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Peter Waterman and *T*-matrix methods



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ABSTRACT

This paper summarizes the scientific legacy of Peter C. Waterman (1928–2012) who introduced concepts and theoretical techniques that have had a major impact on the fields of scattering by particles and particle groups, optical particle characterization, radiative transfer, and remote sensing. A biographical sketch is also included.

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1. Introduction

At a wedding reception in 1998, Peter Waterman, feeling shy in an unfamiliar crowd, approached the only well recognizable figure in the room: an old, grand piano. Without meaning to, he became the entertainment of the night, playing original scores for the guests, filling the air with graceful energy. After the reception, the hotel manager followed Peter and his wife Karen into the elevator and asked if Peter would be interested in a job playing the piano at the hotel. Peter looked at Karen, smiled, and politely refused. Although music was an essential part of his life, his professional allegiance was to mathematical physics.

A quiet and unassuming man of habits, Peter Cary Waterman (Fig. 1) was well known to many but personally acquainted with a few. During his more than 50 years in science, he had (co)authored only 39 peer-reviewed

publications; these are listed below in Section 8 and cited as [Wxx]. Yet his research has had a dramatic and long-lasting impact and to a large degree has guided the progress in the disciplines of electromagnetic, acoustic, and elastic wave scattering by obstacles.

2. Early years

Born in Albany, New York, on 14 June 1928 to Frederick Collins and Gertrude Melissa (Cary) Waterman, Peter Waterman graduated from Bethlehem High School, Delmar, New York, in 1946 (Fig. 2). He received a BS degree in physics from Syracuse University in 1951, and from 1952 to 1953, served as a physics instructor at Union College in Schenectady, New York. Waterman was subsequently appointed to an Alcoa Research Fellowship in the Division of Applied Mathematics at Brown University. There, his first few papers [W02, W03, W04, W06] and his MS thesis [W01] were studies of elastic wave propagation through anisotropic materials.

In 1954, Peter Waterman married Katherine Adella Dearstyne, with whom he had three sons: Diedrich, Jonathan, and Jeremy. Peter and Katherine divorced in 1975.

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Fig. 1. Peter Cary Waterman (1928–2012).

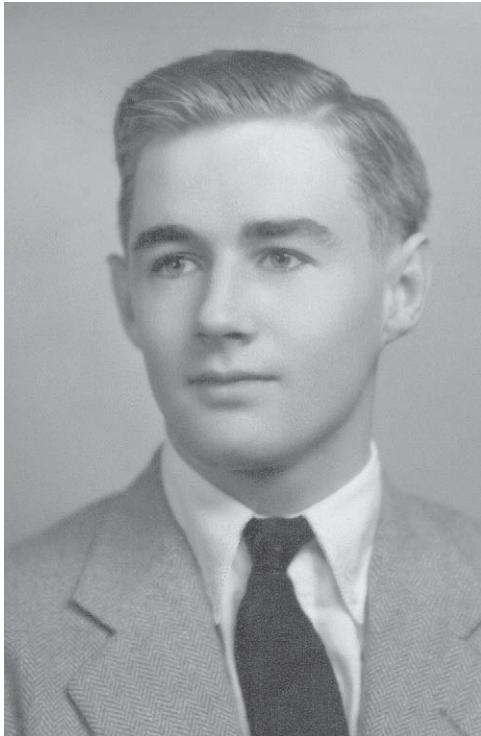


Fig. 2. Peter Waterman, a 1946 graduate of Bethlehem High School, Delmar, New York.

Waterman received his PhD degree in applied mathematics from Brown in 1958. His thesis on “Multiple scattering of waves” [W05] was supervised by Rohn Truell. This work will be discussed in Section 3.

After graduation, Waterman worked briefly for the Linde Company (1958), in Tonawanda, New York. There, he investigated the unmixing of gases in Laval nozzles, leading to two papers [W07, W08].

3. Multiple scattering

From 1959 to 1965, Waterman worked in the Research and Advanced Development Division of the Avco Corporation in Wilmington, Massachusetts. He extended his thesis work with Truell on three-dimensional acoustic scattering (governed by the Helmholtz equation, $(\nabla^2 + k^2)u = 0$, k being the wavenumber and u the acoustic field) by random arrangements of identical spherical obstacles, in the spirit of earlier work by Leslie Foldy and Melvin Lax. The main result is a formula for the effective wavenumber, K , assuming that the number of scatterers per unit volume, n , is small. Waterman and Truell [W09] obtained the formula

$$(K/k)^2 = [1 + 2\pi nk^{-2}f(0)]^2 - [2\pi nk^{-2}f(\pi)]^2, \quad (1)$$

where $f(\theta)$ is the far-field pattern for scattering of a plane wave by a single sphere. (The scattered field is asymptotically $r^{-1}e^{ikr}f(\theta)$ as $r \rightarrow \infty$, where r and θ are spherical polar coordinates.) Written out, Eq. (1) becomes

$$K^2 = k^2 + 4n\pi f(0) + n^2 \delta_2, \quad (2)$$

where

$$\delta_2 = (2\pi/k)^2 \{[f(0)]^2 - [f(\pi)]^2\}. \quad (3)$$

The term proportional to n in Eq. (2) is well known. The second-order coefficient, δ_2 , had been obtained by Urlick and Ament [1] using a very different argument: the attraction of Waterman and Truell’s derivation is that it is systematic, with testable assumptions, and that it has the potential for extension to other physical situations (such as electromagnetic scattering). In fact, it turned out later that the formula for δ_2 , Eq. (3), is wrong: for a review and references, see Ref. [2]. Nevertheless, the paper [W09] has been very influential: it has been cited about 500 times and it was reprinted in 1996 in the SPIE Milestone volume on “Selected papers on linear optical composite materials,” edited by Akhlesh Lakhtakia.

John Fikioris joined Avco in 1962. He graduated from Harvard University in 1963 with a PhD thesis on “The theory of radially stratified media,” written under Ronald W. P. King’s direction. Fikioris and Waterman [W12] improved and corrected the analysis in Ref. [W09]. They also wrote [W12, p. 1414]:

The vector extension of the present work has been performed, in application to the electromagnetic case. The procedure and results of this extension will be described in a subsequent paper.

That paper did not appear until 2013 [W42]. For a description of how this came about, see Section 7.

4. The extended boundary condition

In 1965, Waterman published a paper [W13] on electromagnetic scattering by a perfectly conducting obstacle in which he introduced the “extended boundary condition”. He was interested in developing a numerical method for three-dimensional obstacles of arbitrary shape, one that does not suffer from spurious irregular frequencies. He describes his formulation as follows:

The induced surface currents on the obstacle must produce a field that, through interference, precisely cancels the incident wave throughout the interior volume. Employing this statement as a *boundary condition* leads to an “extended” integral equation, defined to be one in which the unknown (surface current) appears in an integral over the surface, whereas the equation itself is required to hold throughout the volume bounded by that surface. ... Mathematical tractability of the extended integral equation stems from the observation that one need only make the field vanish in any *portion* of the interior volume. The field can then be shown to vanish *everywhere* in the interior [by analytic continuation].

The portion chosen is a ball, and vector spherical wavefunctions (VSWFs) are introduced leading to a linear algebraic system which is truncated and solved numerically. Results for scattering by an axisymmetric sphere-cone-sphere shape are presented and compared with experimental results obtained by N.E. Pedersen. (Waterman would work with Norman and Jeanne Pedersen for 40 years.) The extension to dielectric scatterers was presented at the 1968 URSI Symposium on electromagnetic theory in Stresa, Italy [W14].

Waterman’s 1965 paper [W13] includes some remarks on computational aspects:

One present drawback of the method is the poor numerical convergence of the truncation procedure which has been found when dealing with more elongated shapes. This is of course not surprising, since one is departing from the nearly spherical shapes most ideally suited to the vector wave functions employed.

This difficulty provoked many further investigations by Waterman and others.

5. The *T*-matrix

From 1965 to 1975, Waterman worked for the MITRE Corporation in Bedford, Massachusetts. His first journal publication from this period [W15] introduces the *transition matrix* T in the context of acoustic scattering. This infinite matrix is defined by $c=Td$, where the column vector c contains the coefficients in an expansion of the scattered field in terms of outgoing spherical wavefunctions and the column vector d contains the coefficients in an expansion of the incident field in terms of regular spherical wavefunctions. Waterman emphasizes that “what is more important” with his new approach is that it “works directly with the transition matrix, in which setting the rôle of reciprocity and energy conservation is explicitly displayed.” He relates T to the scattering matrix S by $S=I+2T$, where I is the unit matrix, and thereby infers properties of T from those of S . He shows that the T -matrix can be built by adapting his earlier use of the extended boundary condition (Section 4); the result is the formula $QT = -\text{Re}(Q)$, where the matrix Q contains the values of certain surface integrals involving spherical wavefunctions. He gives a detailed investigation of the structure of Q and shows that Q is symmetric for

ellipsoidal scatterers. Finally, he presents some numerical results. (For a detailed description of acoustic T -matrix theory, see chapter 7 of Ref. [3].)

Waterman extended his acoustic formulation [W15] to electromagnetic scattering in his famous 1971 paper [W16]. This classic paper and its predecessor [W13] have been cited more than a thousand times. Both were reprinted in the 1988 SPIE Milestone volume, “Selected papers on light scattering,” edited by Milton Kerker.

In the 1970s, Waterman developed T -matrix methods for other situations, including elastodynamic problems [W21] and grating problems (scattering by periodic surfaces) [W20]. Other groups also began using T -matrix methods. Then, in June 1979, Vijay and Vasundara Varadan hosted at The Ohio State University a major international symposium on the T -matrix method attended by 125 participants, including Peter Waterman. The proceedings [4] were published (Waterman contributed a survey article [W24]); the preface includes these words:

A major development in the last ten years has been the T -Matrix method. This method which incorporates certain elegant analytical properties has also proven to be an efficient computational technique. ... The Symposium was focused on the T -Matrix so that its full potential and/or limitations could be properly understood in the context of other well known methods ...

Several databases of peer-reviewed publications advancing, based on, or using the T -matrix method stand as a vivid testimony to the great importance of Waterman’s contribution. The first was published in 1988 and lists 151 publications on various aspects of electromagnetic, acoustic, and elastic wave scattering [5]. A more recent database initiated in 2004 [6] and supplemented by regular updates (the most recent, Ref. [7], is included in this special issue) has a much narrower scope and covers only electromagnetic scattering by finite objects. Yet it includes over 1800 references. Besides the edited collection [4], Waterman’s work on the T -matrix method and the theory of multiple scattering is central to many other books; see, for example, Ref. [3] and Refs. [8–12].

6. Later activities

From 1975 to 1990, Waterman was a consulting physicist, primarily for Digital Equipment and Panametrics, Inc. He also taught a graduate-level course on “New techniques for wave propagation problems with irregular boundaries” at Northeastern University. During the fall of 1980, he held a visiting professorship at the Institute of Theoretical Physics in Göteborg, Sweden, where he worked closely with Staffan Ström and Gerhard Kristensson (Fig. 3).

The paper [W26], published in 1981, appears to be a digression from Waterman’s main course of research. However, it turned out to be a very useful mathematical account of the standard plane-parallel problem of the scalar radiative transfer theory. As with many of Waterman’s journal publications, Ref. [W26] was rooted in the applied-physics work that he had done as a consultant.



Fig. 3. Gerhard Kristensson visiting Peter Waterman in 2008 (West Yarmouth, MA). Courtesy of G. Kristensson.



Fig. 4. Karen and Peter Waterman had a common ancestor, Edward Doty, who came to America in 1620 on the Mayflower. Doty was born in Suffolk, England in 1598 and died in Plymouth, MA in 1655. The picture was taken in October 1999 at Jonathan Waterman's wedding.

The result was an elegant formulation based on the matrix-operator technique and new analytical solutions for the reflection and transmission matrices in the limiting cases of infinitesimal or effectively infinite optical thickness of the scattering layer.

From 1990 through 2000, Waterman was employed by Pedersen Research, Inc., continuing to work in the area of electromagnetic and acoustic scattering. For example, he wrote several papers with Jeanne Pedersen on scattering by thin wires [W33, W35, W36].

In 1996, Peter married Karen Marlene Gates (Fig. 4), with whom he lived happily in West Yarmouth, Massachusetts, until his peaceful death on 3 June 2012. His last paper [W41] was submitted in July 2008 at the age of 80.

Peter Waterman was a member of Sigma Xi, Sigma Pi Sigma, Pi Mu Epsilon, the American Physical Society, and

A BEAT NOTE ISOLATOR FOR USE IN PIANO TUNING

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Summary

Standard procedure in tuning a piano involves listening for beats (periodic intensity variations) between the key being tuned and one already tuned, and adjusting string tension on the new key so as to obtain the correct beat rate. The two keys being struck generate a total of ten or more tones at different frequencies or pitches, and the presence of those tones, many of which can be at higher intensity levels than the two producing the beat, make it difficult for the ear to determine beat rate with any accuracy. The simple device described herein will eliminate this problem. It consists of a microphone, the signal from which is fed into an amplifier which filters out all frequencies except those of the two tones which together make up the beat note. The tuner can then listen to the pure beat note at a convenient volume level using headphones, while he is adjusting beat frequency. This procedure should lead to easier, more rapid, and more accurate tuning, while at the same time the tuner can employ exactly the tuning sequence he feels is best.

Fig. 5. Summary of an unpublished manuscript on piano tuning.

the Electromagnetics Academy. He was on the technical program committee for the joint URSI/IEEE International Symposium several times and chaired the sessions on numerical methods and electromagnetic theory.

While Waterman was active in his career, consulting for the US Army as a mathematical physicist into his 80s, he was also an accomplished pianist and an amateur composer. His research in piano tuning and tempering resulted in several unpublished manuscripts (Fig. 5) as well as a US patent, awarded to him in 1981 [W25]. Peter Waterman's music was recorded and later played during his memorial service on 12 June 2012.

7. Multiple scattering of waves. III

To conclude this brief account of Peter Waterman's life and accomplishments, it is appropriate to provide some background material on Ref. [W42] included in this special issue. This paper dates back to 1964. The manuscript was found in Waterman's professional archive on 1 July 2012 along with printouts of several e-mail messages exchanged by Waterman and John Fikioris in the fall of 2010. Although it remains unknown why this finalized manuscript had remained unpublished, the likely explanation is Waterman's move from Avco to the MITRE Corporation in 1965 and his new interest in the extended boundary condition method. In one of his e-mails of 2010, Fikioris suggested the idea of publishing the 1964 manuscript, perhaps with minor updates. Waterman embraced the idea and suggested developing a plan of actions within two months. Unfortunately, it appears that this was not done as no traces of subsequent correspondence between Waterman and Fikioris have been found. John Fikioris died on 14 February 2012. Peter Waterman died on 3 June 2012.

Given the absence of further information, we decided to leave the 1964 manuscript in its present form. One of us (PAM) has re-typeset the manuscript in LATEX and made a few minor edits, mostly intended to comply with the *JQSRT* format and style.

The "new" paper is the third in a three-part series. The first was published in 1961 [W09]. It has the same title as Waterman's PhD thesis of 1958 [W05]. Fikioris and Waterman published the second part in 1964 [W12]. Both papers are concerned with scalar wave problems (acoustic scattering by random configurations of many identical spheres) but Fikioris and Waterman did state that they had also made the extension to electromagnetic scattering

problems: the paper describing that extension remained unpublished until now.

At the heart of the analysis in the third part [W42] is an exact treatment of electromagnetic scattering by many spheres, combining separated Lorenz–Mie solutions in terms of VSWFs with appropriate addition theorems. For VSWFs, such addition theorems were not available until Cruzan's paper from 1962 [13]. Interestingly, Fikioris and Waterman appear to be the first to find the notorious minus sign error in Ref. [13]. The 1964 manuscript has a handwritten note by Fikioris indicating that Cruzan had acknowledged this error in a letter sent to Fikioris and Waterman.

Ref. [W42] essentially describes what is now known as the superposition T -matrix method [8,9] or the generalized Mie theory [14]. It thus anticipates Refs. [15,16] and all numerous reinventions and modifications of this technique which is now used routinely to solve the problem of electromagnetic scattering by many spheres.

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