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ENERGY EQUATION APPROXIMATIONS IN FLUID MECHANICS

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There is some confusion in the literature of fluid mechanics[1,2] in regard to the correct form of the energy equation for the study of the flow of nearly incompressible fluids. Several forms of the energy equation and their use are therefore discussed in this note.

The general energy equation for a fluid, when the chemical composition is fixed, may be expressed in terms of the entropy S, the enthalpy H, or the internal energy U as

$$\rho T\dot{S} = \rho\dot{H} - \dot{p} = \rho\dot{U} + p\rho\dot{v} = J \qquad (1)$$

where

$\rho = 1/v$, density

T = temperature

p = pressure

J = net rate of heat addition per unit volume, including conduction $(-\bar{\nabla}\cdot k\bar{\nabla}T)$, dissipation (Φ), heat sources (ρQ)

The dot indicates the substantial time derivative, e.g.,

$$\dot{U} = dU/dt = \partial U/\partial t + \bar{V}\cdot\bar{\nabla}U$$

When the energy equation is used in this form, no question arises, but when $\dot{H}$ or $\dot{U}$ is expressed in terms of $\dot{T}$, an error can appear by neglecting the terms $(\partial H/\partial p)_T$ or $(\partial U/\partial v)_T$.

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The commonly used coefficients needed for the discussion are:

$C_p \equiv (\partial H/\partial T)_p = T(\partial S/\partial T)_p$, specific heat, constant pressure

$C_v \equiv (\partial U/\partial T)_v = T(\partial S/\partial T)_v$, specific heat, constant volume

$\alpha \equiv (\partial v/\partial T)_p/v$, coefficient of thermal expansion

$\beta_1 \equiv -(\partial v/\partial p)_T/v$, coefficient of compressibility

$\beta = (\partial p/\partial T)_v/p$, coefficient of tension

The existence of an equation of state between p, v, T implies

$$\alpha = \beta\beta_1 p \qquad (\text{Eq. } (1.05))^3 \tag{2}$$

The reciprocity relation is

$$\left(\frac{\partial S}{\partial v}\right)_T = \left(\frac{\partial p}{\partial T}\right)_v = \beta p \qquad (\text{Eq. } (4.23))^3 \tag{3}$$

or

$$\left(\frac{\partial S}{\partial p}\right)_T = -\left(\frac{\partial v}{\partial T}\right)_p = -\alpha v \qquad (\text{Eq. } (4.46))^3 \tag{4}$$

Also

$$C_p - C_v = \alpha\beta p v T \qquad (\text{Eq. } (4.27))^3 \tag{5}$$

If $H = H(T,p)$ and $U = U(T,v)$, then

$$\dot{H} = C_p\dot{T} + (1 - \alpha T)v\dot{p} \tag{6}$$

$$\dot{U} = C_v\dot{T} - (1 - \beta T)p\dot{v} \tag{7}$$

and with the above definitions, the energy equation may be stated in the two forms

$$J = \rho C_v\dot{T} + \beta T p\rho\dot{v} \tag{8a}$$

$$J = \rho C_p\dot{T} - \alpha T\dot{p} \tag{8b}$$

For a perfect gas, $\alpha T = \beta T = 1$, yielding the familiar forms,

$$\dot{H} = C_p\dot{T},\ \dot{U} = C_v\dot{T},\ J = \rho C_p\dot{T} - \dot{p} = \rho C_v\dot{T} + p\rho\dot{v} \tag{9}$$

With liquids it would seem reasonable to neglect the terms involving $\dot{v}$ in Eqs. (7) and (8a). This, however, is not generally permissible because β can become very large. For liquids of very small compressibility (β_1 small), the value of $\beta(= \alpha/\beta_1 p)$ may be large with a consequence that the term $\beta T p \rho \dot{v}$ in (8a) and $\beta T p \dot{v}$ in (7) might not be negligibly small. The kind of magnitudes involved may be judged from the data of the table:

Liquid	T, ^oC	p, atm	α, $\frac{1}{^oC}$	β_1, $\frac{1}{atm}$	v, $\frac{cm^3}{gm}$	C_p, $\frac{Joule}{gm\text{-}^oC}$	β, $\frac{1}{^oC}$	$\frac{C_p - C_v}{C_p}$
H_2O	220	50	0.001546	102.8×10^{-6}	1.187	4.26	0.301	0.324
Hg	100	1.0	0.000180	4.26×10^{-6}	0.075	0.137	42.2	0.157

The table would suggest that for many liquids $\beta T >> 1$. Eq. (7) may then be expressed as $\dot{U} \approx C_v \dot{T} + \beta T p \dot{v} = \rho T \dot{S} = J$ to the effect that all heat addition increases internal energy and none is available for external work. The energy equations(8a,b) do not permit such general simplification independently of the particular problem. For example, in a problem where $\dot{p}/p$ is of the same order as $\dot{T}/T$, the ratio of the term $\alpha T \dot{p}$ to $\rho C_p \dot{T}$ in Eq. (8b) is of the order of $\alpha p/\rho C_p$, which for water and mercury at conditions cited in the table has values of 2×10^{-3} and 10^{-5}, respectively. For these fluids under such circumstances, Eq (8b) becomes

$$J \approx \rho C_p \dot{T} \qquad (10)$$

On the other hand, for nearly isentropic flows (such as an accoustic wave), J is small relative to both $\rho C_p \dot{T}$ and $\alpha T \dot{p}$, and both these terms must be retained.

In many flow problems, the use of Eq. (8b) is preferable to (8a) because the ranges of p and T are indicated by the boundary conditions, making it possible to decide in advance when either term may be omitted.

It is often assumed for liquids that $C_p \approx C_v$ (e.g., Ref. 3, p. 41) because of the factor αv in Eq. (5). The effect of the additional factor $\beta p T$ may render this assumption inadequate in many cases as shown in the table. In cases where (8b) may be simplified to (10), the corresponding simplification of (8a), namely $J = C_v \rho \dot{T}$, is not admissible unless the liquid satisfies the additional condition $C_p \approx C_v$, which cannot be assumed for all condensed fluids. The particular fluid and problem together will determine whether (8a) or (8b) yields the simplest equation.

The approximation used by Millsaps and Pohlhausen[1] is $J = \rho C_v \dot{T}$ for water at 30° C and unstated pressure, which will be assumed to be 1 atm. For these conditions, $\alpha = 0.0003/^\circ C$, $\beta_1 = 48 \times 10^{-6}/\text{atm}$, from which $(C_p - C_v)/C_p = 0.014$. Since the effect of $\dot{p}$ is small, the correct approximation is $J = \rho C_p \dot{T}$. The discrepancy in the energy equation is minor because of the small value of $(C_p - C_v)/C_p$. The same equation is also used for argon where the error is not small, since $(C_p - C_v)/C_p = 0.4$. The proper approximation with small $\dot{p}$ is $J = \rho C_p \dot{T}$ as was noted by Ostrach and Thornton.[5]

Hidalgo[2] uses the approximation $J = \rho C_p \dot{T} - \dot{p}$ for liquid glass. In principle, the equation is not correct, since, if the $\dot{p}$ term is to be retained, it should be retained as $\alpha T \dot{p}$ (eq. (8b)). In the problem investigated, $\dot{p}$ turns out to be negligibly small, so that a physically significant error would develop only if $\alpha T \gg 1$. For glasses in the temperature range of the problem, αT is not large, so that no substantial numerical errors result from this cause.

REFERENCES

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