ORBIT DETERMINATION OF LEO SATELLITES FOR A SINGLE PASS THROUGH A RADAR: COMPARISON OF METHODS

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1. Introduction

The problem of determining the orbit of a space object from measurements based on one pass through the field of view of a radar is not a new one. Extensive research in this area has been carried out in the USA and Russia since the late 50s when these countries started the development of ballistic missile defense (BMD) and Early Warning systems. In Russia these investigations got additional stimulation in the early 60s after the decision to create a Space Surveillance System, whose primary task would be the maintenance of the satellite catalog. These problems were the focus of research interest until the middle 70s when the appropriate techniques and software were implemented for all radars. Then for more than 20 years no new research papers appeared on this subject. This produced an impression that all the problems of track determination based on one pass had been solved and there was no need for further research.

In the late 90s interest in this problem arose again in relation to the following. It was estimated that there would be greater than 100,000 objects with size greater than 1-2 cm and collision of an operational spacecraft with any of these objects could have catastrophic results. Thus, for prevention of hazardous approaches and collisions with valuable spacecraft the existing satellite catalog should be extended by at least an order of magnitude This is a very difficult scientific and engineering task. One of the issues is the development of data fusion procedures and the software capable of maintaining such a huge catalog in near real time. The number of daily processed measurements (of all types, radar and optical) for such a system may constitute millions, thus increasing the number of measurements by at least an order of magnitude. Since we will have ten times more satellites and measurements the computer effort required for the correlation of measurements will be two orders of magnitude greater. This could create significant problems for processing data close to real time even for modern computers. Preliminary "compression" of data for one pass through the field of view of a sensor can significantly reduce the requirements to computers and data communication. This compression will occur when all the single measurements of the sensor are replaced by the orbit determined on their basis. The single measurement here means the radar parameters (range, azimuth, elevation, and in some cases range rate) measured by a single pulse.

Two types of techniques have been traditionally used for processing single measurements; recursive and joint or batch processing. Recursive procedures convenient for real time processing usually are based on Kalman's recursive filter [1]. Less convenient joint processing

techniques basically use the least squares [2] or least modules methods [3]. When the single measurement errors are time correlated, and when the statistical characteristics of the single measurements are not known completely these techniques do not provide a guaranteed evaluation of the errors of the generated estimates.

This limitation can be avoided when we use the joint method with the guarantee approach when the guarantee ranges of orbital parameters are obtained on the basis of guarantee ranges of the parameters of single measurements [4]. The guarantee approach has one more remarkable feature. With certain limitations on the distribution of the errors of the measurements and with a large enough number of these measurements this approach leads to a much more accurate estimate than the traditional techniques mentioned above.

The general procedure based on the guarantee approach is a linear programming procedure with the amount of computation significantly greater than the least squares procedure. This resulted in a lack of interest to such procedures in 60-70s. However, now the situation is different, the capacity of the computers is significantly greater. Modern sensors have small and rather stable errors. Thus we have the reasons to look again at this promising method.

The current paper presents the comparative analysis of the accuracy characteristics of different procedures using mathematical simulation with the following background data:

- Limitations for the composition of the measured coordinates:

$$\text{local spherical coordinates}: \quad d, \quad \alpha, \quad \beta \tag{1}$$

- Limitations of the field of view of the radar:

$$150\,km < d < 7000\,km \qquad 0 < \beta < 50^{\circ} \tag{2}$$

- Limitations for the accuracy of the not abnormal single measurements:

$$0.01\,km < \sigma_d < 0.05\,km \qquad 0.01^{\circ} < \sigma_\alpha,\ \sigma_\beta < 0.03^{\circ} \tag{3}$$

- Limitations for the time interval between neighboring measurements:

$$1\,s < \Delta < 10\,s \tag{4}$$

- Limitation for the tracking time:

$$\Delta t < 300\,s \tag{5}$$

- Limitations for the observed satellites:

$$\text{no active operations and small atmospheric drag} \tag{6}$$

- Limitations on satellite parameters:

$$i > 30^{\circ} \qquad h_p > 100\,km \qquad e < 0.8 \tag{7}$$

2. The Joint (non-recursive) Algorithm Based on the Least Squares Method

The least squares method search for the $\min_{\mathbf{a}} \Psi(\mathbf{a}) = \Psi(\mathbf{a}_{min})$ of the function $\Psi(\mathbf{a})$ of the form

$$\Psi(\mathbf{a}) = \sum_{k=1}^{n} \left(\frac{1}{\sigma_{d_k}^2}(d_k - d_k(\mathbf{a}))^2 + \frac{1}{\sigma_{\alpha_k}^2}(\alpha_k - \alpha_k(\mathbf{a}))^2 + \frac{1}{\sigma_{\beta_k}^2}(\beta_k - \beta_k(\mathbf{a}))^2 \right), \tag{8}$$

where

- $\mathbf{x}_k = (d_k,\, \alpha_k,\, \beta_k)$ – k-th measurement $(k = 1, 2, ..., n)$;
- t_k – time of the k-th measurement;
- $\sigma_{d_k},\, \sigma_{\alpha_k},\, \sigma_{\beta_k}$ – RMS of the errors $d_k,\, \alpha_k,\, \beta_k$;
- $\mathbf{h}_k(\mathbf{a}) = (d_k(\mathbf{a}),\, \alpha_k(\mathbf{a}),\, \beta_k(\mathbf{a}))$ – values of the parameters of the k-th measurement, calculated using the vector $\mathbf{a} = \mathbf{a}(\bar{t})$ of the orbital parameters referred to certain time $\bar{t}$.

For the case of Gaussian non-correlated errors of the measurements the estimate $\mathbf{a}_{min}$ provides a maximum for the probability density function $p(\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_n)$, thus it is the maximum likelihood estimate having the feature of asympthotical efficiency [5]. The covariation matrix of the errors $\mathbf{K}$ of the estimate $\mathbf{a}_{min}$ in this case can be calculated using the formula

$$\mathbf{K} = 0.5 \cdot \left(\frac{\partial^2 \Psi}{\partial \mathbf{a}^2}(\mathbf{a}_{min}) \right)^{-1} \tag{9}$$

The estimate $\mathbf{a}_{min}$ retains the asymptotic efficiency feature for the case of low level correlation between measurements (correlation interval significantly smaller than the interval of observations). If all the errors of the measurements are Gaussian and all the functional relationships $\mathbf{x}_k(\mathbf{a})$ are linear, then the estimate $\mathbf{a}_{min}$ is unbiased and has the least RMS errors within the class of unbiased estimates having the shape of linear functions of measurements.

These remarkable features of the least squares technique make it attractive for applications in practice. However, implementation of this method sometimes faces several problems. The major ones are:

- choosing the vector $\mathbf{a}$ and the method for its propagation for time interval τ;
- calculation of the initial approximaiton $\mathbf{a}_0$, for minimization of $\Psi(\mathbf{a})$;
- selection of technique for reaching the minimum of $\Psi(\mathbf{a})$.

For the vector of estimated parameters $\mathbf{a}$ it is convenient to use the state vector $(x,\, y,\, z,\, \dot{x},\, \dot{y},\, \dot{z})$ for the time period in the local rectangular coordinate frame (LRCF) related to geographical directions of the sensor. In this coordinate frame the basic plane is the plane of the local horizon, the x axis is in the plane of the local horizon and is directed to the west, the y axis is directed normal to the plane of local horizon upwards and the z axis is in the

plane of local horizon and is directed to the north. Coordinates x, y, z can be obtained from local spherical coordinates d, α, β using the formulas

$$x = -d\sin\alpha\cos\beta \qquad y = d\sin\beta \qquad z = d\cos\alpha\cos\beta \tag{10}$$

The equations of motion in LRCF with accuracy sufficient for solving the task mentioned in the introduction have the form [2]:

$$\ddot{\mathbf{d}} = -\frac{\mu}{r^3}\mathbf{r} - 2\vec{\omega}\times\dot{\mathbf{d}} - \vec{\omega}\times(\vec{\omega}\times\mathbf{r}) - \frac{3J_2\mu R_e^2}{2r^5}\left(\mathbf{r} + 2(\mathbf{r},\vec{\omega}_0)\vec{\omega}_0 - \frac{5(\mathbf{r},\vec{\omega}_0)^2}{r^2}\mathbf{r}\right), \tag{11}$$

where

- $\mathbf{d} = (x, y, z)'$, $\dot{\mathbf{d}} = (\dot{x}, \dot{y}, \dot{z})'$ – position and velocity vectors in local coordinate frame;
- $\mathbf{r} = (x{-}b_x,\ y{-}b_y,\ z{-}b_z)'$ – position vector of an object with respect to the center of the Earth in local rectangular coordinate frame;
- $r = \sqrt{(x{-}b_x)^2{+}(y{-}b_y)^2{+}(z{-}b_z)^2}$ – length of the vector $\mathbf{r}$;
- $\mathbf{b} = (b_x,\ b_y,\ b_z)'$ – coordinates of Earth's center in local rectangular coordinate frame – constants, which values are determined in the following way using the coordinates X_{g0}, Y_{g0}, Z_{g0} of the radar locaiton (the origin of local rectangular frame) in the Greenwich rectangular frame:

$$r_1 = \sqrt{X_{g0}^2{+}Y_{g0}^2} \quad r = \sqrt{X_{g0}^2{+}Y_{g0}^2{+}Z_{g0}^2} \quad \tilde{z} = Z_{g0}/r \quad \tilde{r}_1 = r_1/r \tag{12}$$

$$R = R_e{\cdot}(1{-}\alpha{\cdot}\tilde{z}^2) \qquad b_x = 0 \qquad b_y = -r \qquad b_z = 2{\cdot}\alpha{\cdot}R{\cdot}\tilde{z}{\cdot}\tilde{r}_1 \tag{13}$$

  $\alpha = 0.0033528107$ – is the constant characterizing Earth oblateness

- $\vec{\omega} = \omega_e{\cdot}\vec{\omega}_0$, where ω_e - angular velocity of Earth's rotation, $\vec{\omega}_0$ - unit vector of the angular velocity of Earth's rotation in local rectangular coordinate frame, which components $\omega_{0x}, \omega_{0y}, \omega_{0z}$ are the constants calculated using the formulas:

$$\omega_{0x} = 0 \qquad \omega_{0y} = \tilde{z} + \tilde{r}_1{\cdot}b_z/r \qquad \omega_{0z} = \tilde{r}_1 - \tilde{z}{\cdot}b_z/r \tag{14}$$

- $\mu = 398600.44\,km^3/c^2$ – gravitational constant;
- $J_2 = 0.001083$ – coefficient of the second harmonic of the Earth's potential expansion;
- $R_e = 6378.137\,km$ – equatorial radius of the Earth;
- $(\mathbf{a}, \mathbf{b})$, $\mathbf{a}\times\mathbf{b}$ – scalar and vector products of vectors $\mathbf{a}$ and $\mathbf{b}$;
- the vectors are denoted in small letters, matrices – in bold ones;
- vector with Latin notation are in bold letters, with Greek one – have $\vec{}$ superscript;

• sign $'$ after vector or matrix means transposition;

For solving these equations (propagation of the state vector to the given time τ) we use the numerical 4th order Runge-Kutta method.

For solving the system of differential equations $\dot{\mathbf{a}}(t) = \mathbf{F}(\mathbf{a}(t))$ c $\mathbf{a}(t{=}0) = \mathbf{a}_0$ this method uses a stepwise process $\mathbf{a}_{i+1} = \mathbf{a}_i + \Delta\mathbf{a}_i$, where $\mathbf{a}_{i+1}$ – is the result of propagation of the parameters $\mathbf{a}_0$ for the time $\tau_1{+}\tau_2{+}...{+}\tau_i$ until we reach the time $t = \tau$. For the i-th step the increment $\Delta\mathbf{a}_i$ is calculated by the formula:

$$\Delta\mathbf{a}_i = (\mathbf{k}_1^{(i)} + 2\mathbf{k}_2^{(i)} + 2\mathbf{k}_3^{(i)} + \mathbf{k}_4^{(i)})/6, \tag{15}$$

where

$$\mathbf{k}_1^{(i)}{=}\tau_i{\cdot}\mathbf{F}(\mathbf{a}_i) \quad \mathbf{k}_2^{(i)}{=}\tau_i{\cdot}\mathbf{F}(\mathbf{a}_i{+}0.5\mathbf{k}_1^{(i)}) \quad \mathbf{k}_3^{(i)}{=}\tau_i{\cdot}\mathbf{F}(\mathbf{a}_i{+}0.5\mathbf{k}_2^{(i)}) \quad \mathbf{k}_4^{(i)}{=}\tau_i{\cdot}\mathbf{F}(\mathbf{a}_i{+}\mathbf{k}_3^{(i)}) \tag{16}$$

Selection of the technique for the minimization of $\Psi(\mathbf{a})$ that will have a guaranteed and quick convergence is not an easy task. However, the condition (2) generates the situation when during the tracking interval the major effect of updating by least squares is the improved accuracy of the velocity components $\dot{\mathbf{u}}{=}(\dot{x},\, \dot{y},\, \dot{z})$ of the state vector $\mathbf{a}$. The position parameters $\mathbf{u}{=}(x,\, y,\, z)$ of the vector $\mathbf{a}$ in fact are not updated. Thus we suggest for the minimization of $\Psi(\mathbf{a}) = \Psi(\mathbf{u}, \dot{\mathbf{u}}) = \psi(\dot{\mathbf{u}})$ we use only the vector $\dot{\mathbf{u}}$. Under the limits (3) of the measurement errors and the technique for calculation of the initial approximation selected below the relationship $\mathbf{x}(\dot{\mathbf{u}})$ is close to linear. Thus the iteration process converges very quickly. Eventually for all the cases the search for the minimum $\dot{\mathbf{u}}_{min}$ of the function $\psi(\dot{\mathbf{u}})$ requires only one iteration.

For the initial approximation $\mathbf{a}_0$ We can use the estimate $\hat{\mathbf{a}}_n$, obtained using measurements $\mathbf{x}_1,\ \mathbf{x}_2, ..., \mathbf{x}_n$ by the recursive algorithm described in the next section.

For the point of the minimum $\dot{\mathbf{u}}_{min}$ of the function $\psi(\dot{\mathbf{u}})$ for the case when it is obtained in one iteration we suggest using Newton's formula

$$\dot{\mathbf{u}}_{min} = \dot{\mathbf{u}}_0 - \left(\frac{\partial^2\psi}{\partial\dot{\mathbf{u}}^2}(\mathbf{a}_0)\right)^{-1} \cdot \frac{\partial\psi}{\partial\dot{\mathbf{u}}}(\mathbf{a}_0) \tag{17}$$

where the first and the second derivatives of the function $\psi(\dot{\mathbf{u}})$ with respect to parameters $\dot{x},\, \dot{y},\, \dot{z}$ are calculated using finite differences.

3. Recursive Algorithm

We know that the estimate $\mathbf{a}_{min}$ of the least squares method has a recursive structure: the estimate of the parameters based on k measurements can be written as a function of

the same estimate for $k-1$ measurements and the k-th measurement. This is true when all the functions $\mathbf{h}_k(\mathbf{a})$ are linear. For non-linear problems including our case this is sufficiently accurate when the errors of the measurements are relatively small and a satisfactory initial approximation for orbital parameters is available.

The recursive feature of the least squares estimate can be extended to the more general case [1], when the parameters $\mathbf{a}_k$=$\mathbf{a}(t_k)$ Of the system for the time t_k (in our case the orbital parameters for the time t_k) in addition to deterministic component have a random one which has Markov's feature (the "future" for the fixed "past" depends only on the "present"), i.e.

$$p\left(\mathbf{a}_k|\mathbf{a}_{k-1},\, \mathbf{a}_{k-2}, ...\right) = p\left(\mathbf{a}_k|\mathbf{a}_{k-1}\right) \tag{18}$$

where $p\left(\mathbf{a}_k|....\right)$ - conditional probability density $\mathbf{a}_k$.

The condition (18) is satisfied if

$$\mathbf{a}_k = \mathbf{f}_k\left(\mathbf{a}_{k-1}\right) + \gamma_k \tag{19}$$

where γ_k - "noise of the system" - time independent random perturbations with zero mean and covariation matrices $\mathbf{\Gamma}_k$.

Regarding the problem under consideration these perturbations may be related to the errors of the propagation model (in particular the atmospheric density model) and the calculations (e.g. roundoff) errors.

For (18) and independent measurement errors of the aposteriori (after acquisition of the measurement $\mathbf{x}_k$) probability density of the parameters $\mathbf{a}_k$ we have the following recursive formula

$$p\left(\mathbf{a}_k|\mathbf{x}_k,\, \mathbf{a}_{k-1},\, \mathbf{a}_{k-2}, ...\right) = p\left(\mathbf{a}_k|\mathbf{x}_k,\, \mathbf{a}_{k-1}\right) = p\left(\mathbf{x}_k|\mathbf{a}_k\right)\cdot p\left(\mathbf{a}_k|,\, \mathbf{a}_{k-1}\right)/p\left(\mathbf{x}_k\right), \tag{20}$$

From this relationship under the assumption (19) we can derive the approximate recurrent formulas for the first and second moments $\hat{\mathbf{a}}_k$, $\mathbf{P}_k$ of the aposteriori distribution $\mathbf{a}_k$

$$\mathbf{P}_k^{-1} = \mathbf{P}_{k|k-1}^{-1} + \mathbf{H}_k'\mathbf{R}_k^{-1}\mathbf{H}_k \qquad \mathbf{P}_k^{-1}\hat{\mathbf{a}}_k = \mathbf{P}_{k|k-1}^{-1}\hat{\mathbf{a}}_{k|k-1} + \mathbf{H}_k'\mathbf{R}_k^{-1}\mathbf{x}_k, \tag{21}$$

where

$$\hat{\mathbf{a}}_{k|k-1}=\mathbf{f}_k(\hat{\mathbf{a}}_{k-1}) \quad \mathbf{F}_k=\frac{\partial \mathbf{f}_k(\hat{\mathbf{a}}_{k-1})}{\partial \mathbf{a}_{k-1}} \quad \mathbf{H}_k=\frac{\partial \mathbf{h}_k(\hat{\mathbf{a}}_{k|k-1})}{\partial \mathbf{a}_k} \quad \mathbf{P}_{k|k-1}=\mathbf{F}_k\mathbf{P}_{k-1}\mathbf{F}_k'+\mathbf{\Gamma}_k \tag{22}$$

Equations (21), (22) are exact if the functions $\mathbf{h}_k(\mathbf{a}_k)$, $\mathbf{f}_k\left(\mathbf{a}_{k-1}\right)$, relating the measured parameters to the orbital parameters for the times t_k and t_{k-1}, are linear, and the distributions of the measurement and dynamic model errors are normal.

Equations (21) and (22) present a non-linear generalization of the known recursive Kalman's filter. For our case when the measured parameter is a scalar u^*, the matrix

*We can consequently update the previous estimate using the current measurements of range ($u = d$), azimuth ($u = \alpha$) and elevation ($u = \beta$), since the measurements d, α, β are not correlated

$\mathbf{H}_k$ becomes a vector-column $\mathbf{h}_k$ and (21) are transferred to the form

$$\begin{aligned} \tilde{w}_k &= w_k/(1 + w_k \mathbf{h}_k' \mathbf{P}_{k|k-1} \mathbf{h}_k) \\ \mathbf{P}_k &= \mathbf{P}_{k|k-1} - \tilde{w}_k (\mathbf{P}_{k|k-1} \mathbf{h}_k)(\mathbf{P}_{k|k-1} \mathbf{h}_k)' \\ \hat{\mathbf{a}}_k &= \hat{\mathbf{a}}_{k|k-1} + w_k \mathbf{P}_k \mathbf{h}_k (u_k - \mathrm{h}_k(\hat{\mathbf{a}}_{k|k-1})) \end{aligned} \tag{23}$$

where $w_k = 1/\sigma_{u_k}^2$ – is the weight characteristic of measurement u_k.

The choice of the specific algorithm for processing the measurements assumes that the following should be determined:

- parameters u_k=$u(t_k)$ of the measurements for the time t_k,
- parameters $\mathbf{a}_k$=$\mathbf{a}(t_k)$ of the orbit for the time t_k,
- the operator $\mathbf{f}_k(\mathbf{a}_{k-1})$ for propagating the orbital parameters from the time t_{k-1} to the time t_k,
- operator $\mathbf{F}_k(\mathbf{a}_{k-1})$ for the propagation of the variations of orbital parameters from time t_{k-1} to the time t_k,
- the technique for generating the initial approximaiton $\hat{\mathbf{a}}_0$, $\mathbf{P}_0$ and t_0 for calculating using the recurrent relationships (16),
- the technique for calculating the noise matrix of the system $\mathbf{\Gamma}_k$.

The coordinate frames of the parameters u_k, $\mathbf{a}_k$ and the propagation operator $\mathbf{f}_k(\mathbf{a}_{k-1})$ are the same as selected for the least squares method.

For the operator $\mathbf{F}_k(\mathbf{a}_{k-1})$ we will take the matrix $\mathbf{F}_k$ with dimensions 6×6 with the following non-zero elements: $f_{i,i}$=1 for $i = 1, 2, ..., 6$, $f_{i,i+3}$=τ_k for $i = 1, 2, 3$, where $\tau_k = t_k - t_{k-1}$.

$$\mathbf{F}_n = \begin{pmatrix} 1 & 0 & 0 & \tau_n & 0 & 0 \\ 0 & 1 & 0 & 0 & \tau_n & 0 \\ 0 & 0 & 1 & 0 & 0 & \tau_n \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

The initial approximation $\hat{\mathbf{a}}_0 = (\hat{x}_0, \hat{y}_0, \hat{z}_0, \dot{\hat{x}}_0, \dot{\hat{y}}_0, \dot{\hat{z}}_0)' = (\hat{\mathbf{d}}_0, \dot{\hat{\mathbf{d}}}_0')$ for the vector of orbital parameters for the time t_0 and the matrix $\mathbf{P}_0$ of its errors are determined from the first two measurements $\mathbf{x}_1 = (d_1, \alpha_1, \beta_1)$ and $\mathbf{x}_2 = (d_2, \alpha_2, \beta_2)$, acquired for the times t_1 and t_2. Regarding the limitations (3) and (4) the following procedure will be correct:

- The measurements $\mathbf{x}_1$, $\mathbf{x}_2$ are transferred to $\mathbf{d}_1$, $\mathbf{d}_2$ using formulas (10).
- Then we calculate the state vector $(\tilde{\mathbf{d}}_0, \dot{\tilde{\mathbf{d}}}_0)$, consistent with the limitations (3) for the

measurement errors of the angular coordinates:

$$\tilde{\mathbf{d}}_0{=}0.5(\mathbf{d}_1{+}\mathbf{d}_2){-}\tau^2\ddot{\mathbf{d}}_0/8 \qquad \tilde{\dot{\mathbf{d}}}_0{=}(\mathbf{d}_2{-}\mathbf{d}_1)/\tau \qquad t_0 = 0.5(t_1{+}t_2) \tag{24}$$

where $\tau = (t_2{-}t_1)$, a $\ddot{\mathbf{d}}_0$ is determined using (11) from the state vector without compensating term $\tau_0^2\ddot{\mathbf{d}}_0/8$.

- In the calculated state vector $(\tilde{\mathbf{d}}_0, \tilde{\dot{\mathbf{d}}}_0)$, the parameters $\tilde{d}_0, \tilde{\dot{d}}_0$ are changed to more accurate values $\hat{d}_0{=}\sqrt{\hat{\xi}_0}$, $\hat{\dot{d}}_0{=}\hat{\dot{\xi}}_0/2\hat{d}_0$, adequate to the limitations (3) for the errors of range measurements:

$$\hat{\xi}_0{=}0.5(\xi_1{+}\xi_2){-}\tau^2\ddot{\xi}_0/8 \qquad \hat{\dot{\xi}}_0{=}(\xi_2{-}\xi_1)/\tau{-}\tau^2\xi_0^{(3)}/24 \tag{25}$$

where $\xi{=}d^2$, $\ddot{\xi}_0{=}2(\tilde{\dot{\mathbf{d}}}_0, \tilde{\dot{\mathbf{d}}}_0){+}2(\tilde{\mathbf{d}}_0, \ddot{\mathbf{d}}_0)$, $\xi_0^{(3)}{=}6(\tilde{\dot{\mathbf{d}}}_0, \ddot{\mathbf{d}}_0){+}2(\tilde{\mathbf{d}}_0, \mathbf{d}_0^{(3)})$, and $\mathbf{d}_0^{(3)}$ is determined from the state vector using the formulas by taking time derivatives of the motion equations (11).
After this substitution we get $\hat{\mathbf{a}}_0$.

- The correlation matrix $\mathbf{P}_0{=}\begin{pmatrix} \mathbf{A}_0 & \mathbf{B}_0 \\ \mathbf{B}_0' & \mathbf{C}_0 \end{pmatrix}$ of the initial approximation errors $\hat{\mathbf{a}}_0$ is determined as follows:

$\sigma_{\hat{\alpha}}{=}\sigma_\alpha \qquad \sigma_{\hat{\beta}}{=}\sigma_\beta \qquad \sigma_{\hat{\dot{\alpha}}}{=}2\sigma_\alpha/\tau \qquad \sigma_{\hat{\dot{\beta}}}{=}2\sigma_\beta/\tau$

$v{=}\sqrt{\hat{\dot{x}}_0^2{+}\hat{\dot{y}}_0^2{+}\hat{\dot{z}}_0^2} \qquad \sigma_{\hat{d}}{=}\sigma_d{+}\tau v{\cdot}\max(\sigma_\alpha, \sigma_\beta)/4 \qquad \sigma_{\hat{\dot{d}}}{=}2\sigma_d/\tau$

$$\mathbf{G}_0 = \frac{\partial(x, y, z)}{\partial(d, \alpha, \beta)}(\mathbf{d}_0) \text{ is calculated using formulas following from (10)} \tag{26}$$

$$\mathbf{R}_{0,0} = \begin{pmatrix} \sigma_{\hat{d}}^2 & 0 & 0 \\ 0 & \sigma_{\hat{\alpha}}^2 & 0 \\ 0 & 0 & \sigma_{\hat{\beta}}^2 \end{pmatrix} \qquad \mathbf{R}_{0,1} = \begin{pmatrix} \sigma_{\hat{\dot{d}}}^2 & 0 & 0 \\ 0 & \sigma_{\hat{\dot{\alpha}}}^2 & 0 \\ 0 & 0 & \sigma_{\hat{\dot{\beta}}}^2 \end{pmatrix}$$

$\mathbf{A}_0 = \mathbf{G}_0\mathbf{R}_{0,0}\mathbf{G}_0' \quad \mathbf{C}_0 = \mathbf{G}_0\mathbf{R}_{0,1}\mathbf{G}_0' \quad \mathbf{B}_0 = \mathbf{0}$

Here σ_d, σ_α, σ_β – are the RMS values of typical errors of the measured parameters.

- Accounting for the system noise follows [6]. The matrix $\mathbf{\Gamma}_k$ is calculated in this way:

$$\mathbf{\Gamma}_{11,k} = \begin{pmatrix} 110{\cdot}\sigma_d^2/(\tau_n^2(\Delta t_{\text{ef,d}})^4) & 0 & 0 \\ 0 & 110{\cdot}\sigma_\alpha^2/(\tau_n^2(\Delta t_{\text{ef},\alpha})^4) & 0 \\ 0 & 0 & 110{\cdot}\sigma_\beta^2/(\tau_n^2(\Delta t_{\text{ef},\beta})^4) \end{pmatrix}$$

$$\mathbf{G}_k = \frac{\partial(x, y, z)}{\partial(d, \alpha, \beta)}(\mathbf{d}_{k|k-1}) \qquad \mathbf{\Gamma}_k{=}\begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{\Gamma}_{11,k} \end{pmatrix}$$

where

– σ_d, σ_α, σ_β – are the RMS values of typical values of the errors of the measured parameters,
– $\Delta t_{\text{ef,d}}$, $\Delta t_{\text{ef},\alpha}$, $\Delta t_{\text{ef},\beta}$ – efficient memory of the algorithm for parameters d, α, β (parameters of the algorithm selected on experimental basis).

The cycle of processing the measurements d_k, α_k, β_k (k=3,4,...) uses the formulas (23). They provide the following: using the previous parameters estimate $\hat{\mathbf{a}}_{k-1}$ and the calculated correlation matrix of its errors $\mathbf{P}_{k-1}$, related to the time t_{k-1}, and using the current scalar measurement u_k, acquired for the time $t_k \geq t_{k-1}$, calculate the updated estimate $\hat{\mathbf{a}}_k$ and calculated correlation matrix of its errors $\mathbf{P}_k$. First using (23) we update the previous estimate using $u_k=d_k$, then the estimate is additionally updated using $u_k=\alpha_k$ and finally the estimate is updated using $u_k=\beta_k$. The vectors $\mathbf{h}_k$ used in (23) are calculated using formulas resulting from partial differentiation of relationship (10).

4. Non-recursive Method Based On the Guarantee Method

The basic assumption of the Guarantee approach for non-linear problems is the small level of measurement errors and the known values of the upper limits of these errors. The term "Guarantee" in this case means that the algorithm provides not only the calculated orbital parameters, but also the maximum possible errors of these parameters. The principals of the approach are as follows.

Assume at the times t_k $(t_k \leq t_{k+1};\ k = 1, 2, ...n)$ we acquire measurements u_k of certain functions $h_k(\mathbf{c})$ of the m-vector of parameters $\mathbf{c}$ $(m<n)$, and the errors of the measurements $\delta u_k = u_k - h_k(\mathbf{c})$ are limited from above by the constant $\delta_{k,max}$. The estimate $\bar{\mathbf{c}}_n$ of parameters $\mathbf{c}$ obtained using the Guarantee approach and the vector $\delta\bar{\mathbf{c}}_{n,max}$ of the maximum errors of the components of this estimate have the following geometrical interpretation.

The limits of the measurement errors define in the m-dimensional space of parameters $\mathbf{c} = (\mathbf{c_1}, \mathbf{c_2}, ..., \mathbf{c_m})$ a domain

$$\mathbf{D}_n = \bigcap_{k=1}^{n} \{u_k - \delta_{i,max} \leq h_k(\mathbf{c}) \leq u_k - \delta_{k,max}\} \tag{27}$$

of possible values of $\mathbf{c}$. We project this domain to the coordinate of the components of vector $\mathbf{c}$ and among the projected points for each axis find the most right $\mathbf{c}_{n,r}$ and the most left $\mathbf{c}_{n,l}$. They define the boundaries (maximum and minimum values) for the changes of each component of parameter $\mathbf{c}$. In this case the estimate $\bar{\mathbf{c}}_n$ of parameter and maximum errors $\delta\bar{\mathbf{c}}_{n,max}$ of this estimate are naturally defined as

$$\bar{\mathbf{c}}_n = \frac{1}{2}\cdot(\mathbf{c}_{n,r} + \mathbf{c}_{n,l}) \qquad \delta\bar{\mathbf{c}}_{n,max} = \frac{1}{2}\cdot(\mathbf{c}_{n,r} - \mathbf{c}_{n,l}) \tag{28}$$

If the measured parameters are linearly related to the determined parameter $\mathbf{a}$, i.e. $h_i(\mathbf{a}) = \mathbf{h}_i' \cdot \mathbf{a}$, where the components 6×1 of vector $\mathbf{h}_i$ do not depend on $\mathbf{a}$, this problem can be formulated as a standard problem of linear programming. Its solution is obtained in [22], we will not describe the algorithm here.

If the measured parameters are linearly related to the determined parameter i.e. $h_k(\mathbf{c})=\mathbf{h}'_k\cdot\mathbf{c}$, where components $m\times 1$ of vector $\mathbf{h}_k$ do not depend on $\mathbf{c}$, this problem can be formulated as a standard promlem of linear programming. The solution of the problem is known but the procedure is rather sophisticated and we will not present it here.

For evaluation of the accuracy characteristics of the parameters obtaned using method it will be useful to consider a model case: we measure a scalar parameter c, with $h=1$ and $\delta_{k,max}=\delta_{max}$. In this one dimensional model

$$\mathbf{D}_n=\bigcap_{k=1}^{n}\{[\,u_k-\delta_{max},\,u_k+\delta_{max}\,]\}=[\,\max_k u_k-\delta_{max},\,\min_k u_k+\delta_{max}\,] \tag{29}$$

$$\bar{c}_n=\frac{1}{2}\cdot(\max_k u_k+\min_k u_k)\qquad \delta\bar{c}_{n,max}=\delta_{max}-\frac{1}{2}\cdot(\max_k u_k-\min_k u_k) \tag{30}$$

where $[\,a,\,b\,]$ - denotes an interval with left end a and right end b.

The estimate $\bar{c}_n$ from (30) has some important properties which to a certain extent are retained for the multi-dimensional case.

1) The estimate (30) does not depend on δ_{max}, that means it is not sensitive to the accuracy of the knowledge of the characteristics of the errors – in this case to the accuracy of the value δ_{max}.
2) If we know δ_{max} precisely the value $\delta\bar{c}_{n,max}$ is a correct upper estimate for the error of the estimate $\bar{c}_n$, for any correlations of the errors of different measurements. Correctness in this case means the following. On one hand the true errors of the estimate $\bar{a}_n$ for any n can not be greater than $\delta\bar{a}_{n,max}$. On the other hand – the change of $\delta\bar{c}_{n,max}$ with growth of n correspond to the change in the correlation characteristics of the measurement errors. If the correlation interval of the errors is limited, the value $\delta\bar{c}_{n,max}$ with the growth of n can be made indefinitely small. If this is not the case, i.e., a systematic error is present in the measurements, the value $\delta\bar{c}_{n,max}$ for any n will be limited from below by the value of this error.
3) For uncorrelated measurement errors the estimate $\bar{c}_n$ with regard to accuracy is not inferior and sometimes can be essentially more accurate (!) than the estimate $\hat{c}_n$ of the least squares technique, which in this case has the form

$$\hat{c}_n=\frac{1}{n}\sum_{k=1}^{n}u_k \tag{31}$$

Thus, for example for the most frequent in practice uniform and triangular distributions of the measurement errors within the interval $(-\delta_{max},\delta_{max})$ the estimates $\hat{c}_n$ and $\bar{c}_n$ are unbiased, and the r.m.s. deviation of their errors $\sigma_{\hat{c}_n}$ and $\sigma_{\bar{c}_n}$ for $n>10$ is calculated with a relative error not more than 10% using the following asymptotic formulas

– for uniform distribution:

$$\sigma_{\bar{c}_n} \simeq \frac{1.4{\cdot}\delta_{max}}{n} \qquad\qquad \sigma_{\hat{c}_n} \simeq \frac{0.58{\cdot}\delta_{max}}{\sqrt{n}} \tag{32}$$

– for triangular distribution:

$$\sigma_{\bar{c}_n} \simeq \frac{0.46{\cdot}\delta_{max}}{\sqrt{n}} \qquad\qquad \sigma_{\hat{c}_n} \simeq \frac{0.41{\cdot}\delta_{max}}{\sqrt{n}} \tag{33}$$

that follow from precise relationships, given in [5].

From (32) and (33) one can see that for a triangular distribution the accuracy of the estimates $\hat{a}_n$ and $\bar{a}_n$ is close and for the uniform one the estimate $\bar{a}_n$ is about $\approx 0.4\sqrt{n}$ times more accurate. Thus for the uniform distribution, not satisfying the regularity condition of Duget [5]†, the estimate $\bar{a}_n$ turns to be super-efficient.

These properties make the Guarantee approach more advantageous compared to the least squares method and make it rather attractive for use in applications characteristic for the operation of different radars. This is especially characteristic for rather accurate and stable in performance radars where the abnormal measurements (if any) can be easily selected during preliminary processing of the single measurements and do not enter the process of orbit determination.

Let us treat the considered task in more detail.

Assume that the radar at the times t_k $(k = 1, 2, ..., n)$ measures the range d, azimuth α and elevation angle β in the local spherical coordinate frame and the errors of the measurements do not exceed respectively the values $\delta_{d,max}$, $\delta_{\alpha,max}$, $\delta_{\beta,max}$. The task is to determine at the time $\bar{t}{=}0.5{\cdot}(t_1{+}t_n)$ the six-dimensional vector of orbital parameters $\mathbf{c} = (d,\, \alpha,\, \beta,\, \dot{d},\, \dot{\alpha},\, \dot{\beta})$ and its maximum errors in this coordinate frame.

The following algorithm for solving the task is suggested.

Divide all the measurements into $n/2$ groups‡. The first group includes the measurements acquired at t_1 and $t_{0.5n}$, the second group – the measurements with t_2 and $t_{0.5n+1}$, etc. For each k-th group $(k = 1, 2, .., 0.5{\cdot}n)$ from two position vectors $\mathbf{u}_k = (d_k,\, \alpha_k,\, \beta_k)$ and $\mathbf{u}_{0.5n+k} = (d_{0.5n+k},\, \alpha_{0.5n+k},\, \beta_{0.5n+k})$ in the local spherical coordinate frame we determine the six dimensional vector of orbital parameters $\check{\mathbf{c}}_k = (\check{d},\, \check{\alpha},\, \check{\beta},\, \check{\dot{d}},\, \check{\dot{\alpha}},\, \check{\dot{\beta}})_k$ in the same coordinate frame for the time $\bar{t}$.

Instead of the labor consuming operation of determination of the domain (27) and its projections on the coordinate axes of the phase vector we suggest to project on these axes the

†In this case the break points of the distribution depend on the estimated parameter.

‡We assume that n is even. If n is odd we consider that for the time t_n we have two identical measurements.

domain $\mathbf{D}_k$, corresponding to the k-th group and then for each axis search for the intersections of this projections.

The domain $\mathbf{D}_k$ is approximated by a six-dimensional parallelepiped with the center in the point $\check{\mathbf{c}}_k$ and 64 apexes determined by the formula

$$\check{\mathbf{c}}_k \pm \delta_{d,max}\cdot\mathbf{j}_1 \pm \delta_{\alpha,max}\cdot\mathbf{j}_2 \pm \delta_{\beta,max}\cdot\mathbf{j}_3 \pm \delta_{d,max}\cdot\mathbf{j}_4 \pm \delta_{\alpha,max}\cdot\mathbf{j}_5 \pm \delta_{\beta,max}\cdot\mathbf{j}_6, \tag{34}$$

where $\mathbf{j}_1$, $\mathbf{j}_2$, $\mathbf{j}_3$, $\mathbf{j}_4$, $\mathbf{j}_5$, $\mathbf{j}_6$ – are the lines of the matrix of partial derivatives $\mathbf{J} = \dfrac{\partial(\mathbf{c}_k)}{\partial(\mathbf{u}_k,\, \mathbf{u}_{0.5n+k})}$ of the functional transformation $\mathbf{c}_k{=}\mathbf{f}_k(\mathbf{u}_k,\, \mathbf{u}_{0.5n+k})$ in the point $\check{\mathbf{c}}_k$. The boundary projections $\mathbf{c}_{k,l}$ and $\mathbf{c}_{k,r}$ of these 64 points of the six dimensional phase space to each of its axes determines the boundaries of the vector interval $(\mathbf{c}_{k,l},\, \mathbf{c}_{k,r})$ of the possible values of all six orbital parameters determined by the k-th group of measurements.

After determination of the common part $(\mathbf{c}_l,\, \mathbf{c}_r) = \bigcap_{k=1}^{n}(\mathbf{c}_{k,l},\, \mathbf{c}_{k,r})$ of these intervals for all groups of measurements using eq. (28) we determine the estimate of the orbital parameters and its maximum errors.

The major element of the suggested algorithm is the procedure $\mathbf{c}_k{=}\mathbf{f}_k(\mathbf{u}_k,\, \mathbf{u}_{0.5n+k})$ for orbit determination on the basis of two positions. This procedure is used for determination of the state vector $\mathbf{c}_k$, and the matrix of derivatives $\mathbf{J}$. A rather simple and efficient algorithm for solving this task for Kepler's approximation is given in [7] (algorithms 31 and 51). The algorithm uses the measured position vectors $\mathbf{r} = (X,\, Y,\, Z)$ of the satellite in a fixed coordinate frame for Kepler's orbit. They are obtained from the measurements $\mathbf{u}_k = (d_k,\, \alpha_k,\, \beta_k)$ in the following way:

– For the fixed coordinate frame $X,\, Y,\, Z$ we select the system whose axes for the time $\bar{t} = 0.5(t_1{+}t_n)$ coincide with the axes of the Greenwich system. The measurements $\mathbf{u}_k = (d_k,\, \alpha_k,\, \beta_k)$ are transferred to $\mathbf{r}_k = (X_k,\, Y_k,\, Z_k)$.

– From all the measurements $\mathbf{r}_k$ we substract such values of $\Delta\mathbf{r}_k$, for which $\tilde{\mathbf{r}}_k{=}\mathbf{r}_k{-}\Delta\mathbf{r}_k$ follow (up to the errors of the measurements) Keplerian orbit, whose parameters for the time $\bar{t} = 0.5{\cdot}(t_1{+}t_n)$ coincide with the parameters of real (perturbed) orbit. To obtain $\Delta\mathbf{r}_k$ we solve the equations of motion using Euler's method

$$\Delta\ddot{\mathbf{r}} = -\tilde{\mu}\cdot\Delta\tilde{\mathbf{r}} + 3{\cdot}\tilde{\mu}\cdot(\Delta\tilde{\mathbf{r}},\, \tilde{\mathbf{r}})\cdot\tilde{\mathbf{r}} - 3{\cdot}\tilde{\mu}\cdot\tilde{R}^2{\cdot}J_2{\cdot}(0.5{\cdot}\tilde{\mathbf{r}} + \tilde{z}\vec{\omega}_0 - 2.5{\cdot}\tilde{z}^2{\cdot}\tilde{\mathbf{r}}) \tag{35}$$

for the deviations of $\Delta\mathbf{r}$, $\Delta\dot{\mathbf{r}}$ from the Keplerian orbit for $\Delta\mathbf{r}\,(\bar{t}\,) = \Delta\dot{\mathbf{r}}\,(\bar{t}\,) = 0$ and $\mathbf{r} = \mathbf{r}_k$. Here $r = \sqrt{(\mathbf{r},\, \mathbf{r})}$ $\tilde{\mathbf{r}} = \mathbf{r}/r$ $\Delta\tilde{\mathbf{r}} = \Delta\mathbf{r}/r$ $\tilde{z} = z/r$ $\tilde{\mu} = \mu/r^2$ $\tilde{R}{=}R_e/r$.

5. Conclusions

A new method for determining the orbit of a space object from a single pass through a radar has been presented. This method, called the Guarantee Method, is based on linear programming and offers advantages over the standard least squares and Extended Kalman filter. Under certain conditions this method provides a much more accurate state estimate than the least squares and Kalman filter. It also provides limits on the state estimate errors when the measurement errors have a uniform or triangular distribution. The amount of computation is significantly more for the Guarantee Method than the other methods, but this is not a problem with modern computers. A comparative analysis with the other methods has been presented.

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