C. S. Jacobs et al.: Rotational Alignment Altered by Source Position Correlations, IVS 2010 General Meeting Proceedings, p.305–309
http://ivscc.gsfc.nasa.gov/publications/gm2010/jacobs2.pdf

Rotational Alignment Altered by Source Position Correlations

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Abstract

In the construction of modern Celestial Reference Frames (CRFs) the overall rotational alignment is only weakly constrained by the data. Therefore, common practice has been to apply a 3-dimensional No-Net-Rotation (NNR) constraint in order to align an under-construction frame to the ICRF. We present evidence that correlations amongst source position parameters must be accounted for in order to properly align a CRF at the 5–10 μas level of uncertainty found in current work. Failure to do so creates errors at the 10–40 μas level.

1. Introduction to No-net-rotation Constraint Equations

Since the adoption of the ICRF [4] by the IAU in 1997 [1], the axes which define Right Ascension and declination have been realized by an ensemble of quasar positions. Subsequent frames, most notably the ICRF2 [5], have been aligned to the original ICRF by imposing a 3-dimensional No-Net-Rotation (NNR) constraint in the form of the cross-product (see full derivation in appendix),

$$\sum_{i=1}^{N_{src}} s_{0i} \times \Delta s_i = 0 \tag{1}$$

where for a set of N_{src} sources,

$$s_{0i} = (\cos\alpha_{0i}\cos\delta_{0i},\ \ \sin\alpha_{0i}\cos\delta_{0i},\ \ \sin\delta_{0i}) \tag{2}$$

is the unit vector in the reference source direction for the i^{th} source of coordinates $(\alpha_{0i}, \delta_{0i})$ and Δs_i is the difference of the reference position from the estimated position.

$$\begin{aligned} s_{0i} \times \Delta s_i \ = \ & [-\cos\alpha_{0i}\sin\delta_{0i}\cos\delta_{0i}\ \Delta\alpha_i \ + \ \sin\alpha_{0i}\ \Delta\delta_i], \\ & [-\sin\alpha_{0i}\sin\delta_{0i}\cos\delta_{0i}\ \Delta\alpha_i \ - \ \cos\alpha_{0i}\ \Delta\delta_i], \\ & [\cos^2\delta_{0i}\ \Delta\alpha_i] \end{aligned} \tag{3}$$

The above triplet of constraints affects both the estimated set of α_i and δ_i and their associated covariance including the correlations amongst parameters.

The MODEST linear least squares fit software [7] requires the coefficient with respect to $\Delta\alpha_i$ and $\Delta\delta_i$ of each rotation constraint, $C_j = (\sum_i s_{0i} \times \Delta s_i)_j$ where j goes from 1 to 3. The result is

$$\partial C_1/\partial\Delta\alpha_i = -\cos\alpha_{0i}\sin\delta_{0i}\cos\delta_{0i}, \qquad \partial C_1/\partial\Delta\delta_i = \sin\alpha_{0i} \tag{4}$$

$$\partial C_2/\partial\Delta\alpha_i = -\sin\alpha_{0i}\sin\delta_{0i}\cos\delta_{0i}, \qquad \partial C_2/\partial\Delta\delta_i = -\cos\alpha_{0i} \tag{5}$$

$$\partial C_3/\partial\Delta\alpha_i = \cos^2\delta_{0i}, \qquad \partial C_3/\partial\Delta\delta_i = 0 \tag{6}$$

Note that the constraint Eq. (1) is unweighted in the position parameters, s. The MODEST software implements constraints by treating them as pseudo-observations with given uncertainties. Specifying a vanishingly small uncertainty is equivalent to using an absolute constraint.

Using the preceding partials (Eqs. 4, 5, 6), each C_j was separately constrained to be zero to within a tight constraint uncertainty of $\sigma_j = 10^{-10}$ radians.

2. Sensitivity of S/X, K, and X/Ka Frame Alignment to Correlations

Our procedure was first to estimate source positions under the influence of unweighted NNR constraints (Eq. 1) to the ICRF2 defining sources. We used only sources with at least three sessions, ten delay observations, and a formal error ellipse major axis smaller than 5 nrad, and which were also in the ICRF2 defining list.

Second, we did an after-the-fact verification that the NNR constraints indeed kept the positions from rotating with respect to the defining sources by estimating a 3-D rotation with respect to the ICRF2 defining sources.

This second step was done using alternately diagonal-only parameter covariance and full covariance for the Celestial Reference Frame (CRF) being constrained. The reference, ICRF2, always used diagonal covariances because the off-diagonal correlations are not published.

The results for three data sets are given in Table 1 with descriptions of the sets as follows:

- The S/X data set included observations from October 1978 to September 2009 totalling 5.3M observations from 1709 sources. The NNR constraint used 267 ICRF2 defining sources.
- The K-band data set [3] covers 2002 to 2009 with over 100,000 observations of 275 sources. The NNR constraint used 125 ICRF2 defining sources.
- The X/Ka-band data set [2] covers 2005 to 2010 with over 10,000 observations of 387 sources. The NNR constraint used 153 ICRF2 defining sources.

Table 1. 3-D Rotations (μas) estimated *with* vs. *without* correlations.

CRF	N_{Def}	R_1	σ_{R_1}	R_2	σ_{R_2}	R_3	σ_{R_3}
S/X with corr.	267	1.4	±5.4	2.4	±5.6	2.0	±4.4
without corr.	267	−25.8	±4.9	9.4	±4.9	−1.6	±4.3
K with corr.	125	0.1	± 6.8	1.2	± 7.4	−0.1	±5.2
without corr.	125	−20.5	±11.5	−17.7	±11.9	−8.2	±7.9
X/Ka with corr.	153	0.3	± 6.3	2.9	± 6.6	− 0.1	± 4.9
without corr.	153	−35.1	±17.9	31.1	±18.3	38.6	±11.2

3. Discussion

NNR Constraint Weighting: Given that the NNR constraint to the ICRF2 was unweighted, there may be problems caused by using a diagonal weighting for our after-the-fact check of rotational alignment. Our alignment check used the sum of the covariance of the frame under study and the diagonal ICRF2 covariance (note that the ICRF2 off-diagonal terms have not been made available). For the three cases studied above, the X/Ka frame has the largest uncertainties, which are expected to dominate the combined X/Ka + ICRF2 covariance. The K-band uncertainties are somewhat larger than the ICRF2 and will partially dominate the combined covariance. However, because the S/X frame comes from almost the same data as the ICRF2, its uncertainties are comparable to those of the un-inflated ICRF2. Once the ICRF2 diagonal covariances are inflated by a scale factor of 1.5 and then have a 40 μas noise floor RSS'ed into each position coordinate, the ICRF2 covariance dominates the diagonal uncertainties.

These remaining nuances in weighting may explain why—even with full covariance—our after-the-fact check shows a few μas residual non-alignment. We have not yet studied these issues closely and can only say that the subject deserves more careful investigation.

Nutation constraints: One important lesson learned in the course of our comparisons is that other constraints can distort the desired effect of the NNR constraint. In the case of our K-band solution, our a priori nutation model was the IAU 2000 standard of Matthews, Herring, Buffet (MHB) [6]. Because there was evidence external to our study that the MHB model had 100–150 μas errors, we at first estimated nutation corrections with "weak" 200 μas a priori uncertainties on the nutation angles for each of twelve day-long sessions. Noting that a priori constraints act as pseudo-measurements, this effectively added 12 observations of the nutation pole each with 200 μas uncertainty. Thus, in retrospect, we realized that we had added a global constraint on the nutation pole of 200 μas $/\sqrt{12} = 58$ μas which was strong enough to bias the 3-D rotational alignment of our celestial reference frame. This example is shared as a reminder that celestial frame alignment at the level of 10 μas requires careful attention to many details.

4. Conclusions

The celestial frame No-Net-Rotation constraint, $\sum s_{0i} \times \Delta s_i = 0$, alters the α-δ full covariance produced by a CRF solution. The correlations contain significant information which is needed to correctly estimate 3-D rotations and associated sigmas. At the 5–10 μas levels of axial alignment precision found in current CRF work, these correlations must be accounted for in order to avoid statistically significant misalignment of a given CRF.

5. Acknowledgements

The research described herein was performed at the Jet Propulsion Laboratory of the California Institute of Technology, under a contract with the National Aeronautics and Space Administration.

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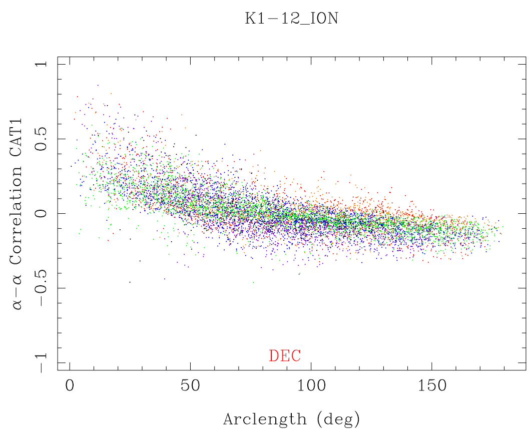

Figure 1. K-band (24 GHz) 12-session distribution of α-α correlations vs. arclength for all combinations of two sources. The correlations are systematically positive for short arcs and become negative for the longest arcs. These trends alter both the estimated rotations and their uncertainties. Color coding of 22.5° dec. bands: low to high (orange, red, green, blue, purple, black).

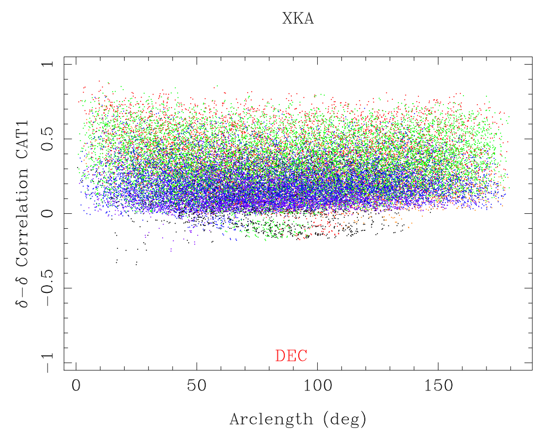

Figure 2. X/Ka (8.4/32 GHz) CRF's distribution of δ-δ correlations vs. arclength. The correlations remain systematically positive over the full range of arc lengths, thus altering both the estimated rotations and their uncertainties. Color coding is the same as in the preceding figure. Note the tendency for low declination to be more strongly correlated.

6. Appendix: Derivation of NNR Constraint Equation

Given two sets of sources forming two frames, one frame is a priori assigned the role of the absolute reference and the other frame we wish to estimate from our data. Next, we establish the mathematical formula which describes the angular rotational offset between these two frames. Then we estimate the positions from data by least squares such that the rotational offset is zero.

Let s_{0i} be the reference position vector of the i^{th} source, and let s_i be the estimated position. Denote the difference as $\Delta s_i = s_{0i} - s_i$. Suppose an approximately infinitesimal rotation, $R(\vec{\epsilon})$, is applied to s_i. Then the rotated position is

$$R(\vec{\epsilon})\ s_i \ = \ s_i \ + \ \vec{\epsilon} \times s_i \tag{7}$$

We want the rotation $R(\vec{\epsilon})$ that minimizes the mean square angular position difference:

$$(\Delta\theta)^2 \ = \ \frac{1}{N_{src}} \sum_{i=1}^{N_{src}} \ (\Delta s_i \ - \ \vec{\epsilon} \times s_i) \cdot (\Delta s_i \ - \ \vec{\epsilon} \times s_i) \tag{8}$$

We minimize with respect to variations in ϵ_j by setting the derivative, $\partial\ (\Delta\theta)^2/\ \partial\epsilon_j = 0$

$$\sum_{i=1}^{N_{src}} [\frac{\partial}{\partial\epsilon_j}(\Delta s_i - \vec{\epsilon} \times s_i)] \cdot (\Delta s_i - \vec{\epsilon} \times s_i) \ + \ (\Delta s_i - \vec{\epsilon} \times s_i) \cdot [\tfrac{\partial}{\partial\epsilon_j}(\Delta s_i - \vec{\epsilon} \times s_i)] \ = \ 0 \tag{9}$$

$$\sum_{i=1}^{N_{src}} \ (-(\frac{\partial}{\partial\epsilon_j}\vec{\epsilon}) \times s_i) \cdot (\Delta s_i - \vec{\epsilon} \times s_i) \ + \ (\Delta s_i - \vec{\epsilon} \times s_i) \cdot (-(\tfrac{\partial}{\partial\epsilon_j}\vec{\epsilon}) \times s_i) \ = \ 0 \tag{10}$$

Since dot products commute, this can be re-written:

$$\sum_{i=1}^{N_{src}} \ (-2)(\Delta s_i - \vec{\epsilon} \times s_i) \cdot ((\tfrac{\partial}{\partial\epsilon_j}\vec{\epsilon}) \times s_i) \ = \ 0 \tag{11}$$

We want the above derivative to be zero, $\partial(\Delta\theta)^2/\partial\epsilon_j = 0$ when evaluated at $\vec{\epsilon} = 0$, thus

$$\sum_{i=1}^{N_{src}} \ (\Delta s_i) \cdot ((\tfrac{\partial}{\partial\epsilon_j}\vec{\epsilon}) \times s_i) \ = \ 0 \qquad \text{for j} = 1, 2, 3 \tag{12}$$

Using the general vector property $v_1 \cdot (v_2 \times v_3) = v_2 \cdot (v_3 \times v_1)$ we have:

$$\sum_{i=1}^{N_{src}} \ (\tfrac{\partial}{\partial\epsilon_j}\vec{\epsilon}) \ \cdot (s_i \times \Delta s_i) \ = \ 0 \qquad \text{for j} = 1, 2, 3 \tag{13}$$

Since $\frac{\partial}{\partial\epsilon_j}\vec{\epsilon}$ is just the elementary unit vector on the j^{th} axis, this represents three equations which can be combined into the vector equation:

$$\sum_{i=1}^{N_{src}} \ s_i \ \times \ \Delta s_i \ = \ 0 \tag{14}$$

Noting that $s_i \times \Delta s_i = (s_{0i} - \Delta s_i) \times \Delta s_i = s_{0i} \times \Delta s_i - \Delta s_i \times \Delta s_i = s_{0i} \times \Delta s_i$ since for any vector, v, $v \times v = 0$, we have:

$$\sum_{i=1}^{N_{src}} \ s_{0i} \ \times \ \Delta s_i \ = \ 0 \tag{15}$$