Stochastic neural network dynamics: synchronisation and control

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Stochastic Neural Network
Dynamics: Synchronisation and
Control

Scott Michael Dickson

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Abstract

Biological brains exhibit many interesting and complex behaviours. Understanding of the mechanisms behind brain behaviours is critical for continuing advancement in fields of research such as artificial intelligence and medicine. In particular, synchronisation of neuronal firing is associated with both improvements to and degeneration of the brain’s performance; increased synchronisation can lead to enhanced information-processing or neurological disorders such as epilepsy and Parkinson’s disease. As a result, it is desirable to research under which conditions synchronisation arises in neural networks and the possibility of controlling its prevalence.

Stochastic ensembles of FitzHugh-Nagumo elements are used to model neural networks for numerical simulations and bifurcation analysis. The FitzHugh-Nagumo model is employed because of its realistic representation of the flow of sodium and potassium ions in addition to its advantageous property of allowing phase plane dynamics to be observed. Network characteristics such as connectivity, configuration and size are explored to determine their influences on global synchronisation generation in their respective systems. Oscillations in the mean-field are used to detect the presence of synchronisation over a range of coupling strength values. To ensure simulation efficiency, coupling strengths between neurons that are identical and fixed with time are investigated initially. Such networks where the interaction strengths are fixed are referred to as homogeneously coupled. The capacity of controlling and altering behaviours produced by homogeneously coupled networks is assessed through the application of weak and strong delayed feedback independently with various time delays. To imitate learning, the coupling strengths later deviate from one another and evolve with time in networks that are referred to as heterogeneously coupled. The intensity of coupling strength fluctuations and the rate at which coupling strengths converge to a desired mean value are studied to determine their
impact upon synchronisation performance.

The stochastic delay differential equations governing the numerically simulated networks are then converted into a finite set of deterministic cumulant equations by virtue of the Gaussian approximation method. Cumulant equations for maximal and sub-maximal connectivity are used to generate two-parameter bifurcation diagrams on the noise intensity and coupling strength plane, which provides qualitative agreement with numerical simulations. Analysis of artificial brain networks, in respect to biological brain networks, are discussed in light of recent research in sleep theory.
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My family have always encouraged me and guided me along the right path. I cannot thank them enough for their continued encouragement and am forever proud to know that they are in my life; I hope to be able to reciprocate the support that they have provided.

Finally, it is to my beloved Tanya that I dedicate this work. Meeting her has completely changed my life and has made me happier than I ever imagined to be possible. Tanya inspires me to make myself better in everything I attempt; I hope to be able to provide her with a fulfilling life with all of the rewards that she deserves.
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List of Abbreviations

Adenosine Triphosphate (ATP)
Adenosine Monophosphate (AMP)
α-Amino-3-Hydroxy-5-Methyl-4-Isoxazolepropionic Acid (AMPA)
Autocorrelation Function (ACF)
Autonomic Nervous System (ANS)
Average Path Length (APL)
Central Nervous System (CNS)
Cyclic Alternating Pattern (CAP)
Cyclic AMP Response-Element Binding Protein (CREB)
Cytoplasmic Polyadenylation Element-Binding Protein (CPEB)
Electroencephalogram (EEG)
Electromyographic (EMG)
Excitatory Postsynaptic Potential (EPSP)
Gamma-Aminobutyric Acid (GABA)
Inhibitory Postsynaptic Potential (IPSP)
Long-Term Potentiation (LTP)
Longest Established Path (LEP)
Mitogen-Activated Protein (MAP)
N-Methyl-D-aspartic Acid (NMDA)
Non-Rapid Eye Movement (NREM)
Number of Disconnected Pairs (NDP)
Ordinary Differential Equations (ODE’s)
Parasympathetic Nervous System (PSNS)
Percentage of Connectivity (PC)
Peripheral Nervous System (PNS)
Probability Density Distribution (PDD)
Rapid Eye Movement (REM)
Ribonucleic Acid (RNA)
Signal-to-Noise Ratio (SNR)
Slow Wave Activity (SWA)
Slow Wave Sleep (SWS)
Somatic Nervous System (SoNS)
Sudden Unexpected Death in Epilepsy (SUDEP)
Sympathetic Nervous System (SNS)
Synaptic Homeostasis Hypothesis (SHH)
Chapter 1

Introduction

1.1 Motivation

There has been a long history of mathematical application into the field of biology. With an increase in availability and power of computational tools, advances in neuroscience have been made particularly prominent. The brain has long been considered a sophisticated organic computing machine; computational neuroscience dates back to 1907, where the integrate and fire model of a neuron was first introduced [1]. Since then, neuronal models of varying complexity have been proposed to characterise a number of different behaviours with varying degrees of accuracy. Currently, it is not readily possible to simulate the actual size and complexity of the system; a simplified model of the brain is often constructed in order to concentrate on a specific element of behaviour, such as synchronisation.

Synchronisation in neural networks is associated with improved brain processing capabilities [2, 3, 4, 5] in addition to neurological disorders, such as epilepsy and Parkinson’s disease [6, 7]. Efforts have been made to understand under which conditions synchronisation arises in large networks. Studies using stochastic FitzHugh-Nagumo models have largely concentrated on systems coupled through the mean-field [8, 9, 10]; however, the assumption of coupling through the mean-field is unrealistic in its representation of the brain, where there is relatively sparse connectivity [11].

As such, this research has placed emphasis upon analysing systems where connectivity is sub-maximal by using numerical simulations to identify the impact the following network parameters impart on synchronisation: network connectivity, configuration, coupling strength and size. Delayed feedback mechanisms are also applied to investigate the
possibilities of altering synchronisation in light of improving artificial brain processing capabilities and developing treatments for pathological abnormalities.

1.2 Thesis Outline

An overview of existing literature, methods and results that introduces the field of stochastic neuron-like network modelling is given in Chapters 2, 3 and 4. The impact of establishing a network by connecting individual units and the origins of “neuronal noise” are discussed in Chapter 3; the configuration of biological brains and its relation to neural functioning and behaviour is also considered. A selection of historical neuronal models are outlined in Chapter 4; dynamics of the single unit FitzHugh-Nagumo model are described and its application is justified. Chapter 5 provides an introduction to the concept of synchronisation and outlines some of the research performed previously that is relevant to artificial neural networks of the kind studied here. An ensemble of stochastic FitzHugh-Nagumo elements is coupled through their local mean-field to generate a network consisting of \( N \) identical units in Chapter 6; the extent of influence of network connectivity, configuration and size upon synchronisation is determined when all units in the network have equal interaction strengths (homogeneous coupling); attention is directed towards the degree of synchronisation achieved. Subsequently, attempts are made to control synchronisation using global and neuron-specific delayed feedback mechanisms in Chapter 7. Heterogeneous coupling strengths that evolve with time, according to the Ornstein-Uhlenbeck Process, are investigated in Chapter 8; attention is directed to the influence of coupling strength noise intensity and convergence rate parameters. In Chapter 9, cumulant equations are derived from the set of stochastic delay differential equations used for numerical simulations by following the Gaussian approximation method; the resultant deterministic equations give approximations to the behaviour of the simulated networks, allowing for bifurcation analysis to be conducted on the coupling strength and noise intensity plane. Cases where delayed feedback is present and absent are also dis-
cussed. Chapter 10 provides a brief summary of the previous topics and conclusions; future recommendations are given to encourage further work in the research field. Supplementary material and details of extensive calculations are provided in Chapter 11; background information is also provided in relation to neuronal patterns that are observed during neural processing, sleep and in neurological disorders such as epilepsy and Parkinson’s disease.
Chapter 2

The Biological Structure of Individual Neurons

Before one can design a mathematical model aimed at simulating aspects of neural network behaviour, it is essential to understand the foundation of biological principles and the functions underlying their structure and behaviour. A pioneer in neurobiology, Santiago Ramón y Cajal initially concluded neurons (nerve cells) are individual units that interact with one another to form a network, during the late 19th century [12, 13]. Neurons establish the major pathways of communication, creating a network capable of processing and integrating electrical and chemical information. The brain consists of approximately $10^{11}$ (100 billion) neurons, which are interconnected to a certain degree [14, 15]; there are approximately $10^5$ neurons in 1mm$^3$ of cortical tissue [16]; the average adult human brain is approximately 1350cm$^3$ in volume. An estimated 5% of cells in the brain are neurons and an estimated 90% are glial cells (Appendix 11.1) [17].

Figure 2.1: A canonical model of two connected nerve cells (neurons) and their typical features. Reprinted from http://www.docstoc.com/docs/80653935/Dendrites-Cell-body-Nucleus–Axon-hillockAxon-Signal-direction.
One common feature of all cells is the surrounding surface membrane that is differentially permeable; these membranes selectively exchange specific nutrients and gases between the cell’s interior and its surrounding fluid. Membranes encompass a nucleus within an intracellular fluid called the cytoplasm; the nucleus stores genetic material, such as information that controls protein synthesis within the cell [18]. As neurons display vast heterogeneity, many different types exist with variations in anatomical structure and/or electrical properties. Despite the diversity of neurons, a canonical neuron based on shared features can be used to understand fundamental elements underlying all neurons (Fig. 2.1): dendrite, soma, and axon respectively relate to input, processing and transmission.

2.1 Dendritic Branches

The idea of neurons receiving information in the dendrites, and it flowing through the soma (main body) (Section 2.2) and axon (Section 2.3), was proposed by Ramón y Cajal who called it “the rule of dynamic polarisation” [12, 13]. Dendrites are branches that usually extend from one extremity of the soma and are primarily devoted to receiving electrical signals from other neurons and transporting them to the soma. Dendritic trees show extreme diversity in their shape and can be characterised by their order, degree and asymmetry index (Appendix 11.2) [19].

The dendritic branches grow from the soma of a neuron during early brain development; genetic factors and activity levels affect their augmentation and expansion. However, a fully matured dendritic tree can constitute up to 90% of the neuron’s surface area [20]. Despite the large proportion of space occupied, dendritic trees are very compact in order to maintain short wiring lengths. Such a feature is critical for energy efficiency since electrical signals diffuse through the dendrites in a passive and decremental manner with distance. Even with their compact structure, the amplitude of dendrite signals still decreases by approximately 80% when diffusing towards the soma [21]. However,
the greatest contributions to the dendritic surface area occur as a result of extensive branching; branching maximises the number of dendritic tips and spines (Appendix 11.3) available for the reception of synaptic input 2.4. The configuration of dendritic trees is very intricate to avoid formation of closed loops within the global structure.

2.2 Somatic Integration

The soma is the neuron’s cell body containing the nucleus, many organelles and most of the protein synthesising material of the neuron. The soma predominantly processes and integrates synaptic inputs (Section 2.4), determining whether the neuron becomes active and transmits electrical signals to other neurons. Inputs can be excitatory, promoting active responses in subsequent neurons, or inhibitory, encouraging inactive responses [22]; approximately 80% of neurons are excitatory in contrast to 20% of inhibitory neurons [23].

The large number of synaptic inputs per neuron gives rise to temporal and spatial summation. Temporal summation occurs when two incoming pulses from a single dendritic branch arrive at the soma in quick succession [24, 25]. Given that the first pulse has not completely faded, the second pulse will be accumulated to the remaining signal of the first pulse. Spatial summation is characterised by accrued incoming pulses arriving from different dendritic branches almost simultaneously. Consequently, inputs must arrive within a short time period to significantly raise the electrical potential at the soma; the timing within this interval affects the magnitude of contribution from each input.

When the overall electrical input falls below a designated threshold, the membrane voltage of the neuron will elicit small, input-graded oscillations around its stable (resting) state; when the designated threshold is exceeded, a high spike of electrical current known as an action potential will occur in a nonlinear fashion [18, 24, 25]. Action potentials are initiated at the axon hillock, where the axon emerges from the soma. The shape and duration (approximately 1ms) of the high spike is invariable when input values supersede
the threshold; when this occurs, the neuron is said to be spiking or firing (Fig. 2.2). Following an action potential, the membrane voltage returns to the resting state. The summation of inputs at the soma therefore results in an output following an all or nothing principle [26].

According to the membrane hypothesis [27], the inside of the neuron is approximately 70mV more negative than the outside in its resting state. The ionic imbalance is caused by an uneven distribution of positive sodium, $Na^+$, and potassium, $K^+$, ions on either side of the membrane and a large number of negatively charged protein anions inside the cell (Fig. 2.3). Uneven allocation of ions results from the existence of some non-gated potassium channels. Sodium ions only diffuse into the neuron when its voltage-gated channels are open; due to this restriction, it is not possible to neutralise the voltage difference across the membrane when resting. Although potassium ions follow a high to low concentration gradient, the difference in charge applies a force in the opposing direction. These forces are equal when the membrane potential is -70mV; resultanty, no net movement occurs.

The ionic hypothesis [28] explains that the suprathreshold summation of excitatory and inhibitory inputs reduces the cellular resting membrane potential (to approximately -55mV), triggering the voltage-gated sodium channels to open at the axon hillock. The axon hillock has a high density of sodium channels allowing sodium ions to diffuse into the cell rapidly down both concentration and electrochemical gradients; this generates an
action potential causing depolarisation with an approximate magnitude of 100 - 110mV. Thus, the membrane’s interior becomes positive in contrast to its exterior. A sufficient voltage difference causes sodium channels to close or (deactivate) and voltage-gated potassium channels to open (activate). Rearrangement of the gate allows potassium ions to flow out of the cell, hyperpolarising the neuron beyond its original voltage. Protein pumps in the membrane return potassium and sodium ions to their original positions, by active transport, to re-establish the -70mV membrane difference; this process requires energy in the form of adenosine triphosphate (ATP). A stimulus is incapable of eliciting another action potential for a period of 1ms, due to the inactive state of sodium ion channels; this is known as the absolute refractory period. Subsequently, a relative refractory period occurs when spikes are initiated if an increased threshold is reached [29].

Although a single action potential displays the same characteristics for any given suprathreshold stimulus, the neuron is able to distinguish certain features, such as intensity, duration and type of stimulus. The intensity of the stimulus is calculated using the frequency of action potential generation; stimulus duration can be derived from the period of time over which action potentials are elicited; the pathway of transmission chosen can distinguish the type of stimulus. A back-propagating spike may be sent from the axosomatic region to the dendrites through passive decremental diffusion, alerting the dendrites to the activity of the neuron [30, 31]. Feedback from the soma to the dendrites
through back-propagation is supported by the existence of voltage-dependent channels in the dendritic tree [32, 33]. The notion of Hebbian learning (Section 4.1) is illustrated through back-propagation: information concerning which signals initiate action potentials and which synapses (Section 2.4) should be strengthened is sent to the dendrites.

## 2.3 Axonal Propagation

An axon is an extension that is typically located on the opposing side of the soma to the dendrites. Axon length can substantially vary in size, from several micrometres to beyond a metre; the diameter of an axon ranges from 1µm to 1mm. As mentioned in the Neuron Doctrine [12, 13], axons only carry electrical signals away from the soma through dynamic polarisation whereby adjacent regions of the axon become excited as the signal travels. The initiation of an action potential at the axon hillock spreads a wave of depolarisation along the axon’s length as the electrical signal is conducted; the electrical signal propagates along the axon, travelling at velocities of up to 10m/s without decreasing in strength. Propagation is an active process, accumulating excessive metabolic demands; it allows axons to reach greater lengths than dendrites, but causes longer axons to experience increased delays in signal exchange as signals are transmitted at finite speeds. To lessen this metabolic strain, many axons display the intermittent presence of myelin sheaths (Fig. 2.1) to allow greater signal transmission speeds by the process of saltatory conduction [18, 24, 25]. A myelin sheath enables faster transmission because the signal jumps between the gaps in the sheath at positions containing high densities of sodium channels called nodes of Ranvier. Each axon incrementally branches, developing many axon terminals, which transfer the signal to the dendrites of many neurons.

## 2.4 Synapses

The term “synapse” was coined by the neurophysiologist Sir Charles Sherrington [22] in 1897; it refers to a structure that allows electrical signals to be transferred between
neurons. There are many types of synapse, such as axoaxonic, axosomatic, somato-axonic, somatodendritic, dendroaxonic, and dendrodendritic. The majority of synapses are axodendritic and are considered to be unidirectional chemical synapses (Fig. 2.4); they are situated between a transmitting neuron and a receiving dendrite. The axonal signal (action potential) is converted from an electrical to a chemical form at the presynaptic terminal and reverted back to an electrical signal at the postsynaptic terminal.

When an action potential arrives at the presynaptic terminal of an axodendritic synapse, also known as a bulb or bouton, it triggers the opening of voltage-gated calcium, $Ca^{2+}$, channels where calcium ions enter [34]. The influx of calcium causes membranous sacks (approximately 30 - 40nm in diameter), vesicles, to fuse with the presynaptic membrane; vesicles secrete neurotransmitter chemicals into a synaptic cleft, a 20 - 40nm gap found between the axon terminal of the presynaptic neuron and the terminal tip of the postsynaptic neuron’s dendrite, through a process known as exocytosis [31, 35]. Neurotransmitter molecules diffuse across this gap taking approximately 10µs, binding to specific receptors on the postsynaptic membrane; at this point, the signal reverts back from a chemical state to an electrical form. Larger synaptic clefts cause increased delays in the signal transmission process between neurons.

Different transmitter-gated ion channels will open depending on the type of neurotransmitter attached to the receptor; the main excitatory neurotransmitter in the brain
Figure 2.5: (a) Schematic representation of an electrical synapse. (b) Schematic representation of channels located at gap junctions that allow electrical transmission between neurons. Reprinted from http://www.mun.ca/biology/desmid/brian/BIOL2060/BIOL2060-13/13_15.jpg.

is glutamate. Glutamate opens sodium channels, developing an excitatory postsynaptic potential (EPSP); this causes individual membrane potentials to be altered by 0.06 - 2mV with an average of 0.27mV \[29\]. Collectively, EPSP’s cause a change in membrane potential from -70mV to 0mV. Unlike action potentials, synaptic potentials elicit corresponding responses to input values. It is possible that the dendrites act in a non-linear fashion, able to perform logical operations upon inputs (e.g. AND, NOT, XOR, AND-NOT etc) when wired properly, thus providing a rich repertoire of local operations \[30, 31, 36, 37\].

Gamma-aminobutyric acid (GABA) is the most prominent inhibitory transmitter; the binding of GABA to the postsynaptic membrane receptors elicits an inhibitory postsynaptic potential (IPSP), further hyperpolarising the membrane to -75mV. Many inhibitory synapses are proximal to the soma, effectively suppressing input from more distal excitatory synapses; synapses strategically block specific dendritic tree regions, leaving other areas unaffected \[38, 39\].

Dale’s Principle \[40\] is useful for simplifying mathematical models simulating synaptic transmission; the most widely accepted interpretation states that a neuron releases the same set of neurotransmitter substances at all of its presynaptic terminals \[41\]. For instance, if only excitatory transmitter substances are found at an axon terminal of a neuron, all other axon terminals of this neuron are assumed to contain only excitatory neurotransmitters.
In addition to chemical synapses, there are electrical synapses, which allow electrical signals to simply jump across a small gap between the presynaptic and postsynaptic membranes (Fig. 2.5); these gap junctions are approximately 3nm in length. By comparison to the synaptic cleft, which has an approximate 2ms delay of synaptic transmission, the smaller electrical synapse has a reduced delay of around 0.02ms [18, 25]. However, electrical synapses suffer from decrement of the signal strength with distance, whereas, chemical synapses fully restore the action potential at the postsynaptic dendrite providing that the receiving neuron is sufficiently stimulated. Connexin proteins that form the electrical synapse channel allow ions to flow directly between neuronal cytoplasms; these channels have a diameter of approximately 1 - 2nm, allowing all major ions bidirectional movement with minimal resistance and high conductance in either direction. Ions that enter the postsynaptic cell cause depolarisation; if the threshold value is exceeded, the voltage-gated ion channels open provoking an action potential.

Genetic and developmental processes dictate the neuronal connections; the strength of these connections is regulated by experience. The cellular mechanisms of learning and memory result from the collective interaction of neurons, as opposed to unique properties of a singular unit [18]. Over time, the human body activates dormant synapses and synthesises new axon terminals to accommodate short-term and long-term memories.

Potentiation is the development of short-term memories, brought about by increases in strength of nerve impulses along recently used pathways. The strengths of synaptic connections are altered by modulatory interneurons, which form synapses with presynaptic bulbs; upon a single instance of stimulation, serotonin is released into and diffuses across the synaptic cleft. As a result, serotonin is bound to the metatropic receptors on the mediatory presynaptic neuron (Fig. 2.6), leading to the production of cyclic adenosine monophosphate (AMP). This chemical is responsible for activating protein kinase A, an enzyme that encourages neurotransmitter release and closes the non-gated potassium channels; resultantly, there is slower action potential generation at the presynaptic membrane, allowing more time for calcium influx and neurotransmitter secretion [18].
In contrast to short-term memories, where potentiation causes functional changes in synaptic efficacy, long-term memories are formed through structural changes, such as the development of new axon terminals and synapses, as a result of long-term potentiation (LTP). LTP is a lasting enhancement of signal transmission between two neurons that results from repeated electrical stimulation; it is widely considered to play a critical role in synaptic consolidation [42]. An influx of calcium ions through N-Methyl-D-aspartic acid (NMDA) receptors induces LTP; blocking these receptors prevents LTP from spreading in the hippocampus [18, 43]. LTP is believed to consist of two stages [44]; early stage LTP does not synthesise protein, creating a vulnerability to interference; late stage LTP triggers protein synthesis, altering the structure of synapses and dendrites.

Late stage LTP begins approximately 4 – 5 hours following the onset of early stage LTP. Repeated firing activity of a neuron releases repeated pulses of serotonin (Fig. 2.6), resulting in higher concentrations of cyclic AMP; this process initiates the return of pro-
tein kinase A from the axon to the cell’s nucleus [18]. Within the nucleus, protein kinase A activates cyclic AMP response-element binding protein-1 (CREB)-1 and recruits mitogen-activated protein (MAP) kinase, inactivating CREB-2. Subsequently, CREB-1 binds to a promoter gene, enabling messenger ribonucleic acid (RNA) code to synthesise the appropriate proteins; the RNA molecules are transported to all axonal terminals established by the neuron. Only synapses that initiated the messenger RNA are able to activate the molecules from a dormant state; activation is induced by cytoplasmic polyadenylation element-binding protein (CPEB), which is converted to its dominant form by repeated serotonin pulses. In its dominant form, CPEB self-perpetuates and converts recessive forms into dominant forms; CPEB is only able to activate messenger RNA in this condition, initialising protein synthesis and synaptic terminal growth [18].

2.5 Types of Neuron

It is believed that over a hundred different types of neuron exist, which significantly vary in structure, function and/or size; many inconsistencies can be found between neurons within the same class, such as the number of dendrite branches contained within a particular nerve cell. The easiest way to differentiate between neurons is through their polarity, which falls under three categories:

- Unipolar,
- Bipolar,
- Multipolar.

A unipolar neuron has a single process extending from the cell body; both an axon and a dendritic branch may emerge from a single protrusion. This class often contains primary sensory neurons (Fig. 2.7). Bipolar neurons contain two extensions from the soma, which are usually found at opposing ends; examples include the sensory neurons responsible for converting external stimuli from the environment into electrical nerve impulses. Examples
include retinal neurons which respond to visual stimuli and olfactory neurons responding to auditory stimuli. Bipolar neurons may mature into unipolar neurons and are often called pseudo-unipolar neurons. Multipolar neurons contain many dendritic branches and a single axon; they are the most abundant class in the brain, accounting for the majority of motor neurons and interneurons. Motor neurons carry electrical signals from the spinal chord to the muscles, bringing about bodily movement; interneurons provide connective links to other neurons.

There are only two subcategories of multipolar neuron; these subcategories, Golgi type I and Golgi type II, are named after their discoverer [45]. The former contains long axon processes and the latter possess only short axons or none at all. There are many neurons that fall under these two subcategories; a few examples will be discussed to highlight the range of discrepancies found between classes.
Pyramidal cells, named accordingly due to their triangular-shaped soma, are Golgi type I excitable neurons, containing spined dendrites to increase their receptive surface area, especially at distant regions to the cell body [12]. These cells are the most common type of neuron in the brain and contain three subtypes, each eliciting distinctive responses to stimulation:

- Adapting regular spiking,
- Non-adapting regular spiking,
- Intrinsically bursting.

Adapting regular spiking neurons fire individual action potentials with spike frequencies that are adapted by a resultant hyperpolarising effect [46]. Non-adapting regular spiking neuron types respond to stimulation by depicting a train of action potentials without hyperpolarisation. Intrinsically bursting cells fire between 2 – 5 action potentials in quick succession; one subtype of pyramidal cell, the Betz cell, is the largest in the central nervous system (Section 3.1) with a diameter of up to 100µm [47].

Stellate cells can be excitatory or inhibitory; they have several dendrites protruding from their soma and are considered to have a regular firing pattern [48]. These are often involved in the excitation of pyramidal cells and are of Golgi type II structure.

Adversely, the following neurons are believed to inhibit pyramidal neurons upon stimulation:

- Double-Bouquet cells,
- Neurogliaform cells,
- Martinnotti cells.

Double-bouquet cells are inhibitory in their actions and have vertical axonal projections [49]. Neurogliaform cells are late spiking in nature [50], displaying small dendritic and large axonal branching. Martinnotti cells have dense axonal and sparse dendritic branching [51].
Basket cells are inhibitory interneurons and are of Golgi type II structure; they are characterised by dense branching of their axon around the soma of a target cell and their fast spiking firing patterns [23].

Purkinje cells are Golgi type I neurons and have dendrites that are studded with approximately 100,000 spines (10 per µm), which contain actin filaments [30, 31]. Located in the cerebellum of the brain, these neurons receive inhibitory input from basket and stellate cells. These neurons may elicit simple spiking behaviour, firing at frequencies between 17Hz and 150Hz spontaneously or upon stimulation; they also evoke complex spiking patterns between 1-3Hz, whereby initial large amplitude spikes are followed by a high frequency burst of smaller amplitude action potentials [38].

Renshaw cells are inhibitory interneurons that are used as a negative feedback mechanism; they receive electrical input from motor neurons, which are responsible for innervating extrafusal muscle fibres (alpha neurons), leading to skeletal muscle contraction. Upon stimulation, these cells send inhibitory signals back to the initial alpha neuron or to an alternative alpha neuron in proximity; this reduces the likelihood of the receiving alpha neuron firing [52].

Granule cells constitute almost half of the neurons within the central nervous system (Section 3.1) and many send impulses to Purkinje dendrites [53]. Being of Golgi type II classification, granule cells have an extremely small diameter of approximately 10µm.
Chapter 3

The Brain Network and Nervous System

Having introduced the behaviours of individual neurons in Chapter 2, the network patterns that are generated by their interactions will now be considered. The complexity of the neural system induces a diverse range of behaviours. For invertebrates, behaviour diversity arises from single neurons, each possessing numerous extensions and a large cell body that is disconnected from the main stream of information [25]; the intricate structure of these individual units remove the necessity of a complex network. In contrast, vertebrates rely on an abundance of neurons, each relatively simplistic in structure, to increase system complexity as a collective [25].

3.1 Brain Regions

The brain and the spinal cord form the central nervous system (CNS); the nerves outside of the CNS are parts of the peripheral nervous system (PNS). The CNS is responsible for integrating all information received from the body and coordinating the appropriate responses; the PNS, consisting of the somatic nervous system (SoNS) and the autonomic nervous system (ANS), supplies information to the CNS. The SoNS is responsible for conscious movements involving skeletal muscle contractions and the ANS controls involuntary functions that may be regulated consciously to some degree, such as breathing and heart rate [24]. The medulla oblongata (Fig. 3.1 (a)), which directs the ANS, consists of two subsystems that largely coordinate opposing physiological responses: the sympathetic ner-
vous system (SNS) and parasympathetic nervous system (PSNS). These subsystems act independently, maintaining homeostatic levels of critical functions, such as blood pressure and heart rate [24].

The brain can be divided into the following three categories: forebrain, midbrain, and hindbrain. The forebrain has the largest area, containing the cerebrum (cortex), thalamus and hypothalamus. The cerebrum is extensively folded to increase surface area, effectively and efficiently supporting a large number of neurons [54]; responsible for voluntary bodily actions, the cerebrum is divided into two interconnected hemispheres consisting of four lobes (Fig. 3.1 (b)); frontal, parietal, occipital, and temporal.

The frontal lobe dictates conscious thought, decision making, movement, problem solving, planning, and emotions. Broca’s area [55], usually located amid the left hemisphere within the frontal lobe, is associated with speech and language production. Broca’s area was the first brain region to be linked with a specific purpose; in 1861, Paul Broca discovered that lesions in this location caused speech impediment. In certain cases, damage is overcome by the natural transfer of relevant processes to the equivalent region in the alternate hemisphere [56]. Alongside other association areas within the brain, the association cortex digests information received from various sensory receptors, forming relations with knowledge from previous experiences to devise an appropriate response; this cortex in the frontal lobe uniquely organises actions and thoughts. Nerve impulses are transmitted from any association area to the motor cortex, which is also situated in the frontal lobe, where responses are initiated and executed. The motor cortex spans both hemispheres and each hemisphere governs movement of the opposite side of the body [57]; the superior part controls the body’s lower limbs and the inferior section commands the upper body parts. Single neurons in the motor cortex are capable of influencing the force of output generated by many muscles [58].

The parietal lobe is essential to integrating sensory information and important to the awareness of spatial orientation, especially during movement. If the association cortex in this region is damaged, the ability to recognise objects by touch becomes impaired;
although damage impairs recognition since association areas are critical in recalling particular object features, touch sensitivity will be unaffected. The somatosensory cortex integrates sensory information, received from the body, relating to touch [59]; the sense of touch involves many different receptors that monitor a broad spectrum of data (Appendix 11.4).

The occipital lobe is dedicated to processing information relating to the sense of sight [60]; after initially integrating visual information, the data is sent to the parietal and temporal lobes. Colour discrimination, depth perception and motion detection are primary functions of this region.

The temporal lobe manages data involved with smelling and hearing, in addition to helping retention of visual stimuli concerning objects and people; it influences the interpretation and recognition of future visual memories [61]. Understanding language, detecting sounds, reasoning, speech, and emotion are also primary functions of this lobe. The hippocampus, located in the temporal lobe, is necessary to the conversion and relocation of short-term memories; consequently, it is significant in forming and encoding long-term memories. Long-term potentiation (Section 2.4) in the hippocampus is widely accepted to be the neural mechanism underlying memory storage within the brain [62]. The existence of “place cells” reflects the hippocampal attempts in forming neural representations of
external features and their location/orientation; these cells fire bursts of action potentials when the body passes through or looks at a particular part of the environment. The hippocampus consists of organised layers of various neurons, of which pyramidal and granule cells constitute the largest proportion. Wernicke’s area digests language and sounds [63]; unsurprisingly, it is connected to Broca’s area and commonly found in the left hemisphere. The connected paths between these two areas allows for conversation to be understood and reciprocated. The auditory cortex carries out the fundamental operations of hearing, such as differentiating volume, pitch, different sounds, and location of origin [64]; this is partially achieved due to the order of neurons, accordingly organised to the frequencies that they most astutely detect [65].

Independent of the lobes, the thalamus and hypothalamus (Fig. 3.1 (a)) are situated in the forebrain. The former relays sensory information to different brain regions; it regulates sleep, consciousness, alertness and activity [66], acting as a intermediary hub (Section 3.2) to indirectly link various regions. The inclusion of many reciprocal connections at the thalamus indicates the involvement of a feedback mechanism [67]. The hypothalamus regulates the endocrine system and controls most of the signals sent to the pituitary gland, influencing its hormone secretion activity; it is involved in homeostasis, circadian rhythms, and the ANS [68]. Diverse connectivity to numerous brain regions allows the hypothalamus to rapidly receive data on changes to the body and issue timely corrections.

The midbrain consists of the tectum and tegmentum; the former moderates visual and auditory reflexes using its extensions to the spinal cord [69]; the latter manages autonomic procedures and is involved in motor processes. The substantia nigra is located within the red nucleus of the tegmentum and produces dopamine; this neurotransmitter is critical to synaptic transmission, influencing on mood, sleep, and memory. The onset of Parkinson’s disease (Appendix 11.10) is caused by large numbers of dopamine-producing neurons dying in the pars compacta, a portion of the substantia nigra [70, 71, 72, 73, 74].

The hindbrain is composed of the cerebellum, pons, and medulla oblongata. The cerebellum is associated with cognitive aspects and fine-tuning motor control processes
the motor elements largely relate to movement coordination and timing, as opposed to the selection and initiation of actions. Hence, the cerebellum is involved in vestibular activities, concentrating on balance and spatial orientation, and responding to stimuli to the highest degree of accuracy. The cerebellum consists of a highly organised arrangement of mostly Purkinje and granule neurons; despite taking up only 10% of the brain’s volume, the number of neurons in this area exceeds the sum of cells found in the rest of the brain [76]. Densely packed neurons, within the folds, increase surface area; unlike most parts of the brain, spatial efficiency is optimised as almost all connections are unidirectional and sequential, establishing an almost entirely feed-forward network of segregated modules (Section 3.2). The large ratio of inputs to outputs allow modules to often share inputs, seldom influencing one another; subsequently, there is reduced requirement for extensive and complex wiring patterns. The cerebellum’s largely non-recurrent architecture is unable to self-sustain neural oscillations; its extreme levels of synaptic plasticity create flexibility between inputs and outputs, assisting in fine-tuning and precision of movements [77]. The pons relays nerve impulses between the forebrain and the cerebellum; it helps to control sleep, respiration, swallowing, eye-movement, and posture [78]. The medulla oblongata is primarily responsible for autonomic functions involving heart rate, breathing, and blood pressure; these are respectively monitored by its cardiac, respiratory and vasomotor centres [24]. Central chemoreceptors in the brain, as well as the peripheral chemoreceptors in aortic and carotid bodies, provide sensory information, such as pH content and partial pressures of oxygen and carbon dioxide. Baroreceptors detect blood vessel pressure. Stretch receptors in the bronchi and bronchiole walls of the lungs ensure that inspiration limits are not exceeded. The medulla utilises a negative feedback mechanism using the SNS and PSNS; the former dictates increases in heart rate, breathing, and vasoconstriction; the latter invokes vasodilation in response to low partial pressures of oxygen.
3.2 Network Architecture and Characterisation

Having briefly discussed various brain regions and functions (Section 3.1), the aim of this section is to discuss global configuration, achieved through a neural network’s interconnectivity. Properties are introduced that will be utilised when describing stochastic neural-like networks in Chapter 6.

The network’s architecture significantly impacts upon its performance; poor configuration can reduce the efficiency of communication and information processing. The degree of a neuron [79, 80] is calculated by the number of connections made to other neurons (Fig. 3.2 A); neurons with high degrees (usually connecting different modules) are known as hubs (Fig. 3.2 E) to signify their greater influence upon signal transmission performance. In systems where connections are restricted to a particular direction, a neuron can have separate degrees for inbound and outbound connections. The brain contains many directed connections resulting from the transmission of electrical signals through chemical synapses (Section 2.4).

Wiring length corresponds to the cable’s distance between nodes; for example, short wiring lengths connect nearby nodes. Due to the brain’s spatial limitations, most wiring lengths are short, which leads to the development of clustered neurons; these highly interconnected groups are also known as modules (Fig. 3.2 E) and each permutation of interconnectivity is called a motif (Fig. 3.2 C). There are many structural advantages of a cluster; the short wiring lengths reduce signal cross-talk errors [31, 81]; the topological ordering of adjacent neurons with similar functions significantly increases efficiency of local communication and information transfer; the impact of damage to and random failure of a single connection may be minimised as alternate pathways within a module may be available.

The clustering coefficient $C$ (Fig. 3.2 B) for a neuron $i$ measures the extent of a
neuron’s tendency to connect with its nearest neighbours; this can be defined as

\[ C_i = \frac{Q_i}{N_i}. \quad (3.1) \]

\( Q_i \) details the number of connections that neuron \( i \) establishes with its direct neighbours; \( N_i \) is the number of possible direct neighbour connections. The clustering coefficient for the entire network \( C \), for a network of size \( n \), is given by the mean clustering coefficient across all neurons [82]:

\[ C = \frac{1}{n} \sum_{i=1}^{n} C_i. \quad (3.2) \]

Path length is the minimum number of connections that a signal has to bypass to travel from one neuron to another (Fig. 3.2 D). Short path lengths reduce the number of intermediate transmission steps; therefore transmission delays are reduced and signals are less exposed to noise-inflicting elements (Section 3.3). If a network consists of nodes that are connected only to their closest neighbours, in a regular lattice structure, the path length between two distant neurons would be large; consequently, a signal would take considerable time to entirely transmit throughout a vast network. An alternative network
with all-to-all connectivity would produce the optimal path length since all neurons could communicate directly; however, the disadvantage to this structure would be its spatial inefficiency. Since the volume of the actual brain is limited, the neuronal population is too large to accommodate the spatial and material costs of such a complex wiring network [12]. If $l_{ij}$ denotes the path length from neuron $i$ to neuron $j$, the average path length $\bar{l}$ in a network of $n$ neurons is given by

$$\bar{l} = \frac{1}{n(n-1)} \sum_{i \neq j} l_{ij}.$$  

(3.3)

The brain’s network has small-world characteristics, whereby many short connections form local clusters in addition to the occasional distant connections, which allow rapid transmission to distant brain nodes [79, 80, 83, 84].

A probability measure $p$ of randomness in the neural network’s connections can be defined where $p = 0$ corresponds to a regular lattice (non-random) and $p = 1$ relates to a fully random network [82]. When $p = 0$, the network displays high clustering with a disadvantage of high path lengths to distant neurons (Fig. 3.3); when $p = 1$, the system allows rapid communication between distant neurons, due to short path lengths, but establishes weak interconnectivity among clusters. A network is small-world if it has a significantly larger clustering coefficient than a random system counterpart and
a negligible average path length discrepancy. The brain’s configuration has the best attributes of both non-random and random systems \((0 < p < 1)\), though values tend slightly towards zero. Given that axonal propagation occurs without decrement, occasional long-distance neural connections are logical despite the developmental and metabolic costs they incur [83].

Each neuron establishes up to 10,000 synaptic connections to other nodes; an average of 7,000 per neuron contributes to a total of around 100 trillion or \(10^{14}\) connections in the entire network [11]. Though this average per neuron may seem large, it can be argued that the number of connections is sparse in relation to the possible maximum of approximately 100 billion connections per neuron.

### 3.3 Neuronal Noise

Noise, with respect to neural circuits, refers to random fluctuations affecting the transmission of signals regarding timings, strength, space, or any other domain. Many sources produce noise within the human brain with various impacts upon neuronal activity.

Synapses account for a major proportion of the noise in the brain. Different neurotransmitters at chemical synapses vary in availability, depending on the frequency of action potentials previously arriving. The release of available transmitters only occurs with finite probability; often (with probability between 0.5 – 0.9), none is released into the synaptic cleft despite the arrival of an action potential to the presynaptic bulb [85]; at times, it is also secreted randomly despite no incoming action potential. The probability of \(k\) successful releases of neurotransmitter into the synaptic cleft at \(n\) sites can be described by the binomial distribution

\[
P(X = k) = \binom{n}{k} p^k (1 - p)^{n-k},
\]

with probability \(p\) of a successful release at each site [85]. An assumption is that sites have independent releases and uniform size of neurotransmitter molecules. If a variable
\( \gamma \) is introduced representing the magnitude of conductance change brought about by each neurotransmitter molecule, the distribution has mean synaptic conductance \( \gamma np \) and variance \( \gamma^2 np(1 - p) \) [31, 86]. Noise is also produced when synaptic efficacy varies with time through learning and activity, leading to a heterogeneous weight distribution of connections.

Axons and dendrites contribute to noise in the network; lengths of the axons and dendrites widely vary, imposing inhomogeneous transmission delays across the system (Sections 2.1 and 2.3). Variable dendritic branch lengths also cause different magnitudes of signal strength attenuation between neurons [24, 25]; the signal strength arriving to each target neuron is prone to variation even if the initial firing neuron is common to all targets. Since suprathreshold stimulation usually requires contributions from multiple neurons, rather than just one, any delay could mean the difference between an active and quiescent response.

Ion channels (usually sodium, potassium, and calcium) are influenced by and amplify weak thermal noise (Section 2.2). When changing shape to allow specific ions to diffuse through the differentially permeable membrane, ion channels only open with finite probability and are therefore a stochastic process. Some gates can be modelled as a Markov process where the probability of a future state of the channel depends only upon its current state, disregarding any previous state. Gates influenced by their previous states can have transitional probabilities of the form \( 1/t \), where \( t \) is the amount of time elapsed during a given state [31, 86]. Different ion channels also have conflicting (de)activation voltage thresholds or are not dependent upon voltage [32]. Even when ion channels are open, fluctuations in concentration and electrochemical gradients cause ions to move in a seemingly random fashion. Ion restoration is subjected to metabolic noise created from varying ATP supplies.

Gaussian white noise is frequently used to model the random fluctuations inherent in biological neural systems [87]; this is due to noise usually having a continuous distribution and fluctuations occurring at a faster rate than the neuronal response. Due to
its unpredictability, incorporating noise into neuronal models leads to each neuron being considered as a stochastic or random unit rather than a deterministic one (Appendix 11.5).

The abundance of neurons and synapses in the brain undoubtedly bestows a degree of robustness upon the system; the misfiring of individual neurons and synapses decreases in significance due to the law of large numbers, whereby the expected relative error is of the order $\frac{1}{\sqrt{n}}$ ($n$ is the number of input neurons for a particular output neuron). An individual cell's activity tends to be unpredictable, though the network as a collective produces orderly patterns and dynamics [31, 88, 89]; the activity-enhancing and activity-suppressing fluctuations of the ensemble are expected to cancel one another. Even if fluctuations are not wholly nullified through averaging, the ensemble presumably reflects whether the receiving neuron should fire; the signal will rarely be close to the threshold value after summation. Although minor variations should be insignificant, an irregular event, such as the opening of a solitary ion channel, can be amplified, causing a single action potential to result in a cascade of neural activity [31, 88].

The human brain can make logical decisions based upon available information and devises strategies to achieve desired purposes and targets; it is intuitive that these cognitive operations are consciously performed. However, at any time moment, each neuron receives action potentials from a multitude of neurons (Section 3.2); despite their fixed
shapes, the temporal pattern of these signals elicit wide unpredictability. Furthermore, neurons can exhibit ranged responses to the same input signals and occasionally fire spontaneously without any stimulation; the mean of such a large number of uncorrelated signals closely resembles Gaussian white noise behaviour (Fig. 3.4) and is conveniently used for modelling neuronal firing as a random or stochastic process.
Chapter 4

Neuronal Models

This chapter gives an overview of the most popular neuron models used as basic units to imitate neural networks. Section 4.3 describes the model chosen for the research presented in this thesis.

4.1 Neuronal Learning

The basic neuronal model was introduced by McCulloch and Pitts in 1943 [31, 81, 90]; it states that neuron $i$ receives $n$ input signals $\xi_j$ where $(j = 1, \ldots, n)$ with weights $\mu_{ij}$. The total input is given as

$$I_i = \sum_{j=1}^{n} \mu_{ij} \xi_j. \quad (4.1)$$

The output $\eta_i$ can be modelled by

$$\eta_i = C f(I_i - \theta_i), \quad (4.2)$$

for a nonlinear activation or transfer function $f$, threshold $\theta_i$, and constant $C$.

Neurons can have a particular state or output at any moment in time; a network of neurons and the state evolution of each individual neuron can be modelled as a system of differential equations. The Wilson-Cowan model [91] was developed in 1972, extending the work of Beurle [92] in 1956 to accommodate both inhibitory and excitatory neurons with a refractory period; this model is a set of ordinary differential equations (ODE’s), describing the time evolution of the mean level of activity within a neural population; it
Figure 4.1: The sigmoid function $S(x) = (1 + e^{-x})^{-1}$.

is mathematically represented [93] as

$$
\begin{align*}
\mu_x \dot{x}_i &= -x_i + (1 - \tau_x x_i) S(p_{xi} + \sum_{j=1}^{n} a_{ij} x_j - \sum_{j=1}^{n} b_{ij} y_j), \\
\mu_y \dot{y}_i &= -y_i + (1 - \tau_y y_i) S(p_{yi} + \sum_{j=1}^{n} c_{ij} x_j - \sum_{j=1}^{n} d_{ij} y_j). 
\end{align*}
$$ (4.3)

$x$ and $y$ respectively represent excitatory and inhibitory neuron activity; $\mu_x, \mu_y > 0$ are membrane time constants; $\tau_x$ and $\tau_y$ are refractory periods of excitatory and inhibitory neurons, respectively; $a$, $b$, $c$ and $d$ characterise synaptic coefficients when $i \neq j$; $b_{ii}$ and $c_{ii}$ denote synaptic interactions between excitatory and inhibitory neurons; $a_{ii}$ and $d_{ii}$ provide neuronal feedback; $p_{xi}$ and $p_{yi}$ are inputs from external sources; $S$ corresponds to the sigmoid function (Fig. 4.1)

$$
S(x) = \frac{1}{1 + e^{-x}}. 
$$ (4.4)

The perceptron, introduced by Frank Rosenblatt in 1957 [94], is an artificial neural network designed to recognise patterns and provide an algorithm for supervised classification of inputs; following this invention, the 1982 Hopfield model [95] of a recurrent artificial neural network incorporated a set of the McCulloch and Pitts neurons. The Hopfield model became useful in the understanding of memory formation within the brain. The nodes in a Hopfield network generate binary output when the input exceeds a designated threshold. Using the same notation for $\mu_{ij}$ (see above), Hopfield connections abide
by two restrictions; no node can connect to itself, $\mu_{ii} = 0, \forall i$, and connections should be symmetric, $\mu_{ij} = \mu_{ji}, \forall i, j$. Nodes in the Hopfield network are updated at discrete times $t$, according to the following rule:

$$
\xi_i(t + 1) = \begin{cases} 
1, & \text{if } \sum_j \mu_{ij} \xi_j(t) > \theta_i \\
0, & \text{otherwise.}
\end{cases}
$$

(4.5)

Units can be updated individually or simultaneously.

The mechanisms underlying changes in synaptic weights in biological neural networks are still not clarified. The most accepted hypothesis is the synaptic plasticity process of Hebbian learning, whereby repeatedly activated connections are strengthened and unused connections are weakened [96]; when a neuron causes another nerve cell to persistently fire, growth processes or metabolic alterations occur in one or both of the cells and enhances the connection’s efficacy. Hebb’s rule can be described by

$$
\frac{d\mu_{ij}}{dt} = \alpha \eta_i \xi_j,
$$

(4.6)

where $\mu_{ij}$ is the synaptic strength from neuron $j$ to neuron $i$; $\alpha$ is the learning rate parameter; $\eta_i$ is the postsynaptic activity; $\xi_j$ is the presynaptic activity [31, 81]. Structural modifications to connections, resulting from repeated stimulation, may be beneficial or detrimental, depending upon how successfully a neuron performs its tasks; desired firing of a neuron strengthens connections and improves efficacy, whereas undesired firing reinforces detrimental patterns and unnecessary behaviours.

### 4.2 Hodgkin-Huxley Equations

Following their investigation of giant squid axons in 1952, Hodgkin and Huxley developed a biologically-realistic and four-dimensional non-linear set of ODE’s, allowing variables to be fitted to experimental data [28]; these equations have often been modified to account for various parameters involved in the initiation and propagation of neuronal action.
potentials. One popular form of the equations is

\[ C_m \frac{\partial V}{\partial t} = G_{\text{leak}} (V_{\text{leak}} - V) + G_{\text{Na}^+} m^3 h (V_{\text{Na}^+} - V) + G_{\text{K}^+} n^4 (V_{\text{K}^+} - V), \]  

(4.7)

with delayed rectifier currents

\[ \tau_m(V) \frac{dm}{dt} = m_\infty(V) - m, \]
\[ \tau_n(V) \frac{dn}{dt} = n_\infty(V) - n, \]
\[ \tau_h(V) \frac{dh}{dt} = h_\infty(V) - h \]  

(4.8)

and transition rate functions

\[ \tau_m(V) = \frac{1}{a_m(V) + b_m(V)}, \]
\[ \tau_n(V) = \frac{1}{a_n(V) + b_n(V)}, \]
\[ \tau_h(V) = \frac{1}{a_h(V) + b_h(V)}. \]  

(4.9)

\( C_m \) is the membrane capacitance; \( G_{\text{leak}}, G_{\text{Na}^+}, \) and \( G_{\text{K}^+} \) are the maximal conductances of passive leak, transient sodium current, and delayed rectifier potassium current; \( V_{\text{leak}}, V_{\text{Na}^+}, \) and \( V_{\text{K}^+} \) are the respective reversal potentials; \( m, h, \) and \( n \) take values in the interval \([0,1]\) and obey simple relaxation equations with respective equilibrium values of \( m_\infty(V), h_\infty(V), \) and \( n_\infty(V) \); \( m \) and \( n \) are activation variables describing the probability of finding a channel in its open state; \( h \) is an inactivation variable arising from the transient nature of sodium currents. Contributions from other ionic currents are assumed to obey Ohm’s law, namely, \( \text{voltage} = \text{current} \times \text{resistance} \ (V = IR) \). \( a_m, a_h, \) and \( a_n \) and \( b_m, b_h, \) and \( b_n \) are mean transition frequencies; the former represent closed to open states of voltage-gated channels and the latter reflect the reverse. The Hodgkin-Huxley model in Eq. 4.7 illustrates that opening a channel requires activation and recovery from inactivation \([30, 31, 97]\).
4.3 FitzHugh-Nagumo Equations

This section describes the neuronal model chosen as the basic network unit for the research presented in the given thesis.

The FitzHugh-Nagumo model was developed to mathematically represent the properties of neuronal excitability and propagation during sodium and potassium ion activity; it simplifies the Hodgkin-Huxley equations, reducing the dimension of the equations from four to two (Appendix 11.6), which allows tractable analytical solutions to be more readily generated and phase plane analysis conducted. A phase plane is a two-dimensional space encompassing all possible positional values of a system. The FitzHugh-Nagumo model derives from the independent research of Richard FitzHugh in 1961 [98], who initially referred to it as the Bonhoeffer-van der Pol model, and Jin-Ichi Nagumo, an engineer of electronic circuitry, in 1962 [99]. A simple and classic version of the FitzHugh-Nagumo model is:

\[
\begin{align*}
\dot{x} &= x - \frac{x^3}{3} - y, \\
\dot{y} &= x + a + b(t).
\end{align*}
\]  

(4.10)

Here, \( x \) is the membrane potential; \( y \) is the recovery variable; the parameter \( a \) is a constant; \( b(t) \) provides external perturbation, taking the form of a random or deterministic signal; \( \epsilon \) determines the offset in the \( x \) and \( y \) time-scales. At \( \epsilon = 1 \), both variables (\( x \) and \( y \)) evolve according to the same time-scale; where \( |\epsilon| < 1 \), the \( x \)-variable evolves faster than the \( y \)-variable; where \( |\epsilon| > 1 \), the \( y \)-time-scale evolves relatively quickly in comparison to the \( x \)-time-scale. To simulate neural spiking, the parameter \( \epsilon \) is set to be much smaller than 1, \( 0 < \epsilon \ll 1 \), to ensure time-scale separation when \( x \) changes much faster than \( y \). When placed inside a network and subjected to a stochastic input, the FitzHugh-Nagumo model demonstrates a behaviour determined by the location and stability of its fixed point and the location of its nullclines.
In the absence of perturbations, \( b(t) = 0 \), the location and stability of fixed points are easily calculated and provide useful insight into the basic dynamics underlying a system’s behaviour. Setting \( \dot{x} = \dot{y} = 0 \) in Eqs. 4.10, where the nullclines intersect, produces

\[
x - \frac{x^3}{3} - y = 0, \\
x + a = 0.
\] (4.11)

Eqs. 4.11 imply that \( x = -a \), thus

\[-a - \frac{(-a)^3}{3} - y = 0 \quad \Rightarrow \quad y = \frac{a^3}{3} - a.\] (4.12)

A fixed point is located at \((-a, \frac{a^3}{3} - a)\); the stability of the fixed point is determined by the Jacobian matrix evaluated at this point. The Jacobian matrix is given by

\[
\begin{bmatrix}
\frac{\partial \dot{x}}{\partial x} & \frac{\partial \dot{x}}{\partial y} \\
\frac{\partial \dot{y}}{\partial x} & \frac{\partial \dot{y}}{\partial y}
\end{bmatrix} \Rightarrow 
\begin{bmatrix}
1 - x^2 & -1 \\
1 & 0
\end{bmatrix}_{x=a} = 
\begin{bmatrix}
1 - a^2 & -1 \\
1 & 0
\end{bmatrix}.
\]

The determinant \( \Delta \) and trace \( \tau \) evaluated at the fixed point are given by \( \Delta = 1 \) and \( \tau = 1 - a^2 \). The determinant and trace of a matrix can be used to generate a characteristic polynomial representation of the form

\[
\lambda^2 - \tau \lambda + \Delta = \lambda^2 - (1 - a^2)\lambda + 1 = 0,
\] (4.13)

whose solutions by means of the quadratic formula are

\[
\lambda_{1,2} = \frac{-(a^2 - 1) \pm \sqrt{a^4 - 2a^2 - 3}}{2},
\] (4.14)

and can be simplified to

\[
\lambda_{1,2} = \frac{(1 - a^2) \pm \sqrt{(a^2 - 3)(a^2 + 1)}}{2}.
\] (4.15)
Depending on the value of $a$, $\lambda_{1,2}$ can be real or complex. Negative real parts of $\lambda_{1,2}$ imply a stable fixed point, and non-zero imaginary parts imply that the phase trajectory spirals around this point. Bifurcations occur when a system’s behaviour suddenly changes; Andronov-Hopf bifurcations transpire when two complex conjugate eigenvalues simultaneously cross the imaginary axis i.e. when the real part of Eqs. 4.15 is equal to zero [100]. $\lambda_{1,2}$ are purely imaginary when

\[ 1 - a^2 = 0 \quad \Rightarrow \quad a = \pm 1, \quad (4.16) \]

and $\lambda_{1,2}$ become

\[
\lambda_{1,2} = 0 \pm \frac{\sqrt{(1-3)(1+1)}}{2} = \pm \frac{\sqrt{-4}}{2} = \pm 2i. \quad (4.17)
\]

The fixed point is stable when $a > 1$ or $a < -1$ due to a negative real part of $\lambda_{1,2}$. Alternatively, at $-1 < a < 1$ an unstable fixed point exists due to a positive real part of $\lambda_{1,2}$. This proves the requirement of $|a| > 1$ in the FitzHugh-Nagumo model in Eqs. 4.10, preventing random spiking of the neuron without perturbation.

Thus, when $b(t) = 0$ in Eqs. 4.10, there exists a single fixed point, which lies at the intersection of two nullclines (Fig. 4.2). The solitary fixed point represents the resting state of the neuron. In the absence of external perturbations, $b(t) = 0$, from any initial conditions, the system will eventually evolve towards and remain in the resting state. As all phase points located below the cubic nullcline defined by $\dot{x} = 0$ (black line in Fig. 4.2) have $\dot{x} > 0$, an immediate trajectory is cast with a horizontal element pointing east in the phase plane; all phase points above the $x$-nullcline have $\dot{x} < 0$, giving them immediate trajectory with a horizontal element that is directed west; all phase points positioned to the right of the $y$-nullcline defined by $\dot{y} = 0$ (red line in Fig. 4.2) have an
Figure 4.2: Nullclines of Eqs. 4.10 with parameter value $a = 1.05$: $x$-nullcline defined as $\dot{x} = 0$, implying $x = -a$ (vertical red line) and $y$-nullcline defined as $\dot{y} = 0$, implying $y = x - \frac{x^3}{3}$ (black line). Intersection of the two nullclines represents the fixed point where $\dot{x} = \dot{y} = 0$. Dotted lines indicate the shape and direction of limit cycle trajectories exhibited after suprathreshold noise perturbation. Blue arrows show horizontal and vertical trajectory direction for phase points in each segment of the phase plane. Trajectories have leftward horizontal elements above the cubic nullcline and rightward on the segment below. Trajectories have upward vertical elements to the right of the $y$-nullcline and downward on the left hand side.

immediate trajectory whose vertical element is directed towards the north of the phase plane as $\dot{y} > 0$; $\dot{y} < 0$ on the left of the $y$-nullcline, these phase points have an immediate trajectory whose vertical element is directed south. When $|\epsilon| \ll 1$, phase points flow much quicker horizontally than vertically; phase points not lying on the nullclines perform a rapid horizontal “switch” onto the outermost branches of the cubic nullcline, due to the middle branch’s unstable dynamics; subsequently, phase points evolve relatively slowly along the cubic nullcline. On the leftmost branch of the $x$-nullcline, phase points descend along the nullcline towards the fixed point. Phase points ascending the rightmost branch on the cubic nullcline do not reach a fixed point; when a local maximum turning point is reached, the phase point still has upward and western immediate trajectory elements; consequently, the phase point rapidly shifts onto the leftmost branch of the $x$-nullcline (the uppermost dotted green line in Fig. 4.2), descending towards the fixed point at a much slower pace.

The inclusion of external perturbations, $b(t) \neq 0$, causes phase points to deviate from
the system’s natural flow and temporarily move away from the fixed point. Weak perturbations below a designated threshold value (subthreshold) move phase points; however, they stay within the vicinity of the fixed point and the equilibrium position is regained almost immediately; as a result, the probability density (Section 11.5) of Eqs. 4.10 is focused around the fixed point [8]. Strong perturbations above the designated threshold value (suprathreshold) result in the phase point leaving the vicinity of the fixed point; the phase point switches to the eastern branch of the $x$-nullcline (the lowermost dotted green line in Fig. 4.2), performing a long parallelogram-shaped excursion before returning to the fixed point; thus, the probability density becomes mostly concentrated on the slow outermost branches of the cubic nullcline [8]. Suprathreshold excursions in the phase space represent action potentials in neuronal models (Chapter 2).

### 4.4 Other Reduced Hodgkin-Huxley Models

The Hodgkin-Huxley system can be further reduced to the one-dimensional integrate and fire model; Lapicque proposed this model in 1907 [1], which is simply described by the time derivative of the law of capacitance $Q = CV$ \((\text{Charge} = \text{Capacitance} \times \text{Voltage})\),

$$I(t) = C_m \frac{dV_m}{dt}. \quad (4.18)$$

The voltage in this system increases until a threshold value is exceeded; upon suprathreshold stimulation, the system performs a delta function spike and resets the voltage to its resting potential. A refractory period provides an upper bound to the neuron’s firing frequencies, preventing them from firing within this period. The leaky integrate and fire model loses membrane potential through diffusion when a neuron fails to fire; this solves the original model’s memory problem, whereby neurons accumulate membrane potentials over unrealistically large time intervals without decrement. The simple framework of the integrate and fire model is beneficial for simulating large networks.

Another popular derivative of Hodgkin-Huxley equations is the two-dimensional Morris-
Lecar model; developed in 1981 [101], it revolves around two non-inactivating voltage-dependent ionic gates, namely potassium and calcium, representing neural excitability and innervation at muscle fibres at a biological level. The calcium component contributes to neuronal excitation and depolarisation; the potassium element provides a lagging recovery and hyperpolarisation. The Morris-Lecar model can be mathematically described by the equations,

\[
\frac{dV}{dt} = I - G_{\text{leak}}(V - V_{\text{leak}}) - G_{Ca^{2+}}M_{sp}(V - V_{Ca^{2+}}) - G_{K^+}N(V - V_{K^+}), \tag{4.19}
\]

\[
\frac{dN}{dt} = \frac{N - N_{sp}}{\tau_N},
\]

where

\[
M_{sp} = \frac{1}{2} \left\{ 1 + \tanh \left( \frac{V - V_1}{V_2} \right) \right\},
\]

\[
N_{sp} = \frac{1}{2} \left\{ 1 + \tanh \left( \frac{V - V_3}{V_4} \right) \right\}, \tag{4.20}
\]

\[
\tau_N = \tau_0 \text{sech} \left( \frac{V - V_3}{2V_4} \right).
\]

\(N\) corresponds to the recovery variable for potassium; \(I\) is the applied current; \(V\) is the membrane potential; \(C\) represents membrane capacitance; \(G_{\text{leak}}, G_{Ca^{2+}},\) and \(G_{K^+}\) correspond to leak, calcium, and potassium conductances through the membrane channel; \(V_{\text{leak}}, V_{Ca^{2+}},\) and \(V_{K^+}\) are the equilibrium potentials for leak, calcium, and potassium ion channels. \(N(t)\) is the instantaneous probability that a potassium ion channel is in an open state, describing the conformational transitions that occur in the membrane; \(V_1, V_2, V_3,\) and \(V_4\) correspond to steady state tuning parameters. Further to the FitzHugh-Nagumo system in Eqs. 4.10, the Morris-Lecar model described in Eqs. 4.19 and 4.20 enables phase plane analysis.
4.5 Selection of the FitzHugh-Nagumo Model

This chapter has discussed some of the most prolific models used to simulate dynamics at a neuronal level. When constructing a network capable of simulating the behaviour of many connected neurons by introducing coupling, the most appropriate model depends upon the behavioural properties that a researcher wishes to measure, and on their available resources. Due to computational limitations, neural network models are restricted in their complexity and ability to replicate biological networks. Care must be taken to ensure that the model chosen enables the desired level of relevance and accuracy to be achieved. In addition, one must also consider the tractability of analytical solutions and the ability to generate visual representations of the present dynamics.

The FitzHugh-Nagumo model has been identified as the most appropriate research foundation; it has been selected as there are numerous advantages and comparatively few disadvantages. Benefits include the ability to reduce the dimension of individual neuron equations, as opposed to the Hodgkin-Huxley model, which halves four dimensions to two; this reduction permits tractable approximate analytical solutions. Moreover, a two-dimensional system enables phase plane analysis and visual representation of system dynamics. Another advantage is how well-established the FitzHugh-Nagumo model is; it will be useful for comparison and reference to previous studies, as well as contribution to future developments. These benefits have minimal impact upon the qualitative accuracy of results, in comparison to biological data; however, quantitative agreement is sacrificed.
Chapter 5

Synchronisation

Synchronised behaviour in the biological brain is displayed when neurons synchronise their firing patterns; it is the complex dynamic process where uncoordinated rhythms or time-scales of oscillating objects adjust according to their interactions [102, 103]. Widely believed to enhance information-processing capabilities (Appendix 11.8) [2, 3, 4, 5], synchronisation is also associated with neurological disorders, including epilepsy (Appendix 11.9) and Parkinson’s disease (Appendix 11.10) [6, 7]; the first observation and description of synchronisation was believed to be made by Christiaan Huygens in 1665 during his study of pendulum clocks [102, 103, 104]. The oscillations of two pendulums hanging from a common supporting beam were found to coincide perfectly and swing in opposite directions; Huygens concluded that the beam was a form of coupling that allowed interaction between the two clocks, enabling anti-phase synchronisation through mutual adjustment of their rhythms. Consequently, two oscillators with differing oscillation periods can begin to synchronise when their coupling strength is sufficiently large; this mechanism is known as frequency entrainment or locking. For a phenomenon to be classified as synchronisation, the following conditions of oscillating systems [103] are necessary:

- the systems can generate their own rhythms,
- the oscillating rhythms adjust upon weak interaction,
- if one oscillating frequency slowly varies, the second system adopts this variation.

Synchronisation of different orders $m : n$ are possible, where $m$ and $n$ are integers; during the same time interval, one unit makes exactly $m$ oscillations while the other makes $n$ full oscillations. One-to-one synchronisation between two interacting neurons can be
characterised by the phase shift between their firings and can be in-phase, anti-phase, or out-of-phase. The phase shift represents the time delay between the firing of neurons and when the phase shift is equal or close to zero, in-phase synchronisation occurs; anti-phase synchronisation is described by phase shifts equal or close to $\pi$; out-of-phase synchronisation encompasses all other phase shift values.

An oscillator’s phase is a quantity that increases by $2\pi$ during a cycle, proportional to the fraction of the period [103]; two phases that differ by $2\pi$ correspond to the same physical state. Let the firing neuron’s phase space excursion (Section 4.3) be represented by a loop; let $\Phi(t)$ denote the phase, which is the system’s location along the loop at any time moment. One can assume that $\Phi(t) = 0$ corresponds to the phase point in equilibrium and $\Phi(t) = 2\pi n$ denotes subsequent phase point returns, where $n = 1, 2, \ldots, N$. Two neurons, $i$ and $j$, with different firing frequencies within time interval $[0, t)$ have respective phases $\Phi_i$ and $\Phi_j$; if an action potential is performed simultaneously at time $t$ and at all subsequent time moments, neurons $i$ and $j$ are in-phase synchronised. Synchronisation only occurs when there is direct or indirect interaction between two neurons; the firing times of independent neurons may coincidentally correlate without synchronisation. When spike occurrences repeatedly match over a sustained period of time, neurons are assumed to be synchronised from their coupling interactions. Neurons can be considered synchronised, even if their firing times do not coincide, when the phase difference

$$\Delta\Phi(t) = |\Phi_i(t) - \Phi_j(t)|$$

(5.1)

is bounded and does continuously grow with time. Mathematically, synchronisation is present if

$$\Delta\Phi(t) < 2\pi$$

(5.2)

is satisfied [103, 104]. A phase difference equal to or exceeding $2\pi$ is evidence of growth since phases of two neurons will always begin on the same revolution; for two neurons with fixed and marginally different firing frequencies, the phase difference accumulates
with each revolution of $\Phi(t)$. Neurons with phase differences showing unbounded growth are asynchronous.

The definitions for phase difference and synchronisation in Eqs. 5.1 and 5.2 can be generalised to include relationships where a neuron’s firing frequency is a rational proportion of another neuron’s spike rate. Consider that neuron $i$ performs exactly $k$ spikes in time interval $[t, t+l]$, the first spike is initiated at time $t$, completing the $k$th spike at time $t+l$; let neuron $j$ perform exactly $m$ spikes in the same time interval $[t, t+l]$, initiating the first spike at time $t$ and completing the $m$th spike at time $t+l$. Under these conditions, neurons $i$ and $j$ are considered $k : m$ synchronised; the condition for $k : m$ synchronisation [103, 104, 105] can be expressed as

$$\Delta \Phi_{km}(t) = |\Phi_i(t) - \frac{k}{m} \Phi_j(t)| < 2\pi.$$  (5.3)

Synchronisation does not impose any restrictions upon the amplitudes achieved by oscillators.

Random perturbations within a system, also known as noise (Section 3.3), can breakdown synchronisation by counteracting coupling between units, causing local instability. Although the weakening of interaction can disrupt rhythms underlying synchronisation [106], noise may also enhance and induce synchronisation when it is imperfect or absent in a deterministic system; enhancement has been observed in many systems, such as weakly coupled chaotic oscillators [106], globally coupled phase oscillators [107], and excitable FitzHugh-Nagumo units [108]. Excitable systems under the influence of noise may experience the phenomena of stochastic resonance [21, 109, 110] or coherence resonance [8, 111, 112, 113], which can accompany synchronisation.

Stochastic resonance is characterised by the enhancement of subthreshold signal transmission due to the presence of noise [21]; it is exhibited when a system contains some form of threshold, a source of noise, and a source of input. The addition of noise allows some subthreshold signals to overcome the threshold, leading to the amplification of the signal
and signal-to-noise ratio; upon reaching excessive noise intensities, noise dominates the signal and causes it to display increasingly random behaviour. The stochastic resonance effect can be witnessed in bistable systems, perturbed by periodic low-frequency forcing and additive Gaussian white noise [21], monostable nonlinear systems [114, 115], and excitable nonlinear systems [116, 117, 118].

Coherence resonance may be achieved if an optimal noise intensity is reached, producing optimal regularity of noise-induced oscillations; nonlinear properties of the system are activated, inducing motion that was absent in the deterministic model [8, 104]. Coherence resonance occurs within a system of coupled stochastic FitzHugh-Nagumo oscillators [8]. For weak noise, spike trains display random intervals of time between successive excitations; at intermediate noise intensities, spikes are more regular, indicating little variance between spike intervals; spike frequencies elevate at large noise intensities, although the time interval between spikes increase in irregularity once again. The effect of coherence resonance illustrates that noise can significantly regulate a system; synchronous oscillations can be observed within an optimal range of coupling strengths.

Synchronisation may be present within excitable neural networks that contain large ensembles of units; it is dependent upon network properties, such as connectivity, coupling strength, configuration, size, noise, and the properties of individual units. In particular, noise can influence and induce synchronisation within a large network of coupled excitable units [9, 10, 119].

In networks where excitable stochastic FitzHugh-Nagumo elements are coupled through the mean-field, studies have been performed where oscillations are only possible due to noise in the system [9, 10]; units collectively change from asynchronous to synchronous firing, before returning back to asynchrony upon increasing noise intensity [10]. Adding delayed feedback, where units receive feedback signals that are proportional to the difference between mean-field values over a specified time delay, can suppress, enhance, or induce synchrony in certain networks [9, 120, 121].
The predominant aim of this thesis is to extend Patidar and Zaks’s previous results of FitzHugh-Nagumo neural networks that are coupled through the mean-field [9, 10] to sub-maximal connectivity; the findings will be significant as the mean-field scenario currently fails to represent the sparseness of connectivity relative to the biological brain’s size (Section 3.2).

Coupling through global mean-field $\overline{M}_x$ assumes that all neurons receive the same input, which is approximated by the average membrane potential of the neuronal ensemble at any given time:

$$ \overline{M}_x = \frac{1}{N} \sum_{i=1}^{N} x_i. \quad (5.4) $$

The mean-field coupling scenario is unrealistic since neurons do not have all-to-all connectivity or connect to themselves. Universally using the mean-field as the input for all neurons significantly benefits calculation and numerical simulation efficiency; the mean-field’s behaviour provides useful insights into the global dynamics of a network. Although examining individual neuronal behaviour in a large ensemble would be useful to a degree, it is still be difficult to ascertain global network behaviour. The mean-field is more accessible and practical for extracting and visualising the ensemble’s dynamics; however, mean-field coupling only serves as a close approximation with all-to-all network connectivity and as connectivity is reduced, the model’s accuracy also decreases.

For the remainder of this work, the following assumptions about synchronisation will apply. The global firing activity of different neurons within a network can occur in three separate patterns: all in unison, all independently, or partake in both aforementioned activities to a degree (i.e. spiking simultaneously occasionally). When neurons repeatedly fire together with aligned timings, it is assumed they are synchronised.

The global mean-field’s behaviour is applicable as a measure of synchrony within sparsely coupled networks and those coupled through the mean-field. Asynchronous behaviour occurs when mean-field oscillations are small in amplitude and neurons randomly fire at different time moments (Fig. 5.1 (a)); in contrast, large amplitude oscillations correspond to synchronisation. Oscillations in the mean-field can form almost periodic
or irregular (chaotic) patterns; the former is regarded as the strongest form of synchrony, when many neurons spike with aligned timings (Fig. 5.1 (c)), and the latter occurs when neurons fire together at random times (Fig. 5.1 (b)).

Upon observation of the mean-field’s behaviour, transitions between asynchronous and synchronous activity can be identified. Units with different natural frequencies, which are not synchronised initially, can synchronise as time progresses through the Kuramoto transition [107]. It is assumed that there is a distribution of natural firing frequencies for coupled neurons. When all units are asynchronous at time $t = 0$, the mean-field demonstrates random fluctuations with small magnitude. A certain number of units will have similar natural firing frequencies; due to coupling, these neurons are likely to synchronised rapidly and their combined oscillation frequency contributes to the mean-field. As a result, the mean-field acquires a small-amplitude component that oscillates with the same frequency as the synchronised units; neurons with frequencies that are in a certain range of the mean-field are influenced by its activity, becoming entrained to its frequency. Consequently, mean-field oscillations become progressively pronounced, influencing a wider range of units; this process is accompanied by the increasing amplitude of mean-field oscillations. Saturation is eventually reached when all neurons fire in unison, although some neurons with firing frequencies outside of the entrainment range may remain; in the former scenario, the mean-field oscillates identically to the individual neurons; in the lat-
ter, remaining neurons contribute to separate frequencies, not to the common frequency of the mean-field. The degree of influence that remaining neurons have over the mean-field’s behaviour is determined by their relative quantity to the network size. Kuramoto’s transition originally describes self-synchronisation as a large ensemble of globally coupled periodic oscillators, in a scenario without noise; however, the same type of transition can occur in a network of stochastic units performing noise-induced oscillations [9, 10].
Chapter 6

Synchronisation in Homogeneously Coupled FitzHugh-Nagumo Networks

6.1 Homogeneous Coupling Model

The brain’s vast landscape makes comprehensive computational modelling extremely intricate and time-consuming. In a sense, the human brain is a recurrent network since its connections can form loops, aiding in the generation of various behaviours. Due to its large size, the biological brain produces a multitude of activity patterns, ranging from highly irregular asynchronous firing, asynchronous periodic firing, bursting, (non-)oscillatory synchronous states and hot spots of activity [31, 123]. The most sophisticated and large-scale model to date incorporates 22 basic types of neurons, performing simulations with one million neurons and half a billion synapses [23]. Since the understanding of biological brain behaviours is extremely significant to the development of neurological treatments, it is necessary to utilise simpler, scaled-down, and faster models that capture the fundamental brain behaviours and dismiss the fine details generated by more realistic brain models. When altering the size of a network, one must consider behavioural changes and which network parameters must be modified to counteract these. Currently, it is unknown whether behaviour is more influenced by the number of inputs per neuron or the relative proportion of connections in the network compared to its size. Coupling enables the possibility for individual rhythms of oscillating units to be adjusted and under
certain circumstances synchronisation may be achieved.

Let a network consisting of $N$ excitable units be coupled with each element receiving the mean input from an equal proportion of the remaining elements. Each element $i$ is modelled as a FitzHugh-Nagumo system influenced by noise; the network obeys the following system of equations:

$$
\epsilon \dot{x}_i = x_i - \frac{x_i^3}{3} - y_i + \gamma (M_{x_i} - x_i),
$$

$$
\dot{y}_i = x_i + a + \sqrt{2T} \xi_i(t),
$$

$$
M_{x_i} = \left\lfloor \frac{100}{(N-1)v} \right\rfloor \sum_{j=1}^{(N-1)v} x_{g(i,j)},
$$

Eqs. 6.1 couple units in accordance with their local mean-field, as opposed to previous studies that have observed coupling through the global mean-field [9, 10]; coupling through the local mean-field has the advantage of enabling neurons to be connected to a proportion of neurons within the population instead of the entire ensemble. In Eqs. 6.1, $\xi_i(t)$ is Gaussian white noise with zero mean, $\langle \xi_i(t) \rangle = 0$; values of $\xi_i(t)$ follow a normal distribution. It is assumed that the noise processes $\xi_i(t)$ are stationary and ergodic; therefore, time and ensemble averages coincide. Sources of noise in different neurons are uncorrelated. $\xi_i(t)$ could represent the random opening of ion channels in neurons (or other random events occurring at a neuronal level), stochastically changing the membrane conductivity [10]. Since the Gaussian white noise terms are additive, the Itô and Stratonovich interpretations of stochastic calculus are equivalent. The Itô method of stochastic integration is adopted in numerical simulations using a stochastic Runge-Kutta 4th order technique. $T$ corresponds to noise intensity and $\gamma$ is the fixed strength of coupling between neurons within the network. The scenario where coupling strengths are identical and fixed in time is referred to as homogeneous coupling; heterogeneous coupling, where coupling strengths can deviate from one another and evolve with time, is discussed in Chapter 8. The introduction of positive coupling strengths between a network of FitzHugh-Nagumo units,
irrespective of their size, do not alter the stability (Section 4.3) of the system’s fixed point in Eqs. 6.1 [10]. $\epsilon$ is a small number where $0 < \epsilon \ll 1$; it separates time-scales between the slow $y$-variable and fast $x$-variable. The slow $y$-time-scale is the system’s recovery and negative feedback mechanism, returning the phase point back to equilibrium, and $x$ represents the membrane potential. Each neuron in the network receives input from a percentage $v$ of other neurons. $N$ is the network size and $1 \leq g(i,j) \leq N$ is a random function taking integer values from a uniform distribution; this function determines the neurons that provide input to neuron $i$ (Fig. 6.1). $g(i,j) \neq i$ is a required condition, preventing neurons from connecting to themselves. All other neurons may be chosen as an input to neuron $i$ with equal probability, according to a random uniform distribution; therefore, all units are equally likely to be connected. Bidirectionally coupled networks require a reciprocal assignment of a connection between two neurons; such a condition is not imposed upon unidirectionally coupled networks where connections can be established in one direction only. In cases where the percentage of connectivity prevents all neurons from containing the same number of inputs, the number of connections is rounded to the nearest allocation. The number of inputs per neuron must be consistent to reduce the risk of behavioural discrepancies, which result from significant configuration changes rather than system connectivity alterations. Duplications of connections are also not valid, $g(i,k) \neq g(i,l)$ for $k \neq l$. $M_x$ is the mean of all inputs to neuron $i$ at a given time moment; the network coupling is introduced through the local mean-field. Brackets $[\ ]$ represent the nearest integer function giving the integer value closest to its argument [124].

In accordance with previous studies [9, 10], the parameter values $T = 3.1 \times 10^{-4}$, $\epsilon = 0.01$, and $a = 1.05$ are fixed throughout simulations, unless otherwise specified. $|a| > 1$ is necessary to ensure that uncoupled units do not spike without external perturbation; however, value $a = 1.05$ is close enough to 1 to perturb the dynamical system beyond the threshold, generating excitable spiking behaviour. The integration time-step is $h = 0.0005$ and points are recorded every 0.025 time units; for 100 neuron networks, simulations are performed...
Figure 6.1: Schematic representation of the random neuron numbers allocated \( g(i,j) \) as input for neuron \( i = 4 \) (dashed blue line) in a network of size \( N = 10 \). \( j \) denotes the order in which connections are assigned: \( j = 1 \) is the first connection established as input to neuron \( i \). The percentage of connectivity is \( v = 100 \), indicating that the maximum number of possible connections per neuron \( N - 1 \) are established. All \( g(i,j) \) are unique and \( g(i,j) \neq i \).

for 4,000 dimensionless time units; for larger networks, 1,000 dimensionless time units are simulated. Larger networks provide sufficient averaging of the ensemble of neurons, even with sparse connectivity, compensating for limited time-averaging. A relaxation period of 300 dimensionless time units is provided before recording measurements; this period settles the system’s dynamics, ignoring the trajectories caused by the arbitrary initial conditions given to each neuron. The initial conditions are set with \( x_i = -1 \) and \( y_i = -0.8 \), which are in the vicinity of the fixed point.

Unidirectionally and bidirectionally connected networks consisting of 100, 1,000, and 10,000 neurons are investigated using numerical simulations. Varying numbers of input connections per neuron are explored, examining the effects of coupling strength, connection density, system configuration, and network size parameters upon the global state. Understanding the impact of these parameters is imperative when making accurate interpretations and assumptions about the complex biological brain using the mathematical model. Each neuron receives an equal number of inputs to prevent any neuron from complete isolation and increase the likelihood of forming a single global network connecting all units at low connectivities. Reducing the likelihood of isolated neurons is important when assuming the brain should be modelled as a single entity.
Bidirectional connections can be formed in the unidirectionally coupled networks; they are particularly common in unidirectionally coupled networks at high connectivities as they are forced to exist if the network connectivity is greater than 50%. By nature, unidirectionally and bidirectionally connected networks with 100% connectivity are identical; greater disparity is expected between unidirectionally and bidirectionally coupled systems with low connectivity. No other constraints are imposed upon the network’s configuration; it is expected that networks may inherit other synchronisation-influencing characteristics, such as clusters and path lengths, to a varying degree with random variation (Section 3.2); thus, repeating simulations with the same number of connections and a different random generator sequence may yield conflicting results. However, it is assumed that any formed characteristics are partially attributed to the number of established connections, as they are naturally manifested but prone to deviation. Another assumption is that discrepancies minimally alter synchronisation capabilities. As connection strengths are fixed, either a connection is formed between two neurons, $i$ and $j$, or not; in the former case, the strength of the connection is $\gamma$; in the latter, no input is transferred without a connection. Hence, one can describe the probability density distribution (PDD) of coupling strengths in the network as a sum of two Dirac delta-functions (Fig. 6.2). The convention is to mark a delta peak with an arrowhead, stating the corresponding area under the curve; one peak is located at $\gamma = 0$ and the other is located at the designated system coupling strength. The magnitudes of delta peaks are determined by the network connectivity; low connectivity produces a large delta peak at $\gamma = 0$ and a small peak at the designated system coupling strength; high connectivity produces the opposing effect. In reality, connection strengths in biological systems vary from neuron to neuron and change with time, depending on their activity levels; however, for simplicity, equal and fixed coupling strengths are employed. The results are consistent for maximal network connectivity and previously-simulated mean-field coupling [9].

Eqs. 6.1 generate networks, which produce the properties described in Tab. 6.1. The percentage of connectivity (PC) reflects the proportion of established connections out of
Figure 6.2: PDD of coupling strengths $P_1(\gamma)$ over the ensemble of neurons in a 100 neuron network. (a) 30% connectivity (30 inputs per neuron). (b) 70% connectivity (69 inputs per neuron). All connections formed are homogeneous in strength ($\gamma = 0.1$ in this example). The area of each delta function is given by the number next to the arrowhead.

the network’s total capacity. The number of disconnected pairs (NDP) represents any neuronal pairs that do not establish a direct or indirect connection with one another. The average path length (APL) corresponds to the mean number of connections that are bypassed when transmitting signals between two neurons in the network using the shortest route (Section 3.2); out of all the shortest paths, the largest path length is the longest established path (LEP). The LEP could alternatively be viewed as the required number of transmission steps to guarantee indirect transmission of any elicited signal throughout the entire network or as many neurons the paths allow. The LEP has not been used in any previous studies to the authors knowledge.

The NDP provides a quantitative measure of the proportion of neurons with communicative abilities; if the NDP is zero, all neurons can interact (at least indirectly) and the network can be considered as a single entity; if the NDP is small, in comparison to the total number of neuronal pairs, most neurons can be considered as contributing to the global state.

APL reflects signal transmission speed throughout the network; lower APL values indicate rapid spread of information. If the NDP is not zero, the APL is infinite as at least one pair of neurons, which are not connected by a path of finite size, exists. The
LEP helps to identify the maximum number of neurons bypassed when traversing between the most distant neurons, excluding disconnected pairs; it also provides insight into chain and cluster size when disconnected pairs are present in the network.

From this research, it appears that disconnected neuronal pairs are prevalent in all networks when there is only one input per neuron; the APL is infinite with one input per neuron as the NDP is always large. In the bidirectional networks, with one input per neuron, the configuration is completely determined, forcing neurons into isolated pairs; as a result, the LEP is equal to one in all bidirectional cases. In the unidirectionally coupled network consisting of 100 neurons, the LEP is eighteen when there is one input per neuron; this is suggestive of a cluster consisting of at least nineteen neurons. Increasing the network size with this level of connectivity increases the LEP’s absolute value; however, the relative quantity decreases in comparison to the network size decreases. Although the number of allocated connections propagate with the increased network size, the probability of allocating a connection to a particular cluster is reduced.

Two inputs per neuron are sufficient to generate networks without disconnected pairs when the system is unidirectionally coupled and the number of neurons is 100 or 1,000. The APL is higher in the larger network since the increased number of cells decreases the probability of a direct connection with the target neuron; the LEP decreases as a larger network has more connections and all units belong to the same (only) cluster. When all connections are bidirectional, some neuronal pairs disconnect when there are two inputs per neuron due to insufficient diversity of connection arrangement. In networks consisting of 10,000 neurons, more inputs per neuron are required to prevent disconnection of neuronal pairs; the probability of a connection being assigned to a particular neuron decreases; therefore, a larger sample in relation to the network’s size guarantees the selection of all units. Following the removal of all disconnected neuronal pairs, the LEP and APL decrease when the connectivity increases. In the unidirectionally coupled 100 neuron network, 20% or higher connectivity allows any two neurons to communicate directly or bypass only one intermediate neuron; the network has an LEP equal to two which is the
Table 6.1: Network configuration results: (a) Unidirectional 100 neuron network, (b) bidirectional 100 neuron network, (c) unidirectional 1,000 neuron network, (d) bidirectional 1,000 neuron network, (e) unidirectional 10,000 neuron network and (f) bidirectional 10,000 neuron network. The number of possible network connections is given by \( N(N - 1) = 9,900 \) for 100 neurons, 999,000 for 1,000 neurons and 99,990,000 for 10,000 neurons. PC = Percentage of Connectivity, NC = Number of Connections in the Network, NDP = Number of Disconnected Pairs, APL = Average Path Length, LEP = Longest Established Path.

<table>
<thead>
<tr>
<th>PC</th>
<th>NC</th>
<th>NDP</th>
<th>APL</th>
<th>LEP</th>
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</thead>
<tbody>
<tr>
<td>1%</td>
<td>100</td>
<td>8,883</td>
<td>∞</td>
<td>18</td>
</tr>
<tr>
<td>2%</td>
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minimum possible value for a fully formed sub-maximally connected network. The term “fully formed” denotes connections between all neuronal pairs (at least indirectly). The number of intermediary neurons utilised decreases when connectivity further increases. When the configuration is bidirectional, higher connectivity is required to achieve a fully formed sub-maximally connected network with a minimal LEP; the APL becomes identical to the unidirectionally coupled equivalent networks. Upon an increase in network size, a larger number of inputs per neuron is necessary to generate a system where the LEP is equal to two; however, the required connectivity percentage is reduced.

Network behaviours can often be explained by the underlying structure and arrangement of the units they contain. As such, the properties described above allow descriptive analysis of the networks studied to supplement numerical calculations. It is possible that there are other factors contributing to network dynamics that are not captured by the above properties; further studies of network behaviours will aid the identification of such factors, allowing suitable methods of detection to be devised.

6.2 Homogeneous Coupling Results

The stochastic neural network’s global synchronisation behaviour in Eqs. 6.1 can be measured using the variance $\sigma^2_{M_x}$ of the mean-field $M_x$, calculated from a single realization as

$$\sigma^2_{M_x} = \frac{1}{L} \sum_{i=1}^{L} M_x^2(t_i) - \left( \frac{1}{L} \sum_{i=1}^{L} M_x(t_i) \right)^2.$$  

$L$ denotes the number of discrete time moments when measurements are recorded. The typical behaviour of $\sigma^2_{M_x}$ is displayed in Fig. 6.9; values of $\sigma^2_{M_x}$ close to zero indicate the insignificant oscillation of the global mean-field, depicting asynchronous behaviour or the absence of individual neuronal spiking. Large values of $\sigma^2_{M_x}$ show significant oscillations in the global mean-field, suggesting the presence of synchrony; actual recordings of $\sigma^2_{M_x}$ for systems with various connectivities are provided in Figs. 6.3 and 6.4. Results with 100% connectivity are consistent with those observed previously for systems coupled through
Figure 6.3: Variance $\sigma^2_{Mx}$ of the mean-field as a function of coupling strength $\gamma$ in unidirectionally coupled networks obeying Eqs. 6.1 with different combinations of size $N$ and connectivity percentage $v$. (a) $N = 100$, $v = 1 - 10$ (1 - 10 inputs per neuron). (b) $N = 100$, $v = 10 - 100$ (10 - 99 inputs per neuron). (c) $N = 1,000$, $v = 0.1 - 1$ (1 - 10 inputs per neuron). (d) $N = 1,000$, $v = 1 - 10$ (10 - 100 inputs per neuron). (e) $N = 1,000$, $v = 10 - 100$ (100 - 999 inputs per neuron). (f) $N = 10,000$, $v = 0.01 - 1$ (1 - 100 inputs per neuron).
Figure 6.4: Variance $\sigma^2_{\bar{M}_x}$ of the mean-field as a function of coupling strength $\gamma$ in bidirectionally coupled networks obeying Eqs. 6.1 with different combinations of size $N$ and connectivity percentage $v$. (a) $N = 100$, $v = 1 - 10$ (1 - 10 inputs per neuron). (b) $N = 100$, $v = 10 - 100$ (10 - 99 inputs per neuron). (c) $N = 1,000$, $v = 0.1 - 1$ (1 - 10 inputs per neuron). (d) $N = 1,000$, $v = 1 - 10$ (10 - 100 inputs per neuron). (e) $N = 1,000$, $v = 10 - 100$ (100 - 999 inputs per neuron). (f) $N = 10,000$, $v = 0.01 - 1$ (1 - 100 inputs per neuron).
the mean-field \[9\].

One can represent the output of a neuron, \( \sigma(t) \), as a train of delta spikes,

\[
\sigma(t) = \sum_{i=1}^{L} \delta(t - t_i),
\]

(6.3)

where \( t_i \) corresponds to the time of each particular spike \[8\]. The spike count, \( n(t) \), is the number of action potentials elicited within a window, \((0, t)\). The mean firing rate, \( \langle r \rangle \), is the number of action potentials elicited per time unit and is retrieved by dividing the spike count by the length of the time window:

\[
\langle r \rangle = \frac{n(t)}{t}.
\]

(6.4)

An interspike interval, \( \psi \), is the time between successive suprathreshold spikes; \( \psi \) is the time the phase point spends in both active and refractory states (excursion time) and the time taken to sufficiently perturb the phase point from its equilibrium (activation time). The mean interspike interval \( \langle \psi \rangle \) is the average length of time between consecutive action potentials \[8\]; it is given by the inverse of the mean firing rate:

\[
\langle \psi \rangle = \frac{1}{Z} \sum_{i=1}^{Z} (t_{i+1} - t_i)
= \frac{1}{Z} \sum_{i=1}^{Z} \psi
= \frac{1}{\langle r \rangle}
= \frac{t}{n(t)},
\]

(6.5)

where \( Z \) corresponds to the number of interspike intervals. The variance of the interspike interval, \( \sigma_{\psi}^2 \), can be calculated as

\[
\sigma_{\psi}^2 = \langle \psi^2 \rangle - \langle \psi \rangle^2.
\]

(6.6)
Figure 6.5: Mean interspike interval $\langle \psi \rangle$ of the mean-field as a function of coupling strength $\gamma$ in unidirectionally coupled networks obeying Eqs. 6.1 with different combinations of size $N$ and connectivity percentage $v$. (a) $N = 100$, $v = 1 - 10$ (1 - 10 inputs per neuron). (b) $N = 100$, $v = 10 - 100$ (10 - 99 inputs per neuron). (c) $N = 1,000$, $v = 0.1 - 1$ (1 - 10 inputs per neuron). (d) $N = 1,000$, $v = 1 - 10$ (10 - 99 inputs per neuron). (e) $N = 1,000$, $v = 10 - 100$ (100 - 999 inputs per neuron). (f) $N = 10,000$, $v = 0.01 - 1$ (1 - 100 inputs per neuron).
Figure 6.6: Mean interspike interval $\langle \psi \rangle$ of the mean-field as a function of coupling strength $\gamma$ in bidirectionally coupled networks obeying Eqs. 6.1 with different combinations of size $N$ and connectivity percentage $v$. (a) $N = 100$, $v = 1 - 10$ (1 - 10 inputs per neuron). (b) $N = 100$, $v = 10 - 100$ (10 - 99 inputs per neuron). (c) $N = 1,000$, $v = 0.1 - 1$ (1 - 10 inputs per neuron). (d) $N = 1,000$, $v = 1 - 10$ (10 - 99 inputs per neuron). (e) $N = 1,000$, $v = 10 - 100$ (100 - 999 inputs per neuron). (f) $N = 10,000$, $v = 0.01 - 1$ (1 - 100 inputs per neuron).
Figure 6.7: Interspike interval variance $\sigma^2_\psi$ of the mean-field as a function of coupling strength $\gamma$ in unidirectionally coupled networks obeying Eqs. 6.1 with different combinations of size $N$ and connectivity percentage $v$. (a) $N = 100$, $v = 1 - 10$ (1 - 10 inputs per neuron). (b) $N = 100$, $v = 10 - 100$ (10 - 99 inputs per neuron). (c) $N = 1,000$, $v = 0.1 - 1$ (1 - 10 inputs per neuron). (d) $N = 1,000$, $v = 1 - 10$ (10 - 100 inputs per neuron). (e) $N = 1,000$, $v = 10 - 100$ (100 - 999 inputs per neuron). (f) $N = 10,000$, $v = 0.01 - 1$ (1 - 100 inputs per neuron).
Figure 6.8: Interspike interval variance $\sigma^2_\psi$ of the mean-field as a function of coupling strength $\gamma$ in bidirectionally coupled networks obeying Eqs. 6.1 with different combinations of size $N$ and connectivity percentage $v$. (a) $N = 100, v = 1 - 10$ (1 - 10 inputs per neuron). (b) $N = 100, v = 10 - 100$ (10 - 99 inputs per neuron). (c) $N = 1,000, v = 1 - 10$ (1 - 10 inputs per neuron). (d) $N = 1,000, v = 10 - 100$ (10 - 100 inputs per neuron). (e) $N = 1,000, v = 0.1 - 1$ (1 - 10 inputs per neuron). (f) $N = 10,000, v = 0.01 - 1$ (1 - 100 inputs per neuron).
It is possible to consider mean interspike intervals and interspike interval variance values for individual neurons or the mean-field. The quantities \( \langle \psi \rangle \) and \( \sigma^2_\psi \) are used here to depict the mean interspike interval and interspike interval variance of the mean-field. Spikes are detected in numerical simulations by zero crossings of the mean-field. When there is an absence of spiking behaviour, the mean interspike interval, \( \langle \psi \rangle \), and interspike interval variance, \( \sigma^2_\psi \), of the mean-field do not exist; thus the length of time between spikes is infinite. Smaller variances in the interspike interval of the mean-field indicate a more unified and rhythmical spiking behaviour, leading to enhanced synchronisation. The actual recordings for the mean interspike interval, \( \langle \psi \rangle \), in the numerical simulations are displayed in Figs. 6.5 and 6.6; measurements for the variance of the mean-field interspike interval are provided in Figs. 6.7 and 6.8.

![Figure 6.9: Typical behaviour of the mean-field variance \( \sigma^2_\psi \) in Eqs. 6.1 as a function of coupling strength \( \gamma \). The general shape is qualitatively the same for any network connectivity if the number of inputs per neuron is larger than one.](image)

A general pattern occurs across all simulations with more than one input per neuron as the coupling strength \( \gamma \) increases; in the uncoupled and very low connection strength scenarios, individual neurons fire with moderate frequency. Neurons spike according to
Figure 6.10: Illustration of a unidirectionally coupled stochastic network from Eqs. 6.1 with 90% connectivity (89 inputs per neuron) where \( \gamma = 0.02 \) and \( N = 100 \). (a) Mean-field realization. (b) Mean-field phase portrait. (c) and (d) Realizations of individual neurons within the network displaying contrasting firing frequencies. (e) and (f) Corresponding individual neuron phase portraits.

their individual rhythms, causing the network to be out of synchrony (stage I in Fig. 6.9). Slightly larger values of \( \gamma \) elevate the individual neuron’s firing frequency; neurons begin to entrain to other neurons’ frequencies as the coupling strength increases (stage II in Fig. 6.9). Upon saturation, almost all neurons continuously spike with aligned timings (stage III in Fig. 6.9). Increasing \( \gamma \) eventually leads to a gradual reduction in the individual neuron’s firing rate and their coordinated spike timings (stage IV in Fig. 6.9); beyond certain coupling strength values, most individual neuronal spiking ceases, which is the most prominent factor in the attenuation of synchronised activity (stage V in Fig. 6.9).

The major differences between networks of different sizes, configuration, and connectivity are the quantitative ranges of coupling strength exhibiting these behaviours.

An illustration comparing global network behaviour to individual units is given in Fig. 6.10 for \( N = 100 \) and 90% connectivity, where all units are coupled with strength \( \gamma = 0.02 \). The mean-field, \( \bar{M}_x \), remains relatively constant, only producing very small subthreshold oscillations; the realization for the mean-field does not show any spike generation or limit cycle phase trajectories (Fig. 6.10 (a), (b)). Mean-field trajectories remain within close vicinity of an individual unit’s fixed point (Section 4.3). Neurons are weakly coupled, independently firing according to their own rhythms, and unable to influence other
neurons’ activities (Fig. 6.10 (c) - (f)). All units continue this behaviour for coupling strengths $\gamma < 0.03$; the mean-field variance, $\sigma^2_{\psi}$, is small, whereas the mean interspike interval, $\langle \psi \rangle$, and interspike interval variance, $\sigma^2_{\psi}$, of the mean-field do not exist; the time between spikes is considered to be infinitely long. The strength of coupling is insufficient in allowing neurons to significantly influence another neuron’s behaviour, and entrain to a common firing frequency; weakly coupled and uncoupled networks behave in a similar fashion to each other.

For one input per neuron, there is little behavioural change for weak and no coupling, regardless of increases to coupling strength. Although mean-field variance slightly increases at various coupling strengths, synchronous behaviour is absent due to a large proportion of disconnected neurons (Section 6.1); this absence of synchrony is verified by the lack of plateau in the mean interspike interval of the mean-field, compared to networks of greater connectivity, and a non-zero interspike interval variance throughout the range of tested coupling strengths. The absence of direct or indirect connections between neurons causes the APL to become infinite; a large number of disconnected units indicates many neurons’ inabilities to influence other neurons’ firing patterns; consequently, many small clusters of neurons form within the network, synchronising to rhythms of different phases or firing frequencies. Evidence of the developed clusters is given by many small but
Figure 6.12: Illustration of a unidirectionally coupled stochastic network from Eqs. 6.1 with 0.02% connectivity (two inputs per neuron), where $\gamma = 0.03$ and $N = 10,000$. (a) Mean-field realization. (b) Mean-field phase portrait. (c) and (d) Realizations of individual neurons within the network displaying matching rhythms. Spike timings in mean-field realizations coincide with those displayed by the individual neurons.

distinct mean-field limit cycle trajectories (Fig. 6.11); segregated clusters appear for networks of all sizes with one input per neuron, though smaller networks display favourable synchronisation performance.

For bidirectionally connected networks, the stipulation of one input per neuron completely determines the network’s configuration since only pairs of neurons can form connections; thus, performance is drastically impaired and neuronal pairs often synchronise to unique rhythms. The simulated results for one input per neuron show the necessity of comparison of simulations by number of inputs per neuron, as opposed to relative percentage connectivity compared to the network size; this is due to an additional input per neuron prompting globally synchronised behaviour in all tested network sizes. Smaller networks outperform larger networks, for one input per neuron, as the probable number of disconnected clusters is reduced; the network’s number of conflicting rhythms also decreases, aiding synchronisation magnitude; strong synchrony may be possible at one input per neuron, if the network size is small enough. Intuitively, if the network size is $N = 2$ and there is one input per neuron, each neuron must be connected; they would be influencing each other and synchronise, given a sufficient and reasonable coupling strength; thus, no other conflicting rhythms would affect global synchronisation. One could experi-
ment with small network sizes to discover if, and at which network sizes, synchronisation can be achieved with one input per neuron; smaller networks may be easier for achieving synchrony as it is easier to get two people to clap their hands simultaneously than 10,000 (when each person can only see one other individual).

As few as two inputs per neuron sufficiently develop strong synchrony in unidirectionally coupled networks of all sizes (Fig. 6.12); the critical coupling strength for inducing synchronisation is \( \gamma \approx 0.03 \). The onset of synchrony is signified by a sharp increase in mean-field variance, indicating the simultaneous firing of many neurons; individual neurons spike in an almost continuous manner, showing a marginally elevated firing rate. In the unidirectionally coupled 100 neuron network with 2% connectivity (two inputs per neuron), global synchronisation is marginally hindered by a high APL and LEP; the same is true for two inputs per neuron (0.2% connectivity) with 1,000 neurons. For the 100 and 1,000 neuron bidirectionally connected networks, the impact is much more severe due to many units disconnecting; there are many disconnected neuronal pairs for 0.02% and 0.03% connectivity (two and three inputs per neuron) in the unidirectionally coupled 10,000 neuron network. Disconnected pairs of neurons, leading to an infinite APL (Section 6.1), marginally affects synchronisation since there are few disconnections in comparison to the numerous neuronal pair combinations. The many established con-
Figure 6.14: Illustration of a unidirectionally coupled stochastic network from Eqs. 6.1 with 90% connectivity (89 inputs per neuron) where $N = 100$. (a) - (b) $\gamma = 0.03$. (a) Mean-field realization. (b) Mean-field phase portrait. (c) - (d) Corresponding results for $\gamma = 0.04$.

Connections and synchronised units in the main cluster significantly outweigh the quantity of disconnected units; in bidirectional networks of this size, synchronisation is further impaired as the NDP is greater at any given network connectivity. Even when the number of disconnected neuron pairs is relatively low, in comparison to the number of established (connections 0.04 - 0.05% connectivity), the reductions in synchrony are more pronounced in the bidirectionally coupled system; this could be due to the requirement of a larger number of neuron connections to produce strong global synchrony in bidirectionally coupled systems as opposed to unidirectionally connected networks. A larger number of connections is required to generate strong synchrony since the bidirectional configuration reduces the diversity in the connection distribution; fewer neurons are capable of sufficient interaction to entrain the network to a common frequency of oscillations.

For connectivities 3 – 7%, the mean-field’s variance noticeably increases when the connection strength reaches $\gamma = 0.03$, most notably for the lower percentages. In addition to synchronisation, a periodic orbit manifests in the mean-field phase portrait and there is a non-zero plateau in the mean interspike interval, displaying little variance; little difference exists between the plateau values attained by neurons in networks of different connectivities. Neurons with higher connectivity tend to settle at slightly greater mean interspike interval values, spiking less frequently on average.

Increasing system connectivity eventually shifts synchronisation onset to a higher coupling strength ($\gamma \approx 0.04$) despite the increased connectivity decreasing the APL between neurons; a reduced APL increases neuronal interaction which could be expected to enable easier synchronisation at lower coupling strengths. However, the simulated results sug-
suggest the opposite: an increase in neuronal interactions eventually makes synchronisation more difficult to induce with coupling strength. A similar effect is also observed with the bidirectional configuration (Fig. 6.13); for higher connectivities, increasing network size induces synchronisation at $\gamma \approx 0.03$, not $\gamma \approx 0.04$. Prior to achieving full synchronisation, an intermediate state of chaotic synchronisation occurs. During chaotic synchronisation, a constantly fluctuating number of neurons simultaneously fire; graphical representations depict various peak amplitudes in the mean-field realization and a smeared limit cycle in phase portraits (Fig. 6.14).

Synchronisation strength initially increases upon an increase in coupling strength; after a peak value occurs, the degree of synchronisation diminishes; at this peak, the mean-field and individual neurons display their fastest and most periodic spike trains. Further increments in coupling strength lead to a complete loss of synchronisation. Transitions between synchrony onset and demise are more abrupt in larger networks, reducing parameter windows observing chaotic synchronisation (Fig. 6.15). Synchrony decrements are always attributed to the coupling-induced noise reduction phenomenon [125]: with strong coupling, the interaction between individual units binds the network to the the fixed point [10]. Decreased noise results in less neuronal stimulation and more subthreshold inputs; this increase causes the individual neuron’s firing rate to attenuate and eventually cease
Figure 6.16: Illustration of stochastic networks from Eqs. 6.1 with 40% connectivity (forty inputs per neuron) where $\gamma = 0.19$ and $N = 100$. (a) - (d) Unidirectional coupling. (a) Mean-field realization. (b) Mean-field phase portrait. (c) and (d) Realizations of different individual neurons. (e) - (h) Corresponding results for bidirectional coupling.

(Figs. 6.16 and 6.17). The mean interspike interval of the mean-field escalates back towards infinity as firing rate and synchrony attenuate; this is also common of the interspike interval variance of the mean-field. Higher connected networks begin to experience these effects at the lowest coupling strength values (Fig. 6.18); desynchronisation at large coupling strengths is completely different to an absence of synchronisation in uncoupled networks. Desynchronisation refers to the absence of neuronal firing; absence of synchronisation is caused by individual neurons firing rapidly firing to their own rhythms. In some cases, especially at low connectivities, an increased realization duration may need to be examined to confirm whether synchronisation and neuronal firing have completely attenuated; exploring further coupling strength values could attest that synchronisation does not re-emerge.

In spite of simulations with one input per neuron never achieving strong synchronisation, a non-zero mean-field variance may be maintained for the largest coupling strength values. Sparsely connected networks are noisier than higher connected systems as more variable input values are expected, decreasing coupling-induced noise reduction effects; greater coupling strengths are necessary to completely subdue individual neuron firing at lower connectivities. If a network can achieve synchronisation, the lowest value of connectivity is optimal for maintaining synchrony at the largest coupling strength values;
the largest path length seems to prolong synchronisation regarding coupling strength. Alternatively, the law of large numbers suggests the received average of input values by each neuron are more consistent with an increased number of inputs; this is indicated by the system’s reduction in noise. Larger networks are more robust against the effects of coupling-induced noise reduction at low connectivities than smaller networks, but less robust at high connectivities; larger networks vary in noise depending on network connectivity. Random fluctuations are more likely to occur at low connectivities in larger networks; the law of large numbers is more prominent at high connectivities, eliminating noise imparted to neurons through mean inputs. Consequently, larger networks with high connectivity are more susceptible to the effects of coupling-induced noise reduction at lower coupling strength values than smaller networks; larger networks with low connectivity are less vulnerable to coupling-induced noise reduction effects than smaller systems.

Providing synchrony can be achieved, a larger network with low connectivity is more beneficial than smaller networks for synchronisation; more neurons can be entrained to the same firing frequency and the few disconnected neurons are insignificant. A crossover point occurs at mid-range number of inputs per neuron; the benefits and disadvantages of increasing network size upon synchronisation performance are counterbalanced. The
results indicate that the crossover point is close to 100 inputs per neuron in networks of 100, 1,000, and 10,000 neurons (Fig. 6.19).

The bidirectional configuration increases the network’s robustness and resistance to neuronal firing and desynchronisation, which are caused by coupling-induced noise reduction; increased robustness to noise reduction could be a result of the configuration forcing a feedback mechanism into the system. Neuron A’s firing induces neuron B to transmit a signal back to it (and any other neurons receiving its output), which encourages neuron A to spike again; a transmission loop is produced, encouraging the neuronal pair to continuously fire at an increased frequency on a much shorter time-scale. The pair’s connections to other neurons enable communication and modification of firing rates to a common frequency; although bidirectional configuration makes whole network integration more difficult to achieve, local cyclic behaviour is easily developed.

### 6.3 Discussion

Stochastic ensembles of FitzHugh-Nagumo elements are investigated using numerical simulations; network characteristics such as connectivity, configuration, and size are explored to determine the influence imparted upon global synchronisation generation in their re-
Figure 6.19: Variance $\sigma^2_{N_\gamma}$ of the the mean-field as a function of coupling strength $\gamma$ in unidirectionally coupled networks obeying Eqs. 6.1 with different combinations of size $N$ and connectivity percentage $v$. At 1 input per neuron, $\sigma^2_{N_\gamma}$ is largest in smaller networks. At 10 inputs per neuron, $\sigma^2_{N_\gamma}$ is largest in larger networks. At 100 inputs per neuron, $\sigma^2_{N_\gamma}$ has a similar pattern for $N = 100, 1,000$ and 10,000.

Spective homogeneously coupled systems. Two inputs per neuron sufficiently generates strong synchronisation, provided an adequate coupling strength is chosen. Synchronisation can be achieved if the number of neurons connected in the main cluster severely outweighs the disconnected neurons. In homogeneously coupled networks, the studied parameters have a greater impact upon the attenuation of synchrony than its onset; provided that synchronisation is possible, a longer APL leads to a greater range of coupling strength for synchronisation. The number of inputs received by each neuron is more prominent when determining the coupling strength range for synchronisation than the relative proportion of inputs regarding network size.

Coupling-induced noise reduction invokes attenuation of individual neuronal firing at large coupling strength values, resulting in desynchronisation; desynchronisation at large coupling strengths is completely different to the absence of synchronisation in uncoupled or weakly coupled networks. In the former case, desynchronisation is a result of the absence of neuronal firing; in the latter, absence of synchronisation is caused by the individual neuron’s rapid fire to their own rhythms. Bidirectional configuration imparts
some resistance to the effects of coupling-induced noise reduction.

Realistic coupling strengths of biological brains can be deduced from the qualitative robustness of synchronisation across the coupling strength plane; the synaptic homeostasis hypothesis (SHH), which observes that synaptic connections decrease in strength and number during sleep cycles and increase during wakefulness, also helps to determine realistic coupling strengths (Appendix 11.7). Given that neural activity is largely asynchronous during wakefulness and more synchronous during sleep, the possible range of coupling strength parameters is significantly narrowed. Only one phase on the synchronisation and coupling strength plane can simultaneously achieve the above factors (stage IV in Fig. 6.9); the wakeful state lies on the downslope immediately preceding complete loss of neural firing and synchronisation. The exact coupling strength range depends on the network size and connectivity; increasing the number of inputs per neuron shifts the synchronisation range to lower values of coupling strength. When there is 100% connectivity in the unidirectionally coupled 1,000 neuron network, the synchronisation downslope is located at approximately $0.09 \leq \gamma \leq 0.125$. Biological brains have significantly more inputs per neuron on average ($\approx 7,000$), which should shift the synchronisation range towards lower coupling strength values. The shift is expected to be small in magnitude, given that shift sizes decrease with increasing connectivity; the shifts are already small in magnitude even at 1,000 inputs per neuron.

The unexplained heterogeneities should counteract and shift the coupling strength range to higher values. The actual gradient of the downslope parameter range is largely unaffected by connectivity, network size, and heterogeneities in this investigation; upon simulating a more realistic number of neurons in the network, there may not be a significant difference in synchronisation range. Real-world values should lie in the mid-range segment of the downslope since artificial neuronal firing, at larger values of coupling strength, is too infrequent to represent biological behaviour; at smaller values of coupling strength, synchronisation is stronger than the expected value for healthy individuals. Higher degrees of synchronisation in patients with epilepsy and Parkinson’s disease may result from
malfunctioning synaptic homeostasis mechanisms, where the quantity or strength of connections are excessively downscaled; this is illustrated when epileptic seizures occur more frequently during sleep than wakefulness (Appendix 11.7 and 11.9). In addition, seizures occurring during wakefulness could result from alterations to specific connections during sleep, which are only utilised during waking periods.
Chapter 7

Control of Synchronisation in Homogeneously Coupled FitzHugh-Nagumo Networks

7.1 Global Delayed Feedback Model

Although network behaviour under normal time progression has been observed, it remains unknown whether synchronisation can be modified to evoke more desirable responses; it should be possible to improve or destroy synchrony due to its associations with enhanced information processing capabilities [2, 3, 4, 5] and neurological disorders such as epilepsy and Parkinson’s disease [6, 7]. A mechanism able to sufficiently alter amplitude, timescale, or regularity of oscillations is required to increase or suppress synchrony; the desired control method is non-invasive and it is not necessary to modify network structure, configuration, or connections since procedures are are unlikely to be able to safely adapt these in practical real-world scenarios. The method employed in this investigation is delayed feedback control [121, 126]; an additional controlling signal is applied to the system’s elements; this signal is usually proportional to the difference between the system’s current and previous state. Previous research has shown that delayed feedback can control synchronisation by adjusting the system’s dynamics for coupled neurons [127] and networks coupled through the mean-field [9]; thus, focus has been directed to sparsely connected systems. Delayed feedback is applied to unidirectionally connected networks of various connectivities for 100 and 1,000 neurons; global feedback is applied ensuring all neurons
receive feedback proportional to the difference between the system’s current mean-field value and its value from \( \tau \) moments ago. Eqs. 6.1 can be modified to include global feedback:

\[
\epsilon \dot{x}_i = x_i - \frac{x_i^3}{3} - y_i + \gamma (M_{x_i} - x_i),
\]

\[
\dot{y}_i = x_i + a + K(M_{y\tau} - M_y) + \sqrt{2T} \xi_i(t),
\]  

(7.1)

\[
M_{x_i} = \left[ \frac{100}{(N - 1)\nu} \right] \sum_{j=1}^{(N-1)\nu} x_{g(i,j)}, \quad M_y = \frac{1}{N} \sum_{i=1}^{N} y_i.
\]

The additional term in the \( y \)-variable equation corresponds to delayed feedback, which has been introduced for consistency with previous studies [9, 120]. \( K \) represents the feedback strength; \( \tau \) is the time delay; \( M_y \) signifies the mean-field of variable \( y \) at the current time moment; \( M_{y\tau} \) corresponds to its value \( \tau \) moments ago. Feedback strengths are fixed to explore the \((\tau, \gamma)\) plane using a relatively weak feedback strength, \( K = 0.3 \), and a stronger value, \( K = 1 \). Colour coded mappings and surface plots (Figs. 7.1 - 7.8) have been generated by data interpolation from simulations of the various coupling strength values, \( \gamma \), and system time delays, \( \tau \); data points are gathered for incremental time delays of 0.2 between values 0 - 3.8; recorded coupling strength values are 0.01, 0.03, 0.04, 0.10, 0.15, and 0.19. Although gathering more data points yields a more accurate profile, the computational time would be exhaustive; the current amount of data sufficiently detects the major gradient transitions. Differences between simulations with altered connectivity and delayed feedback strength are also compared. Blue and black regions represent high mean-field variance and strong synchronisation; yellow regions denote an absence of mean-field spiking, indicating asynchronous behaviour; intermediate colours refer to partial (chaotic) synchrony states. The \( \tau = 0 \) state does not show any neuronal firing at high connectivity and coupling strengths; the concept of synchronisation is not applicable as there are no comparable rhythms in the system. If individual units do not fire or oscillate then feedback, which is the system’s delayed firing, remains constant and does not promote firing; delayed feedback cannot initiate neuronal firing or synchronisation when individual neurons do not spike.
7.2 Global Delayed Feedback Results

For one input per neuron, delayed feedback cannot create full network synchrony (blue or black regions) in the simulated range of parameters; however, it is possible to achieve strong chaotic synchronisation from asynchrony (Fig. 7.9). The largest increases in synchronisation are invoked when \( K = 1 \); complete destruction of synchronisation is only possible with large networks \((N = 1,000)\) and strong feedback values \((K = 1)\). Even strong feedback requires high values of coupling strength \((\gamma > 0.12)\) and precise values of time delay for synchrony destruction to be achieved. Further investigation is required to discover the critical parameter values of \(N\) and \(K\) that enable complete loss of synchronisation and explain pattern of synchrony destruction.

At low connectivity (excluding one input per neuron), where synchrony is initially \((\tau = 0)\) high for a broad range of coupling strength values, a larger region of parameter values is available to moderately decrease synchrony. The region of initial synchrony becomes narrower as connectivity increases; this reduces available parameter options decreasing synchrony. The magnitude of synchronisation loss, induced by delayed feedback, increases with larger connectivity at certain time delay values; this is shown by areas of moderate synchrony overtaken by regions of lower synchrony at intermediate time delay values between peaks of synchronisation; this intrusion begins at low coupling strengths and progresses through higher values as network connectivity elevates. With enough connectivity, regions of lower synchrony could fully penetrate the area of moderate synchrony; consequently, multiple regions of strong synchronisation would form and be divided by asynchronous boundaries. The segregation of synchronisation regions is especially likely in larger networks accommodating larger numbers of connections and potentially increases the ability to manipulate synchronisation. In the studied parameter ranges, complete destruction of synchronisation is impossible when the initial state shows strong synchronisation; however, weak forms of chaotic synchronisation are achievable (Fig. 7.10). Total destruction of synchrony may be possible at higher connectivities since the boundaries
Figure 7.1: Illustrations of the effects of delayed feedback in stochastic neural networks described by Eqs. 7.1 with sparse coupling $v = 1 - 50$ and $N = 100$. All neurons receive global feedback with strength $K = 0.3$ that is proportional to the difference between the current mean-field value of the system and its value from $\tau$ moments ago. The mean-field variance $\sigma_{\mathcal M}^2$ is given as a function of the time delay $\tau$ and coupling strength $\gamma$. 
Figure 7.2: Illustrations of the effects of delayed feedback in stochastic neural networks described by Eqs. 7.1 with sparse coupling $v = 1-50$ and $N = 100$. All neurons receive global feedback with strength $K = 0.3$ that is proportional to the difference between the current mean-field value of the system and its value from $\tau$ moments ago. The mean-field variance $\sigma^2_{M_x}$ is given as a function of the time delay $\tau$ and coupling strength $\gamma$. Representation of the surface in Fig. 7.1.
Figure 7.3: Illustrations of the effects of delayed feedback in stochastic neural networks described by Eqs. 7.1 with sparse coupling $\nu = 1 - 50$ and $N = 100$. All neurons receive global feedback with strength $K = 1$ that is proportional to the difference between the current mean-field value of the system and its value from $\tau$ moments ago. The mean-field variance $\sigma^2_{\mathcal{M}_x}$ is given as a function of the time delay $\tau$ and coupling strength $\gamma$. 

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Figure 7.4: Illustrations of the effects of delayed feedback in stochastic neural networks described by Eqs. 7.1 with sparse coupling $v = 1 - 50$ and $N = 100$. All neurons receive global feedback with strength $K = 1$ that is proportional to the difference between the current mean-field value of the system and its value from $\tau$ moments ago. The mean-field variance $\sigma^2_{Mx}$ is given as a function of the time delay $\tau$ and coupling strength $\gamma$. Representation of the surface in Fig. 7.3.
Figure 7.5: Illustrations of the effects of delayed feedback in stochastic neural networks described by Eqs. 7.1 with sparse coupling $v = 0.1 - 10$ and $N = 1,000$. All neurons receive global feedback with strength $K = 0.3$ that is proportional to the difference between the current mean-field value of the system and its value from $\tau$ moments ago. The mean-field variance $\sigma^2_{Mx}$ is given as a function of the time delay $\tau$ and coupling strength $\gamma$. 
Figure 7.6: Illustrations of the effects of delayed feedback in stochastic neural networks described by Eqs. 7.1 with sparse coupling $v = 0.1 \sim 10$ and $N = 1,000$. All neurons receive global feedback with strength $K = 0.3$ that is proportional to the difference between the current mean-field value of the system and its value from $\tau$ moments ago. The mean-field variance $\sigma^2_{Mx}$ is given as a function of the time delay $\tau$ and coupling strength $\gamma$. Representation of the surface in Fig. 7.5.
Figure 7.7: Illustrations of the effects of delayed feedback in stochastic neural networks described by Eqs. 7.1 with sparse coupling $v = 0.1 - 10$ and $N = 1,000$. All neurons receive global feedback with strength $K = 1$ that is proportional to the difference between the current mean-field value of the system and its value from $\tau$ moments ago. The mean-field variance $\sigma_{M_x}^2$ is given as a function of the time delay $\tau$ and coupling strength $\gamma$. 

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Figure 7.8: Illustrations of the effects of delayed feedback in stochastic neural networks described by Eqs. 7.1 with sparse coupling $v = 0.1 - 10$ and $N = 1,000$. All neurons receive global feedback with strength $K = 1$ that is proportional to the difference between the current mean-field value of the system and its value from $\tau$ moments ago. The mean-field variance $\sigma_M^2$ is given as a function of the time delay $\tau$ and coupling strength $\gamma$. Representation of the surface in Fig. 7.7.
surrounding synchronisation summits are likely to be distinguished. Significant loss of 
synchronisation is easier to achieve at feedback strength $K = 0.3$ (except for one input 
per neuron) since synchronisation strength fluctuates less with strong feedback ($K = 1$). 
For $K = 1$, there is a decrease in regions of the strongest synchronisation (blue) at high 
values of time delay; this is most evident at low connectivities.

The opposite is not completely valid. At high connectivities where the region of initial 
synchrony is narrow, there is a very small region of parameter values available for which 
synchrony can be greatly increased. However, in this case, extremely low synchronisation 
values can be converted into very high values. Substantial gains are limited to low 
coupling strengths according to the data acquired here. The greatest gains are restricted 
to networks with high enough connectivity such that the onset of synchronisation has 
shifted from $\gamma \approx 0.03$ to $\gamma \approx 0.04$ when $\tau = 0$ (Fig. 7.11). It should be noted that feedback 
consisting of small time delay values ($\tau \approx 0.2$) almost invariably increases synchronisation 
strength to some degree. This is true even when the initial behaviour of the system is

Figure 7.9: Illustration of a unidirectionally coupled stochastic network from Eqs. 7.1 with 0.1% connectivity (one 
input per neuron), where $\gamma = 0.01$ and $N = 1,000$. (a) - (b) No feedback $K = 0$. (a) Mean-field realization. (b) Mean-field 
phase portrait. (c) - (d) Corresponding results for delayed feedback with parameter values $K = 0.3$ and $\tau = 0.2$. An initially 
asynchronous state is converted into a state of strong chaotic synchronisation.

Figure 7.10: Illustration of a unidirectionally coupled stochastic network from Eqs. 7.1 with 0.2% connectivity (two 
inputs per neuron) where $\gamma = 0.04$ $N = 1,000$. (a) - (b) No feedback $K = 0$. (a) Mean-field realization. (b) Mean-field 
phase portrait. (c) - (d) Corresponding results for delayed feedback with parameter values $K = 0.3$ and $\tau = 2.8$. An initially 
synchronised state is converted into a state of weak chaotic synchrony.
Figure 7.11: Illustration of a unidirectionally coupled stochastic network from Eqs. 7.1 with 5% connectivity (fifty inputs per neuron) where $\gamma = 0.03 N = 1,000$. (a) - (b) No feedback $K = 0$. (a) Mean-field realization. (b) Mean-field phase portrait. (c) - (d) Corresponding results for delayed feedback with parameter values $K = 0.3$ and $\tau = 0.2$. An initially asynchronous state is converted into a state of strong synchronisation.

already strongly synchronous. The only opportunities for synchronisation enhancement at high coupling strength values occur in the large network ($N = 1,000$) with sufficiently high connectivity (5% and 10%) and strong feedback ($K = 1$). This window of opportunity is found at coupling strengths $\gamma \approx 0.15$. For 5% connectivity, synchronisation is extended to higher coupling strengths with the introduction of a time delay before saturating at $\tau \approx 1$. However, for 10% connectivity there is only a small window of time delay values ($\tau \approx 2.1$) that allow synchronisation to be increased. Reducing connectivity increases the region of initial synchrony (excluding one input per neuron) and therefore reduces the possibility of using delayed feedback as a synchrony improving mechanism. Gains in synchrony are easier to achieve with feedback strength $K = 1$ from the results found here.

Regions of synchrony fade before re-emerging with increasing time delay values, indicating a recurring synchronisation periodicity. When the system is synchronised at $\tau = 0$, the mean interspike interval (the average length of time between consecutive action potentials) is usually $\langle \psi \rangle \approx 4$, which is close to the $\tau$ value when the second region of synchrony is most prominent; when the time delay is aligned or at least related to the average interspike interval of the initial synchronised state, the most regular network output appears to be achieved. Synchronisation peaks are expected to appear for a series of time delay values, $\tau \approx 8, 12, 16, \cdots$, which can be mathematically defined as $\tau = \langle \psi_s \rangle \times z$ where $z \in \mathbb{N}$. $\mathbb{N}$ refers to the list of natural numbers; $\langle \psi_s \rangle$ represents the time-averaged interspike interval at the initial synchronised state where $\tau = 0$. The points with the greatest loss of synchrony closely occur to time delay values that are furthest away from $\tau = \langle \psi_s \rangle \times z$;
these occur at \( \tau = (\psi_s \times z) + \frac{\psi_s}{2} \) where \( \tau \approx 2, 6, 10, \ldots \). The second synchronisation peak is narrower and weaker than the first summit and the expected peak values could all continue to dissipate to an indistinguishable state; this second synchronisation peak dissipates less with larger networks. Results suggest network connectivity impacts position, size, and magnitude of these synchronisation regions.

### 7.3 Neuron-Specific Delayed Feedback Model

Global feedback can modify synchronisation in networks to an extent (Section 7.1); manipulating synchronisation to a significant magnitude is restricted to narrow parameter ranges in many cases or impossible in others. Greater control over the displayed behaviour is desired; this section assesses the possibility of utilising a feedback mechanism that is specific to individual neuronal behaviour, instead of global behaviour. Neuron-specific feedback entails each neuron to receive feedback stimulation, proportional to the difference between its current and previous activity. Though studying the impact of neuron-specific feedback is currently impractical, it is useful for theoretical purposes; as present technology is unable to measure singular neuronal activity, influencing individual neuronal behaviour is not an immediate prospect. Existing resources only capture the activity of large neuronal clusters through electroencephalogram (EEG) recordings; studying neuron-specific feedback may identify further advantageous behavioural alterations, which could be capitalised by future technological advances. Neuron-specific feedback results should also determine whether global feedback, which is more practical, manipulates behaviour to a similar effect. Comparing global and neuron-specific feedback can provide insight into global delayed feedback modifications, which may enhance desirable impacts.

Introducing neuron-specific feedback requires modification to global feedback equations (Eqs. 7.1) and can be represented by

\[
\epsilon \dot{x}_i = x_i - \frac{x_i^3}{3} - y_i + \gamma (M_{x_i} - x_i),
\]
\[ y_i = x_i + a + K(M_{y_i} - M_{y_i}) + \sqrt{2T} \xi_i(t), \] (7.2)

\[ M_{x_i} = \left[ \frac{100}{(N-1)v} \right] \sum_{j=1}^{(N-1)v} x_{g(i,j)}, \quad M_{y_i} = \left[ \frac{100}{(N-1)v} \right] \sum_{j=1}^{(N-1)v} y_{g(i,j)}. \]

In Eqs. 7.2, \( M_{y_i} \) signifies the average of inputs to neuron \( i \) at the current time moment; \( M_{y_i} \) corresponds to value \( \tau \) time moments ago. Colour coded mappings and surface plots (Figs. 7.13 - 7.20) are generated by similar analysis to global feedback (Section 7.1).

### 7.4 Neuron-Specific Delayed Feedback Results

Neuron-specific delayed feedback prevents increases to synchrony for one input per neuron for the studied network sizes and feedback strengths. The absence of synchronisation enhancements for neuron-specific feedback with one input per neuron contrasts to corresponding results for global feedback; in addition, neuron-specific feedback enables further reductions and (almost) complete destruction of synchrony with certain parameter values for coupling strength and time delay (Fig. 7.12). The largest region of synchronisation loss occurs at high values of \( \tau \) and low values of \( \gamma \); elevating feedback strength or network size increases the range of coupling strength and time delay values where synchrony can be (almost) completely destroyed.

When there are at least two inputs per neuron and feedback strength is weak (\( K = 0.3 \)), similar results to global feedback are produced (Section 7.1); this is particularly true at high network connectivities. Convergence between neuron-specific and global feedback

![Figure 7.12](image-url)

**Figure 7.12:** Illustration of a unidirectionally coupled stochastic network from Eqs. 7.2 with 0.1% connectivity (one input per neuron), where \( \gamma = 0.1 N = 1,000 \). (a) - (b) No feedback \( K = 0 \). (a) Mean-field realization. (b) Mean-field phase portrait. (c) - (d) Corresponding results for delayed feedback with parameter values \( K = 0.3 \) and \( \tau = 1.8 \). Synchronisation has been suppressed further.
Figure 7.13: Illustrations of the effects of delayed feedback in stochastic neural networks described by Eqs. 7.1 with sparse coupling $v = 1 - 50$ and $N = 100$. All neurons receive specific feedback with strength $K = 0.3$ that is proportional to the difference between its own current input value and its input value from $\tau$ moments ago. The mean-field variance $\sigma^2_{\mathcal{M}_x}$ is given as a function of the time delay $\tau$ and coupling strength $\gamma$. 

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Figure 7.14: Illustrations of the effects of delayed feedback in stochastic neural networks described by Eqs. 7.1 with sparse coupling $v = 1 - 50$ and $N = 100$. All neurons receive specific feedback with strength $K = 0.3$ that is proportional to the difference between its own current input value and its input value from $\tau$ moments ago. The mean-field variance $\sigma_{Mx}^2$ is given as a function of the time delay $\tau$ and coupling strength $\gamma$. Representation of the surface in Fig. 7.13.
Figure 7.15: Illustrations of the effects of delayed feedback in stochastic neural networks described by Eqs. 7.1 with sparse coupling $v = 1 - 50$ and $N = 100$. All neurons receive specific feedback with strength $K = 1$ that is proportional to the difference between its own current input value and its input value from $\tau$ moments ago. The mean-field variance $\sigma^2_{\tau}$ is given as a function of the time delay $\tau$ and coupling strength $\gamma$. 

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Figure 7.16: Illustrations of the effects of delayed feedback in stochastic neural networks described by Eqs. 7.1 with sparse coupling $v = 1 - 50$ and $N = 100$. All neurons receive specific feedback with strength $K = 1$ that is proportional to the difference between its own current input value and its input value from $\tau$ moments ago. The mean-field variance $\sigma^2_{M_x}$ is given as a function of the time delay $\tau$ and coupling strength $\gamma$. Representation of the surface in Fig. 7.15.
Figure 7.17: Illustrations of the effects of delayed feedback in stochastic neural networks described by Eqs. 7.1 with sparse coupling $\nu = 0.1 - 10$ and $N = 1,000$. All neurons receive specific feedback with strength $K = 0.3$ that is proportional to the difference between its own current input value and its input value from $\tau$ moments ago. The mean-field variance $\sigma_{Mx}^2$ is given as a function of the time delay $\tau$ and coupling strength $\gamma$. 
Figure 7.18: Illustrations of the effects of delayed feedback in stochastic neural networks described by Eqs. 7.1 with sparse coupling $\nu = 0.1 - 10$ and $N = 1,000$. All neurons receive specific feedback with strength $K = 0.3$ that is proportional to the difference between its own current input value and its input value from $\tau$ moments ago. The mean-field variance $\sigma_{Mx}^2$ is given as a function of the time delay $\tau$ and coupling strength $\gamma$. Representation of the surface in Fig. 7.17.
Figure 7.19: Illustrations of the effects of delayed feedback in stochastic neural networks described by Eqs. 7.1 with sparse coupling $v = 0.1 - 10$ and $N = 1,000$. All neurons receive specific feedback with strength $K = 1$ that is proportional to the difference between its own current input value and its input value from $\tau$ moments ago. The mean-field variance $\sigma_{Mx}^2$ is given as a function of the time delay $\tau$ and coupling strength $\gamma$. 
Figure 7.20: Illustrations of the effects of delayed feedback in stochastic neural networks described by Eqs. 7.1 with sparse coupling \( r = 0.1 - 10 \) and \( N = 1,000 \). All neurons receive specific feedback with strength \( K = 1 \) that is proportional to the difference between its own current input value and its input value from \( \tau \) moments ago. The mean-field variance \( \sigma_{Mx}^2 \) is given as a function of the time delay \( \tau \) and coupling strength \( \gamma \). Representation of the surface in Fig. 7.19.
Figure 7.21: Illustration of a unidirectionally coupled stochastic network from Eqs. 7.2 with 2% connectivity (two inputs per neuron), where $\gamma = 0.1, N = 100$. (a) - (b) No feedback $K = 0$. (a) Mean-field realization. (b) Mean-field phase portrait. (c) - (d) Corresponding results for delayed feedback with parameter values $K = 1$ and $\tau = 3$. An initially synchronised state is converted into an asynchronous state.

Effects are expected with increasing connectivity as $v \rightarrow 100$, $M_{yi} \rightarrow \overline{M}_y$, and $M_{y,\tau} \rightarrow \overline{M}_{y,\tau}$; at low connectivities (two and three inputs per neuron), low synchrony begins to invade areas of high synchronisation at low coupling strength values as well as at large time delays $(2.9 \lesssim \tau \lesssim 3.8)$. The encroachment is most prominent at two inputs per neuron, which is the lowest connectivity without disconnected neuronal clusters (Section 6.1). The magnitude of invasion becomes more severe in larger networks ($N = 1,000$); the extended regions of low synchronisation provide more parameter values than global feedback for destroying synchronisation. Strong feedback ($K = 1$) further spreads low synchronisation across the $(\tau, \gamma)$ plane. For 100 neurons, low synchrony visibly extends into areas of strong synchronisation when connectivity is at 5% and 10% (five and ten inputs per neuron); this does not occur for feedback strength $K = 0.3$. Convergence between neuron-specific and global feedback happens at higher connectivity when feedback is stronger ($K = 1$).

With two inputs per neuron, the low synchronisation region completely penetrates the entire range of studied coupling strengths; the initial state ($\tau = 0$) of synchronisation may be significantly reduced and (almost) destroyed for all coupling strengths when an appropriate time delay is used (Fig. 7.21).

For $K = 0.3$, delayed feedback rarely increases weak synchronisation to a strong state. Networks require sufficiently high connectivity to shift synchronisation at $\tau = 0$ to coupling strength values $\gamma \approx 0.04$ (Chapter 6); enhancements may be found when $\gamma \approx 0.03$ and $\tau \approx 0.3$. For $K = 1.0$, synchronisation can be improved at three inputs per neuron as the second synchrony peak is unusually elevated in regards to coupling strength values and
synchrony magnitude (Fig. 7.22). In the 100 neuron network, enhancing synchronisation is only possible when the system is initially chaotically synchronised. Further coupling strength values need to be researched for 0.3% connectivity in the 1,000 neuron network to discover whether weaker synchronised initial states ($\tau = 0$) transform upon implementing neuron-specific delayed feedback. Larger networks offer the strongest magnitude of synchrony and largest range of parameter values to increase synchronisation through neuron-specific delayed feedback. In the 1,000 neuron network, initially low synchronisation may be increased to a moderate strength at high coupling strengths for 5% and 10% connectivities. In the former case, increasing the feedback time delay value elevates the coupling strength values where synchronisation occurs, eventually leading to a saturation point where synchronisation does not improve further; saturation is maintained if the time delay $\tau$ continues to increase, reflecting findings for global feedback. The latter case has similar results to the 5% connectivity example, though the synchronisation boundary is not immediately elevated as a lag of $\tau \approx 1.25$ occurs. With global feedback, the lag duration is longer and the synchronisation boundary is temporarily elevated for a small parameter window.

7.5 Discussion

Control and manipulation of homogeneously coupled network behaviour is examined for global and neuron-specific feedback by independent application of weak and strong delayed feedback signals and utilisation of various time delays. The former case indicates reduc-
tions in synchrony are largely more achievable than increases, although the most extreme alterations in magnitude occur with increases; the latter case almost enables complete destruction of synchronisation at certain parameter values when initial synchronisation is high, almost completely suppressing synchronisation magnitude when connectivity is low and feedback strength is great. Synchronisation can be increased through delayed feedback in a similar manner across global and neuron-specific feedback types.

The delayed feedback results suggest that even weakly and globally applied delayed feedback may reduce excessive degrees of synchronisation to more moderate levels; it is also possible to strengthen weak synchronisation to improve processing efficiency. Further research would determine the critical level of synchronisation, where further increases would be detrimental to brain processing and develop pathological activity.
Synchronisation in Heterogeneously Coupled FitzHugh-Nagumo Networks

8.1 Heterogeneous Coupling Model

Heterogeneities in neural networks are integral to the breakdown of rhythms; too few driven cells could lead to synaptic interactions that are not strong or homogeneous enough to support synchronisation. Previous studies have shown that parameter values where synchronisation is destroyed in heterogeneous networks can be accurately predicted by studying homogeneous networks [128]. Synchronisation can break down if synaptic strength is reduced, numbers of cells participating in the rhythm are too few, or neuronal connections are too lattice-like in structure (Section 3.2). Decreasing the number of driven cells reduces the quantity of synaptic input per cell, increasing the significance of heterogeneities; firstly, sparse and random connectivity causes different neurons to receive varying numbers and magnitudes of synaptic input, which decreases when connectivity increases due to the law of large numbers; secondly, when neurons are only coupled to local units with low probability of establishing distant connections, neurons near the centre of the spatial domain receive more synaptic input than those located on the edge or outside of the domain. The model will be modified to account for biological variations in neuronal connection strength regarding the impact of additional heterogeneities upon
synchronisation.

In addition to neuronal connections possessing unequal coupling strengths, their strength may vary with time; this is known as the heterogeneous coupling scenario. Variable coupling strengths are important when identifying the effects of increased heterogeneity upon synchronisation robustness (Section 6.2) and simulating the biological brain’s learning processes. An element of control over the ensemble of connection strengths is maintained by approximately specifying the desired average value. Eqs. 6.1 are changed to:

\[
\epsilon \dot{x}_i = x_i - \frac{x_i^3}{3} - y_i + \left[ \frac{100}{(N-1)v} \right] \left( \frac{(N-1)v}{100} \right) \sum_{j=1}^{(N-1)v} \gamma_{ig(i,j)} x_{g(i,j)} - \bar{\gamma}_i x_i,
\]

\[
\dot{y}_i = x_i + a + \sqrt{2T} \xi_i(t).
\]  \hspace{1cm} (8.1)

\(\gamma_{ig(i,j)}\) corresponds to the connection strength from neuron \(g(i,j)\) to neuron \(i\); \(\bar{\gamma}_i\) is the average connection strength applied to neuron \(i\). Each interneuronal connection is modelled after the Ornstein-Uhlenbeck Process, where connection strengths are accordingly updated at every time moment to the following stochastic differential equation:

\[
\dot{\gamma}_{ig(i,j)} = \theta(\bar{\mu} - \gamma_{ig(i,j)}) + \sqrt{2D} \xi_{ig(i,j)}(t).
\]  \hspace{1cm} (8.2)

In Eq. 8.2, \(\xi_{ig(i,j)}(t)\) is a Gaussian white noise term fluctuating value \(\gamma_{ig(i,j)}\) with intensity \(D\); \(\bar{\mu}\) corresponds to the desired long-term mean value for all coupling strengths; \(\theta\) determines the rate at which \(\gamma_{ig(i,j)}\) overcomes fluctuations and reverts to the mean value. The Ornstein-Uhlenbeck Process is often referred to as a “mean reverting process” as the drift term relates to its current value; this process acts as a homeostatic, negative feedback mechanism: if the current value exceeds the desired mean, the drift term turns negative, and vice versa. The mean reverting property enables the Ornstein-Uhlenbeck Process to generate an oscillatory pattern for connection strengths with time resembling the sleep-wake cycle, according to the synaptic homeostasis hypothesis (Appendix 11.7).

The Ornstein-Uhlenbeck Process in Eq. 8.2 is Gaussian with bounded variance; vari-
ability is represented as a Gaussian curve centred at value $\gamma \approx \mu$ in the connection strength probability distribution. A delta peak remains at $\gamma = 0$ due to its correspondence with the network’s proportion of non-existing connections that have a fixed value. Even if coupling strengths are initially non-homogeneous, they revert towards the mean value $\mu$ when there is no coupling strength noise ($D = 0$); the values do not deviate upon establishment. If connections are initially identical, coupling strengths do not vary with time in a similar manner to fixed homogeneous coupling (Section 6.2); due to their increased complexity, Eqs. 8.1 have poorer computational efficiency, leaving only unidirectionally coupled networks consisting of 100 and 1,000 neurons to be considered.

### 8.2 Coupling Strength Fluctuations

The effects induced upon synchronisation are examined when the coupling strength noise intensity parameter $D$ is non-zero, causing connection strengths to differ and change with time; the specific parameter value tested is $D = 0.005$. The chosen value of the convergence rate parameter is $\theta = 1$, which quickly reverts coupling strengths to the mean value $\mu$ after deviation due to random fluctuations. Although individual couplings considerably vary in strength over the given time period, the ensemble mean connection strength closely remains near the desired mean value (Fig. 8.1); however, certain connections may be significantly above or below the mean value at any time moment.

The distribution of coupling strengths across the ensemble of neurons at a particular time is displayed in Fig. 8.2; the coupling strength distributions follow a Gaussian curve.
Figure 8.2: PDD of coupling strengths $P_1(\gamma)$ over the ensemble of neurons in a 100 neuron network after the relaxation period of the system with parameter values $\theta = 1$, $D = 0.005$, and $\mu = 0.1$. (a) 30% connectivity (30 inputs per neuron). (b) 80% connectivity (79 inputs per neuron).

centred at $\gamma \approx \mu$, with the exception of the missing connections at $\gamma = 0$. At noise value $D = 0.005$, connection strengths span values of approximately $\mu \pm 0.2$; the prominence of the peak at $\gamma \approx \mu$ is dependent upon system connectivity.

In the case of approximately no net coupling ($\mu = 0$ and $\langle \gamma \rangle \approx 0$), a small degree of synchrony is evident in many simulations, which is portrayed by a non-zero mean-field variance (Fig. 8.3) and a finite mean interspike interval of the mean-field (Fig. 8.4). The results contrast from homogeneously coupled networks, where no synchrony is evident in the uncoupled scenario, since many individual connections in the heterogeneously coupled networks have non-zero strength despite a zero global average. Non-zero coupling strengths allow interaction between certain neurons; fluctuations are large enough that connection strengths stray to sufficiently interact, causing a small number of neurons to entrain to an identical firing phase and/or frequency. Coupling strengths temporarily remain beyond the critical interaction strength prior to individual connections reverting to the mean value, ceasing neuronal interaction; due to the intermittent nature of interaction, only a small magnitude of network synchrony is achieved. Optimal connectivity for maximising synchronisation magnitude, when $\langle \gamma \rangle \approx 0$, occurs at approximately five
Figure 8.3: Variance $\sigma^2_{\bar{M}_x}$ of the the mean-field as a function of the mean coupling strength $\langle \gamma \rangle$ in unidirectionally coupled networks obeying Eqs. 8.1 with different combinations of size $N$ and connectivity percentage $v$. (a) $N = 100$, $v = 1 - 30$ (1 - 30 inputs per neuron). (b) $N = 100$, $v = 30 - 100$ (30 - 99 inputs per neuron). (c) $N = 1,000$, $v = 0.1 - 0.5$ (1 - 5 inputs per neuron). (d) $N = 1,000$, $v = 0.5 - 10$ (5 - 100 inputs per neuron). $\theta = 1$ and $D = 0.005$ are the coupling strength convergence rate and fluctuation parameters respectively.

inputs per neuron in the studied networks. Differences between heterogeneous and homogeneous coupling (Chapter 6) are illustrated in Fig. 8.5; the former displays chaotic synchronisation with varying amplitude and sub-maximal mean-field spiking, resulting in a smeared and slightly constricted limit cycle; the latter does not depict mean-field spiking, limit cycles, or synchronisation. Furthermore, the firing rate of individual neurons is comparatively elevated in the heterogeneously coupled network.

Despite an initially higher level of synchronisation, the variation of coupling strength values delays the onset of significant synchronisation (if present), in comparison to homogeneously coupled networks; this delay is due to a slower rise in synchrony and a higher
average of coupling strength must be reached for its onset. Larger numbers of connections enhance the network’s robustness, preventing delays to synchronisation onset. Fig. 8.6 depicts the homogeneously coupled network nearly reaching perfect synchronisation, with almost uninterrupted repeated spiking of the mean-field, prior to $\overline{\mu} = 0.04$; the amplitude of oscillations is almost constant, clearly defining the limit cycle in the phase plane. The heterogeneously coupled network contrastingly portrays chaotic synchronisation with variable and sub-maximal amplitude mean-field spiking, where trajectories form a smeared limit cycle. Despite the individual neurons firing at a similar rate, the above network discrepancies occur. When there is no net coupling, synchronisation is better in
heterogeneously coupled systems; by definition, all connection strengths are equal to zero in the homogeneous networks when there is no net coupling and synchronisation does not exist. However, homogeneously coupled networks have a quicker onset of synchronisation, leading to superior synchronisation magnitude. In heterogeneously coupled networks, a minority of individual couplings synchronise even when the mean coupling strength is below the critical synchronisation threshold, due to their higher coupling strengths; these connections have a greater impact in networks with fewer connections. In densely connected networks, there are many inputs per neuron; the numerous subthreshold coupling strengths override the effect of minority suprathreshold connections and prevent coupling-induced spikes. When the number of inputs per neuron is small, the suprathreshold coupling strength sufficiently increases a receiving neuron’s input and the neuron performs a coupling-induced spike; spikes occur at specific time moments due to the interaction between connected neurons, as opposed to a neuron’s individual rhythm. Synchronisation is higher in sparsely connected heterogeneous networks, when the average coupling strength is small, than in homogeneously coupled networks. The coupling strength noise intensity, $D = 0.005$, causes a few connections to obtain suprathreshold strengths even when $\langle \tau \rangle \approx 0$; thus, there is immediate evidence of synchronisation. A similar effect occurs when the
mean system coupling strength marginally exceeds the interaction threshold required for synchronisation onset; the minority of connections have a subthreshold value and limits synchronisation growth in sparsely connected networks. In homogeneously coupled systems, all connections are the same and are above, below, or equal to the critical coupling strength; therefore, the onset of synchronisation switches almost instantaneously and has a faster transition than in heterogeneously coupled systems.

Large connectivity leads to a similar decline in synchronisation found in heterogeneously and homogeneously coupled networks at high connection strengths. Heterogeneously coupled networks, with few inputs per neuron, prolong synchronisation to larger mean coupling strength values compared to homogeneously coupled networks; prolonged synchronisation in systems with low connectivity is indicated by a large mean-field variance in addition to a small and finite mean interspike interval of the mean-field at higher average coupling strength values. The mean coupling strength and input received by neurons fluctuate more in sparsely connected networks than densely connected systems, which counteract variation with the law of large numbers; variability neutralises the coupling-induced noise reduction phenomenon, enabling individual neurons to fire at higher average coupling strengths. Heterogeneities in coupling strengths elevate individual neuron firing
rates (Fig. 8.7); neurons are less likely to cease firing, requiring a larger value of mean coupling strength to initiate coupling-induced noise reduction.

Introducing heterogeneous coupling strengths with chosen parameter values has resulted in a shallower ascent of synchronisation with few inputs per neuron; when system connectivity is high, simulations are similar to homogeneously coupled networks (Section 6.2). Although heterogeneous coupling strengths have little impact upon the achieved magnitude of the synchronisation peak, maximal synchronisation in mean network coupling strengths tends to occur at higher values when connectivity is low; similar to homogeneous coupling, one input per neuron does not generate significant global synchronisation.

8.3 Convergence Rate

Having analysed the effects of introducing noise and heterogeneity into coupling strengths through variable $D$, the impact the convergence rate parameter $\theta$ imparts upon the model in Eqs. 8.1 must be understood. Coupling strength noise is reduced to $D = 0.00031$ and the convergence rate parameter is set to $\theta = 0.02$ to slowly revert coupling strengths to the mean value following perturbations. The connection strengths’ fluctuations, which move above and below the desired mean value, $\overline{\mu}$, are reduced due to the slower convergence
The cycle’s tendency to increase mean coupling strengths simulates the potentiation and learning of an awake brain; a long-term net decrease in connection strengths represents the homeostatic action of sleep, which prevents infinite potentiation (Appendix 11.7).

Fig. 8.8 exemplifies the time series for mean and individual coupling strengths, where the ensemble average of coupling strengths $\bar{\gamma}$ slowly oscillates around the desired mean value (here $\bar{\mu} = 0.2$); this provides a distinct contrast to the coupling strengths in Section 8.2, which quickly alternated upon a rapid convergence rate. The proximity between the mean coupling strength and its desired mean value $\bar{\mu}$ has a similar magnitude to its equivalent in Section 8.2, despite significantly slower fluctuations of coupling $D$. The slow convergence indicates that the individual coupling strengths are less magnetised to $\bar{\mu}$, providing a more diverse range of values; individual neurons display mean values that do not correspond to the desired mean value over short time frames.

The distribution of coupling strengths, within networks of differing connectivity, at a particular time moment is provided in Fig. 8.9; although the time evolution of connection strengths greatly differs from the large coupling noise intensity and rapid convergence scenario, their distributions are very similar. The distribution of coupling strengths is Gaussian and centred approximately at $\bar{\mu}$, spanning a comparable range, with additional unmade connections registered at $\gamma = 0$. A common property of all individual coupling strengths is their desired mean value; however, their slow convergence rates reduce their association with this mean value. The loss of mean value association largely varies between
connections since values can fluctuate above or below the desired mean value to a degree; individual coupling strengths lose association to each other and their desired mean.

Global synchronisation is most affected at lower connectivities with slow coupling convergence (Fig. 8.10), except at one input per neuron, which does not yield strong synchronisation due to the disconnected clusters of neurons within the network (Section 6.1). Highly connected networks are more stable as a result of increased connectivity, which reduces variance between the ensemble mean coupling strength and desired mean value through the law of large numbers.

Despite considerable changes in coupling strength parameters, slow convergence displays similar effects to rapid convergence in Section 8.2; the magnitude of some effects is greater at the current parameter values, in comparison to homogeneously coupled networks, despite lower intensities of coupling strength noise $D$. Given that couplings are homogeneous when $D = 0$, reducing $D$ should cause heterogeneous networks to behave more similarly to homogeneous systems. Considering the only other altered parameter depicts convergence rate, the lowering of variable $\theta$ appears to be responsible for their similarity (Section 8.4); thus, lowering convergence rate and increasing coupling strength
Figure 8.10: Variance $\sigma^2_{M_x}$ of the mean-field as a function of the mean coupling strength $\langle \gamma \rangle$ in unidirectionally coupled networks obeying Eqs. 8.1 with different combinations of size $N$ and connectivity percentage $v$. (a) $N = 100$, $v = 1 - 30$ (1 - 30 inputs per neuron). (b) $N = 100$, $v = 30 - 100$ (30 - 99 inputs per neuron). (c) $N = 1,000$, $v = 0.1 - 0.5$ (1 - 5 inputs per neuron). (d) $N = 1,000$, $v = 0.5 - 10$ (5 - 100 inputs per neuron). $\theta = 0.02$ and $D = 0.00031$ are the coupling strength convergence rate and fluctuation parameters respectively.

noise intensity have similar effects on network synchronisation. When there is approximately no net coupling ($\bar{\mu} = 0$ and $\langle \gamma \rangle \approx 0$), the elevated mean-field variance values are indicative of synchronisation in some of the 100 neuron simulations; maximal synchrony occurs at 5% connectivity (five inputs per neuron), which correlates to results for rapid convergence rate and high noise. However, the magnitude of synchrony elevation is smaller where there is slow convergence rate and low noise. There is insignificant synchrony elevation for $\bar{\mu} = 0$ in all 1,000 neuron simulations, which is an unexpected consequence of slow coupling strength evolution and not a result of a reduction in $D$ (Section 8.4).

Transition towards synchronisation onset (if synchrony exists) follows a more shal-
Figure 8.11: Mean interspike interval \( \langle \psi \rangle \) of the mean-field as a function of the mean coupling strength \( \langle \gamma \rangle \) in unidirectionally coupled networks obeying Eqs. 8.1 with different combinations of size \( N \) and connectivity percentage \( v \). (a) \( N = 100, v = 1 - 30 \) (1 - 30 inputs per neuron). (b) \( N = 100, v = 30 - 100 \) (30 - 99 inputs per neuron). (c) \( N = 1,000, v = 0.1 - 0.5 \) (1 - 5 inputs per neuron). (d) \( N = 1,000, v = 0.5 - 10 \) (5 - 100 inputs per neuron). \( \theta = 0.02 \) and \( D = 0.00031 \) are the coupling strength convergence rate and fluctuation parameters respectively.

low gradient than in homogeneous coupling (Section 6.2) and rapid convergence (Section 8.2); the shallow ascent of synchronisation is most apparent at low connectivity, delaying acquisition of synchronisation peaks to higher average coupling strength values than in corresponding homogeneously coupled networks (Fig. 8.12). The heterogeneously coupled system only has chaotic spiking of the mean-field and very low synchronisation, whereas a homogeneously coupled network with equal parameter values is very strongly synchronised; these differences occur despite a similar frequency of individual neuronal firing. Global synchronisation is easier to achieve when there is more homogeneity and connections between neurons share the same (or similar) coupling strength values. When
connection strengths slowly converge to their desired mean, there are fewer alternations above and below the mean value. Individual connections have less frequent collisions of connection strengths with their desired mean value; thus, higher mean coupling strength values are required to generate optimal synchrony. The eastern shift in the synchronisation peak is subsequently converted into a delayed attenuation of neuronal firing and synchrony, though the rate of neuronal firing attenuation is insignificantly affected compared to the previous findings. A comparison of firing rates between a heterogeneously coupled network and its homogeneous counterpart at a high mean coupling strength is displayed in Fig. 8.13. A faster firing rate is maintained in the heterogeneously coupled network; the rarity of individual neuronal spiking in the homogeneously coupled network causes greater attenuation in the synchronisation indicator $\sigma^2_M$, despite firing simultaneously on most occasions that spikes occur. The instant decrements to synchronisation and neuronal firing highlight the principle that synchronisation cannot be present when individual neuronal firing ceases. Thus, the heterogeneously coupled system synchronises at extended mean coupling strengths as it maintains a higher rate of individual neuronal firing; it is responsible for delaying the coupling-induced noise reduction phenomenon.

Low coupling strength noise intensity with slow convergence rate displays many sim-
Figure 8.13: Illustration of unidirectionally coupled stochastic networks with 4% connectivity (two inputs per neuron), where \(N = 100\). (a) - (d) Heterogeneous coupling using Eqs. 8.1 and 8.2 with parameter values \(\theta = 0.02\), \(D = 0.00031\), and \(\mu = 0.2\). (a) Mean-field realization. (b) Mean-field phase portrait. (c) and (d) Realizations of different individual neurons. (e) - (h) Corresponding results for homogeneous coupling (Chapter 6).

Similarities to high coupling strength noise intensity with fast convergence (Section 8.2); lowering convergence rate can indirectly increase the coupling strength noise intensity.

### 8.4 Noisy and Slow Couplings

The qualitative patterns and behaviours of homogeneously coupled networks are retained when substantial variability is introduced to coupling strength values (Sections 8.2 and 8.3). The quantitative values of behavioural changes are especially robust when connectivity is large; they lose their alignment at low connectivity. The properties of large coupling strength fluctuations and rapid convergence rate will be combined so their simultaneous impact upon network behaviour can be examined; synchronisation behaviour is analysed for convergence rate parameter value \(\theta = 0.02\) and coupling strength noise intensity \(D = 0.005\) in order to validate the conclusions in Sections 8.2 and 8.3; the extent of extreme variability induced by the parameter values hindering synchronisation in highly connected networks can be examined.

Realisations for the ensemble mean and a selection of individual connection strengths (Fig. 8.14) emphasise the slow oscillation of coupling strengths around their desired mean value, \(\overline{\mu}\); however, the oscillation amplitude exceeds the values found in Section 8.3, due
to increased fluctuations caused by the higher value $D$, enabling the ensemble mean and individual coupling strengths to further stray from their desired mean value.

The distribution of coupling strengths within the network in Fig. 8.15 successfully demonstrates greater heterogeneity; the underlying Gaussian distribution (with additional unmade connections at $\gamma = 0$) still centres around the desired mean value. However, the distribution is broader and the tails increasingly encompass more distant values on each side of the mean; system connectivity continues to dictate the peak’s amplitude at $\gamma \approx \bar{\mu}$ and $\gamma = 0$.

**Figure 8.14**: Coupling strength evolution induced by parameter values $\theta = 0.02$, $D = 0.005$, and $\bar{\pi} = 0.2$ for a network with 1% connectivity (ten inputs per neuron) in a 1,000 neuron network. (a) Fluctuations of the ensemble mean coupling strength $\tau$ as time progresses. (b) - (d) Realizations for three different individual connections within the network.

**Figure 8.15**: PDF of coupling strengths $P_1(\gamma)$ over the ensemble of neurons in a 100 neuron network after the relaxation period of the system with parameter values $\theta = 0.02$, $D = 0.005$, and $\bar{\pi} = 0.1$. (a) 30% connectivity (30 inputs per neuron). (b) 80% connectivity (79 inputs per neuron).

Mean-field variance behaviour across a range of ensemble mean coupling strengths
(Fig. 8.16) also clearly affects synchronisation capabilities in networks with dense connectivity; even at 100% (all to all) connectivity with 99 inputs per neuron in the 100 neuron network, the synchronisation range is greatly extended compared to homogeneously coupled networks, at the expense of a more shallow onset. The extended synchronisation window is attributed to slower attenuation of individual neuronal firing at large mean coupling strengths; the mean-field and individual neurons rapidly spike in the heterogeneously coupled network at parameter values where homogeneously coupled networks do not display either activity (Fig. 8.17). The severity of effects are larger at low connectivity, where synchronisation onset is significantly delayed, and the peak value is likely
Figure 8.17: Illustrations of unidirectionally coupled stochastic networks with 100% connectivity (99 inputs per neuron), where $N = 100$. (a) - (d) Heterogeneous coupling using Eqs. 8.1 and 8.2 with parameter values $\theta = 0.02$, $D = 0.005$, and $\mu = 0.2$ (a) Mean-field realization. (b) Mean-field phase portrait. (c) and (d) Realizations of different individual neurons. (e) - (h) Corresponding results for homogeneous coupling (Chapter 6).

to be outside many of the studied parameter ranges; when there is 1% connectivity in a 1,000 neuron network (ten inputs per neuron), global synchronisation is in development at $\mu = 0.15$, approximately peaking only at $\mu = 0.3$ (Fig. 8.18). Onset of global synchronisation has a steeper gradient in networks with larger number of neurons; these results are consistent with the previously studied networks. The slow convergence rate prevents elevated mean-field variance values at $\gamma = 0$ with 1,000 neurons, despite the same coupling strength noise intensity parameter $D$ used in Section 8.2, which elevated synchronisation due to a rapid convergence rate.

Although the pattern of network synchronisation concerning mean coupling strength range remains robust, even at extreme parameter values, the quantitative offset is substantially heightened, compared to homogeneously coupled systems (Section 6.2); the tested parameter values in this network sufficiently extend the observed effects at low connectivities in Sections 8.2 and 8.3 to more densely connected networks. A more realistic representation of coupling strength heterogeneities is likely to be achieved using the Ornstein-Uhlenbeck Process described in Eq. 8.2; upon implementation of this process, the distribution of coupling strengths depends on the interaction between convergence rate parameter $\theta$ and coupling strength noise intensity $D$. To acquire the most accu-
rate model, further research to determine appropriate values of $\theta$ and $D$ is necessary; increased knowledge of these values may be derived from biological investigations, literature regarding connection strengths, and examining discrepancies in connections between sleep and waking states. The applied parameter choices provide useful insight, illustrating the robustness of qualitative behaviour against significant modifications. Evolution of the brain’s connection strengths is undoubtedly slow to an unknown extent. Value $D = 0.005$ is too large since the distribution of coupling strength values exceed a practical range; realistic brain behaviour is likely to be less affected by heterogeneities in the coupling strengths than the observations detailed in this section.

8.5 Discussion

The impact of heterogeneous coupling strengths changing with time is explored in this section; qualitative behaviours of homogeneously coupled networks are exemplified even with coupling strength heterogeneities that are likely to exceed biological values. Heterogeneous coupling causes a proportion of connection strengths to rise above zero when the net coupling strength of the network is zero; the interaction between small clusters of neurons in relation to the network size leads to traces of low magnitude synchronisation
being evident from the mean-field variance. In contrast, when the net coupling strength is equal to zero in the homogeneously coupled networks, there is an absence of interaction between all individual units and synchronisation does not transpire.

However, the onset of synchronisation becomes less abrupt with increments in coupling strength when noise fluctuations are strong and the convergence rate is slow in comparison to equivalent homogeneously coupled networks; the synchronisation zone shifts to higher coupling strength values. In homogeneously coupled networks, all individual coupling strengths are equal to the mean coupling strength value. Consequently, all connection strengths are either above, below, or equal to the critical threshold and the transition to global synchronisation is almost instantaneous. Heterogeneously coupled networks attain a proportion of individual connection strengths that are above the critical threshold even when the mean coupling strength is below this level, resulting in a fraction of the units interacting sufficiently to synchronise. Since, some neurons have coupling strengths below the network’s mean value, a higher mean coupling strength is required for all units to interact above the critical threshold where global synchronisation is maximised. The subsequent decay of synchronisation due to coupling-induced noise reduction is delayed to higher coupling strength values; the firing rate of individual units is elevated in comparison to counterpart homogeneously coupled networks with the same coupling strength during synchrony attenuation. Reducing the number of inputs per neuron increases the magnitude of effects providing that the the network is a complete single entity.

The results may indicate that homogeneous coupling is a reasonable predictor of heterogeneous coupling if the same number of inputs per neuron is applied and the heterogeneous coupling parameters are small; this is particularly true in the case of a large number of inputs per neuron. Despite the relatively low percentage of connectivity, real brains consist of an extensive number of inputs per neuron. Therefore, it appears that the homogeneously coupled networks described would be close in accuracy to the heterogeneously coupled networks when simulating connection quantities and proportions typical of those in biological brains; this is advantageous since homogeneous scenarios are simpler and
more computationally efficient.

Further research would determine whether adding heterogeneities through the numbers of inputs per neuron causes the mechanism of synchronisation to breakdown; in such a scenario, neuronal interaction may be insufficiently strong or homogeneous to entrain their rhythms to each other. The surprising presence of synchronisation, across the series of numerical simulations (for as little as two inputs per neuron), is attributed to the homogeneous quantity of inputs in all cases, in addition to the randomness of connection allocations, enabling rapid signal transmission throughout the entire system.
Chapter 9

FitzHugh-Nagumo Neural Network
Analysis

9.1 Generation of Cumulant Equations

In this chapter, a more analytical approach is directed towards supporting the numerical results discussed in Chapters 6, 7, and 8. The homogeneous system with neuron-specific feedback is considered using Eqs. 7.2, leaving the heterogeneous case for future analysis.

Global behaviour in a stochastic neural network, consisting of \( N \) units, is depicted by its joint PDD, \( P_N(x_1,y_1,x_2,y_2,\cdots,x_N,y_N,t) \); this outlines the state of all variables at all time moments, \( t \) \([9]\). When \( N \) is large, the calculation and visualisation of a function with \( 2N + 1 \) variables is often impractical.

In order to create a practical model, certain assumptions must be made about a network’s properties and units, such as the approximation of molecular chaos \([129]\); in the limit \( N \rightarrow \infty \), it is assumed that elements within the network are independent and uncorrelated. In this case, the \( N \)-dimensional system is decomposed into the product of \( N \) two-dimensional density distributions:

\[
P_N(x_1,y_1,x_2,y_2,\cdots,x_N,y_N,t) = P(x_1,y_1,t)P(x_2,y_2,t)\cdots P(x_N,y_N,t).
\] (9.1)

As all neurons are identical in this model, all \( N \) two-dimensional PDD’s, \( P(x,y,t) \), are also identical; the time-dependent mean-field is equated to the average of the two-
dimensional one-particle probability density, therefore \( \bar{M}_x = \int xP(x,y,t)dx\,dy \) and \( \bar{M}_y = \int yP(x,y,t)dx\,dy \) [10]. The assumption of molecular chaos has been considered in previous studies for systems coupled through the mean-field [10, 130, 131], noise driven bistable elements [132, 133, 134], coupled noisy self-sustained oscillators [135], and coupled phase oscillators [107, 136].

Many of the above principles can be applied to networks with sparse connectivity since the quantity of neurons and coupled units is considered to be infinite; an identical number of inputs per neuron is critical in providing consistent interaction across the network, ultimately truncating the \( N \)-body problem to one-body. Truncation of equations would be much more complicated when using heterogeneous input quantities as the discussed assumptions become invalid; a uniform number of inputs per neuron also ensures consistency between the order of fluctuations for all units. In the limit \( N \to \infty \), these fluctuations are expected to cancel each other; the units’ probability densities decouple in the limit \( N \to \infty \), leading to independent and uncorrelated oscillations. It is necessary to alter the notation depicting couplings when applying the theory of molecular chaos; the term \( \left( \frac{N-1}{100} \right) \), which relates to the proportion of connectivity within the network, is replaced by integer variable \( q \), which references the number of inputs per neuron. To successfully prevent infinity from appearing in the equations, the couplings must be represented by a parameter that depicts the number of inputs per neuron instead of the percentage of connectivity.

Using the above assumptions, the time evolution of the global network state can be represented by a Fokker-Planck equation for \( P(x,y,t) \) [9, 10, 130, 135]. Eqs. 7.2 govern the homogeneously coupled system with neuron-specific delayed feedback and are of the form

\[
\frac{dx}{dt} = A(x,y,t)dt + B(x,y,t)dW_1, \tag{9.2}
\]

\[
\frac{dy}{dt} = C(x,y,t)dt + D(x,y,t)dW_2, \tag{9.3}
\]
where

\[ A = \frac{1}{\epsilon} \left( x - \frac{x^3}{3} - y + \gamma (M_x - x) \right), \]

\[ B = 0, \]

\[ C = x + a + K (M_{yr} - M_y), \]

\[ D = \sqrt{2T}, \]

and index “i” is omitted as compared to Eqs. 7.2. The Gaussian white noise terms are additive and therefore the Itô and Stratonovich interpretations of stochastic calculus are equivalent. It is possible to convert Eqs. 7.2 into a Fokker-Planck Equation [87] using the formula for the Itô interpretation:

\[
\frac{\partial P(x,y,t)}{\partial t} = -\frac{\partial}{\partial x} \left[ A(x,y,t)P(x,y,t) \right] - \frac{\partial}{\partial y} \left[ C(x,y,t)P(x,y,t) \right] + \frac{1}{2} \frac{\partial^2}{\partial x^2} \left[ B^2(x,y,t)P(x,y,t) \right] + \frac{1}{2} \frac{\partial^2}{\partial y^2} \left[ D^2(x,y,t)P(x,y,t) \right].
\]

(9.5)

Eqs. 7.2 can be rewritten with probability density \( P = P(x,y,t) \):

\[
\frac{\partial P}{\partial t} = -\frac{\partial}{\partial x} \left\{ \frac{1}{\epsilon} \left( x - \frac{x^3}{3} - y + \gamma (M_x - x) \right) \right\} P - \frac{\partial}{\partial y} \left\{ \left( x + a + K (M_{yr} - M_y) \right) \right\} P + \frac{1}{2} (\sqrt{2T})^2 \frac{\partial^2 P}{\partial y^2} \\
= -\frac{\partial}{\partial x} \left\{ \frac{1}{\epsilon} \left( x - \frac{x^3}{3} - y + \frac{100}{(N-1)v} \sum_{j=1}^{(N-1)\alpha} x_{g(j)} - \gamma x \right) \right\} P \\
- \frac{\partial}{\partial y} \left\{ \left( x + a + \frac{100}{(N-1)v} \sum_{j=1}^{(N-1)\alpha} y_{rg(j)} - \frac{100}{(N-1)v} \sum_{j=1}^{(N-1)\alpha} y_{g(j)} \right) \right\} P + T \frac{\partial^2 P}{\partial y^2} \\
= -\frac{\partial}{\partial x} \left\{ \frac{1}{\epsilon} \left( (1-\gamma)x - \frac{x^3}{3} - y + \gamma \sum_{j=1}^{a} x_{g(j)} \right) \right\} P \\
- \frac{\partial}{\partial y} \left\{ \left( x + a + \frac{K}{q} \sum_{j=1}^{a} (y_{rg(j)} - y_{g(j)}) \right) \right\} P + T \frac{\partial^2 P}{\partial y^2}. \]

(9.6)

In the case of global feedback described by Eqs. 7.1, Eqs. 9.6 are simplified using \( E[M_y] = \).
\[
\bar{M}_y = \int y P(x, y, t) dx dy;
\]

\[
\frac{\partial P}{\partial t} = - \frac{\partial}{\partial x} \left\{ \frac{1}{\epsilon} \left( (1 - \gamma) x - \frac{x^3}{3} - y + \frac{\gamma}{q} \sum_{j=1}^{q} x_{g(j)} \right) P \right\}
- \frac{\partial}{\partial y} \left\{ \left( x + a + K \left[ \int y' P(x', y', \tau) dx' dy' \right] - \int y' P(x', y', t) dx' dy' \right) P \right\}
+ T \frac{\partial^2 P}{\partial y^2}.
\]

(9.7)

Currently, Eq. 9.7 is a non-linear partial differential equation with delay without a known method of deducing exact solutions; advances may be possible by expanding the probability density using Hermite polynomials [130].

The moment generating function can be used to specify the probability distribution of a random variable, offering an alternative approach to directly using the PDD’s; this method determines all the distribution moments using a series expansion [137]. The moment generating function, \( \Lambda_Z(r) \), of a random variable, \( Z \), is given by

\[
\Lambda_Z(r) = E[e^{rZ}]
= 1 + rE[Z] + \frac{r^2E[Z^2]}{2!} + \frac{r^3E[Z^3]}{3!} + \ldots + \frac{r^nE[Z^n]}{n!} + \ldots
= 1 + r\lambda_1 + \frac{r^2\lambda_2}{2!} + \frac{r^3\lambda_3}{3!} + \ldots + \frac{r^n\lambda_n}{n!} + \ldots
= 1 + \sum_{n=1}^{\infty} \frac{r^n\lambda_n}{n!},
\]

(9.8)

where \( E \) corresponds to the expectation and \( \lambda_n \) is the \( n \)th order moment; the \( n \)th-order moment can be obtained by evaluating the \( n \)th derivative of the moment generating function at \( r = 0 \) [138]:

\[
\lambda_n = \left. \frac{\partial^n}{\partial r^n} \Lambda_Z(r) \right|_{r=0}.
\]

(9.9)

Properties of cumulants are discovered by moments of the distribution as two distributions with equal moments have identical cumulants; the cumulant generating function
$\Upsilon_Z(r)$ is defined as the logarithm of the moment generating function

$$
\Upsilon_Z(r) = \log(E[e^{rZ}]) = \sum_{n=1}^{\infty} \frac{r^n \kappa_n}{n!}, \quad (9.10)
$$

where $\kappa_n$ is the $n$th-order cumulant. Similar to the moments in Eq. 9.9, the $n$th-order cumulants are obtained through the $n$th derivative of the cumulant generating function, evaluated at $r = 0$ [138]:

$$
\kappa_n = \frac{\partial^n}{\partial r^n} \Upsilon_Z(r) \bigg|_{r=0}. \quad (9.11)
$$

One drawback of the method utilising moments and cumulants is the incorporation of infinite sums in Eqs. 9.8 and 9.10; unlike moments, higher order cumulants advantageously carry information of decreasing significance.

A Gaussian approximation involves approximating the distribution of the state variables of the system by a Gaussian function. Making this assumption simplifies the analysis by allowing only the first two cumulants to be considered since all remaining cumulants (of which there are infinitely many) can be assumed to vanish; its reliability has been considered with the FitzHugh-Nagumo network that is coupled through the mean-field [10]. Although there were restrictions to quantitative correspondence at small coupling strength values, Gaussian approximation extracted the most important qualitative features; this approximation corrects some of the approximations made by molecular chaos [139]. Therefore, Gaussian approximation is useful for truncating systems to make equations more tractable, even if actual PDD’s are not Gaussian (as in this case); the Fokker-Planck equation (Eq. 9.7) can be converted into a finite set of ODE’s using the property of ergodicity, which states that ensemble averages equate to time average moments of a non-linear Fokker-Planck equation in a stationary system. The cumulant analysis and Gaussian approximation approaches have been adopted in this investigation, due to their successful application in previous studies of FitzHugh-Nagumo systems [9, 10, 131], noisy self sustained oscillators [135], bistable units [132, 133, 134], and phase oscillators.
the approximation is appropriate because the intensity of random fluctuations in the system is small.

Expressions for cumulants [140] can be calculated by splitting a cumulant of order $n$ into $n$ terms; the terms are then partitioned, using angular brackets $\langle \rangle$ to denote averaged values, up to $n-1$ times, producing as many unique expressions as possible. The number of unique terms per partition is expressed using Stratonovich symmetrisation brackets, $\{\}$, which represent the sum of all possible permutations for arguments inside the brackets [87, 141]. For instance,

$$3\{\langle Z_1 \rangle \langle Z_2 Z_3 \rangle \}_s = \langle Z_1 \rangle \langle Z_2 Z_3 \rangle + \langle Z_2 \rangle \langle Z_1 Z_3 \rangle + \langle Z_3 \rangle \langle Z_1 Z_2 \rangle,$$

(9.12)
gives the sum of all possible partitions for form $\langle Z_1 \rangle \langle Z_2 Z_3 \rangle$. Let $C_p(Z_1, Z_2, \cdots, Z_n)$ equal the sum of all terms obtained for a particular number of partitions $p$; for example, there are two ways of applying a single partition to a fourth-order cumulant, namely $\langle \rangle \langle \rangle \langle \rangle \langle \rangle$ or $\langle \rangle \langle \rangle \langle \rangle \langle \rangle$. For partition type $\langle \rangle \langle \rangle \langle \rangle \langle \rangle$ one has,

$$\langle Z_1 \rangle \langle Z_2 Z_3 Z_4 \rangle + \langle Z_2 \rangle \langle Z_1 Z_3 Z_4 \rangle + \langle Z_3 \rangle \langle Z_1 Z_2 Z_4 \rangle + \langle Z_4 \rangle \langle Z_1 Z_2 Z_3 \rangle = J_1$$

(9.13)
and for partition type $\langle \rangle \langle \rangle \langle \rangle \langle \rangle$,

$$\langle Z_1 Z_2 \rangle \langle Z_3 Z_4 \rangle + \langle Z_1 Z_3 \rangle \langle Z_2 Z_4 \rangle + \langle Z_1 Z_4 \rangle \langle Z_2 Z_3 \rangle = J_2.$$

(9.14)
Hence,

$$C_1(Z_1, Z_2, Z_3, Z_4) = J_1 + J_2.$$  
(9.15)
An expression for the $n$th order cumulant can be obtained using the formula

$$\langle \langle Z_1 Z_2 \cdots Z_n \rangle = \sum_{p=0}^{n-1} (-1)^p p! C_p(Z_1, Z_2, \cdots, Z_n),$$

(9.16)
where double angular brackets, $\langle \rangle$, denote cumulant terms equivalent to $\kappa_n$ in Eq. 9.11.
The first order cumulant does not have any partitions:

\[ \langle Z \rangle = \langle Z \rangle = m_Z. \quad (9.17) \]

The first order cumulant of a variable, \( Z \), is equal to its mean value, \( m_Z \); the second order cumulant includes a term, resulting from one partition:

\[ \langle Z_1Z_2 \rangle = \langle Z_1Z_2 \rangle - \langle Z_1 \rangle \langle Z_2 \rangle. \quad (9.18) \]

Setting \( Z_2 = Z_1 \) allows one to write the following:

\[ \langle Z^2 \rangle = \langle Z^2 \rangle - \langle Z \rangle^2 = D_Z. \quad (9.19) \]

The second order univariate cumulant equals variance \( D_Z \) of variable \( Z \); assigning \( Z_2 = B \) provides the cross variance, \( D_{ZB} \), of variables \( Z \) and \( B \):

\[ \langle ZB \rangle = \langle ZB \rangle - \langle Z \rangle \langle B \rangle = D_{ZB}. \quad (9.20) \]

Consequently, the system of stochastic differential equations (Eqs. 7.2) can be replaced by a system of deterministic equations that corresponds to \( \epsilon \dot{m}_x, \epsilon \dot{m}_y, \epsilon \dot{D}_x, \dot{D}_y \), and \( \epsilon \dot{D}_{xy} \), where \( m_x = E[x_i] = \langle x_i \rangle \) and \( m_y = E[y_i] = \langle y_i \rangle \) are respectively defined as mean values for \( x_i \) and \( y_i \) with variances \( D_x = E[(x_i - m_x)^2] = \langle (x_i - m_x)^2 \rangle \) and \( D_y = E[(y_i - m_y)^2] = \langle (y_i - m_y)^2 \rangle \). The covariance term is \( D_{xy} = E[(x_i - m_x)(y_i - m_y)] = \langle (x_i - m_x)(y_i - m_y) \rangle \). In the above, \( E \) denotes the expected value of the corresponding argument and angular brackets represent time-averages. By the property of ergodicity, it is assumed that averages over the ensemble and time coincide. Subscript \( i \) will be omitted, leaving \( x \) to denote \( x_i \) and \( y \) to denote \( y_i \), unless otherwise stated. Derivations of evolution equations for the above terms describing the network given by Eqs. 7.2 follow. The notations \( \frac{d}{dt} \) and \( \dot{z} \) are used interchangeably where \( \frac{dz}{dt} = \dot{z} \). In the following expressions, neurons are assumed to be
identical and the time-dependent mean-field becomes equal to the average over the one-particle distribution density.

1) 

\[ \epsilon \frac{dm_x}{dt} = E \left[ \frac{dx}{dt} \right] = E \left[ x - \frac{x^3}{3} - y + \gamma (M_x - x) \right] = E[x] - E \left[ \frac{x^3}{3} \right] - E[y] + \gamma E[M_x] - \gamma E[x] = m_x - \frac{m_x^3 + 3m_xD_x}{3} - m_y + \gamma m_x - \gamma m_x. \tag{9.21} \]

In the above, properties

\[ E[x] = m_x, \]
\[ E[M_x] = m_x, \]
\[ E[x^3] = m_x^3 + 3m_xD_x, \]

have been used (Eqs. 11.30 and 11.35 in Appendix 11.11). Consequently, Eq. 9.21 simplifies to

\[ \epsilon \frac{dm_x}{dt} = m_x - \frac{m_x^3}{3} - m_y - m_xD_x. \tag{9.22} \]

2) 

\[ \frac{dm_y}{dt} = E \left[ \frac{dy}{dt} \right] = E[x + a + K(M_y - M_y) + \sqrt{2T}\xi(t)] = E[x] + E[a] + E[K(M_y - M_y)] + E[\sqrt{2T}\xi(t)] = m_x + a + KE[M_y] - KE[M_y] + \sqrt{2T}E[\xi(t)]. \tag{9.23} \]

Since Gaussian white noise has zero mean (i.e. \( E[\xi(t)] = 0 \)), the following result is
obtained where \( m_{\tau} \) represents the mean-field \( \tau \) moments ago:

\[
\frac{dm_y}{dt} = m_x + a + K(m_{\tau} - m_y). \quad (9.24)
\]

3)

\[
\epsilon \frac{dD_x}{dt} = E \left[ \epsilon \frac{d}{dt} (x - m_x)^2 \right] = \epsilon E \left[ 2(x - m_x) \frac{d}{dt} (x - m_x) \right] \quad \text{(using chain rule)}
\]

\[
= 2E \left[ (x - m_x) \left( \epsilon \frac{dx}{dt} - \epsilon \frac{dm_x}{dt} \right) \right]. \quad (9.25)
\]

\( \epsilon \frac{dx}{dt} \) is substituted from Eqs. 7.2 and \( \frac{dm_x}{dt} \) from Eq. 9.22 into Eq. 9.25 to give:

\[
\epsilon \frac{dD_x}{dt} = 2E \left[ (x - m_x) \left\{ \left( x - \frac{x^3}{3} - y + \gamma(M_x - x) \right) - \left( m_x - \frac{m_x^3}{3} - m_y - m_x D_x \right) \right\} \right]
\]

\[
= 2E \left[ (x - m_x) \left( x - \frac{x^3}{3} - y + \gamma(M_x - x) \right) \right.
\]

\[
- (x - m_x) \left( m_x - \frac{m_x^3}{3} - m_y - m_x D_x \right) \]

\[
= 2E \left[ x^2 - \frac{x^4}{3} - xy + \gamma(M_x - x) - m_x x + \frac{m_x x^3}{3} + m_x y - m_x \gamma(M_x - x) \right.
\]

\[
- \left\{ x m_x - \frac{x m_x^3}{3} - x m_y - x m_x D_x - m_x^2 + \frac{m_x^2}{3} + m_x y + m_x^2 D_x \right\} \]

\[
= 2 \left\{ E[x^2] - E \left[ \frac{x^4}{3} \right] - E[xy] + \gamma E[x M_x] - \gamma E[x^2] - m_x E[x] + m_x \gamma E[M_x] \right. \]

\[
+ m_x \gamma E[x] + m_x \gamma E[x] - m_x E[x] + \frac{m_x^2}{3} E[x] + m_y E[x] \]

\[
+ m_x D_x E[x] + m_x^2 - \frac{m_x^4}{3} - m_x m_y - m_x^2 D_x \right\}. \quad (9.26)
\]

Eq. 9.26 can be simplified using properties

\[
E[x] = m_x,
\]

\[
E[x^2] = D_x + m_x^2,
\]

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from Eqs. 11.30 and 11.32 in Appendix 11.11. After cancellation, the equation simplifies to:

\[
\epsilon \frac{dD_x}{dt} = 2 \left\{ D_x - D_{xy} - E \left[ \frac{x^4}{3} \right] - \gamma D_x + \gamma E[xM_x] + m_x E \left[ \frac{x^3}{3} \right] - \gamma m_x^2 \right\}.
\] (9.27)

The properties for a normal variable are given by

\[
E[x^3] = m_x^3 + 3m_x D_x,
\]
\[
E[x^4] = m_x^4 + 6m_x^2 D_x + 3D_x^2.
\]

from Eqs. 11.35 and 11.38 in Appendix 11.11 to produce

\[
\epsilon \frac{dD_x}{dt} = 2 \left\{ D_x - D_{xy} - \left( \frac{m_x^4 + 6m_x^2 D_x + 3D_x^2}{3} \right) - \gamma D_x + m_x \left( \frac{m_x^3 + 3m_x D_x}{3} \right) \right. \\
\left. + \gamma E[xM_x] - \gamma m_x^2 \right\} \\
= 2 \left\{ D_x - D_{xy} - \gamma D_x - \frac{m_x^4}{3} - 2m_x^2 D_x - D_x^2 + \frac{m_x^4}{3} + m_x^2 D_x + \gamma E[xM_x] - \gamma m_x^2 \right\} \\
= 2 \left\{ D_x - D_{xy} - \gamma D_x - m_x^2 D_x - D_x^2 + \gamma D_{xM_x} \right\}.
\] (9.28)

In Eq. 9.28, \(D_{xM_x} = E[xM_x] - m_x^2\). Although \(E[M_x] = E[x] = m_x\), \(E[xM_x]\) does not necessarily equal \(E[x^2]\); Eq. 9.28 can be rewritten as

\[
\epsilon \frac{dD_x}{dt} = 2 \left\{ D_x (1 - \gamma - m_x^2 - D_x) - D_{xy} + \gamma D_{xM_x} \right\}.
\] (9.29)

Eq. 9.29 contains the cross variance term \(D_{xM_x}\); it is preferable to purely write the expression in terms of variables \(x\) and \(y\). The Cauchy-Schwartz inequality [142, 143] states that if \(\langle \alpha, \beta \rangle\) corresponds to the inner product of vectors \(\alpha\) and \(\beta\),

\[
|\langle \alpha, \beta \rangle|^2 \leq \langle \alpha, \alpha \rangle \cdot \langle \beta, \beta \rangle,
\] (9.30)
or

$$|\langle\alpha, \beta\rangle| \leq \|\alpha\| \cdot \|\beta\|.$$ \hspace{1cm} (9.31)

$\|\alpha\| = \sqrt{\langle\alpha, \alpha\rangle}$ and $\|\beta\| = \sqrt{\langle\beta, \beta\rangle}$ correspond to the norms of the respective vectors.

The inequalities in Eqs. 9.30 and 9.31 apply to probability theory, giving $|E[\alpha\beta]|^2 \leq E[\alpha^2]E[\beta^2]$, which derives a very useful property. Assuming that $E[\alpha] = \mu$ and $E[\beta] = \lambda$,

$$D^2_{\alpha\beta} = |E[(\alpha - \mu)(\beta - \lambda)]|^2$$
$$= |(\alpha - \mu, \beta - \lambda)|^2. \hspace{1cm} (9.32)$$

Using the Cauchy-Schwartz inequality, the above equation can be rewritten as

$$D^2_{\alpha\beta} \leq (\alpha - \mu, \alpha - \mu)(\beta - \lambda, \beta - \lambda)$$
$$= E[(\alpha - \mu)^2]E[(\beta - \lambda)^2]$$
$$= D_{\alpha}D_{\beta}, \hspace{1cm} (9.33)$$

giving

$$D_{\alpha\beta} \leq \sqrt{D_{\alpha}D_{\beta}}. \hspace{1cm} (9.34)$$

$M_x$ is assumed to be the mean value of a sample with size $q$, from the population of $x$ values; the population is assumed to have $N \to \infty$ values. Since the number of samples contributing to the behaviour of Eqs. 7.2 equals population size $N$, the number of samples, $S$, tends to infinity. The standard error of mean tends to zero:

$$\frac{\sigma}{S} \to 0. \hspace{1cm} (9.35)$$

$\sigma$ corresponds to the standard deviation of the population; the sample mean is an unbiased estimate of the population mean, derived from the central limit theorem [144, 145].
Since the original population is under the influence of Gaussian white noise, the Central Limit Theorem allows variances to be approximated, even with small samples, such as

\[ \sigma_{q,N} = \frac{\sigma}{\sqrt{q}} \Rightarrow D_{Mx} = \left( \frac{\sigma_x}{\sqrt{q}} \right)^2 = \frac{\sigma_x^2}{q} = \frac{D_x}{q} . \]  \tag{9.36} \]

Using Eq. 9.34, Eq. 9.36 can result in the following property:

\[
D_{xMx} \leq \sqrt{D_x D_{Mx}} \\
= \sqrt{\frac{D_x \sigma_x^2}{q}} \\
= \frac{\sqrt{D_x^2}}{\sqrt{q}} \quad \text{(since } \sigma_x^2 = D_x \text{)} \\
= \frac{D_x}{\sqrt{q}} . \tag{9.37}
\]

The result in Eq. 9.37 can be substituted for \( D_{xMx} \) in Eq. 9.29 to give

\[
\epsilon \frac{dD_x}{dt} \leq 2 \left\{ D_x (1 - \gamma - m_x^2 - D_x) - D_{xy} + \gamma \frac{D_x}{\sqrt{q}} \right\} . \tag{9.38}
\]

4)

\[
\frac{dD_y}{dt} = \frac{d}{dt} E[(y - m_y)^2] \\
= \frac{d}{dt} \left( E[y^2] - E[y]^2 \right) \\
= \frac{d}{dt} \left( E[y^2] \right) - \frac{d}{dt} \left( E[y]^2 \right) \\
= E \left[ \frac{dy^2}{dt} \right] - 2 E[y] \frac{d}{dt} (E[y]) \\
= E \left[ \frac{dy^2}{dt} \right] - 2 m_y \frac{dm_y}{dt} . \tag{9.39}
\]

A new expression for \( \frac{dy^2}{dt} \), containing the above cumulant variables, would simplify Eq.
9.39. Let processes $\zeta(t)$ and $\eta(t)$ obey stochastic differential equations,

\[
\frac{d\zeta(t)}{dt} = \mu dt + \sigma dW, \\
\frac{d\eta(t)}{dt} = \lambda dt + \nu dW,
\]

where $dW = \sqrt{dt}$. Since the Gaussian white noise terms are additive, the Itô and Stratonovich interpretations of stochastic calculus are equivalent. Using the Itô product rule \[87\], the following can be deduced:

\[
\frac{d}{dt}(\zeta(t)\eta(t)) = \zeta(t)\frac{d\eta(t)}{dt} + \eta(t)\frac{d\zeta(t)}{dt} + \sigma\nu dt. \tag{9.40}
\]

If both processes are the same (i.e. $\zeta(t) = \eta(t)$), Eq. 9.40 becomes

\[
\frac{d}{dt}(\eta(t)\eta(t)) = \frac{d\eta(t)^2}{dt} = \eta(t)\frac{d\eta(t)}{dt} + \eta(t)\frac{d\eta(t)}{dt} + \nu\nu dt = 2\eta(t)\frac{d\eta(t)}{dt} + \nu^2 dt. \tag{9.41}
\]

Substituting the result in Eq. 9.41 with $\frac{d\eta^2}{dt}$ in Eq. 9.39, using $\nu = \sqrt{2T}$, yields

\[
\frac{dD_y}{dt} = 2E\left[y\frac{dy}{dt} + T\right] - 2m_y \frac{dm_y}{dt}. \tag{9.42}
\]

The expression for $\dot{y}$ was included in the original stochastic delay differential system in Eqs. 7.2; a cumulant expression was derived for $\dot{m}_y$ in Eqs. 9.24. Substitution of $\dot{y}$ and
\( \dot{m}_y \) into Eq. 9.42 gives

\[
\frac{dD_y}{dt} = 2E\left[ y\{x + a + K(M_{yr} - M_y) + \sqrt{2T}\xi(t)\} + T \right] - 2m_y\left\{ m_x + a + K(M_{yr} - M_y) \right\}
\]

\[
= 2E\left[ xy + ya + Ky(M_{yr} - M_y) + \sqrt{2T}\xi(t) + T \right] - 2m_xm_y - 2am_y - 2m_yK(M_{yr} - M_y)
\]

\[
= 2E[xy] + 2aE[y] + 2\sqrt{2T}E[\xi(t)] + 2KE[yM_{yr}] - 2KE[yM_y] + 2E[T] - 2m_xm_y
\]

\[
- 2am_y - 2m_yKm_{yr} + 2Km_y^2.
\]  

(9.43)

Using Eq. 11.43 in Appendix 11.11, one can use the following properties:

\[
E[\xi(t)] = 0,
\]

\[
E[yM_{yr}] = D_{yM_{yr}} + m_ym_{yr},
\]

\[
E[yM_y] = D_{yM_y} + m_y^2,
\]

\[
E[xy] = D_{xy} + m_xm_y.
\]

The expression in Eq. 9.43 simplifies to

\[
\frac{dD_y}{dt} = 2\left\{ D_{xy} + T + K(D_{yM_{yr}} - D_{yM_y}) \right\}.
\]  

(9.44)

Eq. 9.44 provides the \( \dot{D}_y \) cumulant expression for neuron-specific feedback (Section 7.3).

With global feedback restrictions (Section 7.1), the terms \( 2KE[yM_{yr}] \) and \( 2KE[yM_y] \) in Eq. 9.43 are replaced with \( 2Km_{yr}E[y] \) and \( 2Km_yE[y] \), enabling further simplifications; global feedback drops \( D_{yM_{yr}} \) and \( D_{yM_y} \) from Eq. 9.44 to give

\[
\frac{dD_y}{dt} = 2(D_{xy} + T),
\]  

(9.45)

which is identical to the equation found in previous studies of global feedback [9] and research excluding a feedback mechanism [10]. Upon reverting to Eq. 9.44, which focuses on individual neuronal feedback, and using the property in Eq. 9.34, the following
expressions are derived:

\[
D_{yM_{yr}} \leq \sqrt{D_y D_{M_{yr}}}
\]

\[
= \sqrt{\frac{D_y \sigma^2_{yr}}{q}}
\]

\[
= \sqrt{\frac{D_y D_{y\tau}}{q}}, \quad \text{(since } \sigma^2_{yr} = D_{y\tau}) \quad (9.46)
\]

and

\[
D_{yM_y} \leq \sqrt{D_y D_{M_y}}
\]

\[
= \sqrt{\frac{D_y \sigma^2_y}{q}}
\]

\[
= \sqrt{\frac{D^2_y}{q}} \quad \text{(since } \sigma^2_y = D_y) \quad (9.47)
\]

\[
= \frac{D_y}{\sqrt{q}}. \quad (9.48)
\]

Substituting the result from Eq. 9.46 into Eq. 9.44 gives

\[
\frac{dD_y}{dt} \leq 2 \left( D_{xy} + T + K \left( \sqrt{\frac{D_y D_{y\tau}}{q}} - D_{yM_y} \right) \right). \quad (9.49)
\]

However, substituting the result from Eq. 9.48 into Eq. 9.44 gives

\[
\frac{dD_y}{dt} \geq 2 \left( D_{xy} + T + K \left( D_{yM_{yr}} - \frac{D_y}{\sqrt{q}} \right) \right). \quad (9.50)
\]

It can be deduced from Eqs. 9.49 and 9.50 that the following condition is true:

\[
D_{yM_{yr}} - \frac{D_{y}}{\sqrt{q}} \leq \sqrt{\frac{D_y D_{y\tau}}{q}} - D_{yM_y}. \quad (9.51)
\]

The condition in Eq. 9.51 may prove useful for further investigations as upper and lower boundaries are provided for \( \frac{dD_y}{dt} \).
\[ \epsilon \frac{dD_{xy}}{dt} = \epsilon E \left[ \frac{d}{dt} \left( (x - m_x)(y - m_y) \right) \right] \]

\[ = \epsilon E \left[ (x - m_x) \frac{d}{dt} (y - m_y) + (y - m_y) \frac{d}{dt} (x - m_x) \right] \]

\[ = \epsilon E \left[ (x - m_x) \left( \frac{dy}{dt} - \frac{dm_y}{dt} \right) \right] + E \left[ (y - m_y) \left( \frac{dx}{dt} - \epsilon \frac{dm_x}{dt} \right) \right]. \quad (9.52) \]

The two sets of bracketed terms in Eq. 9.52 can be considered separately.

(a)

Substituting \( \dot{y} \) in Eq. 7.2 and \( \dot{m_y} \) in Eq. 9.24 into Eq. 9.52 gives

\[ \frac{dy}{dt} - \frac{dm_y}{dt} = \left\{ x + a + K(M_{y\tau} - M_y) + \sqrt{2T}\xi(t) \right\} - \left\{ m_x + a + K(m_{y\tau} - m_y) \right\} \]

\[ = x + \sqrt{2T}\xi(t) - m_x + K(M_{y\tau} - M_y) - K(m_{y\tau} - m_y). \quad (9.53) \]

This expression is multiplied by \( \epsilon(x - m_x) \) and the expectation is calculated, giving

\[ \epsilon E \left[ (x - m_x) \left( \frac{dy}{dt} - \frac{dm_y}{dt} \right) \right] = \epsilon E \left[ (x - m_x) \left\{ x + \sqrt{2T}\xi(t) - m_x + K(M_{y\tau} - M_y) \right\} \right. \]

\[ - \left. K(m_{y\tau} - m_y) \right\} \]

\[ = \epsilon \left\{ E[(x - m_x)^2] + \sqrt{2T}E[x\xi(t)] - \sqrt{2T}m_x E[\xi(t)] \right\} \]

\[ + KE[xM_{y\tau}] - KE[xM_y] - Km_x E[M_{y\tau}] + Km_x E[M_y] \]

\[ - Km_{y\tau} E[x] + Km_y E[x] + Km_x m_{y\tau} - Km_x m_y \right\}. \quad (9.54) \]

Many simplifications can be made using the properties from Eqs. 11.32 and 11.43 in
Appendix 11.1. The following properties

\[ E[(x - m_x)^2] = D_x, \]

\[ E[x\xi(t)] = E[\xi(t)] = 0, \]

\[ E[xM_{yr}] = D_{xM_{yr}} + m_xm_{yr}, \]

\[ E[xM_y] = D_{xM_y} + m_xm_y, \]

simplify Eq. 9.54 to

\[ \epsilon E\left[ (x - m_x)\left( \frac{dy}{dt} - \frac{dm_y}{dt} \right) \right] = \epsilon \left\{ D_x + K(D_{xM_{yr}} - D_{xM_y}) \right\}. \] (9.55)

When global feedback is applied (Section 7.1), covariance terms \( D_{xM_{yr}} \) and \( D_{xM_y} \) are cancelled, leaving

\[ \epsilon E\left[ (x - m_x)\left( \frac{dy}{dt} - \frac{dm_y}{dt} \right) \right] = \epsilon D_x. \] (9.56)

(b)

The second bracketed term in Eq. 9.52 can now be considered; substituting the known results for \( \epsilon \dot{x} \) from Eqs. 7.2 and \( \epsilon m_x \) from Eq. 9.22 into this term gives

\[ \epsilon \frac{dx}{dt} - \epsilon \frac{dm_x}{dt} = \left\{ x - \frac{x^3}{3} - y + \gamma(M_x - x) \right\} - \left\{ m_x - \frac{m_x^3}{3} - m_y - m_mD_x \right\} \]

\[ = x - \frac{x^3}{3} - y + \gamma(M_x - x) - m_x + \frac{m_x^3}{3} + m_y + m_mD_x. \] (9.57)

This expression is multiplied by \((y - m_y)\) and the expectation is calculated, giving

\[ E\left[ (y - m_y)\left( \epsilon \frac{dx}{dt} - \epsilon \frac{dm_x}{dt} \right) \right] = E\left[ (y - m_y)\left( x - \frac{x^3}{3} - y \right) - \gamma(x - M_x)(y - m_y) \right. \]

\[ - m_x(y - m_y) + \frac{m_x^3}{3} (y - m_y) + m_y(y - m_y) \]

\[ + m_xD_x(y - m_y) \]
\[ E \left[ xy - x m_y - \frac{x^3 y}{3} + m_y \frac{x^3}{3} - y^2 + y m_y - \gamma (x - M_x) (y - m_y) - m_x y + m_x m_y \right] \\
+ y \frac{m_y^3}{3} - m_y \frac{m_x^3}{3} + y m_y - m_y^2 + y m_x D_x - m_x m_y D_x \]

\[ = E [xy] - m_y E [x] - E \left[ \frac{x^3 y}{3} \right] + m_y E \left[ \frac{x^3}{3} \right] - E [y^2] + m_y E [y] - \gamma E [(x - M_x) (y - m_y)] \\
- m_x E [y] + m_x m_y + \frac{m_x^3}{3} E [y] - m_y \frac{m_x^3}{3} + m_y E [y] - m_x^2 + m_x D_x E [y] - m_x m_y D_x \]

\[ = D_{xy} - D_y - \gamma \left( E [xy] - E [y M_x] - m_y E [x] + m_y E [M_x] \right) - E \left[ \frac{x^3 y}{3} \right] + m_y E \left[ \frac{x^3}{3} \right]. \quad (9.58) \]

Properties \( D_{xy} = E [xy] - E [x] E [y] \) and \( D_x = E [x^2] - E [x]^2 \) are used in the above calculations; using properties from Eqs. 11.35, 11.43, and 11.49 in Appendix 11.11,

\[
\begin{align*}
E [xy] &= D_{xy} + m_x m_y, \\
E [y M_x] &= D_{y M_x} + m_y m_x, \\
E [x^3] &= m_x^3 + 3 m_x D_x, \\
E [x^3 y] &= m_x^3 m_y + 3 m_x^2 D_{xy} + 3 m_x m_y D_x + 3 D_x D_{xy},
\end{align*}
\]

the remaining expectations are evaluated as

\[
\begin{align*}
E \left[ (y - m_y) \left( \epsilon \frac{dx}{dt} - \epsilon \frac{dm_x}{dt} \right) \right] &= D_{xy} - D_y - \gamma (D_{xy} - D_{y M_x}) \\
&\quad - \left( m_y \frac{m_x^3}{3} + m_x^2 D_{xy} + m_x m_y D_x + D_x D_{xy} \right) \\
&\quad + m_y \left( \frac{m_x^3 + 3 m_x D_x}{3} \right) \\
&= D_{xy} - D_y - \gamma (D_{xy} - D_{y M_x}) - m_y \frac{m_x^3}{3} - m_x^2 D_{xy} - m_x m_y D_x \\
&\quad - D_x D_{xy} + m_y \frac{m_x^3}{3} + m_x m_y D_x \\
&= D_{xy} - D_y - \gamma (D_{xy} - D_{y M_x}) - m_x^2 D_{xy} - D_x D_{xy}. \quad (9.59)
\end{align*}
\]
Eq. 9.59 can be rearranged as

\[ E \left[ (y - m_y) \left( \epsilon \frac{dx}{dt} - \epsilon \frac{dm_x}{dt} \right) \right] = D_{xy} (1 - \gamma - m_x^2 - D_x) - D_y + \gamma D_{yM_x}. \]  

(9.60)

Combining the results from (a) (Eq. 9.55) and (b) (Eq. 9.60) reveals solution

\[ \epsilon \frac{dD_{xy}}{dt} = \epsilon \left\{ D_x + K (D_{xM_y} - D_{xM_y}) \right\} + D_{xy} (1 - \gamma - m_x^2 - D_x) - D_y + \gamma D_{yM_x}. \]  

(9.61)

Applying global feedback provides the alternative solution

\[ \epsilon \frac{dD_{xy}}{dt} = \epsilon D_x + D_{xy} (1 - \gamma - m_x^2 - D_x) - D_y + \gamma D_{yM_x}. \]  

(9.62)

Using Eq. 9.34 can give

\[ D_{yM_x} \leq \sqrt{D_y D_{M_x}} \]

\[ = \sqrt{\frac{D_y \sigma_x^2}{q}} \]

\[ = \sqrt{\frac{D_x D_y}{q}}, \quad \text{(since } \sigma_x^2 = D_x) \]  

(9.63)

Substituting Eq. 9.63 into Eq. 9.61 results in

\[ \epsilon \frac{dD_{xy}}{dt} \leq \epsilon \left\{ D_x + K (D_{xM_y} - D_{xM_y}) \right\} + D_{xy} (1 - \gamma - m_x^2 - D_x) - D_y + \sqrt{\frac{D_x D_y}{q}}. \]  

(9.64)

Eq. 9.34 can derive

\[ D_{xM_y} \leq \sqrt{D_x D_{M_y}} \]

\[ = \sqrt{\frac{D_x \sigma_{y}^2}{q}} \]

\[ = \sqrt{\frac{D_x D_{y}^2}{q}}, \quad \text{(since } \sigma_{y}^2 = D_{y}^2) \]  

(9.65)
and

\[ D_{xM_y} \leq \sqrt{D_x D_{M_y}} \]
\[ = \sqrt{\frac{D_x \sigma_y^2}{q}} \]
\[ = \sqrt{\frac{D_x D_y}{q}} \] (since \( \sigma_y^2 = D_y \)) \quad (9.66)
\[ = D_{yM_x}. \] \quad (9.67)

Substituting Eq. 9.65 into Eq. 9.61 results in

\[ \epsilon \frac{dD_{xy}}{dt} \leq \epsilon \left\{ D_x + K \left( \sqrt{\frac{D_x D_y \tau}{q}} - D_{xM_y} \right) \right\} + D_{xy}(1 - \gamma - m^2_x - D_x) - D_y + \gamma D_{yM_x}. \] \quad (9.68)

However, substituting Eq. 9.67 into Eq. 9.61 gives

\[ \epsilon \frac{dD_{xy}}{dt} \geq \epsilon \left\{ D_x + K \left( D_{xM_y} - \sqrt{\frac{D_x D_y}{q}} \right) \right\} + D_{xy}(1 - \gamma - m^2_x - D_x) - D_y + \gamma D_{yM_x}. \] \quad (9.69)

The system of deterministic differential equations (Eqs. 9.22, 9.24, 9.29, 9.44 and 9.61), which represents the dynamics of the cumulants developed from the stochastic differential equations (Eqs. 7.2), reads as

\[ \frac{\epsilon}{dt} \frac{dm_x}{dx} = m_x - \frac{m^3_x}{3} - m_y - m_x D_x, \]
\[ \frac{\epsilon}{dt} \frac{dm_y}{dx} = m_x + a + K(m_{y\tau} - m_y), \]
\[ \frac{\epsilon}{dt} \frac{dD_x}{dx} = 2 \left\{ D_x(1 - \gamma - m^2_x - D_x) - D_{xy} + \gamma D_{xM_y} \right\}, \] \quad (9.70)
\[ \frac{\epsilon}{dt} \frac{dD_y}{dx} = 2 \left\{ D_{xy} + T + K(D_{yM_y} - D_{yM_y}) \right\}, \]
\[ \frac{\epsilon}{dt} \frac{dD_{xy}}{dx} = \epsilon \left\{ D_x + K(D_{xM_y} - D_{xM_y}) \right\} + D_{xy}(1 - \gamma - m^2_x - D_x) - D_y + \gamma D_{yM_x}. \]
When global feedback is applied, the system in Eqs. 9.70 can be simplified to

\[
\begin{align*}
\epsilon \frac{dm_x}{dt} &= m_x - m_x^3 - m_y - m_D D_x, \\
\frac{dm_y}{dt} &= m_x + a + K(m_y - m_y), \\
\epsilon \frac{dD_x}{dt} &= 2\left\{D_x(1 - \gamma - m_x^2 - D_x) - D_{xy} + \gamma D_{xM_x}\right\}, \\
\frac{dD_y}{dt} &= 2(D_{xy} + T), \\
\epsilon \frac{dD_{xy}}{dt} &= \epsilon D_x + D_{xy}(1 - \gamma - m_x^2 - D_x) - D_y + \gamma D_{yM_x}.
\end{align*}
\]  
(9.71)

The above derived systems in Eqs. 9.70 and 9.71 are the same as those previously found in the mean-field coupling scenario [9, 10] with the exception of extra terms \(D_{xM_x}, D_{yM_x}\), and \(D_{M_x}\); as sub-maximal connectivity is considered, the additional terms \(D_{xM_x}, D_{yM_x}\), and \(D_{M_x}\) must also be derived.

6)

\[
\epsilon \frac{dD_{xM_x}}{dt} = \epsilon E\left[\frac{d}{dt}\left((x - m_x)(M_x - m_x)\right)\right].
\]  
(9.72)

The differentiation product rule can be utilised to expand the above equation, giving

\[
\epsilon \frac{dD_{xM_x}}{dt} = \epsilon E\left[\frac{d}{dt}\left((x - m_x)\frac{d}{dt}(M_x - m_x) + (M_x - m_x)\frac{d}{dt}(x - m_x)\right)\right]
= E\left[\left(x - m_x\right)\left(\epsilon \frac{dM_x}{dt} - \epsilon \frac{dm_x}{dt}\right) + (M_x - m_x)\left(\epsilon \frac{dx}{dt} - \epsilon \frac{dm_x}{dt}\right)\right].
\]  
(9.73)

For Eq. 9.73, one must find an expression for \(\dot{M}_x\). The following description can be deduced from Eq. 6.1:

\[
\frac{dM_x}{dt} = \frac{d}{dt}\left(\frac{1}{q} \sum_{j=1}^{q} x_{g(i,j)}\right)
= \frac{1}{q} \sum_{j=1}^{q} \frac{dx_{g(i,j)}}{dt}.
\]  
(9.74)
When \( q \) is small
\[
\frac{dM_x}{dt} \approx \frac{dx}{dt},
\] (9.75)
and when \( q \) is large
\[
\frac{dM_x}{dt} \approx \frac{dm_x}{dt}.
\] (9.76)

where \( \frac{dx}{dt} \) is the same as \( \frac{dx}{dt} \) from Eqs. 7.2 and \( \frac{dm_x}{dt} \) occurs in the expression from Eq. 9.22.

The cases for both small and large \( q \) will now be considered.

(a) For small \( q \),
\[
\epsilon \frac{dD_x M_x}{dt} \approx E \left[ (x - m_x) \left( \epsilon \frac{dx}{dt} - \epsilon \frac{dm_x}{dt} \right) + (M_x - m_x) \left( \epsilon \frac{dx}{dt} - \epsilon \frac{dm_x}{dt} \right) \right].
\] (9.77)

Substituting \( \epsilon \dot{x} \) from Eqs. 7.2 and \( \epsilon \dot{m}_x \) from Eq. 9.22 into the above equation yields
\[
\epsilon \frac{dD_x M_x}{dt} \approx E \left[ (x - m_x) \left( \left( x - \frac{x^3}{3} - y + \gamma (M_x - x) \right) - \left( m_x - \frac{m_x^3}{3} - m_y - m_x D_x \right) \right) 
+ (M_x - m_x) \left( \left( x - \frac{x^3}{3} - y + \gamma (M_x - x) \right) - \left( m_x - \frac{m_x^3}{3} - m_y - m_x D_x \right) \right) \right].
\] (9.78)

Expanding the brackets leaves
\[
\epsilon \frac{dD_x M_x}{dt} \approx E \left[ x^2 - \frac{x^4}{3} - xy + \gamma x M_x - \gamma x^2 - x m_x + \frac{x^3}{3} m_x + y m_x - \gamma M x m_x + \gamma x m_x - x m_x 
+ x \frac{m_x^3}{3} + x m_y + x m_x D_x + m_x^2 - \frac{m_x^4}{3} - m_x m_y + m_x^2 D_x + x M_x - \frac{x^3}{3} - y M_x 
+ \gamma M_x^2 - \gamma x M_x - x m_x + \frac{x^3}{3} m_x + y m_x - \gamma M_x m_x + \gamma x m_x - M_x m_x + M_x \frac{m_x^3}{3} 
+ M_x m_y + M_x m_x D_x + m_x^2 - \frac{m_x^4}{3} - m_x m_y + m_x^2 D_x \right],
\] (9.79)
which may be simplified to

\[
\epsilon\frac{dD_{xM_x}}{dt} \approx E[x^2] - \frac{1}{3}E[x^4] - E[xy] - \gamma E[x^2] - 3m_x E[x] + \frac{2}{3}m_x E[x^3] + 2m_x E[y] \\
- 2\gamma m_x E[M_x] + 2\gamma m_x E[x] + \frac{1}{3}m_x^3 E[x] + m_y E[x] + m_x D_x E[x] + 2m_x^2 - \frac{2}{3}m_x^3 \\
- 2m_x m_y - 2m_x^2 D_x + E[xM_x] - \frac{1}{3}E[x^3 M_x] - E[yM_x] + \gamma E[M_x^2] - m_x E[M_x] \\
+ \frac{1}{3}m_x^3 E[M_x] + m_y E[M_x] + m_x D_x E[M_x]. \\
\]  

(9.80)

The following polynomials found in Eqs. 11.30, 11.32, 11.35, 11.38, 11.43, and 11.49 in Appendix 11.11 can be used to express various expectations in Eq. 9.80:

\[
E[x] = m_x, \\
E[y] = m_y, \\
E[M_x] = m_x, \\
E[x^2] = D_x + m_x^2, \\
E[M_x^2] = D_{M_x} + m_x^2, \\
E[x^3] = m_x^3 + 3m_x D_x, \\
E[x^4] = m_x^4 + 6m_x^2 D_x + 3D_x^2, \\
E[xy] = D_{xy} + m_x m_y, \\
E[xM_x] = D_{xM_x} + m_x^2, \\
E[yM_x] = D_{yM_x} + m_x m_y, \\
E[x^3 M_x] = m_x^4 + 3m_x^2 D_{xM_x} + 3m_x^2 D_x + 3D_x D_{xM_x}. \\
\]
These expressions can be substituted for the raw moments into Eq. 9.80 to give

\[ \epsilon \frac{dD_{z,M_x}}{dt} \approx D_x + m_x^2 - \frac{1}{3}(m_x^4 + 6m_x^2D_x + 3D_x^2) - (D_{xy} + m_xm_y) - \gamma(D_x + m_x^2) - 3m_x^2 \\
+ \frac{2}{3}m_x(m_x^3 + 3m_xD_x) + 2m_xm_y - 2\gamma m_x^2 + 2\gamma m_x^2 + \frac{1}{3}m_x^4 + m_xm_y + m_x^2D_x + 2m_x^2 \\
- \frac{2}{3}m_x^4 - 2m_xm_y - 2m_x^2D_x + (D_{z,M_x} + m_x^2) \\
- \frac{1}{3}(m_x^4 + 3m_x^2D_{z,M_x} + 3m_x^2D_x + 3D_xD_{z,M_x}) - (D_{y,M_x} + m_xm_y) + \gamma(D_{z,M_x} + m_x^2) \\
- m_x^2 + \frac{1}{3}m_x^4 + m_xm_y + m_x^2D_x. \] (9.81)

After cancellation, the equation can be rewritten as

\[ \epsilon \frac{dD_{z,M_x}}{dt} \approx D_x - m_x^2D_x - D_x^2 - D_{xy} - \gamma D_x + D_{z,M_x} - m_x^2D_{z,M_x} - D_xD_{z,M_x} - D_{y,M_x} + \gamma D_{z,M_x}. \] (9.82)

Thus, for small \( q \), the following approximate equation is obtained:

\[ \epsilon \frac{dD_{z,M_x}}{dt} \approx D_x(1 - \gamma - m_x^2 - D_x) - D_{xy} + D_{z,M_x}(1 - m_x^2 - D_x) - D_{y,M_x} + \gamma D_{z,M_x}. \] (9.83)

(b) For large \( q \),

\[ \epsilon \frac{dD_{z,M_x}}{dt} \approx E \left[ (x - m_x) \left( \epsilon \frac{dm_x}{dt} - \epsilon \frac{dx}{dt} \right) + (M_x - m_x) \left( \epsilon \frac{dx}{dt} - \epsilon \frac{dm_x}{dt} \right) \right] \\
= E \left[ (M_x - m_x) \left( \epsilon \frac{dx}{dt} - \epsilon \frac{dm_x}{dt} \right) \right]. \] (9.84)
Substitute $\epsilon \dot{x}$ from Eqs. 7.2 and $\epsilon m_x$ from Eq. 9.22 into the above equation to give

$$
\epsilon \frac{d D_x M_x}{dt} \approx E \left[ (M_x - m_x) \left( x - \frac{x^3}{3} - y + \gamma (M_x - x) - \left\{ m_x - \frac{m^3}{3} - m_y - m_x D_x \right\} \right) \right]
$$

$$
= E \left[ x M_x - \frac{1}{3} x^3 M_x - y M_x + \gamma M_x^2 - \gamma x M_x - M_x m_x + \frac{1}{3} M_x m_x^3 + M_x m_y + M_x m_x D_x 
- x m_x + \frac{1}{3} x^3 m_x + y m_x - \gamma M_x m_x + \gamma x m_x + m_x^2 - \frac{1}{3} m_x^3 - m_x m_y - m_x D_x \right]
$$

$$
= E[x M_x] - \frac{1}{3} E[x^3 M_x] - E[y M_x] + \gamma E[M_x^2] - \gamma E[x M_x] - m_x E[M_x]
+ \frac{1}{3} m_x^3 E[M_x] + m_y E[M_x] + m_x D_x E[M_x] - m_x E[x] + \frac{1}{3} m_x E[x^3] + m_x E[y]
- \gamma m_x E'[M_x] + \gamma m_x E'[x] + m_x^2 - \frac{1}{3} m_x^4 - m_x m_y - m_x D_x.
$$

(9.85)

The following polynomials found in Eqs. 11.30, 11.32, 11.35, 11.43, and 11.49 in Appendix 11.11 can be used to express various expectations in Eq. 9.85:

$$
E[x] = m_x,
$$
$$
E[y] = m_y,
$$
$$
E[M_x] = m_x,
$$
$$
E[M_x^2] = D_{M_x} + m_x^2,
$$
$$
E[x^3] = m_x^3 + 3m_x D_x,
$$
$$
E[x M_x] = D_{x M_x} + m_x^2,
$$
$$
E[y M_x] = D_{y M_x} + m_x m_y,
$$
$$
E[x^3 M_x] = m_x^4 + 3m_x^2 D_{x M_x} + 3m_x^2 D_x + 3D_x D_{x M_x}.
$$
These expressions can be substituted for the raw moments into Eq. 9.85 to give

\[
\epsilon \frac{dD_{xM_x}}{dt} \approx D_{xM_x} + m_{x}^2 - \frac{1}{3}(m_{x}^4 + 3m_{x}^2D_{xM_x} + 3m_{x}^2D_x + 3D_{x}D_{xM_x}) - (D_{yM_x} + m_xm_y) \\
+ \gamma(D_{xM_x} + m_{x}^2) - \gamma(D_{xM_x} + m_{x}^2) - m_{x}^2 + \frac{1}{3}m_{x}^4 + m_xm_y + m_{x}^2D_x - m_{x}^2 \\
+ \frac{1}{3}m_x(m_{x}^3 + 3m_xD_x) + m_xm_y - \gamma m_{x}^2 + \gamma m_{x}^2 + m_{x}^2 - \frac{1}{3}m_{x}^4 - m_xm_y - m_{x}^2D_x \\
= D_{xM_x} - m_{x}^2D_{xM_x} - D_{x}D_{xM_x} - D_{yM_x} + \gamma D_{M_x} - \gamma D_{zM_x}. \tag{9.86}
\]

Thus, for large \( q \), the approximate equation for \( D_{xM_x} \) reads as

\[
\epsilon \frac{dD_{xM_x}}{dt} \approx D_{xM_x}(1 - \gamma - m_{x}^2 - D_x) - D_{yM_x} + \gamma D_{M_x}. \tag{9.87}
\]

7) \[
\epsilon \frac{dD_{yM_x}}{dt} = \epsilon E \left[ \frac{d}{dt} \left( (y - m_y)(M_x - m_x) \right) \right] \\
= \epsilon E \left[ (y - m_y) \frac{d}{dt}(M_x - m_x) + (M_x - m_x) \frac{d}{dt}(y - m_y) \right] \\
= E \left[ (y - m_y) \left( \frac{dM_x}{dt} - \frac{dm_x}{dt} \right) + \epsilon(M_x - m_x) \left( \frac{dy}{dt} - \frac{dm_y}{dt} \right) \right]. \tag{9.88}
\]

The cases for small and large \( q \) will again be considered separately.

(a) For small \( q \), \[
\epsilon \frac{dD_{yM_x}}{dt} \approx E \left[ (y - m_y) \left( \frac{dx}{dt} - \frac{dm_x}{dt} \right) + \epsilon(M_x - m_x) \left( \frac{dy}{dt} - \frac{dm_y}{dt} \right) \right]. \tag{9.89}
\]
Substituting $\epsilon \dot{x}$ from Eqs. 7.2 and $\epsilon \dot{m}_x$ from Eq. 9.22 into the above equation provides

$$\epsilon \frac{dD_{yM_x}}{dt} \approx E \left[ (y - m_y) \left( x - \frac{x^3}{3} - y + \gamma (M_x - x) - \left( m_x - \frac{m_x^3}{3} - m_y - m_x D_x \right) \right) \right.$$

$$+ \epsilon (M_x - m_x) \left( x + a + K (M_{yt} - M_y) + \sqrt{2T} \xi (t) - \{m_x + a + K (m_{yt} - m_y)\} \right) \right]$$

$$= E \left[ xy - \frac{1}{3} x^3 y - y^2 + \gamma y M_x - \gamma xy - y m_x + \frac{1}{3} y m_x^3 ight.$$ 

$$+ y m_y + m_y x D_x - x m_y + \frac{1}{3} x^3 m_y + y m_y - \gamma M_x m_y + \gamma x m_y + m_x m_y - \frac{1}{3} m_x^3 m_y$$

$$- m_y^2 - m_x m_y D_x + \epsilon (x M_x + a M_x + K M_x M_{yt} - K M_y M_y) + \sqrt{2T} M_x \xi (t)$$

$$- M_x m_x - a M_x - K M_x m_{yt} + K M_y m_y - x m_x - a m_x - K M_{yt} m_x + K M_y m_x$$

$$- \sqrt{2T} m_x \xi (t) + m_x^2 + a m_x + K m_x m_{yt} - K m_x m_y \right]$$

$$= E [xy] - \frac{1}{3} E [x^3 y] - E [y^2] + \gamma E [y M_x] - \gamma E [xy] - m_x E [y] + \frac{1}{3} m_x^3 E [y]$$

$$+ m_y E [y] + m_x D_x E [y] - m_y E [x] + \frac{1}{3} m_y E [x^3] + m_y E [y] - \gamma m_y E [M_x]$$

$$+ \gamma m_y E [x] + m_x m_y - \frac{1}{3} m_x^3 m_y - m_y^2 - m_x m_y D_x + \epsilon E [x M_x] + a e E [M_x]$$

$$+ \epsilon K E [M_x M_{yt}] - \epsilon K E [M_t M_y] + \sqrt{2T} e E [M_x \xi (t)] - e m_x E [M_x] - a e E [M_y]$$

$$- \epsilon K m_{yt} E [M_x] + \epsilon K m_y E [M_x] - e m_x E [x] - a e m_x - \epsilon K m_x E [M_{yt}]$$

$$+ \epsilon K m_x E [M_y] - \sqrt{2T} e m_x E [\xi (t)] + e m_x^2 + a e m_x + \epsilon K m_x m_{yt} - \epsilon K m_x m_y$$.

(9.90)

The following polynomials found in Eqs. 11.30, 11.32, 11.35, 11.43, and 11.49 in Appendix 11.11 can be used to express various expectations in Eq. 9.90:

$$E [x] = m_x,$$

$$E [y] = m_y,$$

$$E [M_x] = m_x,$$

$$E [M_y] = m_y,$$

$$E [M_{yt}] = m_{yt}.$$
\[ E[\xi(t)] = 0, \]
\[ E[y^2] = D_y + m_y^2, \]
\[ E[x^3] = m_x^3 + 3m_xD_x, \]
\[ E[xy] = D_{xy} + m_xm_y, \]
\[ E[xM_x] = D_{xM_x} + m_x^2, \]
\[ E[yM_x] = D_{yM_x} + m_xm_y, \]
\[ E[M_xM_y] = D_{M_xM_y} + m_xm_y, \]
\[ E[M_x\xi(t)] = D_{M_x\xi(t)} = 0, \]
\[ E[x^3y] = m_x^3m_y + 3m_x^2D_{xy} + 3m_xm_yD_x + 3D_xD_{xy}. \]

These expressions can be substituted for the raw moments into Eq. 9.90 to give

\[
\epsilon \frac{dD_{yM_x}}{dt} = D_{xy} + m_xm_y - \frac{1}{3}(m_x^3m_y + 3m_x^2D_{xy} + 3m_xm_yD_x + 3D_xD_{xy}) - (D_y + m_y^2)
+ \gamma(D_{yM_x} + m_xm_y) - \gamma(D_{xy} + m_xm_y) - m_xm_y + \frac{1}{3}m_x^3m_y + m_y^2 + m_xm_yD_x
- m_xm_y + \frac{1}{3}m_y(m_x^3 + 3m_xD_x) + m_y^2 - \gamma m_xm_y + \gamma m_xm_y + m_xm_y - \frac{1}{3}m_x^3m_y
- m_y^2 - m_xm_yD_x + \epsilon(D_{xM_x} + m_y^2) + aem_x + \epsilon K(D_{M_xM_y} + m_xm_y)
- \epsilon K(D_{M_xM_y} + m_xm_y) - em_x^2 - aem_x - \epsilon Km_xm_y + \epsilon Km_xm_y - em_x^2 - aem_x
- \epsilon Km_xm_y + \epsilon Km_xm_y + em_x^2 + aem_x + \epsilon Km_xm_y - \epsilon Km_xm_y
= D_{xy} - m_x^2D_{xy} - D_y + \gamma D_{yM_x} - \gamma D_{xy} + \epsilon D_{xM_x} + \epsilon K D_{M_xM_y} - \epsilon K D_{M_xM_y}.
\] (9.91)

For small \( q \),

\[
\epsilon \frac{dD_{yM_x}}{dt} \approx D_{xy}(1 - m_x^2 - D_x) - D_y + \gamma D_{yM_x} + \epsilon D_{xM_x} + \epsilon K (D_{M_xM_y} - D_{M_xM_y}). \] (9.92)
(b) For large \( q \),

\[
\epsilon \frac{dD_{yM}}{dt} \approx E \left[ (y - m_y) \left( \epsilon \frac{dm_y}{dt} - \epsilon \frac{dm_x}{dt} \right) + \epsilon(M_x - m_x) \left( \frac{dy}{dt} - \frac{dm_y}{dt} \right) \right] \\
= E \left[ \epsilon(M_x - m_x) \left( \frac{dy}{dt} - \frac{dm_y}{dt} \right) \right].
\]  

(9.93)

Substitute \( \epsilon \dot{y} \) from Eqs. 7.2 and \( \epsilon \dot{m}_y \) from Eq. 9.24 into the above equation to give

\[
\epsilon \frac{dD_{yM}}{dt} \approx E \left[ \epsilon(M_x - m_x) \left( x + a + K(M_{yT} - M_y) + \sqrt{2T} \xi(t) - \{m_x + a + K(m_{yT} - m_y)\} \right) \right] \\
= E \left[ \epsilon x M_x + aM_x + KM_x M_{yT} - KM_x M_y + \sqrt{2T} M_x \xi(t) - M_x m_x - aM_x \\
- KM_x m_{yT} + KM_x m_y - x m_x - a m_x - KM_y m_x + KM_y m_x - \sqrt{2T} \xi(t) m_x + m_x^2 \\
+ a m_x + K m_x m_{yT} - K m_x m_y \right] \\
= \epsilon \left( E[x M_x] + aE[M_x] + K E[M_x M_{yT}] - K E[M_x M_y] + \sqrt{2T} E[M_x \xi(t)] - m_x E[M_x] \\
- a E[M_x] - K m_{yT} E[M_x] + K m_y E[M_x] - m_x E[x] - a m_x - K m_x E[M_{yT}] \\
+ K m_x E[M_y] - \sqrt{2T} m_x E[\xi(t)] + m_x^2 + a m_x + K m_x m_{yT} - K m_x m_y \right). 
\]  

(9.94)

The following polynomials found in Eqs. 11.30 and 11.43 in Appendix 11.11 can be used to express various expectations in Eq. 9.94:

\[
E[x] = m_x, \\
E[M_x] = m_x, \\
E[M_y] = m_y, \\
E[M_{yT}] = m_{yT}, \\
E[\xi(t)] = 0, \\
E[x M_x] = D_{xM_x} + m_x^2, \\
E[M_x M_y] = D_{M_x M_y} + m_x m_y.
\]
\[ E[M_x M_y^\tau] = D_{M_x M_y^\tau} + m_x m_y^\tau, \]
\[ E[M_x \xi(t)] = 0. \]

These expressions can be substituted for the raw moments into Eq. 9.94 to give

\[
\epsilon \frac{d D_{y M_x}}{d t} \approx \epsilon \left( D_{x M_x} + m_x^2 + am_x + K(D_{M_x M_y^\tau} + m_x m_y^\tau) - K(D_{M_x M_y} + m_x m_y) - m_x^2 - am_x \right. \\
\left. - K m_x m_y - K m_x^2 - m_x^2 - am_x - K m_x m_y + K m_x m_y + m_x^2 + am_x \right) \\
\left. + K m_x m_y - K m_x m_y \right) \\
= \epsilon \left( D_{x M_x} + K D_{M_x M_y^\tau} - K D_{M_x M_y} \right). \tag{9.95}
\]

For large \( q \),

\[
\epsilon \frac{d D_{y M_x}}{d t} \approx \epsilon \{ D_{x M_x} + K (D_{M_x M_y^\tau} - D_{M_x M_y}) \}. \tag{9.97}
\]

When global feedback is applied, the \( D_{M_x M_y^\tau} \) and \( D_{M_x M_y} \) terms are dropped. Thus, for small \( q \), the approximate equation is

\[
\epsilon \frac{d D_{y M_x}}{d t} \approx D_{x y} (1 - \gamma - m_x^2 - D_y) - D_y + \gamma D_{y M_x} + \epsilon D_{x M_x}, \tag{9.98}
\]

and for large \( q \),

\[
\epsilon \frac{d D_{y M_x}}{d t} \approx \epsilon D_{x M_x}. \tag{9.99}
\]
\[ \epsilon \frac{dD_{M_x}}{dt} = \epsilon E \left[ \frac{d}{dt} (M_x - m_x)^2 \right] \]
\[ = \epsilon E \left[ 2(M_x - m_x) \frac{d}{dt} (M_x - m_x) \right] \]
\[ = E \left[ 2(M_x - m_x) \left( \epsilon \frac{dM_x}{dt} - \epsilon \frac{dm_x}{dt} \right) \right]. \quad (9.100) \]

For large \( q \) where \( \dot{M}_x \approx \dot{m}_x \), \( \epsilon \dot{D}_{M_x} \approx 0 \) can be derived; for small \( q \) where \( \dot{M}_x \approx \dot{x} \),
\[ \epsilon \frac{dM_x}{dt} \approx E \left[ 2(M_x - m_x) \left( \frac{dx}{dt} - \epsilon \frac{dm_x}{dt} \right) \right]. \quad (9.101) \]

Substituting \( \epsilon \dot{x} \) from Eqs. 7.2 and \( \epsilon \dot{m}_x \) from Eq. 9.22 into the above equation gives
\[ \epsilon \frac{dD_{M_x}}{dt} \approx E \left[ 2(M_x - m_x) \left( \frac{dx}{dt} - \epsilon \frac{dm_x}{dt} \right) \right] \]
\[ = 2 \epsilon \left[ xM_x - \frac{1}{3} x^3 M_x - yM_x + \gamma M_x^2 - \gamma xM_x - M_x m_x + \frac{1}{3} M_x m_x^3 + M_x m_y + M_x m_x D_x \right. \]
\[ - \left. x m_x + \frac{1}{3} x^3 m_x + y m_x - \gamma M_x m_x + \gamma x m_x + m_x^2 - \frac{1}{3} m_x^4 - m_x m_y - m_x^2 D_x \right] \]
\[ = 2 \left( E[x M_x] - \frac{1}{3} E[x^3 M_x] - E[y M_x] + \gamma E[M_x^2] - \gamma E[x M_x] - m_x E[M_x] \right. \]
\[ + \frac{1}{3} m_x E[M_x] + m_y E[M_x] + m_x D_x E[M_x] - m_x E[x] + \frac{1}{3} m_x E[x^3] + m_x E[y] \]
\[ - \gamma m_x E[M_x] + \gamma m_x E[x] + m_x^2 - \frac{1}{3} m_x^4 - m_x m_y - m_x^2 D_x \right). \quad (9.102) \]

The following polynomials found in Eqs. 11.30, 11.32, 11.35, 11.43, and 11.49 in Appendix 11.11 can be used to express various expectations in Eq. 9.102:
\[ E[x] = m_x, \]
\[ E[y] = m_y, \]
\[ E[M_x] = m_x, \]
\[ E[M_x^2] = D_{M_x} + m_x^2. \]
\[ E[x^3] = m_x^3 + 3m_x D_x, \]
\[ E[xM_x] = D_{xM_x} + m_x^2, \]
\[ E[yM_x] = D_{yM_x} + m_x m_y, \]
\[ E[x^3 M_x] = m_x^4 + 3m_x^2 D_{xM_x} + 3m_x^2 D_x + 3D_x D_{xM_x}. \]

These expressions can be substituted for the raw moments into Eq. 9.102 to give

\[
\epsilon \frac{dD_{M_x}}{dt} \approx 2 \left\{ D_{xM_x} + m_x^2 - \frac{1}{3}(m_x^4 + 3m_x^2 D_{xM_x} + 3m_x^2 D_x + 3D_x D_{xM_x}) - (D_{yM_x} + m_x m_y) \right.
\]
\[ + \gamma(D_{xM_x} + m_x^2) - m_x^2 + \frac{1}{3}m_x^4 + m_x m_y + m_x^2 D_x - m_x^2 \]
\[ + \frac{1}{3}m_x(m_x^3 + 3m_x D_x) + m_x m_y - \gamma m_x^2 - \gamma m_x^2 + m_x^2 - \frac{1}{3}m_x^4 - m_x m_y - m_x^2 D_x \Bigg) \]
\[ = 2(D_{xM_x} - m_x^2 D_{xM_x} - D_x D_{xM_x} - D_{yM_x} + \gamma D_{M_x} - \gamma D_{xM_x}). \quad (9.103) \]

For small \( q \),

\[ \epsilon \frac{dD_{M_x}}{dt} \approx 2 \left\{ D_{xM_x} (1 - \gamma - m_x^2 - D_x) - D_{yM_x} + \gamma D_{M_x} \right\}. \quad (9.104) \]

### 9.2 Solutions of Cumulant Equations

The cumulant equations described in Eqs. 9.70 apply to all values of \( q \). Additional estimates for \( \epsilon \dot{D}_{xM_x} \), \( \epsilon \dot{D}_{yM_x} \), and \( \epsilon \dot{D}_{M_x} \) have been generated with formulae that depend on system connectivity. For small \( q \),

\[
\epsilon \frac{dD_{xM_x}}{dt} \approx D_x (1 - \gamma - m_x^2 - D_x) - D_{xy} + D_{xM_x} (1 - m_x^2 - D_x) - D_{yM_x} + \gamma D_{M_x}, \quad (9.105) \\
\epsilon \frac{dD_{yM_x}}{dt} \approx D_{xy} (1 - \gamma - m_x^2 - D_x) - D_y + \gamma D_{yM_x} + \epsilon D_{xM_x} + \epsilon K(D_{M_x M_y} - D_{M_x M_y}), \quad (9.106) \\
\epsilon \frac{dD_{M_x}}{dt} \approx 2 \left\{ D_{xM_x} (1 - \gamma - m_x^2 - D_x) - D_{yM_x} + \gamma D_{M_x} \right\}. \quad (9.107) 
\]
For large $q$,

$$\frac{dD_{xM_x}}{dt} = D_{xM_x}(1 - \gamma - m_x^2 - D_x) - D_{yM_x} + \gamma D_{M_x}, \quad (9.108)$$

$$\frac{dD_{yM_x}}{dt} = \epsilon \{ D_{xM_x} + K(D_{M_x,M_y} - D_{M_x,M_y}) \}, \quad (9.109)$$

$$\frac{dD_{M_x}}{dt} = 0. \quad (9.110)$$

The solutions to Eqs. 9.70 will be considered before conditions of Eqs. 9.105, 9.106, 9.107, 9.108, 9.109, and 9.110 are added. It will be shown that in the case of mean-field connectivity, the solutions converge exactly towards those obtained previously for this scenario [10].

The stationary points of cumulant equations (Eqs. 9.70) can be analysed by setting derivatives to zero:

$$\epsilon = m_x - \frac{m_x^3}{3} - m_y - m_x D_x, \quad (9.111)$$

$$0 = m_x + a + K(m_y - m_y), \quad (9.112)$$

$$\epsilon = 2 \left\{ D_{x}(1 - \gamma - m_x^2 - D_x) - D_{xy} + \gamma D_{xM_x} \right\}, \quad (9.113)$$

$$0 = 2 \left\{ D_{xy} + T + K(D_{M_xM_y} - D_{M_y}) \right\}, \quad (9.114)$$

$$\epsilon = \epsilon \left\{ D_x + K(D_{M_y} - D_{xM_y}) \right\} + D_{xy}(1 - \gamma - m_x^2 - D_x) - D_y + \gamma D_{yM_x}. \quad (9.115)$$

Rearranging terms in Eq. 9.112 and simplifying using the property that $m_{y\tau} = m_y$ when $\dot{m}_y = 0$ leaves

$$m_x = -a, \quad (9.116)$$

whereas rearranging terms in Eq. 9.111 yields

$$m_y = m_x - \frac{m_x^3}{3} - m_x D_x. \quad (9.117)$$
Substituting $m_x$ from Eq. 9.116 into the above equation gives

$$m_y = -a + \frac{a^3}{3} + aD_x. \tag{9.118}$$

From Eq. 9.114,

$$D_{xy} = -T - K(D_yM_{y\tau} - D_yM_y). \tag{9.119}$$

Rearranging terms in Eq. 9.115 yields

$$D_y = \epsilon \left\{ D_x + K(D_{xM_{y\tau}} - D_{xM_y}) \right\} + D_{xy}(1 - a^2 - aD_x) + \gamma D_yM_x. \tag{9.120}$$

Substitute $m_x$ using Eq. 9.116 and $D_{xy}$ using Eq. 9.119 into the above equation to provide

$$D_y = \epsilon \left\{ D_x + K(D_{xM_{y\tau}} - D_{xM_y}) \right\} - \left\{ T + K(D_yM_{y\tau} - D_yM_y) \right\} (1 - a^2 - aD_x) + \gamma D_yM_x. \tag{9.121}$$

After rearranging,

$$D_y = \epsilon D_x + T(\gamma + a^2 + D_x - 1) + \gamma D_yM_x + \epsilon K(D_{xM_{y\tau}} - D_{xM_y})$$

$$+ K(D_yM_{y\tau} - D_yM_y)(\gamma + a^2 + D_x - 1). \tag{9.122}$$

Eq. 9.113 can be rewritten as

$$0 = D_x(1 - a^2 - aD_x) - D_{xy} + \gamma D_{xM_x}. \tag{9.123}$$

$D_{xy}$ from Eq. 9.119 can be substituted into this equation to give

$$0 = D_x \left\{ 1 - \gamma - a^2 - D_x \right\} + T + K(D_{yM_{y\tau}} - D_yM_y) + \gamma D_{xM_x}. \tag{9.124}$$
Rearranging Eq. 9.124 into the form $0 = AD_x^2 + BD_x + C$ gives

$$0 = D_x^2 - D_x(1 - \gamma - a^2) - T - K(D_{yM_y} - D_{yM_y}) - \gamma D_{xM_x}.$$ (9.125)

Applying quadratic formula $D_x = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$ derives the following solution:

$$D_x = \frac{1 - \gamma - a^2 \pm \sqrt{(1 - \gamma - a^2)^2 + 4\{T + K(D_{yM_y} - D_{yM_y}) + \gamma D_{xM_x}\}}}{2}.$$ (9.126)

Fixed points can be found when all of the following conditions are satisfied:

$$m_x = -a,$$

$$m_y = a\left(\frac{a^2}{3} + D_x - 1\right),$$

$$D_x = \frac{1 - \gamma - a^2 \pm \sqrt{(1 - \gamma - a^2)^2 + 4\{T + K(D_{yM_y} - D_{yM_y}) + \gamma D_{xM_x}\}}}{2},$$

$$D_y = \varepsilon D_x + T(\gamma + a^2 + D_x - 1) + \gamma D_{yM_x} + \varepsilon K(D_{xM_y} - D_{xM_y}) + K(D_{yM_y} - D_{yM_y})(\gamma + a^2 + D_x - 1)$$

$$D_{xy} = -T - K(D_{yM_y} - D_{yM_y}).$$ (9.127)

The scenarios for both small and large $q$ need to be considered. Firstly, for small $q$, setting $\dot{D}_{xM_x} = 0$ in Eq. 9.105 gives

$$D_{xy} = D_x(1 - \gamma - m_x^2 - D_x) + D_{xM_x}(1 - m_x^2 - D_x) - D_{yM_x} + \gamma D_{M_x},$$ (9.128)

whereas setting $\dot{D}_{yM_x} = 0$ in Eq. 9.106 leaves

$$D_y = D_{xy}(1 - \gamma - m_x^2 - D_x) + \gamma D_{yM_x} + \varepsilon D_{xM_x} + \varepsilon K(D_{M_x} - D_{M_x}M_y).$$ (9.129)
When $\dot{D}_{M, x} = 0$ in Eq. 9.107,

$$D_{yM, x} = D_{xM, x}(1 - \gamma - m^2 + D) + \gamma D_{M, x}. \quad (9.130)$$

For large $q$, setting $\dot{D}_{xM, x} = 0$ in Eq. 9.108 leaves

$$D_{yM, x} = D_{xM, x}(1 - \gamma - m^2 - D) + \gamma D_{M, x}, \quad (9.131)$$

whereas setting $\dot{D}_{yM, x} = 0$ in Eq. 9.109 provides

$$\epsilon D_{xM, x} = -\epsilon K (D_{M, x}M_y - D_{M, x}M_y). \quad (9.132)$$

and simplifies to

$$D_{xM, x} = -K (D_{M, x}M_y - D_{M, x}M_y). \quad (9.133)$$

For large $q$ and $\epsilon \neq 0$, when there is global or no feedback,

$$D_{xM, x} = 0. \quad (9.134)$$

Using the above equation, Eq. 9.131 simplifies to

$$D_{yM, x} = \gamma D_{M, x}. \quad (9.135)$$

When there is no delayed feedback ($K = 0$), the cumulant equations in Eqs. 9.70
reduce to

\begin{align*}
\frac{dm_x}{dt} &= m_x - \frac{m_y^3}{3} - m_y - m_x D_x, \\
\frac{dm_y}{dt} &= m_x + a, \\
\frac{dD_x}{dt} &= 2\left(D_x(1 - \gamma - m_x^2 - D_x) - D_{xy} + \gamma D_{xM_x}\right), \\
\frac{dD_y}{dt} &= 2(D_{xy} + T), \\
\frac{dD_{xy}}{dt} &= \epsilon D_x + D_{xy}(1 - \gamma - m_x^2 - D_x) - D_y + \gamma D_{yM_x}. 
\end{align*}

(9.136)

The fixed points can be found by setting \( \frac{d}{dt} = 0 \) in Eqs. 9.136 resulting in

\begin{align*}
m_x &= -a, \\
m_y &= a\left(\frac{a^2}{3} + D_x - 1\right), \\
D_x &= 1 - \gamma - a^2 \pm \frac{\sqrt{(1 - \gamma - a^2)^2 + 4(T + \gamma D_{xM_x})}}{2}, \\
D_y &= \epsilon D_x + T\left(\gamma + a^2 + D_x - 1\right) + \gamma D_{yM_x}, \\
D_{xy} &= -T. 
\end{align*}

(9.137)

The solutions in Eqs. 9.137 are very similar to previous results found for mean-field coupling [10], with the exception of extra terms \( \gamma D_{xM_x} \) in the \( D_x \) equation and \( \gamma D_{yM_x} \) in the \( D_y \) equation; these additional terms are a result of the introduction of different network connectivities. Substituting the known inequalities for \( D_{xM_x} \) and \( D_{yM_x} \) using Eqs. 9.37 and 9.63 into the system in Eqs. 9.137 gives

\begin{equation}
D_x \leq \frac{1 - \gamma - a^2 \pm \sqrt{(1 - \gamma - a^2)^2 + 4(T + \gamma D_{xM_x})}}{2},
\end{equation}

(9.138)
and

\[ D_y \leq \epsilon D_x + T \left( \gamma + a^2 + D_x - 1 \right) + \gamma \sqrt{\frac{D_x D_y}{q}}. \]  

(9.139)

The previous results for mean-field connectivity are derived when \( q \to \infty \) [10]. When \( q \) is large, at the fixed points, \( D_{xM_x} = 0 \) in Eq. 9.134 and \( D_{yM_x} = \gamma D_{M_x} \) in Eq. 9.135. In Eq. 9.36,

\[ D_{M_x} = \frac{D_x}{q}. \]  

(9.140)

As \( q \to \infty \),

\[ D_{M_x} \to 0, \]  

(9.141)

and

\[ D_{yM_x} \to 0. \]  

(9.142)

For large connectivity, the following fixed points are present:

\[ m_x = -a, \]

\[ m_y = a \left( \frac{a^2}{3} + D_x - 1 \right), \]

\[ D_x = \frac{1 - \gamma - a^2 \pm \sqrt{(1 - \gamma - a^2)^2 + 4T}}{2}, \]  

(9.143)

\[ D_y = \epsilon D_x + T \left( \gamma + a^2 + D_x - 1 \right), \]

\[ D_{xy} = -T; \]

these states are identical to the previously obtained results [10].
9.3 Bifurcation Analysis

Having derived the cumulant equations (Eqs. 9.71) to replace the initial system of stochastic delay differential equations with global feedback (Eqs. 7.1), focus is given to bifurcation analysis of these equations. This section offers support to the findings of numerical simulations in Chapters 6 and 7. All bifurcation diagrams are generated using the software XPPAUT [146]. Bifurcation analysis of neuron-specific feedback is omitted and remains open to future investigation.

![Bifurcation Diagrams](image)

Figure 9.1: Bifurcation diagrams of Eqs. 9.71 on the noise $T$ and coupling strength $\gamma$ plane with fixed parameter values $\alpha = 1.05$ and $\epsilon = 0.01$. Curves represent Andronov-Hopf bifurcation whose inside regions represent unstable equilibrium states. Fixed points in zones outside of the curves are stable. (a) Bifurcation diagram for no feedback $K = 0$, where $q$ ranges from one to mean-field connectivity $q \to \infty$. (b) Bifurcation diagram for delayed feedback applied globally with parameters $K = 0.1$ and $\tau = 0.2$.

Bifurcation diagrams (Fig. 9.1) cover a two-parameter plane $(T, \gamma)$. Eqs. 9.71 depict mean-field activity, where spiking reflects synchronisation; curves represent the Andronov-Hopf bifurcation points for various quantities of inputs per neuron $q$. Regions inside the curves correspond to unstable equilibrium states and external areas depict stable fixed point behaviour; the former is characterised by the tendency of a mean-field phase point to perform excursions in the phase plane, indicating the presence of synchronisation. Synchronisation does not occur in regions outside the line of Andronov-Hopf bifurcation, where a stable fixed point exists and mean-field phase points remain below threshold activity levels.
At the very small noise value, where $T = 0.00031$, the bifurcation diagram (Fig. 9.1 (a)) highlights the results obtained from previous numerical simulations (Chapter 6). Increasing the coupling strength from $\gamma = 0$ results in systems with larger values of $q$ crossing the Andronov-Hopf bifurcation line first; these types of networks enter the stable equilibrium region, where synchronisation is lost, before networks of weaker connectivity, which have smaller values of $q$. The results for one input per neuron contrast with those found in Chapter 6 by indicating that synchronisation is possible and achievable for the widest range of coupling strengths for the chosen parameter values; however, these results assume a fully connected network, ignoring the possibility of disconnected clusters. A fully integrated system is assumed to have complete neuronal communication, at least indirectly; the bifurcation diagram should be considered as a best case scenario of synchronisation performance. The results indicate that if all simulated networks do not have any disconnected clusters, one input per neuron provides the largest range of synchronisation, all-to-all connectivity would display the shortest range, and intermediate connectivities would follow this linear order. Bifurcation curves for mean-field coupling are consistent with previous results [10]; the graph implies that networks with low connectivity, where $q$ is smaller, are better equipped for mean-field firing when noise intensity is low. A wider synchronisation region in sparsely connected networks is attributed to coupling-induced noise reduction impairing neuronal firing more severely in largely connected networks; however, highly connected networks with larger values of $q$ display a broader range of synchronisation at high noise intensity values. Sparsely connected networks are innately noisier than densely connected networks as their mean inputs display higher amplitudes of fluctuations; increasing noise intensity overpowers coupling strength parameters quicker in networks with low connectivity. Phase points begin to randomly fluctuate to a high degree, where the interaction level is unable to sufficiently bind in synchrony. Sparsely connected systems have narrower ranges of synchronisation at elevated noise intensities; highly connected systems are helped by an initial increase in noise intensity, which generates the broadest region of unstable equilibrium states on the ($T$, $\gamma$) plane.
The contrast between the bifurcation curves, shown in Figs. 9.1 (a) and (b), marks the impact produced by delayed feedback of weak strength ($K = 0.1$) and a small time delay ($\tau = 0.2$); the delayed feedback parameters shift the range of coupling strengths, with unstable fixed point behaviour, to more positive values. The intersection of all bifurcation curves is shifted to lower noise values; transitions to stable dynamics through Andronov-Hopf bifurcations occur at lower values of noise intensity when neurons are uncoupled ($\gamma = 0$). For sparsely connected networks, delayed feedback with the given parameter values cause transitions to stable dynamics to occur at lower noise intensities across the entire range of coupling strength; densely connected networks display unstable dynamics for larger noise intensities at high coupling strengths.

The effect of altering feedback strength is displayed for mean-field connectivity in Fig. 9.2. Increasing feedback strength shifts the range of coupling strengths, where unstable fixed points can occur, to more positive values; the areas of unstable equilibrium with synchronisation are also increased. Increased regions of synchronisation are caused by stretching of the Andronov-Hopf bifurcation curve in noise intensity and coupling strength directions.
9.4 Discussion

The system of stochastic differential equations used for the numerical simulations in Chapter 6 is transformed into a deterministic set of cumulant equations; they are used at maximal and sub-maximal connectivity to generate two-parameter bifurcation diagrams along noise intensity and coupling strength planes. In mean-field coupling, solutions converge exactly towards previously generated results [10]. Andronov-Hopf bifurcation curves are compared for different network connectivities in addition to scenarios that include and omit global delayed feedback; the analysis supports the previous results from the tested numerical simulations. Networks with larger connectivity show greater synchronisation robustness regarding noise intensities; in conjunction with simulated findings, mean-field spiking persists for a higher range of coupling strengths at low noise intensity values in networks with low connectivity. The only contradiction between the numerical and analytical conclusions lie in expectations regarding the presence of synchronisation for one input per neuron; this discrepancy is due to the assumption of one fully integrated cluster in the analytical calculations. In contrast, the numerical simulations display an abundance of disconnected clusters, resulting in an absence of global synchronisation. Applying global delayed feedback shifts Andronov-Hopf bifurcation lines to higher coupling strength values; unstable dynamics of the mean-field are restricted regarding noise intensity in sparsely connected networks; densely connected networks are robust against the effects of strong noise intensities at large coupling strengths.
Chapter 10

Conclusions

This thesis initially introduces the biological foundations of the research undertaken, dedicating particular attention to the structure, function, and types of individual neurons; knowledge and understanding of the individual elements of the brain are integral to the discussion of related networks. The brain’s efficiency and performance are examined from an architectural perspective; fluctuations and the randomness of signals present within the neural system are also considered.

A selection of the most prominent neural network models in the field of neurobiology are briefly highlighted, leading to the justification of employing the FitzHugh-Nagumo model as the basis for studying certain network elements. Synchronisation in neural networks has recognised implications of improved brain processing capabilities in addition to association with pathological disorders of epilepsy and Parkinson’s disease.

A stochastic FitzHugh-Nagumo model is introduced to simulate synchronisation patterns in neural networks with varying degrees of connectivity. Synchronisation activity is analysed for a range of coupling strengths, which indicate the degree of interaction between neurons in the network. Initially, the scenario where all interactions are equal is considered; synchronisation is achieved for an optimal range of coupling strengths with two or more inputs per neuron in the system. Coupling strengths above and below the synchronisation range prevent synchrony for different reasons; in the former case, individual neurons cease firing, leading to the destruction of synchronisation; the latter shows frequent firing of units to their individual rhythms. Declines in individual neuronal firing at high coupling strengths are attributed to the coupling-induced noise reduction phenomenon. Lower connectivity is shown to provide resistance to the effects of coupling-
induced noise reduction, enabling neuronal firing and synchronisation to be maintained to higher coupling strengths. Synchronisation follows a consistent pattern throughout a range of coupling strengths, network connectivities, and system sizes; aided by recent insight into advances in sleep research, these patterns enable realistic predictions to be made regarding brain behaviour. The networks studied here contain a random architecture. The randomness of connections could be partly responsible for the prominence of synchronisation observed; it is expected that a regular lattice structure of connections would significantly reduce synchronisation capabilities, however, synchronisation is expected to remain strong and perhaps even improve under a small-world architecture. Systems with only bidirectional connections are shown to have synchronisation onset and decay occurring at higher coupling strength values. Increasing the size of the network expands the synchronisation region at low connectivities but constricts it at high connectivities; transitions between synchronous and asynchronous behaviour occur more rapidly in larger network sizes.

Control of synchronisation behaviour is assessed using both global and neuron-specific applications of the delayed feedback mechanism; global feedback generally is able to alter the degree of synchronisation to a limited extent, especially when feedback strength is weak; neuron-specific feedback offers the prospect of more severe modifications, particularly at low connectivities. However, current technology is incapable of applying neuron-specific feedback in a non-invasive manner. Strong and weak feedback are shown to be capable of increasing synchronisation magnitude in both network sizes to a moderate degree when there is one input per neuron under global delayed feedback; destruction of synchronisation is limited to stronger feedback in the larger network. Neuron-specific delayed feedback removes opportunities to enhance synchrony but allows total synchrony reduction for one input per neuron. At low connectivities (above one input per neuron), global delayed feedback can cause moderate reductions in synchrony for a broad range of parameters but gains in synchronisation magnitude are limited to low coupling strengths; neuron-specific synchronisation shows displays similar capability of enhancing synchroni-
sation but can allow complete synchrony destruction to be achieved when there is strong feedback. Global and neuron-specific feedback have similar effects at high network connectivity. Synchronisation enhancements are possible at high and low coupling strengths but reductions only moderate reductions in synchrony are observed.

Modifications are made to the model to realistically mirror further heterogeneities found in biological neural networks; although these amendments are likely to increase accuracy, they elevate the complexity of simulations. The effects of varying connection strengths values are considered; coupling strengths are given the freedom to evolve with time; larger mean coupling strength values are required to onset synchronisation peaks when the coupling strengths depict greater amplitudes in fluctuations or slower convergence to the desired ensemble mean value. A consequence of delayed synchronisation onset is the attenuation of neuronal firing at higher mean coupling strength values; results indicate that densely connected systems have greater robustness to such disturbances. Furthermore, there are less discrepancies between homogeneously and heterogeneously coupled networks when the number of inputs per neuron is large.

Analysis and manipulation of the simulated system enable a set cumulant equations to be derived and solved. Bifurcation diagrams on noise and coupling strength planes display satisfactory correspondence to the numerically simulated results. Bifurcation lines predict that networks with lower connectivity show sustained mean-field spiking at higher values of coupling strength. The only discrepancy between the numerical and analytical results occurs at one input per neuron, where the bifurcation line assumes the presence of synchronisation in contrast to numerical simulations, which show an absence of synchronisation; this discrepancy results from the analytical assumption that a single and fully integrated network is generated, even at one input per neuron, which conflicts with numerical simulations that reveal multiple disconnected clusters of neurons. High connectivity provides superior robustness against changes in noise intensity and global delayed feedback.

The impact observed by changing network parameter values can be used to make
predictions about the actual connection strengths in full-scale biological networks. According to the synaptic homeostasis hypothesis, synaptic connections decrease in strength and number during sleep cycles and increase during wakefulness; synchronisation is more prominent during sleep which corresponds to the region of desynchronisation in the results observed here. Biological brains include many bidirectional connections and this type of configuration raises the likely value of realistic coupling strengths. The effect of bidirectional connections will be more than offset by the combination of increased inputs per neuron and size of biological brains causing a reduction in the realistic coupling strength values. Due to the sharper synchronisation transitions in larger network sizes, the desynchronisation region range will be reduced. On the other hand, the heterogeneity and variability in biological brains elevates the realistic coupling strength values by shifting the dysynchronisation zone. The results indicate that synchronisation manipulations are possible in biological brains and large-scale artificial networks using delayed feedback mechanisms. Synchronisation may be sufficiently enhanced to improve network performance through increased transmission speeds; alternatively, it may be possible to use delayed feedback to sufficiently reduce synchronisation and maintain it below pathological levels.

The biological brain is huge in size and vast in complexity. It is likely that an exhaustive list of parameters must be considered to reflect completely accurate neural behaviours. Future models are necessary in determining other factors influencing synchronisation activity; they should include many different neuronal types and introduce a heterogeneous number of inputs per unit, since these features likely hinder the synchronisation capabilities of networks by reducing the system’s homogeneity. In addition, models need to incorporate a mixture of unidirectional and bidirectional connections in vast networks of relatively sparse connectivity; this requirement arises from the coexistence of unidirectional chemical synapses and bidirectional electrical synapses in brain structures; the presence of bidirectional connections naturally manifested in unidirectionally connected network simulations with large connectivity. Hybrid models will help account for the
noticeable non-random neural characteristics, such as the high proportion of reciprocal pathways \cite{79, 80}; another non-random feature that needs to be accounted for is the small-world architecture (Section 3.2), developed by real-world brain networks. Improving the small-world architecture of models expectedly aids synchronisation generation, partially offsetting likely hindrances caused by adding further heterogeneities; increases in synchronisation is also expected as this type of architecture provides an optimal compromise between favourable attributes of random and lattice-like network connectivities. Small-world configurations encourage efficiency in local integration, which increases system robustness against connection damage; moreover, it enables rapid signal transmission to distant regions. A more realistic representation of the brain would also incorporate the existence of differing brain regions (Section 3.1); improved models should appropriately accommodate the existing number and proportion of modules. Future research should further explore different delayed feedback parameter values; longer time delay values are required to determine underlying periodic patterns. These studies should aim to ensure that responses are robust to a wide range of parameter values and discover the critical values where qualitative behavioural changes occur.

In light of recent research involving the SHH, model representations of the sleep-wake cycle could be improved; more appropriate modelling could involve alternating between net phases of downscaling and amplification. The former phase would involve a net decrease in connection strength and quantity, representing the sleep cycle; the latter would focus upon a net increase in connection strength and number to signify the effect of synaptic potentiation during wakefulness. Two Ornstein-Uhlenbeck processes, relating to coupling strength and quantity, could work in conjunction to effectively model the SHH. Extended research would certainly be required to assess the most appropriate fluctuation intensity and convergence rate values for numerical simulations; values are particularly difficult to generalise since environmental and genetic conditions have varying effects upon human behaviour and physiology.

Another possible route of research could analyse the possibility of synaptic failures,
focusing on neurons that only receive input with some pre-determined probability; fur-
ther efforts could be made to account for the numerous forms of randomness occurring
at a molecular level, such as the quantity of receptors receiving neurotransmitters at the
postsynaptic cleft and the timings of signal transmission events. Lengths of axons and
dendritic branch density show wide variation in biological brains; axons generate dif-
ferent signal transmission time delays and signals in different dendrites display diverse
levels of attenuation. Variability in axons and dendrites create noise within the network,
which could counteract the generation of biological rhythms and periodicities underlying
synchronisation.
Chapter 11

Appendix

11.1 Role of Glial Cells

Neurons are supported and protected by glial cells that surround neurons to insulate them from one another, supplying oxygen and necessary nutrients for functioning. Glial cells are known to clean the brain by destroying pathogens, removing damaged cells and forming the blood-brain barrier; this barrier determines which substances are exchanged between the blood and the brain. Their abundant volume allows glial cells to moderate neuronal activity by regulating the exchange of information between neurons at synapses [147]. Glial cells are prominent in assisting neurons during cognitive development.

11.2 Dendrite Characterisation

The order of a dendritic tree corresponds to the number of levels existing within the tree. The first branch extending from the soma has order zero; splitting off directly, its daughter branches have order one. Subsequent divisions are further incremented in order value and continues until terminal dendrites have the highest order. The degree refers to the number of terminal tips found in a dendritic tree, in which there can be approximately four hundred (Fig. 11.1). The diameter of a dendrite within a tree is often an inversely proportional function of the distance from the soma [148]. Consequently, terminal dendrites are usually the smallest in diameter. The asymmetry index describes the probability of an imbalance of branching patterns at any of the \( n - 1 \) branch points.
in a tree of degree $n$. This is usually calculated using the formula

$$A = \frac{1}{n-1} \sum_i A_p(r_i, s_i),$$

where $r_i$ and $s_i$ are the degrees of the two subtrees at branching point $i$ [149]. The partition asymmetry $A_p$ is described by the equation

$$A_p = \frac{|r-s|}{r+s-2} \text{ where } r+s > 2 \text{ and } A_p(1,1) = 0.$$ (11.2)

A completely symmetric tree has the value $A = 0$ while the most imbalanced dendritic trees occur when $A = 1$.

### 11.3 Dendritic Spines

Dendritic spines are small (approximately 1µm) protrusions on the dendritic cable that many axodendritic synapses (Section 2.4) are situated upon [150]. Dendrites can range between very spiny, sparsely spiny and smooth; however, even smooth dendrites usually contain some spines. The spine shapes can vary and may therefore influence properties such as membrane resistance and conductance of electrical signals. Spines also alter over time depending on their activity; consequently, spines are heavily associated with neuronal plasticity and learning. Accumulation of dendritic spines and tips allows the dendritic tree to contain $500 - 200,000$ synaptic contact points [151]; the location of these points
can significantly impact upon the magnitude of the electrical impulse reaching the soma.

11.4 Sensory Receptors

Sensory receptors are structures that respond to internal or external stimuli and transmit nerve impulses across the brain, conveying the gathered information; there are many types within the human body and a selection will be discussed below.

Nociceptors respond to potentially damaging stimuli, invoking feelings of pain [152]. Thermoreceptors report the absolute and relative temperature changes that the body experiences. Chemoreceptors send data regarding values and alterations of specific chemicals; they notify the brain of fluctuations in oxygen and carbon dioxide partial pressures in addition to pH levels [24]. Muscle spindles and the Golgi tendon organs are types of proprioceptor. Muscle spindles lie parallel to muscle fibres, providing useful information regarding alteration of their length [153]; they are enclosed within a capsule and contain many intrafusal muscle fibres, which stretch when the muscle lengthens. Mechanically-gated ion channels open, depolarising a receptor membrane and triggers an action potential; lengths of the intrafusal fibres constantly adjust to maintain sensitivity and responsiveness to stretching stimuli. Golgi tendon organs are positioned in series with muscle fibres, reporting muscular tension based on the tightness of a tendon caused by muscular contraction [154]. The brain generates a description and awareness of the body’s position and orientation based on information provided by the aforementioned proprioceptors. Mechanoreceptors aid proprioception by delivering signals about angles of joints and motion velocity to the brain [155], though they predominantly detect skin pressure and distortion.; there are four major types of mechanoreceptor:

- Pacinian corpuscles,
- Meissner corpuscles,
- Merkel discs,
- Ruffini endings.

A Pacinian corpuscle’s sensitivity to vibrations and pressure enables these receptors to distinguish between rough and smooth surfaces; they rapidly adapt to constant stimuli, causing nerve impulses to quickly attenuate unless stimulus intensity is increased. Higher intensity stimulation increases output degree; Pacinian corpuscles display a burst of activity upon exceeding a certain threshold. Meissner corpuscles are the most sensitive to pressure, reacting strongly to gentle touches; in a similar manner to Pacinian corpuscles, they rapidly adapt to constant stimuli. Meissner corpuscles are binary operators as their main function is to detect contact, deciphering stimuli with high resolution; they do not distinguish magnitudes of pressure due to their small receptive fields. Merkel discs sense pressures and textures, having a rapid initial response to a stimulus, which is followed by irregular firing upon stabilisation of the applied force; they can generate action potentials for a considerable time, in excess of 30 minutes, following constant pressure [156]; sharper surfaces induce accelerated responses compared to large areas that are blunt or flat. Similar to Meissner corpuscles, Merkel discs are densely packed with small receptive fields allowing highly detailed reception. In addition to Merkel discs, Ruffini endings also slowly adapt and respond to sustained pressure; they are sensitive to stretching, allowing for grip modifications that alleviate pain in response to some sustained heat [157].

11.5 Characteristics of a Random Process

A random process is a collection of random variables mapped onto a waveform, which is a function of time. In contrast to a deterministic model, which only has one evolution path from a set of initial conditions [100], a stochastic (random) process can follow numerous and often infinite numbers of trajectories. Variability in trajectories of stochastic models derives from noise, causing the output of governing equations to randomly fluctuate; repeating a random process with identical initial conditions can generate an entirely different outcome. A single observed trajectory of a random process over a period of time
is known as a realization.

A probability density distribution (PDD) calculates the probability of a random process \(X(t)\) falling within an interval \([x, x + \Delta x]\) at a particular time moment \(t\) [104, 158]. To completely define a PDD of \(N\)-dimensions, the ensemble of realizations needs to be described for \(N\) arbitrary time moments

\[
P_N(x_1, t_1, x_2, t_2, \cdots, x_N, t_N) = 
\lim_{\Delta x_1 \to 0} \lim_{\Delta x_2 \to 0} \lim_{\Delta x_N \to 0} \frac{P\{X(t_1) \in [x_1, x_1 + \Delta x_1) \land X(t_2) \in [x_2, x_2 + \Delta x_2) \land \cdots \land X(t_N) \in [x_N, x_N + \Delta x_N)\}}{\Delta x_1 \Delta x_2 \cdots \Delta x_N}.
\]

Different realizations of a random process can provide different output values from the same input; statistical properties can be generated by averaging a sufficiently large ensemble of realizations; if a process is stationary, its statistical properties do not change at different times. If the \(N\)-dimensional PDD does not vary under any time shift \(\tau\) for any \(N\), the process is said to be strict sense stationary; strict sense stationarity for an \(N\)-dimensional PDD is mathematically represented as

\[
P_N(x_1, t_1, x_2, t_2, \cdots, x_N, t_N) = P_N(x_1, t_1 + \tau, x_2, t_2 + \tau, \cdots, x_N, t_N + \tau).
\] (11.3)

Wide-sense stationarity offers a less restrictive definition, providing more opportunity for practical utilisation. Wide-sense stationary processes do not need to be first order stationary since the only requirements are as follows [104, 158]:

- The mean value of the random process must be constant with time,
- The variance of the random process must be constant with time,
- The autocovariance (and therefore its autocorrelation) can be described as a function that is solely dependent on the time shift between time moments.

An ergodic random process dictates that statistical characteristics can be extracted from any of its single realizations; a random process \(X(t)\) is said to be \(N\)th-order ergodic if
any of its single realizations $x(t)$ carry the same information as its $N$-dimensional PDD. If a process is $N$th-order ergodic, it must also be $N$th-order stationary; however, $N$th-order stationary does not necessarily imply $N$th-order ergodicity. If the property of ergodicity holds, averaging over time is equivalent to averaging over the ensemble of realizations [104, 158].

The autocorrelation function (ACF) characterises the degree of mutual influence between values of a random process at two different time moments. An autocorrelation function $K_{XX}(t_1, t_2)$ is given by

$$K_{XX}(t_1, t_2) = \int_{-\infty}^{\infty} x_1 x_2 P_2(x_1, t_1, x_2, t_2) dx_1 dx_2 = \overline{x(t_1)} x(t_2),$$  \hspace{1cm} (11.4)$$

where the upper bar denotes the statistical average over the ensemble of realizations. $P_2(x_1, x_2)$ is the two-dimensional joint PDD; however, it is more convenient to describe the autocorrelation using the notion of the time interval $\tau$,

$$K_{XX}(t, \tau) = \int_{-\infty}^{\infty} x_1 x_2 P_2(x_1, t, x_2, t + \tau) dx_1 dx_2 = \overline{x(t)} x(t + \tau).$$  \hspace{1cm} (11.5)$$

A centred process has a mean value of zero and an equal ACF to its autocovariance since the autocovariance of a process measures the statistical difference between the actual values of a process with its mean values; the autocovariance converts processes with non-zero mean values into centred processes and calculates the resulting ACF [104, 158]. Autocovariance is defined by

$$\Psi_{XX}(t_1, t_2) = \int_{-\infty}^{\infty} (x_1 - \overline{x(t_1)})(x_2 - \overline{x(t_2)}) P_2(x_1, t_1, x_2, t_2) dx_1 dx_2$$

$$= \left(\overline{x(t_1)} - \overline{x(t_1)}\right) \left(\overline{x(t_2)} - \overline{x(t_2)}\right)$$

$$= \overline{x(t_1)} \overline{x(t_2)} - X(t_1) X(t_2).$$  \hspace{1cm} (11.6)$$
Alternatively, autocovariance can be expressed in terms of time interval \( \tau = t_2 - t_1 \) by

\[
\Psi_{XX}(t, \tau) = \int_{-\infty}^{\infty} (x_1 - \bar{X}(t))(x_2 - \bar{X}(t+\tau))P_2(x_1, t, x_2, t+\tau)dx_1dx_2
\]

\[
= \frac{(X(t) - \bar{X}(t))(X(t+\tau) - \bar{X}(t+\tau))}{(X(t)X(t+\tau) - \bar{X}(t)X(t+\tau))}. \tag{11.7}
\]

Power spectrum (or power spectral density) can be used as a measure of coherence of a spike train [8]; the power spectrum identifies the proportion of signal present at different frequencies, discovering any underlying periodicities. Assuming that a process \( X(t) \) is wide-sense stationary, according to the Wiener-Khintchine theorem it has a power spectrum

\[
S_X(\omega) = \int_{-\infty}^{\infty} (x(t)x(t+\tau))e^{i\omega\tau}d\tau
\]

\[
= \Im(K_{XX}(\tau)) = \int_{-\infty}^{\infty} K_{XX}(\tau)e^{-i\omega\tau}d\tau. \tag{11.8}
\]

\( x(t) \) is the value taken by the random process \( X(t) \) on a particular realization; \( \omega \) corresponds to the angular frequency; \( \tau \) is the time interval; the angular brackets \( \langle \rangle \) denote averaging over time. \( \Im \) denotes the Fourier transform and \( K_{XX}(\tau) \) is the autocorrelation function described above.

A Wiener Process is a continuous-time stochastic process used to represent the integral of Gaussian white noise; Brownian motion famously exemplifies this process with the seemingly random particle movement in liquids and gases, resulting from repeated collisions. A Wiener process \( W_t \) obeys the following three properties:

- \( W_0 = 0 \),
- \( W_t \) is almost surely continuous,
- \( W_t - W_s \sim N(0, t-s) \) for \( (0 \leq s < t) \) such that increments in \( W_t \) are independent.

In the above, “almost surely” describes an event with an uncertain outcome, which para-
doxically occurs with probability one; for instance, infinitely tossing a fair coin will almost surely result in at least one heads. \(N(\mu, \sigma^2)\) denotes the normal distribution with mean \(\mu\) and variance \(\sigma^2\) [87].

If one moves from value \(W_t\), at time moment \(t\), to value \(W_{t+\Delta t}\), at time \(t + \Delta t\), a continuous model requires \(\Delta t\) to be as short as possible (i.e. \(\Delta t \to 0\)). The progression from time \(t\) to time \(t + \Delta t\) obeys the following equation [159]:

\[
W_{t+\Delta t} = W_t + q\sqrt{\Delta t},
\]

where \(q\) is a random number from a normal probability distribution. The resultant Wiener process has zero mean and variance \(\Delta t\); a random variable, \(q\), with normal distribution has mean \(E[q] = 0\) and variance \(Var(q) = E[q^2] - (E[q])^2 = 1\). \(E\) denotes the “expected” value, which is the average value of a random variable occurring if a random experiment is infinitely repeated. Thus, \(E[q\sqrt{\Delta t}] = 0\) and \(E[q^2] = 1\); the variance of \(q\sqrt{\Delta t}\) is given by

\[
Var(q\sqrt{\Delta t}) = Var(q) \cdot (\sqrt{\Delta t})^2
\]
\[
= 1 \cdot \Delta t
\]
\[
= \Delta t.
\]

### 11.6 Reduction of the Hodgkin-Huxley Model to FitzHugh-Nagumo Equations

The Hodgkin-Huxley model in Eqs. 4.7 in Section 4.2 reduces to the FitzHugh-Nagumo model form by identifying and combining variables with similar time-scales [31, 160, 161]; Eqs. 4.8 describe the rate at which \(m, n,\) and \(h\) approach their asymptotic values \(m_\infty(V), n_\infty(V),\) and \(h_\infty(V)\); smaller \(\tau\) values converge faster. Sodium channels may instantaneously activate, simplifying the system by taking \(m = m_\infty(V)\) [162]. The Hodgkin-Huxley
model in Eqs. 4.7 is a three variable model and is once again written as:

\[ C_m \frac{\partial V}{\partial t} = G_{\text{leak}}(V_{\text{leak}} - V) + G_{\text{Na}} m^3_h(V) h(V_{\text{Na}} - V) + G_{\text{K}} n^4(V_{\text{K}} - V). \]  \hspace{1cm} (11.11)

\( n \) and \( h \) dynamics evolve slower than \( m \); \( n \) and \( h \) have similar behaviours and their almost linear relationship can be approximately described as

\[ h = \alpha n + \beta. \]  \hspace{1cm} (11.12)

Replacing \( n \) and \( h \) with their asymptotic values incapacitates the model from generating action potentials [162]; \( n \) and \( h \) would counteract action potentials as quickly as \( m \) initiates them; it is necessary for dynamics of \( n \) and \( h \) to lag behind \( m \) dynamics. Converting the system in Eq. 11.11 from three dimensions to two, \( n \) and \( h \) must be redefined as a single variable by substituting in a new variable, \( U \) [162], where

\[ U = U_{\infty}(S_u), \quad U \in \{n, h\}. \]  \hspace{1cm} (11.13)

\( n \) and \( h \) to depict their lagging using auxiliary voltages \( S_n \) and \( S_h \); \( U_{\infty} \) is the equilibrium variable. A new function with variables \( f(V, S) \) can be defined, which reflects a function that only consists of the asymptotic values for \( m, n \) and \( h \):

\[ f(V, S) = F(V, m_{\infty}(V), n_{\infty}(S), h_{\infty}(S)). \]  \hspace{1cm} (11.14)

The time dependencies of the two functions must be consistent for constant \( V \), giving

\[ \frac{\partial F}{\partial t} = \frac{\partial f}{\partial t}. \]  \hspace{1cm} (11.15)

Using the chain rule for partial differentiation, the equation can be rewritten as

\[ \frac{\partial F}{\partial n} \frac{dn}{dt} + \frac{\partial F}{\partial h} \frac{dh}{dt} = \left( \frac{\partial f}{\partial m_{\infty}} \frac{dn_{\infty}(S)}{dS} + \frac{\partial f}{\partial h_{\infty}} \frac{dh_{\infty}(S)}{dS} \right) \frac{dS}{dt}. \]  \hspace{1cm} (11.16)
where partial derivatives for $\frac{\partial f}{\partial t}$ are evaluated at $U = U_\infty(S)$; the resulting two-dimensional set of ODE’s are

$$C_m \frac{dV}{dt} = -f(V, S) + I, \quad (11.17)$$
$$\frac{dS}{dt} = g(V, S).$$

The function can be given as the ratio of two other functions:

$$g(V, S) = \frac{A(V, S)}{B(V, S)}, \quad (11.18)$$

where

$$A(V, S) = \frac{\partial F}{\partial n} \left( \frac{n_\infty(V) - n_\infty(S)}{\tau_n} \right) + \frac{\partial F}{\partial h} \left( \frac{h_\infty(V) - h_\infty(S)}{\tau_h} \right),$$
$$B(V, S) = \frac{\partial f}{\partial n_\infty} \frac{dn_\infty(S)}{dS} + \frac{\partial f}{\partial h_\infty} \frac{dh_\infty(S)}{dS}. \quad (11.19)$$

Eqs. 11.17 have the same form as the FitHugh-Nagumo system in Eqs. 4.10, providing a geometric interpretation of the dynamics observed in the phase plane; the system’s dynamics correspond closely to the Hodgkin-Huxley responses [163]. There are two disadvantages in reducing the more complex Hodgkin-Huxley model, in Eq. 4.7, to the FitzHugh-Nagumo model: the accuracy of minor details decreases and parameter values are incomparable to experimental data.

### 11.7 Sleep and Plasticity

Sleep can be unfairly categorised as a low-activity resting state; in fact, it is a highly structured and dynamic set of processes, which perform many critical homeostatic functions in preparation for the following waking period. Sleep requires more energy during some stages than during wakefulness [164]. Sleep is assumed to carry out critical bodily functions; this concept is only theoretical as animals, which do not sleep or have some form of
compensatory mechanism, are unknown [165, 166]. Additionally, severe sleep deprivation may lead to death, supporting the hypothesis that sleep has restorative capabilities and is physiologically required for survival.

Sleep can be divided into five stages, where the first four are classified as non-rapid eye movement (NREM) and the last is categorised as rapid eye movement (REM); sleep approximately consists of 20% REM sleep and 80% NREM sleep. All stages are repeatable, with the exception of stage 1, forming cycles of an estimated 90 – 110 minutes [44]; each stage demonstrates the following common characteristics: stereotypic electrical activity, neurochemical bases, and activity enhancement and depression in specific brain regions.

NREM sleep typically displays slow waves of electrical activity, which have a high voltage and are synchronised, throughout the cortex. Stage one sleep briefly starts upon sleep onset and is characterised by low voltage waves of mixed frequencies; during this stage, the brain transitions from alpha waves of 8 - 13Hz (during wakefulness) to theta waves of 4 - 7Hz [167]. Stage two sleep displays fast 12 - 14Hz EEG spindles and slower, sometimes spontaneous, K-complex signals; these signals are distinguished by high amplitudes and slow oscillations [168]; this stage generally constitutes 60% of total sleep duration [44]. The third and fourth stages are often referred as slow wave sleep (SWS), where the network reaches maximum levels of synchronisation; combined, they account for approximately 20% of total sleep [44]; early SWS (stage three) consists of 0.5 - 3Hz (delta) EEG waves and higher frequency signals, whereas late stage SWS (stage four) typically portrays higher delta content [44]. Long-term potentiation (Section 2.4) is more difficult to achieve in SWS than other sleep stages, especially compared to REM phases; the amount of SWS decreases from high to low throughout the period of sleep.

In addition to its namesake, REM sleep is highlighted by irregular shallow breathing, increased heart rate, increased cortical blood flow, muscular paralysis, and (4 - 7Hz) theta electroencephalogram (EEG) waves; these waves have low voltage, no synchrony and are reflective of those observed during wakefulness [44, 169, 170]. REM sleep has relatively normal plasticity and induces LTP (Section 2.4); it contains constantly present
tonic events and intermittent phasic events. A phasic event is exemplified by the burst of electromyographic (EMG) activity known as twitching; tonic events include high frequency and low voltage EEG recordings, high awakening threshold, and reduced body temperature [171].

In contrast to SWS, REM sleep stages proportionally increase with sleep length (Fig. 11.2). Although sleep rhythms could occur during wakefulness at certain times, they might be masked by external noise created by the environment. Individual cells depict their own 24 hour circadian rhythm and display specific characteristics at particular times throughout the day; circadian rhythms are partially developed from chemical reactions, which have varying speeds depending on the temperature. Synchronisation patterns during sleep have been linked to increases in brain temperature [172] and may be a factor in epileptic seizures that occur when sleeping; reducing brain temperatures, thus neuronal hyperexcitability, decreases the frequency of seizures [173].

The strongest experimental evidence relating to the purpose of sleep concerns the
regulation of brain plasticity and cognitive performance. Learning occurs mostly during wakefulness and sleep appears to consolidate new memories, facilitating their retention; this proposition is reflected by ongoing changes in individual neurons (e.g. Hebbian learning), interconnections between neurons, and arborisation of dendritic trees [3]. Adequate sleep, prior to and following an event, is associated with high quality recognition; a state of optimal alertness is achieved with sleep prior to an event, enabling recognition and distinction of intricate details of the event; post-event sleep is associated with long-term memory storage. A short sleep during the day, even a few minutes, could lead to better memory retention [170]. Sleeping immediately after learning significantly improves memory retention compared to a delay of ten hours before sleeping [44]; logically, this decreases noise exposure and reduces the possibility of learning new material that could replace newly-developed and vulnerable memories. Memories fall into two categories: procedural (implicit) or declarative (explicit). Procedural memory acquires information during the learning of skills regarding actions; motor, perceptual, and cognitive skills in addition to habit formation, simple classical conditioning, and non-associative learning are all skills. Declarative memory enables conscious access to fact-based memories; there are two subcategories of declarative memory, episodic and semantic memory. The former relates to information regarding personal events and episodes; the latter concerns factual knowledge about the world [44]. During wakefulness and REM sleep, there is little activity in the hippocampus; most cells are silent, though a few units regularly spike [174]. The mean firing rate of neurons increases across the hippocampus [170] during SWS, supporting the association of sleep and memory consolidation, in addition to the notion that sleep is an active state.

Memory consolidation relates to the synaptic or systematic level. The former stabilises information storage at local nodes in the memory-encoding neuronal circuit, through the mechanism of LTP (Section 2.4); the latter corresponds to the conversion process of temporary short-term memories following initial learning; these memories, stored in the hippocampus, are gradually redistributed and reactivated in the neocortex (supposedly)
through SWS. REM sleep allows subsequent synaptic consolidation of memories in the cortex.

The mechanism of consolidating information during sleep is believed to revolve around the memory replay hypothesis [170]; this hypothesis observes that neuronal firing patterns found during an event most commonly reoccur during the first 15–30 minutes of the following NREM SWS period. Occasionally, memory traces can be seen replaying during REM sleep and even during waking periods prior to sleep; they may be a temporary short-term measure of consolidation during wakefulness until sleep, which provides a longer-term solution. The memory consolidation hypothesis proposes that relevant synapses are modified during sleep, increasing efficiency of the encoding of memory traces; a heightened state of plasticity may be created when sleeping, increasing optimisation of memory storage. According to the synaptic homeostasis hypothesis (SHH), memory replay could also protect desirable memories by preventing specific and fragile connections, which are related to the event, from downscaling [170]; however, the mechanisms behind memory replay hypothesis and SHH are not necessarily mutually exclusive.

During wakefulness, sleep oscillations are suppressed by input from various ascend-
ing cholinergic, monoaminergic, and glutaminergic systems; the brain constantly gathers new information from the external environment [175]. During wakefulness, another net increase in synaptic formation and strength is caused by the accumulation of synaptic potentiation; the SHH proposes that a process of global downscaling and pruning takes place during sleep (Fig. 11.3), compensating for this net increase to reduce the number and strength of non-essential connections in a negative feedback manner [170, 176]. Synaptic homeostasis mechanisms are essential in preventing runaway potentiation or depression; aversion of synaptic overload is also economically favourable for energy balance and membrane maintenance. Reduced activity of the noradrenergic system during sleep almost guarantees that downscaling occurs instead of potentiation; although a net downregulation and renormalising in synapse number and dendritic complexity occurs during sleep, additional connections may be established to optimise information storage. The mechanism that rescales synaptic weights during sleep is not clearly understood; possible suggestions include arbitrary reduction, where all synaptic strengths decrease by an equal amount; relative weight reduction, where the imposed reduction is proportional to the neuron’s current strength; Hebbian learning, where synaptic connections adapt according to their frequency of activation.

Slow wave activity (SWA) denotes slow waves, which appear as oscillations between 0.5 and 4Hz in EEG recordings [176, 177]; the amplitude and duration of slow wave activity are closely and positively correlated to wakefulness before sleep. The intensity of SWA oscillations reliably measures an individual’s need for sleep, increasing as a function of prior wake time and declining during sleep [177]; they also defend the notion that sleep has a restorative function. Further justification for the SHH derives from observation of higher synaptic terminal retention, resulting from sleep deprivation. Synaptic protein quantities increase during wakefulness and decrease during sleep, independent of circadian rhythms [170], indicating the use of protein during sleep. The extent of synaptic net downscaling occurring during sleep depends on the activity in the preceding wakefulness period and the system’s current connectivity; if both variables are high, the network requires an
increased magnitude of restoration during subsequent sleep for complete resetting to be achieved.

Similar to the theory SHH, the main function of sleep appears to provide maximal plasticity state, optimising the development and maintenance of neural circuitry. Two age classes showing the highest sleep volumes are babies and the elderly; the former requires more sleep time for developmental processes; the latter need sleep to maintain neuronal circuits. Implementing synaptic modifications during wakefulness, an asynchronous and unpredictable state, may disrupt behaviour and learning capabilities; they could be dangerous and undesirable during wakefulness. Repetitive hyperpolarisations during wake time, induced by depressions in network connection strengths, would be expected to severely interfere with behaviour [176].

Hindered REM sleep duration in epileptic individuals could be indicative of the role REM sleep has in disturbing the synchrony developed in NREM sleep; REM sleep could interrupt NREM sleep, which might be detrimental over long periods of time. Despite the apparent benefits and preferential responses short periods of NREM sleep provide, long durations may cause excessive downscaling; however, studies involving the use of drugs to prevent REM sleep phase suggests that lack of REM sleep does not have adverse effects in humans [178, 179]. Thus, REM sleep could be the brain’s attempt to simulate wakefulness, testing the success of the modifications created during NREM sleep; if the individual is asleep, any discovered faults could be corrected in the subsequent NREM sleep period. The muscle tension during REM phases are comparable to wakefulness [178]. Muscles cannot be activated due to paralysis when sleeping because of the strong emotions that can be reached in this state, such as dreams and nightmares; this paralysis is induced by REM sleep as a defensive mechanism and may continue into wakefulness on rare occasions. Individuals woken during REM sleep rapidly regain alertness compared to those disturbed during deep NREM sleep; the latter approximately requires twenty minutes to restore alertness and blood pressure [178]. As dolphins do not perform REM sleep, this phase may not be a fundamental mechanism in terms of survival [178]; one
hemisphere in a dolphin’s brain remains awake at any given time to maintain swimming motions. The duration of REM phases increase as sleep progresses, possibly due to fewer modifications that need to be made (if any); more time is dedicated to testing brain functionality through REM sleep. The longest proportion of REM sleep is achieved around a human’s birth; this ratio declines with age, supporting the hypothesis that REM sleep artificially stimulates synapses, mimicking environmentally-induced stimuli.

In reality, the mechanisms and principles behind sleep function are likely to be far more complex; further investigations are still being performed to determine whether there the hypothalamus has a centre, which globally assesses synaptic potentiation and accordingly regulates sleep. It is uncertain whether the structural changes occurring during sleep affect neuronal circuit functioning; structural changes appear to solely and compactly store information without losing any major detail, ensuring the availability of more space for future requirements.

11.8 Neural Processing and Synchronisation

Intelligent cognition can selectively detach itself from current stimuli and external events, utilising only relevant inputs to produce actions that are aligned to the system’s intrinsic goals and motivational states [2]; realistically, such computations must be extremely fast and reliable to be of practical use. It is suggested that computational efficiency is achieved by predictions, which are continuously compared to actual signals as observed in the surrounding environment; expectations may lead to simultaneous initiation of all relevant responses to the occurrence of stimuli and can be achieved through synchronised neuronal firings [2]. Neural synchrony is hypothesised to be crucial for response selection, attention, object representation, and motor integration; it occurs in a multitude of cognitive and behavioural tasks [180].

Synchronisation is believed to greatly affect the brain’s ability to process information and communicate between different regions [103]; on a large scale, synchrony may benefit
the computational processing and efficiency of the cortex, leading to improved informa-
tion storage and transmission [3]. The brain’s activity reflects the heterogeneous nature
of the surrounding environment, depicting bursts of seemingly unpredictable activity fol-
lowed by states of relative quiescence [181]. The overwhelming complexity of the external
environment demands versatility and dynamism from the brain regarding the elements
that it utilises. Individual neurons are relatively modest in their computational power
and neuronal networks collectively perform extremely intricate operations [182]. The
abundance of received information often dictates a need for simultaneous information
processing across multiple brain regions; unit synchronisation optimises the clarity of the
signal acceptance and response. However, synchronisation may conflict with other com-
putations if a critical level is reached [3]; thus, healthy brains must prepare preventative
measures against excessive synchrony.

Different stimuli require certain neuronal groups to coordinate their activities to elicit
the required response; synchronised groups enhance task performance efficiency. Flexible
participation and contribution of individual cells and neuronal patterns provide the brain
with a vast repertoire of behaviours [5]; synchronisation is integral for encoding many
cognitive behaviours [4] and enables the brain to link information stored across the cor-
text within milliseconds [183]. The magnitude of success regarding task performance is
reviewed to reinforce or amend the neural response to optimise efficiency and resolution
of neuronal patterns.

11.9 Epilepsy

Evidence of seizure occurrence can be dated back to Hippocrates and Aristotle, who were
respectively born approximately in 460 BC and 384 BC [175]. Epilepsy is characterised
by distinct mechanisms of (hyper)excitability, resulting in spontaneous recurrent seizures,
epileptic spikes, and high frequency oscillations without any metabolic intoxication or
fever [184]; (hyper)excitability is caused by imbalance of excitation and inhibition within
the neural circuitry [185]; an increase in excitation of cells, a decrease in inhibition of cells, or both factors are responsible for this imbalance [184]. An estimated 1.3% of the world’s population suffer from epilepsy, where children are the most affected age-group; 10% of the world’s population have at least one seizure during their lives [186]. Seizure frequencies have been observed to reduce or completely cease as children age, which partially could be due to neuronal changes during early age development; the brain reduces delays in axonal propagation by increasing myelination during maturation, increasing conduction velocities through the saltatory conduction mechanism (Section 2.3); improved efficiency and functionality of neuronal communication from saltatory conduction reduces the risk of pathological behaviour patterns.

Epileptic seizures are classified as abnormal spatio-temporal neural activity, occurring as a result of excessive and synchronous electrical discharges; they can happen within a unique neuronal population relating to a partial/focal seizure or involve both brain hemispheres for generalised seizures. Typical seizure behaviour corresponds to neuronal spiking of approximately 3Hz for an average of twenty seconds and a series of slow waves of roughly 1.3Hz follow. Seizure frequency may vary from as few as one annual episode to several per day; over forty seizure types and thirty categories of epilepsy renders the discovery of a universal precursor difficult. Epilepsy categories are classified by the brain’s affected location called epileptogenic zones; they are classically described by rapid discharges in the upper beta and gamma bands of the electroencephalogram (25 - 100Hz) [187, 188, 189, 190]. Fast ripples are transient events hallmarked by low amplitude but high frequency oscillations in the range 250 - 600Hz; they are hypothesised as unique to brain areas, which can generate these seizures [191].

Seizure symptoms can vary from partial to severe cases; brief lapses in attention, impairment of consciousness, or muscle jerks are typical of partial and petit mal seizures [192]; they generally involve a single brain hemisphere, displaying SWA for long periods (0.5 - 1.5Hz). Severe or convulsive seizures result in violent and involuntary muscular contractions called grand mal seizure types; they usually involve both brain hemispheres,
causing the individual to collapse with tonic chlonic seizures. Partial seizures may develop into petit mal or grand mal types. Convulsions of these types usually occur with intervals of approximately three seconds; partial seizures can be categorised as simple, if the individual remains conscious, or complex, when consciousness is lost. It is common that an individual receives a sensation before a seizure, warning them of the imminent onset of a seizure; surprisingly, a grand mal seizure can induce a feeling of elation. The epiphany is known as an aura, briefly allowing individuals to prepare precautions; however, the individual is often unconscious during the seizure, rendering them unaware of experiencing a seizure after regaining consciousness. Following a seizure, REM sleep (Appendix 11.7) duration reduces in the subsequent sleep period and stage one sleep time increases at the expense of stages two and four [175]. The reduction in REM sleep increases the probability of further seizures as seizures during this phase are extremely rare, due to the typical behaviour of asynchronous spiking. Kindling occurs when an individual temporarily recovers from a string of seizures, due to a lowering of seizure threshold. Models have revealed that sporadic epileptic spikes escalate to a chronic state as a function of time; chaotic spiking is characterised by the repetition of spontaneous seizures [193].

Detecting and measuring seizure activity is best achieved through EEG recordings (Fig. 11.4); detection of abnormalities is unreliable for partial seizures, which have small and localised effects in small brain areas. Apart from seizure activity, an epileptic patient’s brain is similar to non-sufferers, appearing “normal” in contrast to other diseases such as Parkinson’s (Appendix 11.10), where brain dysfunction is usually evident. Current treatment of epilepsy generally involves anticonvulsant medication that is based on seizure type and an individual’s characteristics, though surgery and neurostimulation are also occasionally employed for treatment.

Existing research clearly states that sleep deprivation greatly influences and impacts upon epileptic seizure occurrence; it is likely that the brain is forced to severely change from the long-term potentiation achieved during wakefulness, rendering normal resetting processes insufficient. Seizures may occur during sleep as a result of the brain’s delayed
response, only attempting network modifications after sleep onset; they are also common in the immediate hours following waking, possibly indicating the brain’s inability to switch between sleeping and waking states. Two-thirds of some epileptic seizure types occur at night, usually during stage two sleep [192]; the brain’s actions are more inhibited during wakefulness and more vulnerable during sleep. Different brain pathways are used between sleep and wakefulness [170]; some pathways connect the thalamus, which congregates the most input from sensory organs, to the frontal cortex (Section 3.1); these are responsible for developing the fast and slow spindles (EEG oscillations) generated during light sleep. Abnormalities in specific pathway functioning during sleep are the underlying cause of different seizure types occurring between sleep and waking; since most seizures occur during sleep, many people may be unaware of their condition. It is unsurprising that seizures almost never occur during REM sleep as electrical brain activity is irregular during this stage [192]; many seizures occur during NREM sleep where synchrony has greater prominence. Elevated synchronisation in NREM sleep may activate and enhance propagation of postsynaptic responses including epileptogenic discharges; spikes seen in NREM sleep
are of higher amplitude, longer duration, and are less sharp than in wakefulness; spikes in REM sleep are of lower amplitude, shorter duration, and increased sharpness [195]. Epileptic individuals are most susceptible to nighttime seizures at the beginning of the night (9–11pm) and the end of the sleep cycle (3–5am) [196].

Upon sleep onset, neurons in the midbrain reticular formation and mid-pons (Section 3.1) have a slower firing rate [175]; decreased excitatory input hyperpolarises cortical and thalamocortical neurons, reducing synaptic input. Further hyperpolarisation of thalamic reticular nucleus cells and de-inactivation of low threshold spikes, $Ca^{2+}$, lead to sleep spindles appearing; as the cortex becomes deafferented, slow cortical oscillations (< 1Hz) initially occur in small regions, gradually spreading; resultantly, delta activity continues to synchronise in a typical manner of stages three and four of NREM sleep (Appendix 11.7). Slow oscillations are intimately associated with spike formation and wave discharges (seizures) [175]. Astrocytes are a subtype of glial cell (Appendix 11.1), which have a much higher density of $Na^{+}$ channels in epileptic sufferers, enabling the generation of much larger currents; alterations in the $K^{+}$ channels may affect buffering, leading to excessive build up of $K^{+}$ in the extracellular space [31, 105]. Epileptic seizures result from glial cells failing to regulate synaptic transmission, propagating electrical signals with excessive voltage to above normal activity levels [147].

11.10 Parkinson’s Disease

Parkinsonism is a neurodegenerative disease associated with increased activity in basal ganglia (Section 3.1) nuclei output; increases in burst discharges, oscillatory firing, and synchronous firing patterns throughout the basal ganglia can also be witnessed [197]. Parkinson’s disease is the second most common neurodegenerative disorder with the most common being Alzheimer’s disease [198]. Motor function is most noticeably compromised in Parkinson’s disease; common symptoms include an abnormal absence of voluntary movements (akinesia), slowness of movement, indicating an impairment (bradykinesia),
rigidity, and tremors. The principal frequency of a resting tremor in Parkinsonian patients is 3 - 6Hz and corticocortical synchronisation occurs prior to a tremor; this confirms the sequential activation of the motor cortex and tremor bursts, in addition to supporting the hypothesis of a central oscillator [31, 105]. Persistent beta oscillations (13 - 30Hz) in the basal ganglia are positively correlated with akinesia and bradykinesia symptoms [199]. Sleep abnormalities, fatigue, depression, and posture problems are also common in Parkinson’s disease; this complex disease relates to progressive neuronal degeneration from several locations in the central and peripheral nervous systems, most notably the dopaminergic neurons in the substantia nigra pars compacta (Section 3.1) [70, 71, 72, 73, 74]. The substantia nigra is located in the midbrain (Section 3.1) and is heavily involved in reward, addiction, and movement; the pars compacta primarily provides input to the basal ganglia with its many strong projections. The pars compacta is particularly important in supplying dopamine to the striatum; required for temporal processing, it is also involved in learned responses to stimuli. The pars compacta may regulate the sleep-wake cycle [200], as supported by insomnia and REM sleep disturbances reported from Parkinsonian patients. The cause of death among dopaminergic neurons in the pars compacta can only currently be hypothesised; death could be caused by abnormalities in mitochondrial complex 1 of dopaminergic neurons, which contain less calbindin, a protein involved in calcium ion transport that prevents build up to toxic levels [201]. In all cases, the pars compacta is very robust, requiring 50 – 80% of dopaminergic neurons to be dead before the onset of Parkinsonian symptoms.

The basal ganglia is a group of heavily interconnected subcortical nuclei [202] critical to the selection of appropriate actions; damage to this area impairs its ability to achieve desired responses [203]. Abnormal discharges from the basal ganglia area, beyond firing rate, is an essential intermediate step in the genesis of Parkinson’s disease symptoms; the subthalamic nucleus and globus pallidus are most affected by these symptoms. Neurons in these respective sites of Parkinsonian patients elicit bursting activity of action potentials, which are short epochs of substantially raised firing rates; however, these could
Neuronal behaviour within the basal ganglia of neurologically healthy individuals is typically characterised by spontaneous and asynchronous spiking; synchronous discharges are extremely rare, supporting the notion that the basal ganglia functions as a series of parallel and largely independent modules. In contrast, Parkinson’s disease can cause synchronous firing activity within the basal ganglia [6, 204, 205, 206, 207, 208], causing dopamine levels to reduce (Fig. 11.5), neurons in this area to widen their receptive fields [209], and synapses and dendritic spines to degenerate. Models developed to simulate the basal ganglia’s behaviour must include time delays to produce the system’s inherent oscillatory and unstable nature. Synaptic transmission delays between the subthalamic nucleus and globus pallidus are approximately 6ms, according to studies of rats and monkeys; these cannot be considered negligible compared to other parameters of the population model [199, 210]. Models including time delays are also susceptible to multistability, which is the coexistence of a large number of attractors, each associated with its own finitely-sized basin of attraction that is enclosed by separatrices; these are unstable boundaries, capable of dominating the system’s dynamics [211, 212, 213]. Multistability can lead to the selection of incorrect motor programmes, due to the convergence to an alternate stable attractor, in Parkinsonian patients.

The two most prominent treatments currently used to alleviate Parkinsonian symp-
toms are dopamine receptor agonist and deep brain stimulation. The former lowers the
degree of abnormal synchronisation in basal ganglia neurons [207, 214]; dopamine is im-
plied to be (at least) partially responsible in actively maintaining isolated firing of individual
neurons. The latter has successfully treated Parkinson’s disease symptoms, requiring
the application of high frequency stimulation to the subthalamic nucleus within the basal
ganglia; the treatment theoretically regularises the brain’s activity by overwriting patho-
logical elements [215, 216, 217] and by entraining the firing of target structures [197, 218];
however, it could also diversify the basal ganglia nuclei’s output, leading to an assortment
of increased and decreased firing rates. The therapeutic frequency for deep brain stimula-
tion is above 100Hz due to a step change occurring as the frequency increases to this level;
this strongly skews spike rate and spike regularity, causing transitions from symmetric to
asymmetric effects [219]. Basal ganglia neurons are believed to transmit inhibitory output,
which would spread upon an increase in activation; the neurons receiving the inhibition
must also be basal ganglia cells, reducing the amount of inhibition transmitted to the
following neuron; this reduces inhibition of the target structure, signalling the selection
of motor programmes [203, 220, 221]. Studies have shown that weakening inhibition in
interconnected inhibitory networks disrupts population rhythms [222]; reduced inhibition
in excitatory networks increases synchronisation activity [223].

11.11 Derivation of Raw Moments from Central Mo-
ments
This section details the derivation of the raw moments used in formulating the required
cumulant equations for the analysis in Section 9.1. The results for the $n$th order central
moments can be generated by addition of the products of all possible covariances; if $n$
is odd, the central moment for the $n$th order is zero. The covariance properties are
defined as $D_x = E[(x - m_x)^2]$, $D_y = E[(y - m_y)^2]$ (the covariance with itself), and $D_{xy} =
E[(x - m_x)(y - m_y)]$; $m_x$ and $m_y$ correspond to their respective mean values. Using these
properties, the results for the fourth order central moments can be found.

In the following derivations, subscript notation is used to denote different instances of the same random variable. The fourth order term consisting of one variable only is

\[ E[(x - m_x)^4] = E[(x_1 - m_{x_1})(x_2 - m_{x_2})(x_3 - m_{x_3})(x_4 - m_{x_4})]. \tag{11.20} \]

Splitting Eq. 11.20 into a sum of covariance products leaves

\[
E[(x - m_x)^4] = E[(x_1 - m_{x_1})(x_2 - m_{x_2})]E[(x_3 - m_{x_3})(x_4 - m_{x_4})]
+ E[(x_1 - m_{x_1})(x_3 - m_{x_3})]E[(x_2 - m_{x_2})(x_4 - m_{x_4})]
+ E[(x_1 - m_{x_1})(x_4 - m_{x_4})]E[(x_2 - m_{x_2})(x_3 - m_{x_3})]
= D_x D_x + D_x D_x + D_x D_x. \tag{11.21}
\]

Eq. 11.21 uses the property \( x_1 = x_2 = x_3 = x_4 \), giving

\[ E[(x - m_x)^4] = 3D_x^2. \tag{11.22} \]

The scenario where a new variable is added instead of an order of the initial variable is

\[ E[(x - m_x)^3(y - m_y)] = E[(x_1 - m_{x_1})(x_2 - m_{x_2})(x_3 - m_{x_3})(y - m_y)]. \tag{11.23} \]

Splitting Eq. 11.23 into a sum of covariance products leaves

\[
E[(x - m_x)^3(y - m_y)] = E[(x_1 - m_{x_1})(x_2 - m_{x_2})]E[(x_3 - m_{x_3})(y - m_y)]
+ E[(x_1 - m_{x_1})(x_3 - m_{x_3})]E[(x_2 - m_{x_2})(y - m_y)]
+ E[(x_1 - m_{x_1})(y - m_y)]E[(x_2 - m_{x_2})(x_3 - m_{x_3})]
= D_x D_{xy} + D_x D_{xy} + D_{xy} D_x. \tag{11.24}
\]
simplifying to

\[ E[(x - m_x)^2(y - m_y)] = 3D_xD_{xy}. \]  \hfill (11.25)

The case where two variables have equal order is

\[ E[(x - m_x)^2(y - m_y)^2] = E[(x_1 - m_{x_1})(x_2 - m_{x_2})(y_1 - m_{y_1})(y_2 - m_{y_2})]. \]  \hfill (11.26)

Splitting Eq. 11.26 into a sum of covariance products leaves

\[
E[(x - m_x)^2(y - m_y)^2] = E[(x_1 - m_{x_1})(x_2 - m_{x_2})]E[(y_1 - m_{y_1})(y_2 - m_{y_2})] \\
+ E[(x_1 - m_{x_1})(y_1 - m_{y_1})]E[(x_2 - m_{x_2})(y_2 - m_{y_2})] \\
+ E[(x_1 - m_{x_1})(y_2 - m_{y_2})]E[(x_2 - m_{x_2})(y_1 - m_{y_1})] \\
= D_xD_y + D_{xy}D_{xy} + D_{xy}D_{xy},
\]

simplifying to

\[ E[(x - m_x)^2(y - m_y)^2] = D_xD_y + 2D_{xy}^2. \]  \hfill (11.27)

Having found the central moments, the expressions for raw moments containing one variable can be found. The first raw moment is \( E[x] \) and can be extracted from the first central moment as shown:

\[ E[x - m_x] = E[x] - m_x. \]  \hfill (11.29)

However, the central moment equals zero when the order is odd, resulting in

\[ E[x] = m_x. \]  \hfill (11.30)
The second raw moment, \( E[x^2] \), can be obtained from the second central moment:

\[
E[(x - m_x)^2] = E[x^2 - 2xm_x + m_x^2]
\]
\[
= E[x^2] - 2m_x E[x] + m_x^2
\]
\[
= E[x^2] - 2m_x^2 + m_x^2
\]
\[
= E[x^2] - m_x^2
\]
\[
= D_x,
\] (11.31)

and can be rearranged as

\[
E[x^2] = D_x + m_x^2.
\] (11.32)

In a similar manner, the third raw moment, \( E[x^3] \), can be written as

\[
E[(x - m_x)^3] = E[x^3 - 3x^2m_x + 3xm_x^2 - m_x^3]
\]
\[
= E[x^3] - 3m_x E[x^2] + 3m_x^2 E[x] - m_x^3.
\] (11.33)

Substitute known results for \( E[x^2] \) (Eq. 11.32) and \( E[x] \) (Eq. 11.30) into Eq. 11.33 to obtain

\[
E[(x - m_x)^3] = E[x^3] - 3m_x (D_x + m_x^2) + 3m_x^2 E[x] - m_x^3
\]
\[
= E[x^3] - 3m_x D_x.
\] (11.34)

Since the third central moment is odd, it equals zero, resulting in

\[
E[x^3] = m_x^3 + 3m_x D_x.
\] (11.35)
Continuing in the same manner for the fourth raw moment, $E[x^4]$ provides

$$E[(x - m_x)^4] = E[(x^2 - 2xm_x + m_x^2)(x^2 - 2xm_x + m_x^2)]$$
$$= E[x^4 - 4x^3m_x + 6x^2m_x^2 - 4xm_x^3 + m_x^4]$$
$$= E[x^4] - 4m_xE[x^3] + 6m_x^2E[x^2] - 4m_x^3E[x] + m_x^4. \quad (11.36)$$

Substitute known results for $E[x^3]$ (Eq. 11.35), $E[x^2]$ (Eq. 11.32), and $E[x]$ (Eq. 11.30) into Eq. 11.36 to get

$$E[(x - m_x)^4] = E[x^4] - 4m_x(m_x^3 - 3m_xD_x + 6m_x^2(D_x + m_x^2) - 4m_x^4 + m_x^4$$
$$= E[x^4] - m_x^4 - 6m_x^2D_x. \quad (11.37)$$

Using the property for the fourth order central moment, $E[(x - m_x)^4] = 3D_x^2$ (Eq. 11.22), results in

$$E[x^4] = m_x^4 + 6m_x^2D_x + 3D_x^2. \quad (11.38)$$

The fifth raw moment, $E[x^5]$, gives

$$E[(x - m_x)^5] = E[(x - m_x)(x^4 - 4x^3m_x + 6x^2m_x^2 - 4xm_x^3 + m_x^4)]$$
$$= E[x^5] - 5x^4m_x + 10x^3m_x^2 - 10x^2m_x^3 + 5xm_x^4 - m_x^5]$$
$$= E[x^5] - 5m_xE[x^4] + 10m_x^2E[x^3] - 10m_x^3E[x^2] + 5m_x^4E[x] - m_x^5. \quad (11.39)$$

Substituting the known results for $E[x^4]$ (Eq. 11.38), $E[x^3]$ (Eq. 11.35), $E[x^2]$ (Eq. 11.32), and $E[x]$ (Eq. 11.30) into Eq. 11.39 gives

$$E[(x - m_x)^5] = E[x^5] - 5m_x(m_x^4 + 6m_x^2D_x + 3D_x^2) + 10m_x^2(m_x^3 + 3m_xD_x)$$
$$- 10m_x^3(D_x + m_x^2) + 5m_x^5 - m_x^5$$
$$= E[x^5] - m_x^5 - 10m_x^3D_x - 15m_xD_x^2. \quad (11.40)$$
Since all the odd central moments equal zero, the fifth raw moment is

\[ E[x^5] = m_x^5 + 10m_x^3D_x + 15m_xD_x^2. \]  

(11.41)

Having derived the raw moments up to the fifth order for a single variable, another variable is introduced and the multivariate raw moments are extracted; the above method remains applicable.

The lowest order raw multivariate moment, \( E[xy] \), gives

\[
E[(x - m_x)(y - m_y)] = E[xy - ym_x - xm_y + m_xm_y]
= E[xy] - m_xE[y] - m_yE[x] + m_xm_y
= E[xy] - m_xm_y
= D_{xy},
\]

(11.42)

rearranging to

\[ E[xy] = D_{xy} + m_xm_y. \]

(11.43)

The multivariate raw moment for \( E[x^2y] \) can be generated from

\[
E[(x - m_x)^2(y - m_y)] = E[(x^2 - 2xm_x + m_x^2)(y - m_y)]
= E[x^2y - 2ym_x + m_x^2y - y^2m_y + 2xm_xm_y - m_x^2m_y]
= E[x^2y] - 2m_xE[xy] + m_x^2E[y] - m_yE[x^2] + 2m_xm_yE[x] - m_x^2m_y.
\]

(11.44)
Substituting $E[xy]$ (Eq. 11.43), $E[x^2]$ (Eq. 11.32), $E[x]$ (Eq. 11.30), and $E[y] = m_y$ (Eq. 11.30) into Eq. 11.44 gives

$$E[(x - m_x)^3(y - m_y)] = E[x^2] - 2m_x(D_{xy} + m_xm_y) + m_x^2m_y - m_y(D_x + m_x^2) + 2m_x^2m_y - m_x^2m_y$$

$$= E[x^2] - 2m_xD_{xy} - m_yD_x - m_x^2m_y. \quad (11.45)$$

The third order multivariate central moment equals zero, thus

$$E[x^2] = m_x^2m_y + m_yD_x + 2m_xD_{xy}. \quad (11.46)$$

The fourth order raw moment, $E[x^3y]$, provides

$$E[(x - m_x)^3(y - m_y)] = E[(x^3 - 3x^2m_x + 3xm_x^2 - m_x^3)(y - m_y)$$

$$= E[x^3y] - 3x^2ym_x + 3nym_x^2 - ym_x^3 - x^3m_y + 3x^2m_xm_y - 3xm_x^2m_y$$

$$+ m_x^3m_y$$

$$= E[x^3y] - 3m_xE[x^2y] + 3m_x^2E[xy] - m_x^3E[y] - m_yE[x^3]$$

$$+ 3m_xm_yE[x^2] - 3m_x^2m_yE[x] + m_x^3m_y]. \quad (11.47)$$

Substituting $E[x^2y]$ (Eq. 11.46), $E[x^3]$ (Eq. 11.35), $E[xy]$ (Eq. 11.43), $E[x^2]$ (Eq. 11.32), $E[x]$ (Eq. 11.30), and $E[y]$ (Eq. 11.30) into Eq. 11.47 leaves

$$E[(x - m_x)^3(y - m_y)] = E[x^3y] - 3m_x(2m_xD_{xy} + m_yD_x + m_x^2m_y) + 3m_x^2(D_{xy} + m_xm_y)$$

$$- m_x^3m_y - m_y(m_x^3 + 3m_xD_x) + 3m_xm_y(D_x + m_x^2) - 3m_x^3m_y + m_x^3m_y$$

$$= E[x^3y] - 3m_x^2D_{xy} - 3m_xm_yD_x - m_x^3m_y. \quad (11.48)$$

Applying the property $E[(x - m_x)^3(y - m_y)] = 3D_xD_{xy}$ (Eq. 11.25) rearranges Eq. 11.48 to

$$E[x^3y] = m_x^3m_y + 3m_x^2D_{xy} + 3m_xm_yD_x + 3D_xD_{xy}. \quad (11.49)$$

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The fifth order term, $E[x^4y]$, is given by

\[
E[(x - m_x)^4(y - m_y)] = E[(x^4 - 4x^3m_x + 6x^2m_x^2 - 4xm_x^3 + m_x^4)(y - m_y)]
\]
\[
= E[x^4y - 4x^3ym_x + 6x^2ym_x^2 - 4xym_x^3 + ym_x^4 - x^4m_y + 4x^3m_xm_y
\]
\[
- 6x^2m_x^2m_y + 4xm_x^3m_y - m_x^4m_y]
\]
\[
= E[x^4y] - 4E[x^3y] + 6m_x^2E[x^2y] - 4m_x^3E[xy] + m_x^4E[y]
\]
\[
- m_yE[x^4] + 4m_xm_yE[x^3] - 6m_x^2m_yE[x^2] + 4m_x^3m_yE[x] - m_x^4m_y.
\]

(11.50)

Substituting $E[x^3y]$ (Eq. 11.49), $E[x^4]$ (Eq. 11.38), $E[x^2y]$ (Eq. 11.46), $E[x^3]$ (Eq. 11.35), $E[xy]$ (Eq. 11.43), $E[x^2]$ (Eq. 11.32), $E[x]$ (Eq. 11.30), and $E[y]$ (Eq. 11.30) into Eq. 11.50 provides

\[
E[(x - m_x)^4(y - m_y)] = E[x^4y] - 4(m_x^3m_y + 3m_x^2D_{xy} + 3m_xm_yD_{x} + 3D_{x}D_{xy})
\]
\[
+ 6m_x^2(m^2_{xy}m_y + m_yD_{x} + 2m_xD_{xy}) - 4m_x^3(D_{xy} + m_xm_y) + m_x^4m_y
\]
\[
- m_y(m_x^4 + 6m_x^2D_{x} + 3D_{x}^2) + 4m_xm_y(m_x^3 + 3m_xD_{x})
\]
\[
- 6m_x^2m_y(D_{x} + m_x^2) + 4m_x^4m_y - m_x^4m_y
\]
\[
= E[x^4y] - 4m_x^3m_y - 12m_x^2D_{xy} - 12m_xm_yD_{x} - 12D_{x}D_{xy} + 8m_x^3D_{xy}
\]
\[
+ 3m_x^4m_y + 6m_x^2m_yD_{x} - 3m_yD_{x}^2.
\]

(11.51)

However, central moments with an odd order equal zero; Eq. 11.51 rearranges to

\[
E[x^4y] = 4m_x^3m_y - 3m_x^4m_y - 6m_x^2m_yD_{x} + 12m_xm_yD_{x} + 3m_yD_{x}^2
\]
\[
- 8m_x^3D_{xy} + 12m_x^2D_{xy} + 12D_{x}D_{xy}.
\]

(11.52)
An expression for $E[x^2y^2]$ can be obtained through

$$E[(x - m_x)^2(y - m_y)^2] = E[(x^2 - 2xm_x + m_x^2)(y^2 - 2ym_y + m_y^2)]$$

$$= E[x^2y^2 - 2xy^2m_x + y^2m_x^2 - 2x^2ym_y + 4xym_xm_y - 2ym_x^2m_y + x^2m_y^2$$

$$- 2xm_xm_y^2 + m_x^2m_y^2]$$

$$= E[x^2y^2] - 2m_xE[x^2y^2] + m_x^2E[y^2] - 2m_xE[x^2y] + 4m_xm_yE[xy]$$

$$- 2m_x^2m_yE[y] + m_y^2E[x^2] - 2m_xm_y^2E[x] + m_x^2m_y^2. \quad (11.53)$$

Substituting $E[x^2y]$ (Eq. 11.46), $E[x^2y^2]$ (Eq. 11.46), $E[xy]$ (Eq. 11.43), $E[x^2]$ (Eq. 11.32), $E[y^2]$ (Eq. 11.32), $E[x]$ (Eq. 11.30), and $E[y]$ (Eq. 11.30) into Eq. 11.53 yields

$$E[(x - m_x)^2(y - m_y)^2] = E[x^2y^2] - 2m_x(x_m^2m_y + m_xD_y + 2m_yD_{xy}) + m_x^2(D_y + m_y^2)$$

$$- 2m_y(m_x^2m_y + m_yD_x + 2m_xD_{xy}) + 4m_xm_y(D_{xy} + m_xm_y)$$

$$- 2m_x^2m_y + m_y^2(D_x + m_x^2) - 2m_x^2m_y + m_x^2m_y^2$$

$$= E[x^2y^2] - 4m_xm_yD_x - m_x^2D_y - m_x^2m_y^2. \quad (11.54)$$

As $E[(x - m_x)^2(y - m_y)^2] = D_xD_y + 2D_{xy}^2$ (Eq. 11.28), Eq. 11.54 can be simplified to

$$E[x^2y^2] = m_x^2m_y^2 + m_y^2D_x + m_x^2D_y + D_xD_y + 4m_xm_yD_{xy} + 2D_{xy}. \quad (11.55)$$

The two-variable multivariate raw moment $E[x^3y^2]$ extracts from

$$E[(x - m_x)^3(y - m_y)^2] = E[(x^3 - 3x^2m_x + 3x_m^2m_x^2 - m_x^3)(y^2 - 2ym_y + m_y^2)]$$

$$= E[x^3y^2 - 3x^2y^2m_x + 3xym_x^2m_x^2 - y^2m_x^3 - 2x^3ym_y + 6x^2ym_xm_y$$

$$- 6xym_x^2m_y + 2ym_x^3m_y + x^3m_y^2 - 3x^2m_xm_y^2 + 3xm_x^2m_y^2 - m_x^3m_y^2]$$

$$= E[x^3y^2] - 3m_xE[x^2y^2] + 3m_x^2E[xy^2] - m_x^3E[y^2] - 2m_yE[x^3y]$$

$$+ 6m_xm_yE[x^2y] - 6m_x^2m_yE[xy] + 2m_x^3m_yE[y] + m_y^3E[x^3]$$

$$- 3m_xm_y^2E[x^2] + 3m_x^2m_y^2E[x] - m_x^3m_y^2. \quad (11.56)$$
Substituting $E[x^2y^2]$ (Eq. 11.55), $E[x^3y]$ (Eq. 11.49), $E[x^3]$ (Eq. 11.35), $E[x^2y]$ (Eq. 11.46), $E[xy^2]$ (Eq. 11.46), $E[xy]$ (Eq. 11.43), $E[x^2]$ (Eq. 11.32), $E[y^2]$ (Eq. 11.32), $E[x]$ (Eq. 11.30), and $E[y]$ (Eq. 11.30) into Eq. 11.56 gives

$$E[(x - m_x)^3(y - m_y)^2] = E[x^3y^2] - 3m_xm_y + m_y^2D_x + m_x^2D_y + D_xD_y + 4m_xm_yD_{xy} + 2D_{xy}^2$$
$$+ 3m_x^2m_y^2 + m_xD_y + 2m_yD_{xy} - m_y^3(D_y + m_y)$$
$$- 2m_y(m_x^3m_y + 3m_x^2D_y + 3m_xm_yD_x + 3D_xD_{xy})$$
$$+ 6m_xm_y(m_x^2m_y + m_yD_x + 2m_xD_{xy}) - 6m_y^2m_y(D_{xy} + m_xm_y)$$
$$+ 2m_x^3m_y^2 + m_x^2(m_y^3 + 3m_xD_x) - 3m_xm_y^2D_x + 2m_x^2 + 3m_x^3m_y^2 - 3m_x^3m_y^2$$
$$= E[x^3y^2] - 3m_xD_xD_{xy} - 6m_xD_{xy}^2 - 6m_xm_yD_{xy} - m_x^3D_y - 3m_xm_y^2D_x$$
$$- 6m_yD_xD_{xy} - m_x^3m_y^2.$$

(11.57)

The central moment of order five equals zero, giving

$$E[x^3y^2] = m_x^3m_y + m_y^3D_x + 3m_xm_y^2D_x + 6m_x^2m_yD_{xy}$$

(11.58)
$$+ 6m_yD_xD_{xy} + 3m_xD_xD_{xy} + 6m_xD_{xy}^2.$$

Multivariate raw moments consisting of more than two variables can be extracted; the lowest order three-variable raw moment can be gathered from

$$E[(x - m_x)(y - m_y)(z - m_z)] = E[(xy - yz - zm_z + xz - m_ym_z)(z - m_z)]$$
$$= E[xyz - yzmx - xzmy + zm_xm_y + xym_z + ym_zm_x - zm_xm_ym_z$$
$$- m_xm_ym_z]$$
$$= E[xyz] - m_xE[yz] - m_yE[xz] + m_xm_yE[z] - m_zE[xy]$$
$$+ m_xm_zE[y] + m_ym_zE[x] - m_xm_ym_z.$$

(11.59)

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Substituting the known results for $E[xy]$ (Eq. 11.43), $E[yz]$ (Eq. 11.43), $E[xz]$ (Eq. 11.43), $E[x]$ (Eq. 11.30), $E[y]$ (Eq. 11.30), and $E[z]$ (Eq. 11.30) into Eq. 11.59 gives

$$E[(x - m_x)(y - m_y)(z - m_z)] = E[xyz] - m_x(D_{yz} + m_y m_z) - m_y(D_{xz} + m_x m_z) + m_x m_y m_z - m_z(D_{xy} + m_x m_y) + m_x m_y m_z + m_x m_y m_z - m_x m_y m_z$$

$$= E[xyz] - m_x D_{yz} - m_y D_{xz} - m_z D_{xy} - m_x m_y m_z. \quad (11.60)$$

The central moment has an odd order of three and equals zero; the raw moment can be rewritten as

$$E[xyz] = m_x m_y m_z + m_x D_{yz} + m_y D_{xz} + m_z D_{xy}. \quad (11.61)$$

### 11.12 Numerical Simulation Code

The program below is written in Fortran language and integrates stochastic delay differential equations, specifically equations for the FitzHugh-Nagumo system with globally applied delayed feedback (Section 7.1). The program gathers data for Poincaré maps and phase portraits; mean-field properties and interspike intervals are recorded; quantitative values describing a system’s configuration are generated (Section 6.1).

```fortran
IMPLICIT REAL*8 (a-h,o-z)
PARAMETER (NM=10,NLAG=500000,KLAG=10,NTK=10000)
DIMENSION rk(4,10,NTK),f(10,NTK),yy1(10,NTK),fb(10),yn(10,NTK)
DIMENSION y(10,NTK),y0(10),tau(KLAG),nfl(KLAG),xlag(KLAG,NTK),
,kl(KLAG),flag(KLAG,NLAG,NTK),lfl(KLAG,NTK),t_poi1(50000),
,poi1(2,50000),pp(2,50000,NTK),av_yflag1(50000),n_ran(NTK,NTK),
,av_xf1(50000),av_yf1(50000),x_f(NTK),y_f(NTK),yf_lag(NTK),
,ipq(NTK),n_ran(NTK,NTK),ipath(NTK,NTK),ipts(NTK),
,iptx(NTK),icv(NTK)
CHARACTER*40 fout,froot
CHARACTER*20 name_dir
CHARACTER*60 name_res
CHARACTER*3 k_rand,key_wr_poi,key_wr_pp,k_bid
CHARACTER*4 apar(10)
```

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```
REAL gasdev,ran3
COMMON/NOISE/ix1
COMMON/PAR/pa(10),tt

c=====Reading Parameters====>
open(1,file='fhn_glob_yin.in')
read(1,*)name_res   !filename for statistical averages
read(1,*)name_dir   !folder to store data files
read(1,*)key_wr.poi,key_wr.pp,t_ini,t_end  !poincare map, phase port (y/n)
read(1,*)n,m        !dimension of system, number of parameters
read(1,*)pa(ii), ii=1,n+1)  !parameters (eps,a,K,gamma,D)
read(1,*)y0(jj), jj=1,n)   !initial conditions
read(1,*)t_rel,t_int,h,kti,n.poi  !h(timestep), kti(time modifier)
read(1,*)ipar     !parameter number to be varied
read(1,*)spa,fpa,stpa !starting value, final value, step of parameter
read(1,*)ndel    !number of delays
read(1,*)kl(ii),ii=1,ndel)  !numbers of delayed variables
read(1,*)tau(ii),ii=1,ndel)  !delay times
read(1,*)net, per,k_bid    !network size, percentage connection (0-1)
read(1,*)k_rand,nix1    !random seed for gasdev?, nix1 manual manipulation
close(1)
c=====Reading Parameters====<

c=====Naming Parameters====>
apar(1)="eps1"
apar(2)="a1"
apar(3)="K"
apar(4)="gamma"
apar(5)="D"
c=====Naming Parameters====<

c=====Initialising Random Variables Generator====>
if(k_rand.eq.'yes'.and.pa(n+1).gt.0.)then
itime=time()
if(itime.lt.0)then
print*, "Warning! The function 'time' was compiled with errors!"
stop
endif
do while (abs(itime).gt.300)
itime=abs(itime)-327
enddo
ix1=-itime
```
else
  ix1=nix1
endif
print*, "Seed for gasdev is ",ix1
do i=1, 1.007
  xxx=gasdev(ix1)
enddo

***Initialising Random Variables Generator***

***Establishing Network Connections***

* n_con is the number of connections needed per neuron.
* net is the network size.
* n_ran is the matrix holding all connection assignments.

id=1 !initial ran3 seed
n_con=int((net-1)*per) !net-1 is maximum possible number of connections
do kk=1, net
  do nr=1, n_con
    n_ran(kk,nr)=0 !initialise connection matrix
  enddo
enddo
print*, "n_con=", n_con

IF(n_con.gt.0)then !connections required at all
do kk=1, net
  do nr=1, n_con
    if(n_ran(kk,nr).eq.0)then !nr'th connection not assigned yet
      do while(n_ran(kk,nr).eq.0) !until unique connection established
        id=id+1
        ran_net=ran3(-id)
        n_ran(kk,nr)=int(ran_net*net)+1 !convert ran3 to NTK scale
      ENDIF
      IF((n_ran(kk,nr).lt.1).or.(n_ran(kk,nr).gt.net))then
        n_ran(kk,nr)=0 !invalid neuron number, out of range
      ELSE
        if(kk.eq.n_ran(kk,nr))then
          n_ran(kk,nr)=0 !no self connections
        else
          if(nr.gt.1)then !comparison with previous from 2nd onwards
            do nq=1, nr-1
              IF(n_ran(kk,nr).eq.n_ran(kk,nq))then
                n_ran(kk,nr)=0 !no repeated connections in same direction
              ENDIF
            enddo
          endif
        endif
      endif
    endif
  enddo
endif
endif

if(k_bid.eq.'yes'.and.n_ran(kk,nr).ne.0) then  !bidirectional?
  do nq=1,n_con
    IF(n_ran(n_ran(kk,nr),nq).eq.kk) then
      n_ran(kk,nr)=0  !no repeats for other neuron
    ENDIF
  enddo
endif

if(n_ran(kk,nr).ne.0) then
  nj=1
  do while(nj.gt.0)
    IF(n_ran(n_ran(kk,nr),nj).eq.0) then
      n_ran(n_ran(kk,nr),nj)=kk
      nj=0  !reciprocal connection established
    ELSE
      nj=nj+1
    IF(nj.gt.net) then
      n_ran(kk,nr)=0
      nj=0
    ENDIF
    ENDIF
  enddo
endif
endif
ENDIF

print*, "n_ran(kk,nr)=",n_ran(1,1)
print*, "n_ran(net,n_con)=",n_ran(net,n_con)
c======Establishing Network Connections======>

c========Calculating Path Lengths========>
  do kk=1,net
    do iw=1,net-1
      ipath(kk,iw)=0  !set all path lengths to zero initially
    enddo
  enddo
  ict=1

inp=0
do iw=1,net-1
  inp = inp + 1
  if(kk.eq.inp)then
    inp = inp + 1      !avoid path lengths to the same neuron
  endif
  ix=0
  ixy=0
  ipl=1            !path length counter
  ils=0
  do nr=1,n_con
    if(n_ran(kk,nr).eq.inp)then      !direct connection
      ipath(kk,iw)=ipl
    else                             !no direct connection
      if(n_ran(kk,nr).lt.kk)then    !connected to a neuron of known path length
        ils=ils+1
        icv(ils)=n_ran(kk,nr)
      endif
      if(ix.gt.0)then
        isw=0
        do kkk=1,ix
          if(ipts(kkk).eq.n_ran(kk,nr))then    !search for duplicates
            isw=1
          endif
        enddo
        if(isw.eq.0)then
          ix=ix+1
          ipts(ix)=n_ran(kk,nr)
          ixy=ixy+1
          iqq(ixy)=n_ran(kk,nr)
        endif
      else
        ix=ix+1
        ipts(ix)=n_ran(kk,nr)        !gather list of directly connected neurons
        ixy=ixy+1
        iqq(ixy)=n_ran(kk,nr)
      endif
    endif
  enddo
  if(ils.eq.n_con.and.ipath(kk,iw).eq.0)then !all connections have known path length
    icvmin=net
  endif
  do iop=1,n_con
if((icv(iop).lt.inp).and.(kk.gt.inp))then
  if(ipath(icv(iop),iw-1).lt.icvmin)then !find minimum of known path lengths
    icvmin=ipath(icv(iop),iw-1)
    ikk=icv(iop)
    endif
  else
    if(ipath(icv(iop),iw).lt.icvmin)then
      icvmin=ipath(icv(iop),iw)
      ikk=icv(iop)
    endif
  endif
enddo
if(icvmin.ge.net)then
  ipath(kk,iw)=net
elseif(icvmin.gt.0.and.icvmin.lt.net) then
  ipath(kk,iw)=icvmin+ipl
endif
endif

nnn=ix
ibv=net-1
do while((ipl.lt.ibv).and.(ipath(kk,iw).eq.0))
  iob=ix
  iok=ix
  ipl=ipl+1
  idw=0
  ix=0
  loop1 : do jx=1,nnn
    do nr=1,n_con
      if(ipath(kk,iw).eq.0)then
        if (n_ran(ipts(jx),nr).eq.inp) then !path found
          ipath(kk,iw)=ipl
          iok=iok+1
          exit loop1
        endif
      else !still no path
        if(ix.gt.0)then
          isw=0
          do kkk=1,ix
            if(iqq(kkk).eq.n_ran(ipts(jx),nr))then !search for duplicates
              isw=1
            endif
          enddo
          if(isw.eq.0)then
            icvmin=ipath(kk,iw)
            ikk=kk
            endif
          endif
        endif
      endif
    enddo
  enddo
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ix=ix+1
iptsx(ix)=n_ran(ipts(jx),nr) !store new paths
ixy=ixy+1
iqq(ixy)=n_ran(ipts(jx),nr)
idw=1
iok=iok+1
endif
else
ix=ix+1
iptsx(ix)=n_ran(ipts(jx),nr)
ixy=ixy+1
iqq(ixy)=n_ran(ipts(jx),nr)
idw=1
iok=iok+1
endif
endif
endif
enddo loop1
nnn=ix
do ih=1,nnn
ipts(ih)=iptsx(ih)
enddo
if(iok.eq.iob)then !no new paths to search
ipath(kk,iw)=net
else
iob=iok
endif
if(ipath(kk,iw).ne.0)exit
enddo
enddo
enddo

print*, "out of loop"

iea=0
ieb=0
iec=0
ied=0
i_dis=0
i_ova=0
in_path=0
inph=0
do kk=1,net
  do iw=1,net-1
    if (ipath(kk,iw).eq.net) then
      ipath(kk,iw)=0 ! restore disconnected pairs to zero
    endif
    in_path=in_path+ipath(kk,iw)
    if (ipath(kk,iw).gt.inph) then ! finding longest path
      ied=iec
      iec=ieb
      ieb=iea
      iea=inph
      inph=ipath(kk,iw)
    endif
    if (ipath(kk,iw).eq.net) then
      i_ova=i_ova+1
    endif
    if (ipath(kk,iw).eq.0) then
      i_dis=i_dis+1 ! count disconnected pairs
    endif
  enddo
enddo
pqr=real(net)*(real(net-1))
av_path=abs(real(in_path)/pqr) ! average path length
print*, "av_path=",av_path,"longest path=",inph
print*, "path length sum=",in_path,"possible connections",pqr
print*, "number disconnected=",i_dis,"over=",i_ova
print*, iea,ieb,iec,ied

! Calculating Path Lengths

! Setting Parameter Values
  if(t_rel.le.0.0)t_rel=h*kti
  npstep=(fpa-spa)/stpa +1
  h2=dsqrt(h)
dt=h*kti
  n1=t_int/h/kti

! Setting Parameter Values

! Parameter Loop
  do k=0,npstep
    open(16,file="num")
    c If parameter to vary is less than m+1, loop parameter.
    c If parameter to vary is greater than m+1, loop time delay.
  enddo
if(ipar.le.m+1) then
    pa(ipar)=spa +k*stpa
    write(16,'(f8.4)')pa(ipar)
else
    tau(ipar-m-1)=spa+k*stpa
    write(16,'(f8.4)')tau(ipar-m-1)
endif
close(16)
c=====Impose Initial Conditions=====>
do kk=1,net
  do i=1,n
    y(i,kk)=y0(i)
  enddo
enddo
c=====Impose Initial Conditions=====<
c=====Initialise Lag=====>
c This prevents an error in case of no subsequent lag.
do ii=1, ndel
  do kk=1,net
    flag(ii,1,kk)=y(kl(ii),kk) !flag is delayed variable array
  enddo
enddo

CALL inilag(ndel,tau,h,flag,nfl,lfl,xlag,net)
c=====Initialise Lag=====<
c=====Relaxation Period=====>
t=0. !initialise time
D=pa(m+1) !initialise noise parameter

  do while (t.lt.t_rel)
    t=t+h
    CALL rkin(ndel,n,y,h2,t,m,rk,f,yy1,fb,yn,flag,nfl,kl, * lfl,xlag,net,x_f,y_f,yf_lag,n_ran,n_con,glob)
  enddo

c=====Relaxation Period=====<
print *, 'Relaxation over:',t
if (ipar.le.m+1) then
  print *,apar(ipar),": ",pa(ipar)
else
    print *, "tau: ", tau(ipar-m-1)
endif ! printing looped/varied parameter

* Integration *

av_xf=0.
av_yf=0.
av_yflag=0.

do kk=1,net
    av_xf=av_xf+x_f(kk)
    av_yf=av_yf+y_f(kk)
    av_yflag=av_yflag+yf_lag(kk)
endo

av_xf=av_xf/net ! average over network at current time
av_yf=av_yf/net
av_yflag=av_yflag/net

i_poi1=0
av_xfprev=av_xf
av_yfprev=av_yf

n1_true=0
i_vr=0

do i=1,n1
    do ii=1,kti
        t=t+h
        CALL rkin(ndel,n,y,h2,t,m,rk,yy1,fb,ym,flag,nfl,k1, lfl,xlag,net,x_f,y_f,yf_lag,n_ran,n_con,glob)
    enddo

av_xf=0.
av_yf=0.
av_yflag=0.

do kk=1,net
    av_xf=av_xf+x_f(kk)
    av_yf=av_yf+y_f(kk)
    av_yflag=av_yflag+yf_lag(kk)
endo

av_xf=av_xf/net
av_yf=av_yf/net
av_yflag=av_yflag/net

if(av_xf.ge.0..and.av_xfprev.lt.0.)then
  i_poi1=i_poi1+1

  c=====Linear Interpolation=====>
  CALL lin_int(t,h,av_xfprev,av_xf,tp1)
  CALL poi_int(t,h,tp1,av_yfprev,av_yf,y1p)

  c=====Linear Interpolation=====<
  t_poi1(i_poi1)=tp1

  c=====Storing Poincare Map Data=====>
  if(key_wr_poi.eq.'yes')then
    poi1(1,i_poi1)=0.
    poi1(2,i_poi1)=y1p
  endif

  c=====Storing Poincare Map Data=====<
  if(i_poi1.ge.n_poi)goto 3

  av_xfprev=av_xf
  av_yfprev=av_yf
enddo

  c=====Storing Phase Plane Data=====>
  if(t.ge.t_ini.and.t.le.t_end)then
    i_wr=i_wr+1
    do j=1,n
      do kk=1,net
        pp(j,i_wr,kk)=y(j,kk)
      enddo
    enddo
    av_xf1(i_wr)=av_xf
    av_yf1(i_wr)=av_yf
    av_yflag1(i_wr)=av_yflag
  endif

  c=====Storing Phase Plane Data=====<
  enddo   end of i,n1 loop

  3 continue

  n1_true=i
  n_wr_pp=i_wr
  n_poi1=i_poi1
  print *, 'Time:', t
  print *, 'Planned number of integration steps:', n1
  print *, 'Actual number of integration steps:', n1_true
  print *, 'Number of phase portrait points to write:', n_wr_pp
  print *, 'Total number of Poincare points:', n_poi1
c=====Integration=====

\[ \text{xf\_mn}=0. \quad \text{xf\_mn is average with time} \\
\timesf\_sqmn=0. \\
\]

do i\_wr=1,n\_wr\_pp 
\[ \text{xf\_mn}=\text{xf\_mn}+\text{av\_xf1}(i\_wr) \\
\timesf\_sqmn=\timesf\_sqmn+\text{av\_xf1}(i\_wr)\timesf\_xf1(i\_wr) \]
enddo
\[ \text{xf\_mn}=\text{xf\_mn}/n\_wr\_pp \\
\timesf\_sqmn=\timesf\_sqmn/n\_wr\_pp \\
\timesf\_var=\timesf\_sqmn-(\text{xf\_mn}\timesf\_mn) \\
\timesf\_sd=sqrt(\timesf\_var) \]

open(7, file="num")
read(7, *) froot
close(7)
kspc=index(froot, " ")
kspc1=index(fout, "")

if(key\_wr\_poi.eq.'yes')then
open(5, file=name\_dir//froot(1:kspc-1)\".poi")
print *,'Writing phase portraits to: \\
\text{name\_dir//froot(1:kspc-1)\".poi"}
write(5,*)'# ',(apar(j),': ',pa(j),', ',j=1,5)
write(5,*)'# ',(apar(j),': ',pa(j),', ',j=6,m)
write(5,*)'# k1:',(kl(ii),ii=1,ndel),
', ' tau:',(tau(ii),ii=1,ndel)
write(5,*)'# t\_poi1(i\_poi)-t\_rel,/'
* ' (poi1(j,i\_poi),j=1,n),t\_isi'
do i\_poi=2,n\_poi 
t\_isi=t\_poi1(i\_poi)-t\_poi1(i\_poi-1)
write(5,200)t\_poi1(i\_poi)-t\_rel,t\_isi, 
(poi1(j,i\_poi),j=1,n)
enddo

close(5)
endif

c=====Writing Poincare Map=====

if(key\_wr\_pp.eq.'yes')then
open(4, file=name\_dir//froot(1:kspc-1)\".pp")
print *,'Writing phase portraits to: \\
\text{name\_dir//froot(1:kspc-1)\".pp"}

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write(4,*)'# ',(apar(j),': ',pa(j),', ',j=1,5)
write(4,*)'# ',(apar(j),': ',pa(j),', ',j=6,m)
write(4,*)'# kl:',(kl(ii),ii=1,ndel),
, ' tau:',(tau(ii),ii=1,ndel)
write(4,*)'# t, y(j,kk),j=1,n'
do i_wr=1,n_wr_pp
   write(4,200)t_ini-t_rel+i_wr*dt,av_xf1(i_wr),av_yf1(i_wr)
enddo
close(4)

open(4, file=name_dir//froot(1:kspc-1)//".pp_all")
do i_wr=1,n_wr_pp
201 format(1x,20(g12.5,1x))
   write(4,201)t_ini-t_rel+i_wr*dt,
, ((pp(j,i_wr,kk),j=1,n),kk=1,10)
enddo
close(4)
endif

c=====Calculating Mean Interspike Interval====>
av_isi=0.
sqvav_isi=0.
do i_poi=2,n_poi1
   t_isi=t_poi1(i_poi)-t_poi1(i_poi-1)
   av_isi=av_isi+t_isi
   sqav_isi=sqvav_isi+t_isi*t_isi
enddo

av_isi=av_isi/REAL(n_poi1-1)
sqvav_isi=sqvav_isi/REAL(n_poi1-1)
var_isi=sqvav_isi-av_isi*av_isi
sd_isi=sqrt(var_isi)
c=====Calculating Mean Interspike Interval====>

c=====Writing Statistical Averages====>
open(3,file=name_res,access='append')
if (ipar.le.m+1) then
   write(3,200) pa(ipar),xf_mn,xf_sqmn,xf_var,xf_sd,av_isi,
   * sqav_isi,var_isi,sd_isi,pa(4)
else
   write(3,200) tau(ipar-m-1),xf_mn,xf_sqmn,xf_var,xf_sd,av_isi,
* sqavisi, varisi, sdisi, pa(4)
endif

close(3)

// Writing Statistical Averages

print *,

enddo

// Parameter Loop

CALL SYSTEM("rm num")

200 format(1x,35(g12.5,1x))

print *,char(7)

END

// FitzHugh-Nagumo System

SUBROUTINE fhn(t,y,fb, xlag, net, x_f, y_f, yf_lag, n_ran, * n_con, glob)
PARAMETER (NTK=10000)
IMPLICIT REAL*8(a-h,o-z)
COMMON/PAR/pa(10), tt
COMMON/NOISE/ix1
DIMENSION y(10,NTK), f(10,NTK), fb(10), xlag(10,NTK), x_f(NTK)
DIMENSION y_f(NTK), n_ran(NTK,NTK), yf_lag(NTK)

do kk=1, net
x_f(kk)=0. ! initialise individual fields
y_f(kk)=0.
yf_lag(kk)=0.
enddo

do kk=1, net
    do nr=1, n_con
        x_f(kk)=x_f(kk)+y(1,n_ran(kk,nr))
y_f(kk)=y_f(kk)+y(2,n_ran(kk,nr))
yf_lag(kk)=yf_lag(kk)+xlag(1,n_ran(kk,nr))
    enddo
    x_f(kk)=x_f(kk)/n_con
    y_f(kk)=y_f(kk)/n_con
    yf_lag(kk)=yf_lag(kk)/n_con
enddo

yf_glob=0.
yflag_glob=0.

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do kk=1,net
    yf_glob=yf_glob+y_f(kk)
    yflag_glob=yflag_glob+yf_lag(kk)
enddo
yf_glob=yf_glob/net
yflag_glob=yflag_glob/net
glob=yflag_glob-yf_glob

do kk=1,net
    f(1,kk)=(y(1,kk)-y(1,kk)*y(1,kk)*y(1,kk)/3.d0-y(2,kk)+
              pa(4)*(x_f(kk)-y(1,kk)))/pa(1)
    f(2,kk)=y(1,kk)+pa(2) + pa(3)*glob
enddo
fb(1)=0.
fb(2)=sqrt(2.*pa(5))

RETURN
END

===FitzHugh-Nagumo System===

===Gaussian Noise===

SUBROUTINE fno(x,y,f,k,n,xlag,net)
PARAMETER (NTK=10000)
IMPLICIT REAL*8(a-h,o-z)
REAL gasdev
COMMON/PAR/pa(10)
COMMON/NOISE/ix1
DIMENSION f(10,NTK),y(10,NTK),xlag(10,NTK)

do kk=1,net
    f(2,kk)=gasdev(ix1)
do i=1,3
    xx=gasdev(ix1)
enddo
RETURN
END

===Gaussian Noise===

===Stochastic Integration===

SUBROUTINE rkin(ndel,n,y,h,h2,t,k,rk,f,y1,fb,yn,flag,
,nnf1,k1,lf1,xlag,net,x_f,f,yf_lag,n_ran,n_con,glob)
IMPLICIT double precision (a-h,o-z)

PARAMETER (NM=10,NLAG=500000,KLAG=10,NTK=10000)

DIMENSION y(NM,NTK),rk(4,NM,NTK),f(NM,NTK),yi(NM,NTK),fb(NM),
, flag(KLAG,NLAG,NTK),nfl(KLAG),lfl(KLAG,NTK),xlag(KLAG,NTK),
, yn(NM,NTK),kl(KLAG),x_f(NTK),y_f(NTK),yf_lag(NTK),
, n_ran(NTK,NTK)

COMMON/NOISE/ix1

CALL fhn(t,y,f,fb,k,n,xlag,net,x_f,y_f,yf_lag,n_ran,
*   n_con,glob)
CALL fno(t,y,yn,k,n,xlag,net)
DO 1 i=1,n
  do kk=1,net
    rk(1,i,kk)=h*f(i,kk)+yn(i,kk)*fb(i)*h2
    y1(i,kk)=y(i,kk)+rk(1,i,kk)/2
  enddo
1 continue

DO j=1,ndel
  do kk=1,net
    flag(j,lfl(j,kk),kk)=y(kl(j),kk)
    lfl(j,kk)=lfl(j,kk)+1
    IF(lfl(j,kk).GT.nfl(j)) lfl(j,kk)=1
    xlag(j,kk)=flag(j,lfl(j,kk),kk)
  enddo
END DO

CALL fhn(t+h/2,y1,y,fb,k,n,xlag,net,x_f,y_f,yf_lag,n_ran,
*   n_con,glob)
DO 2 i=1,n
  do kk=1,net
    rk(2,i,kk)=h*f(i,kk)+yn(i,kk)*fb(i)*h2
    y1(i,kk)=y(i,kk)+rk(2,i,kk)/2
  enddo
2 continue

CALL fhn(t+h/2,y1,y,fb,k,n,xlag,net,x_f,y_f,yf_lag,n_ran,
*   n_con,glob)
DO 3 i=1,n
  do kk=1,net
    rk(3,i,kk)=h*f(i,kk)+yn(i,kk)*fb(i)*h2
    y1(i,kk)=y(i,kk)+rk(3,i,kk)
  enddo
3 continue
3 continue

DO j=1,ndel
  do kk=1,net
    flag(j,lf1(j,kk),kk)=y(kl(j),kk)
    lfl(j,kk)=lfl(j,kk)+1
    IF(lfl(j,kk).GT.nfl(j)) lfl(j,kk)=1
    xlag(j,kk)=flag(j,lfl(j,kk),kk)
  enddo
END DO

CALL fhn(t+h,y1,fb,k,n,xlag,net,s_f,y_f,yf_lag,n_ran,
  * n_con,glob)
DO 4 i=1,n
  do kk=1,net
    rk(4,i,kk)=h*f(i,kk)+yn(i,kk)*fb(i)*h2
    y(i,kk)=y(i,kk)+(rk(1,i,kk)+2*(rk(2,i,kk)+rk(3,i,kk))+
    * rk(4,i,kk))/6
  enddo
4 continue
RETURN
END

c=====Stochastic Integration======<

c=====Initialise Delayed Variables===>
SUBROUTINE inilag(ndel,tau,h,flag,nfl,lfl,xlag,net)
IMPLICIT REAL*8(A-H,O-Z)
PARAMETER (NM=10, NLAG=500000,KLAG=10,NTK=10000)
DIMENSION flag(KLAG,NLAG,NTK),tau(KLAG),nfl(KLAG),
  lfl(KLAG,NTK),xlag(KLAG,NTK)

do ii=1,ndel
  nfl(ii)=tau(ii)*2/h
  do i=1,nfl(ii)
    do kk=1,net
      flag(ii,i,kk)=0.
    enddo
  enddo

do kk=1,net
  lfl(ii,kk)=1
  xlag(ii,kk)=flag(ii,1,kk)
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SUBROUTINE lin_int(t,h,x_prev,x,tp)
IMPLICIT REAL*8(a-h,o-z)
a=(x-x_prev)/h
b=x-a*t
tp=-b/a
RETURN
END
SUBROUTINE poi_int(t,h,tp,x_prev,x,xp)
IMPLICIT REAL*8(a-h,o-z)
a=(x-x_prev)/h
b=x-a*t
xp=a*tp+b
RETURN
END
FUNCTION gasdev(idum)
INTEGER idum
REAL gasdev
INTEGER iset
REAL fac,gset,rsq,v1,v2,ran3
SAVE iset,gset
DATA iset/0/
if (iset.eq.0) then
1  v1=2.*ran3(idum)-1.
v2=2.*ran3(idum)-1.
rsq=v1**2+v2**2
if(rsq.ge.1..or.rsq.eq.0.)goto 1
fac=sqrt(-2.*log(rsq)/rsq)
gset=v1*fac
gasdev=v2*fac
iset=1
else
  gasdev=gset
  iset=0
endif
RETURN
END

FUNCTION ran3(idum)
INTEGER idum
INTEGER MBIG,MSEED,MZ
REAL ran3,FAC
PARAMETER (MBIG=1000000000,MSEED=161803398,MZ=0,FAC=1./MBIG)
INTEGER i,iff,ii,inext,inextp,k
INTEGER mj,mk,ma(55)
SAVE iff,inext,inextp,ma
DATA iff /0/
if(idum.lt.0.or.iff.eq.0)then
  iff=1
  mj=MSEED-iaabs(idum)
  mj=mod(mj,MBIG)
  ma(55)=mj
  mk=1
  do 11 i=1,54
    ii=mod(21*i,55)
    ma(ii)=mk
    mk=mj-mk
    if(mk.lt.MZ)mk=mk+MBIG
    mj=ma(ii)
  11 continue
  do 13 k=1,4
    do 12 i=1,55
      ma(i)=ma(i)-ma(1+mod(i+30,55))
      if(ma(i).lt.MZ)ma(i)=ma(i)+MBIG
    12 continue
  13 continue
  inext=0
  inextp=31
  idum=1
endif
inext=inext+1
if(inext.eq.56)inext=1
inextp=inextp+1
if(inextp.eq.56)inextp=1
mj=ma(inext)-ma(inextp)
if(mj.lt.MZ)mj=mj+MBIG
ma(inext)=mj
ran3=mj*FAC
RETURN
END

c=====Random Number Generator=====

Bibliography


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