathbf{B}_2^{-1}\mathbf{G}_0^T\mathbf{W}_0\mathbf{G}_0\mathbf{B}_2^{-1}\mathbf{G}_2\right)^{-1}\mathbf{G}_2^T\mathbf{B}_2^{-1} \\
&\cdot\left[\begin{array}{l}
\mathbf{G}_0^T\mathbf{W}_0\left(\begin{bmatrix}\sigma_1^2\\ \sigma_2^2\\ \vdots\\ \sigma_M^2\end{bmatrix}+4\mathbf{Q}\mathbf{B}_1\mathbf{W}_0\mathbf{G}_0\left(\mathbf{G}_0^T\mathbf{W}_0\mathbf{G}_0\right)^{-1}\begin{bmatrix}0\\0\\1\end{bmatrix}\right) \\
-4\operatorname{Tr}\left(\mathbf{W}_0\left[\mathbf{G}_0\left(\mathbf{G}_0^T\mathbf{W}_0\mathbf{G}_0\right)^{-1}\mathbf{G}_0^T\mathbf{W}_0-\mathbf{I}\right]\mathbf{B}_1\mathbf{Q}\right)\begin{bmatrix}0\\0\\1\end{bmatrix}
\end{array}\right],
\end{aligned}
\tag{9}
$$

and

$$
\begin{aligned}
\Sigma_2 &= E\{\Delta\mathbf{u}_2\Delta\mathbf{u}_2^T\} \\
&\approx \left(\mathbf{G}_2^T\mathbf{W}_2\mathbf{G}_2\right)^{-1}\mathbf{G}_2^T\mathbf{W}_2E\{\Delta\mathbf{h}_2\Delta\mathbf{h}_2^T\}\mathbf{W}_2\mathbf{G}_2\left(\mathbf{G}_2^T\mathbf{W}_2\mathbf{G}_2\right)^{-1} \\
&= \left(\mathbf{G}_2^T\mathbf{W}_2\mathbf{G}_2\right)^{-1} \\
&= 4\left(\mathbf{G}_2^T\mathbf{B}_2^{-1}\mathbf{G}_0^T\mathbf{W}_0\mathbf{G}_0\mathbf{B}_2^{-1}\mathbf{G}_2\right)^{-1}.
\end{aligned}
\tag{10}
$$

Equations (9) and (10) constitute the desired statistical properties for Stage 2 of the algorithm.

For the third and final stage of the algorithm, the estimated values of $\left(x_0^2, y_0^2\right)$ obtained in the vector $\hat{\mathbf{u}}_2$ are used to obtain a final estimate of $\left(x_0, y_0\right)$ based on $\hat{\mathbf{u}}_2$ and $\hat{\mathbf{u}}_1$ combined. The final estimate $\hat{\mathbf{u}}_3$ of $\left(x_0, y_0\right)$ is given by

$$
\hat{\mathbf{u}}_3 = \mathbf{P}\sqrt{\hat{\mathbf{u}}_3}\,,
$$

where, in this case, "$\sqrt{(\cdot)}$" indicates a component-wise square-root operation and $\mathbf{P} = \operatorname{diag}\left(\operatorname{sgn}\left[\hat{\mathbf{u}}_1(1)\right], \operatorname{sgn}\left[\hat{\mathbf{u}}_1(2)\right]\right)$. Assuming that the sign of each coordinate in $\hat{\mathbf{u}}_1$ is correct, it follows that

$$
\begin{aligned}
\Delta\mathbf{u}_3 &= \hat{\mathbf{u}}_3 - \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \approx \frac{1}{2}\mathbf{B}_3^{-1}\Delta\mathbf{u}_2 \\
&\approx \mathbf{B}_3^{-1}\left(\mathbf{G}_2^T\mathbf{B}_2^{-1}\mathbf{G}_0^T\mathbf{W}_0\mathbf{G}_0\mathbf{B}_2^{-1}\mathbf{G}_2\right)^{-1}\mathbf{G}_2^T\mathbf{B}_2^{-1}\mathbf{G}_0^T\mathbf{W}_0\mathbf{G}_0\Delta\mathbf{u}_1,
\end{aligned}
$$

where $\mathbf{B}_3 = \mathrm{diag}\{x_0, y_0\}$. Hence, the final approximations for the bias vector and autocorrelation matrix of the algorithm are given by

$$
\begin{aligned}
\boldsymbol{\mu}_3 &= E\{\Delta\mathbf{u}_3\} \\
&\approx \frac{1}{2}\mathbf{B}_3^{-1}\left(\mathbf{G}_2^T\mathbf{B}_2^{-1}\mathbf{G}_0^T\mathbf{W}_0\mathbf{G}_0\mathbf{B}_2^{-1}\mathbf{G}_2\right)^{-1}\mathbf{G}_2^T\mathbf{B}_2^{-1}\mathbf{G}_0^T\mathbf{W}_0\mathbf{G}_0 E\{\Delta\mathbf{u}_1\} \\
&\approx c^2\mathbf{B}_3^{-1}\left(\mathbf{G}_2^T\mathbf{B}_2^{-1}\mathbf{G}_0^T\mathbf{W}_0\mathbf{G}_0\mathbf{B}_2^{-1}\mathbf{G}_2\right)^{-1}\mathbf{G}_2^T\mathbf{B}_2^{-1} \\
&\quad \cdot \begin{bmatrix} \mathbf{G}_0^T\mathbf{W}_0\left(\begin{bmatrix} \sigma_1^2 \\ \sigma_2^2 \\ \vdots \\ \sigma_M^2 \end{bmatrix} + 4\mathbf{Q}\mathbf{B}_1\mathbf{W}_0\mathbf{G}_0\left(\mathbf{G}_0^T\mathbf{W}_0\mathbf{G}_0\right)^{-1}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) \\ -4\,\mathrm{Tr}\left(\mathbf{W}_0\left[\mathbf{G}_0\left(\mathbf{G}_0^T\mathbf{W}_0\mathbf{G}_0\right)^{-1}\mathbf{G}_0^T\mathbf{W}_0 - \mathbf{I}\right]\mathbf{B}_1\mathbf{Q}\right)\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \end{bmatrix} \qquad (11) \\
&= c^2\mathbf{B}_3^{-1}\left(\mathbf{G}_2^T\mathbf{B}_2^{-1}\mathbf{G}_0^T\mathbf{B}_1^{-1}\mathbf{Q}^{-1}\mathbf{B}_1^{-1}\mathbf{G}_0\mathbf{B}_2^{-1}\mathbf{G}_2\right)^{-1}\mathbf{G}_2^T\mathbf{B}_2^{-1} \\
&\quad \cdot \begin{bmatrix} \mathbf{G}_0^T\mathbf{B}_1^{-1}\mathbf{Q}^{-1}\mathbf{B}_1^{-1}\left(\begin{bmatrix} \sigma_1^2 \\ \sigma_2^2 \\ \vdots \\ \sigma_M^2 \end{bmatrix} + 4\mathbf{B}_1^{-1}\mathbf{G}_0\left(\mathbf{G}_0^T\mathbf{B}_1^{-1}\mathbf{Q}^{-1}\mathbf{B}_1^{-1}\mathbf{G}_0\right)^{-1}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) \\ -4\,\mathrm{Tr}\left(\mathbf{B}_1^{-1}\mathbf{Q}^{-1}\mathbf{B}_1^{-1}\mathbf{G}_0\left(\mathbf{G}_0^T\mathbf{B}_1^{-1}\mathbf{Q}^{-1}\mathbf{B}_1^{-1}\mathbf{G}_0\right)^{-1}\mathbf{G}_0^T\mathbf{B}_1^{-1} - \mathbf{B}_1^{-1}\right)\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \end{bmatrix},
\end{aligned}
$$

and

$$
\begin{aligned}
\Sigma_3 &= E\left\{\Delta\mathbf{u}_3 \Delta\mathbf{u}_3^T\right\} \\
&= \frac{1}{4}\mathbf{B}_3^{-1} E\left\{\Delta\mathbf{u}_2 \Delta\mathbf{u}_2^T\right\}\mathbf{B}_3^{-1} \\
&\approx 4c^2\mathbf{B}_3^{-1}\left(\mathbf{G}_2^T\mathbf{B}_2^{-1}\mathbf{G}_0^T\mathbf{W}_0\mathbf{G}_0\mathbf{B}_2^{-1}\mathbf{G}_2\right)^{-1}\mathbf{B}_3^{-1} \\
&= 4c^2\mathbf{B}_3^{-1}\left(\mathbf{G}_2^T\mathbf{B}_2^{-1}\mathbf{G}_0^T\mathbf{B}_1^{-1}\mathbf{Q}^{-1}\mathbf{B}_1^{-1}\mathbf{G}_0\mathbf{B}_2^{-1}\mathbf{G}_2\right)^{-1}\mathbf{B}_3^{-1}.
\end{aligned}
\tag{12}
$$

The proposed modifications to this algorithm are discussed in the following and final section of this report.

Proposed Algorithm Modification

In an effort to improve the localization and tracking performance, we propose a simple modification to Stage 1 of the algorithm described above. We consider a Stage 1 estimate based upon a weighted *total-least-squares* solution instead of the weighted *least-squares* solution currently used.[1] In order to describe the proposed new Stage 1 algorithm, we let $\mathbf{G}_0$, $\mathbf{G}_1$, $\mathbf{h}_0$, and $\mathbf{h}_1$ be defined as above, but we change the definition of $\Delta\mathbf{G}_1$ just slightly. In particular, we let

$$
\Delta\mathbf{G}_1 = -2\cdot\begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & c\delta_1 \\ \varepsilon_{21} & \varepsilon_{22} & c\delta_2 \\ \vdots & \vdots & \vdots \\ \varepsilon_{M1} & \varepsilon_{M2} & c\delta_M \end{bmatrix},
$$

where $\left\{\varepsilon_{ij} : i,j = 1,2,\ldots M\right\}$ are independent, identically distributed random variables with mean zero and variance σ_ε^2, where $\sigma_\varepsilon^2 \ll \min\left\{\sigma_1^2,\sigma_2^2,\ldots,\sigma_M^2\right\}$. This is merely a device for making $\Delta\mathbf{G}_1$ full rank with probability one while still reflecting the fact that the positions of the sensors $\left\{(x_1,y_1),(x_2,y_2),\ldots,(x_M,y_M)\right\}$ are much more precisely known than the measured TDOA values.

With this change in mind, we now define the new matrices $\bar{\mathbf{G}}_0 = \left[\mathbf{G}_0 \mid \mathbf{h}_0\right]$, $\bar{\mathbf{G}}_1 = \left[\mathbf{G}_1 \mid \mathbf{h}_1\right]$ and $\bar{\Delta}_1 = \left[\Delta\mathbf{G}_1 \mid \Delta\mathbf{h}_1\right]$. Note that Equation (2) can now be rewritten as

$$
\bar{\mathbf{G}}_0\left[\frac{\mathbf{u}_0}{-1}\right] = 0,
$$

and Equation (3) can be rewritten as

[1] For discussions of both total-least-squares solutions and least-squares solutions, see for example [11].

$$\left(\bar{\mathbf{G}}_1 - \bar{\Delta}_1\right)\begin{bmatrix} \mathbf{u}_0 \\ \hline -1 \end{bmatrix} = 0 .$$

Now, let $\Sigma_\mathrm{L} = E\left\{\Delta_1^T \Delta_1\right\}$ and $\Sigma_\mathrm{R} = E\left\{\Delta_1 \Delta_1^T\right\}$. Then it is straightforward to show that

$$\Sigma_\mathrm{R} \approx 8\sigma_\varepsilon^2 \mathbf{I} + 4c^2\left[\mathbf{Q} + \mathbf{RQR} - \mathbf{QR} - \mathbf{RQ}\right] + 2c^4 \mathbf{Q} \circ \mathbf{Q} + c^4 \begin{bmatrix} \sigma_1^2 \\ \sigma_2^2 \\ \vdots \\ \sigma_M^2 \end{bmatrix} \begin{bmatrix} \sigma_1^2 & \sigma_2^2 & \cdots & \sigma_M^2 \end{bmatrix},$$

and

$$\Sigma_\mathrm{L} \approx \begin{bmatrix} 4M\sigma_\varepsilon^2 & 0 & 0 & 0 \\ 0 & 4M\sigma_\varepsilon^2 & 0 & 0 \\ 0 & 0 & 4c^2 \mathrm{Tr}\mathbf{Q} & -4c^2 \mathrm{Tr}(\mathbf{RQ}) \\ 0 & 0 & -4c^2 \mathrm{Tr}(\mathbf{RQ}) & c^2 \mathrm{Tr}\left(2\mathbf{RQR} + 3c^2 \mathbf{Q} \circ \mathbf{Q}\right) \end{bmatrix}.$$

We seek a matrix $\bar{\Delta}$ and an estimate $\hat{\mathbf{u}}_1$ of $\mathbf{u}_0$ such that the equation

$$\left(\bar{\mathbf{G}}_1 - \bar{\Delta}\right)\begin{bmatrix} \hat{\mathbf{u}}_1 \\ \hline -1 \end{bmatrix} = 0 \qquad (13)$$

is satisfied, and the matrix $\bar{\Delta}$ has minimum possible norm of the form

$$\left\|\bar{\Delta}\right\| = \sqrt{\mathrm{Tr}\left(\bar{\Delta}^T \Sigma_\mathrm{R}^{-1} \bar{\Delta} \Sigma_\mathrm{L}^{-1}\right)}.$$

To find $\hat{\mathbf{u}}_1$ and $\bar{\Delta}$, we first factor Σ_L and Σ_R using the Cholesky decomposition to get $\Sigma_\mathrm{L}^{-1} = \mathbf{W}_\mathrm{L}\mathbf{W}_\mathrm{L}^T$ and $\Sigma_\mathrm{R}^{-1} = \mathbf{W}_\mathrm{R}\mathbf{W}_\mathrm{R}^T$ and note that Equation (13) can be rewritten as

$$\mathbf{W}_\mathrm{R}^T\left(\bar{\mathbf{G}}_1 - \bar{\Delta}\right)\mathbf{W}_\mathrm{L}\mathbf{W}_\mathrm{L}^{-1}\begin{bmatrix} \hat{\mathbf{u}}_1 \\ \hline -1 \end{bmatrix} = 0 ,$$

or equivalently

$$\left(\tilde{\mathbf{G}}_1 - \tilde{\Delta}\right)\tilde{\mathbf{v}} = 0 , \qquad (14)$$

where

$$\tilde{\mathbf{G}} = \mathbf{W}_R^T \bar{\mathbf{G}}_1 \mathbf{W}_L,$$
$$\tilde{\Delta} = \mathbf{W}_R^T \bar{\Delta} \mathbf{W}_L,$$
$$\tilde{\mathbf{u}} = \mathbf{W}_L^{-1} \begin{bmatrix} \hat{\mathbf{u}}_1 \\ \hline -1 \end{bmatrix}.$$

Now to solve Equation (13), we look first for $\tilde{\Delta}$ and $\tilde{\mathbf{u}}$ that satisfy Equation (14) such that $\tilde{\Delta}$ has minimum possible *Froebinius norm*, which is given by

$$\|\tilde{\Delta}\|_F = \sqrt{\mathrm{Tr}\left(\tilde{\Delta}^T \tilde{\Delta}\right)}.$$

The desired solution for Equation (14), which is well known [11], is derived as follows. Let the singular value decomposition of $\tilde{\mathbf{G}}$ be given by

$$\tilde{\mathbf{G}} = \mathbf{UDV}^T,$$

where the columns of $\mathbf{V}$ are the orthonormal eigenvectors of $\tilde{\mathbf{G}}^T \tilde{\mathbf{G}}$. Let $\mathbf{v}_{\min}$ be the column of $\mathbf{V}$ corresponding to the smallest eigenvalue $\lambda_{\min}$ of $\tilde{\mathbf{G}}^T \tilde{\mathbf{G}}$. The desired solution to Equation (14) is given by

$$\tilde{\Delta} = \sqrt{\lambda_{\min}}\, \mathbf{v}_{\min} \mathbf{v}_{\min}^T, \quad \tilde{\mathbf{u}} = \alpha \mathbf{v}_{\min},$$

where α is chosen such that

$$\tilde{\mathbf{u}} = \alpha \mathbf{v}_{\min} = \mathbf{W}_L^{-1} \begin{bmatrix} \hat{\mathbf{u}}_1 \\ \hline -1 \end{bmatrix}$$

or equivalently,

$$\begin{bmatrix} \hat{\mathbf{u}}_1 \\ \hline -1 \end{bmatrix} = \alpha \mathbf{W}_L \mathbf{v}_{\min}.$$

Hence, α is the negative of the inverse of the last component of the vector $\mathbf{W}_L \mathbf{v}_{\min}$, and the vector $\hat{\mathbf{u}}_1$ is the desired Stage 1 solution to Equation (13). Stage 2 and Stage 3 of the new algorithm are identical to the original Stage 2 and Stage 3.

References

[1] "PulsON Technology Overview," http://www.timedomain.com/Files/downloads/techpapers/PulsONOverview7_01.pdf.

[2] J. C. Adams, W. Gregorwich, L. Capots, and D. Liccardo, "Ultra-Wideband for Navigation and Communications," *Proceedings of IEEE Aerospace Conference*, 2001.

[3] C. W. Borst, "Telerobotic Ground Control of a Space Free Flyer," *Proceedings of IEEE/RSJ International Conference on Intelligent Robots and Systems*, 1998.

[4] Y. T. Chan and K. C. Ho, "An Efficient Closed-Form Localization Solution From Time Difference of Arrival Measurements," 1994.

[5] H. Choset, et al., "Path Planning and Control for AERCam, a Free-Flying Inspection Robot in Space," *Proceedings of IEEE International Conference on Robotics and Automation*, 1999.

[6] FCC, "First Notice and Order: Revision of Part 15 of the Commission's Rules Regarding Ultra-Wideband Transmission Systems," Feb. 2002.

[7] M. G. M. Hussain, "Principles of Space-Time Array Processing for Ultrawide Band Impulse Radar and Radio Communications," *IEEE Transactions on Vehicular Technology*, vol. 51, pp. 393-403, 2002.

[8] D. Kortenkamp, et al., "Applying a Layered Control Architecture to a Free-Flying Space Camera," *Proceedings of IEEE International Joint Symposium on Intelligence and Systems*, 1998.

[9] Y. C. Loh, et al., "Wireless Video System for Extra Vehicular Activity in the International Space Station and Space Shuttle Orbiter Environment," *Proceedings of IEEE 49th Vehicular Technology Conference*, 1999.

[10] K. Siwiak, et al., "Advances in Ultra-Wide Band Technology," *Proceedings of Radio Solutions 2001*, Commonwealth Conference & Events Centre, London, 2001.

[11] C. W. Therrien, *Discrete Random Signals and Statistical Signal Processing*. Englewood Cliffs, NJ: Prentice-Hall, 1992.

Appendix

In this Appendix, we derive the bias and mean-squared-error (MSE) for the current weighted-least-squares TDOA algorithm for a simple far-field example. In particular, we let $M = 4$, $\mathbf{Q} = \sigma^2\mathbf{I}$, and $r_0 >> \max_{1\le i\le 4}\left(\max\left\{|x_i|,|y_i|,cd_i\right\}\right)$. Then, from Equation (11), we get

$$\begin{aligned}
\boldsymbol{\mu}_3 &= c^2\sigma^2\mathbf{B}_3^{-1}\left(\mathbf{G}_2^T\mathbf{B}_2^{-1}\mathbf{G}_0^T\mathbf{B}_1^{-2}\mathbf{G}_0\mathbf{B}_2^{-1}\mathbf{G}_2\right)^{-1}\mathbf{G}_2^T\mathbf{B}_2^{-1} \\
&\cdot\left[\begin{matrix}\mathbf{G}_0^T\mathbf{B}_1^{-2}\left(\begin{bmatrix}1\\1\\1\\1\end{bmatrix}+4\mathbf{B}_1^{-1}\mathbf{G}_0\left(\mathbf{G}_0^T\mathbf{B}_1^{-2}\mathbf{G}_0\right)^{-1}\begin{bmatrix}0\\0\\1\end{bmatrix}\right)\\ -4\,\mathrm{Tr}\left(\mathbf{B}_1^{-2}\mathbf{G}_0\left(\mathbf{G}_0^T\mathbf{B}_1^{-2}\mathbf{G}_0\right)^{-1}\mathbf{G}_0^T\mathbf{B}_1^{-1}-\mathbf{B}_1^{-1}\right)\begin{bmatrix}0\\0\\1\end{bmatrix}\end{matrix}\right] \\
&\approx c^2\sigma^2 r_0^2\mathbf{B}_3^{-1}\left(\mathbf{G}_2^T\mathbf{B}_2^{-1}\mathbf{G}_0^T\mathbf{G}_0\mathbf{B}_2^{-1}\mathbf{G}_2\right)^{-1}\mathbf{G}_2^T\mathbf{B}_2^{-1}\left[\begin{matrix}\frac{1}{r_0^2}\mathbf{G}_0^T\left(\begin{bmatrix}1\\1\\1\\1\end{bmatrix}+4r_0\mathbf{G}_0\left(\mathbf{G}_0^T\mathbf{G}_0\right)^{-1}\begin{bmatrix}0\\0\\1\end{bmatrix}\right)\\ -\frac{4}{r_0}\mathrm{Tr}\left(\mathbf{G}_0\left(\mathbf{G}_0^T\mathbf{G}_0\right)^{-1}\mathbf{G}_0^T-\mathbf{I}\right)\begin{bmatrix}0\\0\\1\end{bmatrix}\end{matrix}\right] \\
&= c^2\sigma^2\mathbf{B}_3^{-1}\left(\mathbf{G}_2^T\mathbf{B}_2^{-1}\mathbf{G}_0^T\mathbf{G}_0\mathbf{B}_2^{-1}\mathbf{G}_2\right)^{-1}\mathbf{G}_2^T\mathbf{B}_2^{-1}\left(\mathbf{G}_0^T\begin{bmatrix}1\\1\\1\\1\end{bmatrix}+4r_0\begin{bmatrix}0\\0\\1\end{bmatrix}-4r_0(3-4)\begin{bmatrix}0\\0\\1\end{bmatrix}\right) \\
&= 8c^2\sigma^2\mathbf{B}_3^{-1}\left(\mathbf{G}_2^T\mathbf{B}_2^{-1}\mathbf{G}_0^T\mathbf{G}_0\mathbf{B}_2^{-1}\mathbf{G}_2\right)^{-1}\mathbf{G}_2^T\mathbf{B}_2^{-1}\left(\begin{bmatrix}\sum_{i=1}^4 x_i & \sum_{i=1}^4 y_i & \sum_{i=1}^4 cd_i\end{bmatrix}^T+8r_0\begin{bmatrix}0\\0\\1\end{bmatrix}\right) \\
&\approx 8c^2\sigma^2 r_0\mathbf{B}_3^{-1}\left(\mathbf{G}_2^T\mathbf{B}_2^{-1}\mathbf{G}_0^T\mathbf{G}_0\mathbf{B}_2^{-1}\mathbf{G}_2\right)^{-1}\mathbf{G}_2^T\mathbf{B}_2^{-1}\begin{bmatrix}0\\0\\1\end{bmatrix}.
\end{aligned}$$

Continuing to simplify this, we get

$$\mathbf{B}_2^{-1} = \frac{1}{r_0 x_0 y_0}\begin{bmatrix} r_0 y_0 & 0 & 0 \\ 0 & r_0 x_0 & 0 \\ 0 & 0 & x_0 y_0 \end{bmatrix},$$

$$\mathbf{B}_2^{-1}\mathbf{G}_0^T = \frac{1}{r_0 x_0 y_0}\begin{bmatrix} r_0 y_0 x_1 & r_0 y_0 x_2 & r_0 y_0 x_3 & r_0 y_0 x_3 \\ r_0 x_0 y_1 & r_0 x_0 y_2 & r_0 x_0 y_3 & r_0 x_0 y_4 \\ x_0 y_0 c d_1 & x_0 y_0 c d_2 & x_0 y_0 c d_3 & x_0 y_0 c d_4 \end{bmatrix},$$

$$\mathbf{G}_2^T\mathbf{B}_2^{-1}\mathbf{G}_0^T = \frac{1}{r_0 x_0 y_0} \cdot \begin{bmatrix} y_0\left(r_0 x_1 + x_0 c d_1\right) & y_0\left(r_0 x_2 + x_0 c d_2\right) & y_0\left(r_0 x_3 + x_0 c d_3\right) & y_0\left(r_0 x_4 + x_0 c d_4\right) \\ x_0\left(r_0 y_1 + y_0 c d_1\right) & x_0\left(r_0 y_2 + y_0 c d_2\right) & x_0\left(r_0 y_3 + y_0 c d_3\right) & x_0\left(r_0 y_4 + y_0 c d_4\right) \end{bmatrix},$$

$$\mathbf{G}_2^T\mathbf{B}_2^{-1}\mathbf{G}_0^T\mathbf{G}_0\mathbf{B}_2^{-1}\mathbf{G}_2 = \frac{1}{r_0^2 x_0^2 y_0^2} \cdot \begin{bmatrix} y_0^2 \sum_{i=1}^{4}\left(r_0 x_i + x_0 c d_i\right)^2 & x_0 y_0 \sum_{i=1}^{4}\left(r_0 x_i + x_0 c d_i\right)\left(r_0 y_i + y_0 c d_i\right) \\ x_0 y_0 \sum_{i=1}^{4}\left(r_0 x_i + x_0 c d_i\right)\left(r_0 y_i + y_0 c d_i\right) & x_0^2 \sum_{i=1}^{4}\left(r_0 y_i + y_0 c d_i\right)^2 \end{bmatrix},$$

$$\left(\mathbf{G}_2^T\mathbf{B}_2^{-1}\mathbf{G}_0^T\mathbf{G}_0\mathbf{B}_2^{-1}\mathbf{G}_2\right)^{-1}$$

$$= \frac{r_0^2}{\sum_{i=1}^{4}\left(x_i + \frac{x_0}{r_0} c d_i\right)^2 \sum_{i=1}^{4}\left(y_i + \frac{y_0}{r_0} c d_i\right)^2 - \left[\sum_{i=1}^{4}\left(x_i + \frac{x_0}{r_0} c d_i\right)\left(y_i + \frac{y_0}{r_0} c d_i\right)\right]^2} \cdot \begin{bmatrix} \frac{x_0^2}{r_0^2}\sum_{i=1}^{4}\left(y_i + \frac{y_0}{r_0^2} c d_i\right)^2 & -\frac{x_0 y_0}{r_0^2}\sum_{i=1}^{4}\left(r_0 x_i + x_0 c d_i\right)\left(r_0 y_i + y_0 c d_i\right) \\ -\frac{x_0 y_0}{r_0^2}\sum_{i=1}^{4}\left(r_0 x_i + x_0 c d_i\right)\left(r_0 y_i + y_0 c d_i\right) & \frac{y_0^2}{r_0^2}\sum_{i=1}^{4}\left(x_i + \frac{x_0}{r_0} c d_i\right)^2 \end{bmatrix},$$

$$\mathbf{G}_2^T\mathbf{B}_2^{-1}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{r_0}\begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

$$\mathbf{B}_3^{-1}\left(\mathbf{G}_2^T\mathbf{B}_2^{-1}\mathbf{G}_0^T\mathbf{G}_0\mathbf{B}_2^{-1}\mathbf{G}_2\right)^{-1}\mathbf{G}_2^T\mathbf{B}_2^{-1}\begin{bmatrix}0\\0\\1\end{bmatrix}$$

$$=\frac{1}{\sum_{i=1}^{4}\left(x_i+\frac{x_0}{r_0}cd_i\right)^2\sum_{i=1}^{4}\left(y_i+\frac{y_0}{r_0}cd_i\right)^2-\left[\sum_{i=1}^{4}\left(x_i+\frac{x_0}{r_0}cd_i\right)\left(y_i+\frac{y_0}{r_0}cd_i\right)\right]^2}$$

$$\cdot\begin{bmatrix}\frac{x_0}{r_0}\sum_{i=1}^{4}\left(y_i+\frac{y_0}{r_0^2}cd_i\right)^2-\frac{y_0}{r_0}\sum_{i=1}^{4}(r_0x_i+x_0cd_i)(r_0y_i+y_0cd_i)\\ \frac{y_0}{r_0}\sum_{i=1}^{4}\left(x_i+\frac{x_0}{r_0}cd_i\right)^2-\frac{x_0}{r_0}\sum_{i=1}^{4}(r_0x_i+x_0cd_i)(r_0y_i+y_0cd_i)\end{bmatrix}.$$

Now, let $x_0=r_0\cos\theta$, $y_0=r_0\sin\theta$,

$$\tilde{\mathbf{x}}=\begin{bmatrix}x_1+cd_1\cos\theta & x_2+cd_2\cos\theta & x_3+cd_3\cos\theta & x_4+cd_4\cos\theta\end{bmatrix}^T,$$

$$\tilde{\mathbf{y}}=\begin{bmatrix}y_1+cd_1\sin\theta & y_2+cd_2\sin\theta & y_3+cd_3\sin\theta & y_4+cd_4\sin\theta\end{bmatrix}^T,$$

$$\langle\tilde{\mathbf{x}},\tilde{\mathbf{y}}\rangle=\|\tilde{\mathbf{x}}\|\|\tilde{\mathbf{y}}\|\cos\phi.$$

Then, putting this all together, it follows that the bias vector for this example is given by

$$\mu_3\approx\frac{8c^2\sigma^2r_0}{\|\tilde{\mathbf{x}}\|^2\|\tilde{\mathbf{y}}\|^2-|\langle\tilde{\mathbf{x}},\tilde{\mathbf{y}}\rangle|^2}\begin{bmatrix}\|\tilde{\mathbf{y}}\|^2\cos\theta-\langle\tilde{\mathbf{x}},\tilde{\mathbf{y}}\rangle\sin\theta\\ \|\tilde{\mathbf{x}}\|^2\sin\theta-\langle\tilde{\mathbf{x}},\tilde{\mathbf{y}}\rangle\cos\theta\end{bmatrix}$$

$$=\frac{8c^2\sigma^2r_0\left(\|\tilde{\mathbf{x}}\|^2+\|\tilde{\mathbf{y}}\|^2\right)}{\|\tilde{\mathbf{x}}\|^2\|\tilde{\mathbf{y}}\|^2\left(1-\cos^2\phi\right)}\begin{bmatrix}\frac{\|\tilde{\mathbf{y}}\|^2\cos\theta-\langle\tilde{\mathbf{x}},\tilde{\mathbf{y}}\rangle\sin\theta}{\|\tilde{\mathbf{x}}\|^2+\|\tilde{\mathbf{y}}\|^2}\\ \frac{\|\tilde{\mathbf{x}}\|^2\sin\theta-\langle\tilde{\mathbf{x}},\tilde{\mathbf{y}}\rangle\cos\theta}{\|\tilde{\mathbf{x}}\|^2+\|\tilde{\mathbf{y}}\|^2}\end{bmatrix}.$$

Similarly, Equation (12) gives

$$
\begin{aligned}
\Sigma_3 &= 4c^2\sigma^2\mathbf{B}_3^{-1}\left(\mathbf{G}_2^T\mathbf{B}_2^{-1}\mathbf{G}_0^T\mathbf{B}_1^{-2}\mathbf{G}_0\mathbf{B}_2^{-1}\mathbf{G}_2\right)^{-1}\mathbf{B}_3^{-1} \\
&\approx 4c^2\sigma^2 r_0^2\mathbf{B}_3^{-1}\left(\mathbf{G}_2^T\mathbf{B}_2^{-1}\mathbf{G}_0^T\mathbf{G}_0\mathbf{B}_2^{-1}\mathbf{G}_2\right)^{-1}\mathbf{B}_3^{-1} \\
&= \frac{4c^2\sigma^2 r_0^2}{\sum_{i=1}^{4}\left(x_i+\frac{x_0}{r_0}cd_i\right)^2\sum_{i=1}^{4}\left(y_i+\frac{y_0}{r_0}cd_i\right)^2-\left[\sum_{i=1}^{4}\left(x_i+\frac{x_0}{r_0}cd_i\right)\left(y_i+\frac{y_0}{r_0}cd_i\right)\right]^2} \\
&\quad\cdot\begin{bmatrix} \sum_{i=1}^{4}\left(y_i+\frac{y_0}{r_0^2}cd_i\right)^2 & -\sum_{i=1}^{4}(r_0x_i+x_0cd_i)(r_0y_i+y_0cd_i) \\ -\sum_{i=1}^{4}(r_0x_i+x_0cd_i)(r_0y_i+y_0cd_i) & \sum_{i=1}^{4}\left(x_i+\frac{x_0}{r_0}cd_i\right)^2 \end{bmatrix} \\
&= \frac{4c^2\sigma^2 r_0^2\left(\|\mathbf{x}\|^2+\|\mathbf{y}\|^2\right)}{\|\mathbf{x}\|^2\|\mathbf{y}\|^2\left(1-\cos^2\phi\right)}\begin{bmatrix} \frac{\|\mathbf{y}\|^2}{\|\mathbf{x}\|^2+\|\mathbf{y}\|^2} & -\frac{\|\mathbf{x}\|\|\mathbf{y}\|\cos\phi}{\|\mathbf{x}\|^2+\|\mathbf{y}\|^2} \\ -\frac{\|\mathbf{x}\|\|\mathbf{y}\|\cos\phi}{\|\mathbf{x}\|^2+\|\mathbf{y}\|^2} & \frac{\|\mathbf{x}\|^2}{\|\mathbf{x}\|^2+\|\mathbf{y}\|^2} \end{bmatrix}.
\end{aligned}
$$

Hence, the total MSE is given by

$$
\mathrm{MSE} = \mathbf{Tr}\left(\Sigma_3\right) = \frac{4c^2\sigma^2 r_0^2\left(\|\mathbf{x}\|^2+\|\mathbf{y}\|^2\right)}{\|\mathbf{x}\|^2\|\mathbf{y}\|^2\left(1-\cos^2\phi\right)},
$$

which completes the example.