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Effective Demand and Say's Law in Marxist Theory: An Evolutionary Perspective

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Abstract

In this paper I theorize the roles of effective demand and Say's law in the Marxist theory of exploitation and accumulation. I claim that an exogenous rate of exploitation, or an exogenous functional distribution of income between profits and wages, implies deploying Say's law, which leads profit rates not to equalize across sectors. Marx's own procedure in *Capital III* of simultaneously supposing an exogenous rate of exploitation and profit rate equalization was therefore logically inconsistent. Once Keynes' principle of effective demand is introduced, the rate of exploitation and hence the distribution of income between wages and profits become endogenous to aggregate demand. Profit rates then do equalize across sectors and prices of production can function as gravitational centers for market prices in a competitive economy. If we aim at developing a theory that is both empirically relevant and logically consistent, Marxist scholars must therefore drop Say's law and incorporate Keynes' principle of effective demand for a proper understanding of how capital accumulation determines the rate of exploitation, the functional distribution of income, and the equalization of profit rates.

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1. Introduction

The purpose of this paper is to theorize the roles of effective demand and Say's law in the Marxist theory of exploitation and accumulation. I argue that an exogenous rate of exploitation, or an exogenous functional distribution of income between profits and wages, implies deploying Say's law, which leads profit rates not to equalize across sectors. Marx's own procedure in *Capital III* of simultaneously supposing an exogenous rate of exploitation and profit rate equalization was therefore logically inconsistent. Once Keynes' principle of effective demand is introduced, the rate of exploitation and hence the distribution of income between wages and profits become endogenous to aggregate demand. Profit rates then do equalize across sectors and prices of production can function as gravitational centers for market prices in a competitive economy.

Marxist scholars must therefore drop Say's law once and for all and incorporate Keynes' principle of effective demand for a proper understanding of how capital accumulation determines the rate of exploitation, the functional distribution of income, and the equalization of profit rates.

Marx himself was ambiguous in his original writings regarding the role of effective demand. In the three volumes of *Capital*, in the *Grundrisse*, and in the *Theories of Surplus Value*, Marx repeatedly referred to realization problems and to the fact that produced values have no guarantee to be fully validated in the market. Marx even located the potential lack of demand to validate produced values in the function of money as a means of hoarding, of money as an end in itself. The Marxist literature then developed its own branch of inquiry into realization problems and realization crises. Marx, on the contrary, also often assumed Say's law. In the third volume of *Capital* Marx introduced his reproduction scheme and offered novel ideas on profit rate equalization and falling profitability amid technological progress. But in his reproduction model, Say's law is deployed and realization problems play no part in the theory of accumulation and exploitation.

Because of Marx's death in 1883, his drafts of the second and third volumes of *Capital* were left unfinished. Engels later edited and published the manuscripts in the 1890s but the connections between effective demand, exploitation, and accumulation were left incomplete. Since the advent of Keynesian and Kaleckian macroeconomics in the the 1930s, Marxists have attempted to offer new insights into how the theory of effective demand would relate to Marx's theory of capital accumulation. The influential works of Sweezy ([1942]1970) and Baran and Sweezy (1968), for example, have had a major impact in the Political Economy literature for they represented serious attempts to integrate the principle of effective demand into the Marxist framework. Dutt (2011), however, recently argued that the role of effective demand in Marxist theory is yet underdeveloped. Dutt claimed that Marxists still struggle to provide a consistent theory of how effective demand would affect capital accumulation both in the short and long runs.

In this paper I offer new insights into the Marxist theory of accumulation and exploitation regarding the roles of Say's law and Keynes' principle of effective demand. Marx's procedure in the third volume of *Capital* of assuming simultaneously an exogenous rate of exploitation and profit rate equalization was mistaken. With an exogenous rate of exploitation, whatever its level is, we might obtain *equal* profit rates across sectors but, if so, only by fluke and only temporarily. Equal profit rates, however, do not imply profit rate *equalization*. Profit rate equalization requires profit rates to be equal across sectors and also requires this equality to be self-correcting (i.e. to have *asymptotic stability*). Even if profit rates are now and then equal, if the equality is not self-correcting we cannot accurately call it profit rate equalization.

Marx's assumption of an exogenous rate of exploitation implicitly depends on Say's law because the rate of exploitation can be exogenous only if all values produced are always realized. But if Say's law holds, then there is no self-correction mechanism that would equalize profit rates across sectors. Say's law, or an exogenous rate of exploitation, imply that all values (hence all surplus value and profits) produced must be realized, and thus it pre-determines profitability in such a way that profit rates are unresponsive to the amount invested in each sector. Under an exogenous rate of exploitation or under Say's law the prices of production, the prices that correspond to equalized profit rates, no longer function as gravity centers for market prices. Because it predetermines the amount of profits to be realized, an exogenous rate of exploitation further implies that the profit-wage ratio and hence the functional distribution of income between wages and profits are also exogenous and thus unresponsive to the accumulation of capital.

If, however, Say's law does not hold then the rate of exploitation becomes endogenous and dependent on effective demand, and profit rates do equalize across sectors. When effective demand is taken into account the rate of exploitation will automatically become dependent upon it, profit rates will equalize across sectors, and this equalization will be self-correcting (i.e. will be stable asymptotically). Effective demand allows for profit rate equalization in a competitive economy because sector profitability then responds negatively to the amount of investment in each sector. Over-investment in one sector will erode profits in that sector and capital will then gradually move to other sectors. Because the rate of exploitation becomes endogenous to effective demand, the profit-wage ratio and hence the distribution of income between wages and profits also become endogenous to the level of aggregate expenditures.

Profit rates equalize, and hence prices of production can operate as gravitational centers for market prices, only when the rate of exploitation and the functional distribution of income are dependent upon the level of effective demand. When the rate of exploitation is exogenous, or when the distribution of income between profits and wages is taken as given, Say's law prevents profit rate equalization and hence prevents prices of production from operating as gravitational centers for market prices in a competitive environment.

In order to develop my arguments I introduce an evolutionary model of exploitation, accumulation, and technological progress that closely mimics Marx's original insights. I use this model to demonstrate the implications of Keynes' effective demand and Say's law on the accumulation of capital, exploitation, the distribution of income between wages and profits, and the equalization of profit rates.

The Marxist theory of capital accumulation is certainly one of an evolutionary and adaptive economy. Marx in his own time, unfortunately, had no access to the mathematics and computer simulations of evolutionary systems that we have today. These new tools allow us to develop Marx's theory in ways that were not available to him in the nineteenth century.

My Marxist model of capital accumulation draws on previous contributions. From Foley (2003) and Bowles (2006) I take the notion that in classical Political Economy economic growth takes place in an evolutionary system that is most often not in equilibrium. From Duménil and Lévy (2011; 1995) I take the idea of technical change as a stochastic process. Capitalists adopt randomly created techniques if and only if they increase individual profitability. From Prado (2006) I take the idea of formulating Marx's contribution in *Capital III* along the lines of an evolutionary system. Prado's (2006) contribution is interesting for proposing a simple way of modeling many of Marx's insights by employing replicator equations at the macro level. In Prado (2002) he also used a system of replication to approach Duménil and Lévy's classical evolutionary model of technical change at the micro level. Duménil and Lévy (2011; 1995) and Prado (2006; 2002), however, have not presented an integrated model that incorporates all these insights dynamically and at both the micro and macro levels simultaneously. From Dutt (2011) I then take the challenge of presenting a Marxist growth model in which the roles of effective demand and Say's law are explicitly contrasted.

In the sections that follow I firstly formalize the macro inter-sector competition through which the aggregate and growing monetary capital of an economy is continuously redistributed between two sectors: *sector I* producing means of production and *sector II* producing final consumption goods, such that commodities are produced by means of commodities. The continuous redirecting of investment between sectors takes place according to average profit rate differentials. Secondly, I formalize the micro intra-sector competition in which individual firms within each sector compete against each other via cost-reducing technical change. Innovations are gradually adopted based on profit rate differentials within sectors. Competitive selection occurs simultaneously at the micro intra-sector and at the macro inter-sector levels.

To describe both inter- and intra-sector forms of competition I employ replicator dynamics from evolutionary game theory. The replicator equation describes an updating process with random interactions in which behaviors with higher payoffs proliferate. It is a useful device to mimic the competitive struggle for survival in natural and social environments, for it models the process of equilibration by tracking the results of individual interactions. Chronological, not merely logical time is explicitly incorporated. The proposed evolutionary model formalizes key aspects of Marx's theory of accumulation and profitability in an adaptive system in which agents control their actions but not the aggregate consequences of their individual decisions. Micro decisions produce macro outcomes that then feed back again into micro decisions.

I provide two different closures for the model. In the first closure I use an exogenous rate of exploitation, or an exogenous functional distribution of income, that amounts to deploying Say's law. In the second closure I introduce effective demand and make the rate of exploitation and the distribution of income dependent upon it. For both closures I present computer simulations and an analysis of the evolutionary stability of the different long-run equilibria. The crucial result is that the Marxist scholarship must abandon Say's law and incorporate Keynes' principle of effective demand if it aims at a more relevant and logically consistent theory of exploitation and capital accumulation.

2. The Macro Inter-Sector Competition

The economy-wide circuit of monetary capital, which starts and ends with capital in the form of money, can be represented through the following aggregation:

$$M_t - C_t \begin{cases} LP \\ MP \end{cases} \dots P \dots\ C_t' -\ M_t' \qquad (1)$$

In the investment phase an initial amount of money M_t purchases two types of commodities as inputs, C_t: labor power (LP) and means of production (MP). During the subsequent production phase (... P ...) labor power creates more value than its own. The difference between the value that labor power creates and the value of labor power itself is the surplus value. The total value of the gross output C_t' contains the new value added that productive workers create plus the pre-existing value transferred from the usage of the means of production. Embodying both the value of the productive inputs and the surplus value, the gross output exchanges for a sum of money represented by the aggregate gross expenditures or gross aggregate demand M_t'. Time subscripts indicate that investment, production, and sales are temporal events. The extra value that workers create and for which they receive no equivalent is the basis for the gross profits $\Delta M_t = M_t' - M_t$ in the system. Since production does not automatically generate profits, the surplus value created in the productive sphere still needs to be realized in the circulation sphere subject to all sorts of price movements.

The economy comprises two sectors, each producing a single type of output. Sector I employs labor power and means of production to deliver new supplies of a homogenous type of means of production. Sector II employs labor power and means of production to deliver new supplies of a homogenous type of final consumption good. This division of the economy into a final goods sector and a means of production sector, and hence that commodities are produced by means of commodities, is a core feature of Marxist, Kaleckian, and Sraffian models. Because economic events take place temporally, the overlap of any two consecutive circuits of the total monetary capital can be represented as follows:

$$M_t - C_t \begin{cases} LP \\ MP \end{cases} \ldots P \ldots C_t' - M_t'$$

$$M_{t+1} - C_{t+1} \begin{cases} LP \\ MP \end{cases} \ldots P \ldots C'_{t+1} - M_{t+1}' \tag{2}$$

The circuit at period t+1 formally repeats the circuit at period t. The crucial relation is then that between the total value realized in period t, M_t', and the total investment in the next period t+1, M_{t+1}. The economy reproduces itself very differently depending on the relation between M_t' and M_{t+1}, for it is in this relation where effective demand and Say's law have contrasting effects. Say's law means that all values produced in …P… are realized in M_t' such that no realization problems occur, and potentially that the total value realized in M_t' can be reinvested in M_{t+1}. The Keynesian principle of effective demand, on the contrary, means that $M_t' = M_{t+1}$ and that the direction of causality runs from expenditures in M_{t+1} to incomes in M_t' : total demand for consumption at the end of period t and investment at the beginning of period t+1 jointly determine the total value realized at the end of period t. Say's law and Keynes' effective demand therefore imply very different behaviors for the reproduction of the total monetary capital over time.

There is no fixed capital in this economy, implying that all non-labor inputs are circulating capital and that the means of production that enter as inputs in sectors I and II in period t are the previous output of sector I in period t-1. Technology is represented by a linear production structure with fixed coefficients and constant returns to scale. Using a_{ji} to indicate the quantity of input j per unit of output i, the matrix of input-output coefficients is:

$$A = [a_{ji}] = \begin{bmatrix} a_{11} & a_{12} \\ 0 & 0 \end{bmatrix} \quad \text{with} \quad 0 \leq a_{ji} < 1 \tag{3}$$

Using l_i to indicate the quantity of labor hours per unit of output in sector i, $r_{i,t}$ to indicate the within-sector profit rate per unit of output, p_i to indicate the market price per unit of output, and w to indicate the given money wage per work hour (determined exogenously via bargaining between workers and capitalists), then per unit of output we have that $[p_{j,t-1}a_{ji} + wl_i](1 + r_{i,t}) = p_{t,i}$. For each sector the price system is:

$$\begin{aligned} [p_{1,t-1}a_{11} + wl_1](1 + r_{1,t}) &= p_{1,t} \\ [p_{1,t-1}a_{12} + wl_2](1 + r_{2,t}) &= p_{2,t} \end{aligned} \tag{4}$$

The first term inside brackets on the left hand side represents constant capital and the second term represents variable capital (or the value of labor power), both in money terms and both comprising the total circulating capital given the assumption of no fixed inputs in this model. Their summation $[p_{j,t-1}a_{ji} + wl_i]$ is the unit cost. The material content of the means of production that constitute constant capital is produced and priced at the market prices prevailing at the immediate previous period. Competition within each sector then simultaneously determines profit rates and prices. The profit per unit of output is the unit cost times the profit rate: $[p_{1,t-1}a_{1i} + wl_i]r_{i,t} = \pi_{i,t}$.

Even though the nominal wage per work hour w is exogenously given by the bargaining power between workers and capitalists, the real wage $\frac{w}{p_{2,t}}$ in terms of quantities of the consumption good produced in sector II is determined endogenously in this model. Labor supply is assumed not to be a binding constraint on growth.

At every time period t the total monetary capital M_t in the economy must be invested between the two sectors. In each sector there is a collection of several firms and each of them can switch between sectors depending on the expected average profitability $\overline{r_{i,t}}$. The term 'expected' has to be emphasized since once firms flow into a sector aiming at the prevailing $\overline{r_{i,t}}$ they will immediately and unintentionally alter this average profitability. Capital and labor are perfectly mobile. The two shares of total monetary capital are $f_{1,t}$ representing the fraction at time t invested in the sector producing means of production, and $f_{2,t}$ representing the fraction at time t invested in the sector producing final goods. Hence:

$$M_t = f_{1,t}M_t + f_{2,t}M_t = M_{1,t} + M_{2,t} \quad \text{with} \quad f_{1,t} + f_{2,t} = 1 \qquad (5)$$

Each share $f_{i,t}$ is a function of itself in the previous period and is also a function of the average sector profit rates in the previous period. There is therefore an adjustment process that regulates the amount of monetary capital invested across sectors: $f_{i,t+1} = F(f_{i,t}, \overline{r_{1,t}}, \overline{r_{2,t}})$. Within each sector, the invested amount $M_{i,t}$ then purchases labor power and means of production as in equations (1) and (2).

Outputs $x_{i,t}$, or total quantities supplied by each sector are simply sector investments divided by the respective unit costs:

$$x_{i,t} = \frac{f_{i,t} M_t}{[p_{1,t-1} a_{1i} + w l_i]} = \frac{M_{i,t}}{[p_{1,t-1} a_{1i} + w l_i]} \qquad (6)$$

Within each sector, $M'_{i,t}$ indicates the end-of-circuit expenditures or the valorized monetary capitals that comprise the original monetary capitals invested $M_{i,t}$ plus the surplus value that workers create. Market prices $p_{i,t}$ in each sector are simply expenditures divided by quantities supplied in that sector:

$$p_{i,t} = \frac{M'_{i,t}}{x_{i,t}} = \frac{M_{i,t}(1 + \overline{r_{i,t}})}{x_{i,t}} \qquad (7)$$

We must now determine how capitalists decide to invest in either sector. Supposing a very large collection of firms in the economy, we can normalize the total number of firms to unity and then care only about the evolution of population shares. The set of shares $(f_{1,t}, f_{2,t})$ of the total monetary capital invested in each sector constitutes the set of strategies of the macro inter-sector game. Since the set $(f_{1,t}, f_{2,t})$ depends explicitly on how the system evolves over time, we must derive the dynamic equation describing the evolution of this set of strategies.

The decision to invest either in sector I producing means of production or in sector II producing final consumption goods is taken based on a comparison of expected profitability. Capitalists will invest in the sector in which they expect to profit the most. After investing and thus formally entering into one of the two sectors they will then compete for a share of the mass of profits within the chosen sector.

The monetary capital $M_{i,t}$ invested in sector i at time t is valorized on average to $(1 + \overline{r_{i,t}})$ after the output is sold. The fraction $(1 + \overline{r_{i,t}})$ is the sector valorization factor and it includes the replication of the money invested plus average profits. Thus the valorized capital in each sector is $M'_{i,t} = M_{i,t}\ (1 +$

$\overline{r_{i,t}}) \; = \; f_{i,t}\, M_t\, (1 + \overline{r_{i,t}})$. Using $(1 + \; \tilde{r_t})$ to indicate the economy-wide weighted average valorization factor such that $(1 + \; \tilde{r_t}) = \sum_i f_{i,t} (1 + \overline{r_{i,t}})$ at any time t, the aggregate valorized capital for the entire economy then becomes:

$$M'_t = \; M_t\, (1 + \; \tilde{r_t}) = \sum_i M_{i,t}\, (1 + \overline{r_{i,t}}) \; = \; \sum_i f_{i,t}\, M_t\, (1 + \overline{r_{i,t}}) \qquad (8)$$

The economy-wide weighted average profit rate $\tilde{r_t}$ is a linear combination of the average profit rates $\overline{r_{i,t}}$ in each sector. Corresponding weights are given by the time-sensitive shares $f_{i,t}$ representing the division of the total monetary capital in the economy. The shares of the total monetary capital invested in the following period $t + 1$ will then change according to the average profitability obtained in the current period t in each sector:

$$f_{i,t+1} = \frac{M'_{i,t}}{M'_t} = \frac{f_{i,t}\, M_t\, (1 + \overline{r_{i,t}})}{M_t\, (1 + \; \tilde{r_t})} = \; f_{i,t}\, \frac{(1 + \overline{r_{i,t}})}{(1 + \; \tilde{r_t})} \qquad (9)$$

Rewriting it as $\frac{f_{i,t+1}}{f_{i,t}} = \frac{(1+\overline{r_{i,t}})}{(1+\tilde{r_t})}$, subtracting 1 from both sides and using $\Delta f_{i,t+1} \; = f_{i,t+1} - \; f_{i,t}$, we then obtain the replicator equation that formalizes the macro competition between capitalists across sectors:

$$\Delta f_{i,t+1} = f_{i,t} \left(\frac{1}{1 + \tilde{r_t}}\right) [\overline{r_{i,t}} - \; \tilde{r_t}] \qquad (10)$$

Using carets to denote growth rates and indicating the rate of growth of each sector investment share as $\widehat{f_{i,t}}$ we arrive at $\widehat{f_{i,t}} \; = \frac{\overline{r_{i,t}} - \tilde{r_t}}{1+\tilde{r_t}}$. If the average profit rate $\overline{r_{i,t}}$ in a sector is greater than the economy-wide average profit rate $\tilde{r_t}$ then more firms will flow into the more profitable sector, raising the share of this sector's investment in relation to the total monetary capital in the economy. If the sector average profit rate lies below the economy-wide average profitability then some capitalists will flow out of this relatively less profitable sector in order to invest their monetary capital in the other more profitable sector.

The profitability gap in relation to the economy-wide average thus determines the velocity of the changes in $f_{i,t}$. Since there are only two sectors, one replicator equation is enough to describe the adjustment at the macro level.

It is obviously not so easy and smooth for an actual capitalist to move her investments to a completely different type of economic activity. Most firms producing final goods cannot start producing means of production just due to profit rate differentials. To introduce an extra level of realism in the model, only a fraction $\mu \in (0,1]$ of the capitalists in each sector will in fact shift their monetary capital to a different activity that is currently benefitting from higher returns. The complementary fraction $(1-\mu)$ cannot update their investment behavior even when return differentials are an incentive for them to do so. Including this exogenous parameter μ that controls for how many firms can indeed update their behavior leads us to the final replicator equation for the macro flows of monetary capital:

$$\Delta f_{i,t+1} = \mu\, f_{i,t} \left(\frac{1}{1+\widetilde{r_t}}\right)\left[\overline{r_{i,t}} - \widetilde{r_t}\right] \tag{11}$$

In this evolutionary model, the search for *individual* profits at the micro level creates unintentional consequences both in the average profit rates $\overline{r_{i,t}}$ at the *sector* level as well as at the *aggregate* level via the economy-wide average profit rate $\widetilde{r_t}$. Capitalists invest predicated on average profit rates prevailing in each sector, not based on the economy-wide profit rate, but they end up affecting aggregate profitability unintentionally through their decentralized individual actions. The effects on the aggregate profit rate then feed back into individual decisions about where to invest the monetary capital in the following period.

In a decentralized market, individual actions produce unintended aggregate outcomes that then have feedback effects upon individual actions, and so forth. Capitalists control their actions. They do not, however, control the consequences of their actions. The accumulation of capital and its continuous flow to different sectors of the economy are not reducible to the individual deliberate decisions of capitalists, even though it surely depends upon them. The macro outcome is the resulting vector of all individual decisions plus the complex interplay that none of the agents can anticipate.

The equations so far presented describe the growth of output and the evolutionary adjustments that regulate the shares of investments over time at the macro level. In the next section I turn to the competition for profits through cost-reducing technical change that characterize the micro-adjustments within each sector.

3. The Micro Intra-Sector Competition

Large collections of firms compete for profits within each sector. Markets are intensely competitive, forcing firms to sell at prevailing market prices. The way to increase individual profits lies therefore with the adoption of new cost-reducing technologies. New innovations are randomly created and then adopted conditional on enhancing individual profitability. When an individual firm decides to adopt a new technology, its individual unit cost is reduced through new techniques that save on labor or on means of production or even on both at the same time. Competition then drives other firms in the same sector to gradually adopt the new technology. Each sector will thus display a production structure that is a combination of firms producing with the new techniques and firms still producing with the old technique.

When an individual firm decides upon the adoption of a new productive structure it does so taking the prevailing market price as given. But the individual adoption of the newer technique changes the sector cost structure and therefore unintentionally affects the market price. The new market price then operates as a signal for the remaining firms to also adopt the cost-reducing technique. It is possible, though not guaranteed, that profit-seeking firms will end up unintentionally reducing their profitability over time.

The economy has three evolutionary processes taking place concurrently. The first is the evolutionary diffusion of new techniques in the sector producing means of production. The second is the evolutionary diffusion of new techniques in the sector producing final consumption goods. The third is the evolutionary distribution of the growing aggregate monetary capital between sectors. A single decision to simply adopt a new technique thus triggers a complex chain of reactions and feedback effects that no individual capitalist can anticipate. Furthermore, any intra-sector change in average profitability will then trigger further changes in the inter-sector flows of investment. Externalities do exist in this economy given that firms do not fully internalize the social consequences of their individual actions.

Even though the innovation process is exogenous, the adoption of innovations is endogenous. Adoption occurs if the individual profit rate associated with the new technology is greater than the current profit rate associated with the existing technology. The evolution of technological diffusion can then be modeled employing replicator dynamics for each sector. Innovation means simply new process technologies, not product technologies. Innovations are hence restricted to those that affect production costs, not contemplating those that create new types of products.

The prevailing technique of production is represented in the set of four technical parameters (a_{11}^{o}, a_{12}^{o}, l_{1}^{o}, l_{2}^{o}). A random innovation (a_{11}^{n}, a_{12}^{n}, l_{1}^{n}, l_{2}^{n}) can imply the use of more of both inputs, less of both inputs, or even more of one and less of the other input. Superscript o indicates a technical parameter associated with the older technology, while superscript n indicates a technical parameter associated with the newer technology. The productivity of an input rises if less of that input is required to produce the same amount of output. When the productivity of both factors jointly go up the new technique is surely adopted, and when both productivities jointly go down its adoption is rejected. If a technology requires more of one factor and less of another, adoption then depends on the expected profitability.

The relationship between the old and new technical coefficients is summarized in the set (η_{a11}, η_{a12}, η_{l1}, η_{l2}) determining the growth rates of factor productivity, respectively:

$$\eta_{a11} = \frac{a_{11}^o - a_{11}^n}{a_{11}^n} \quad \text{and} \quad \eta_{a12} = \frac{a_{12}^o - a_{12}^n}{a_{12}^n}$$
$$\eta_{l1} = \frac{l_1^o - l_1^n}{l_1^n} \quad \text{and} \quad \eta_{l2} = \frac{l_2^o - l_2^n}{l_2^n} \tag{12}$$

When positive, these growth rates indicate that factor productivities are rising and hence production is saving on those inputs. For example, $\eta_{a12} = 10\%$ implies that the production of each unit of the final consumption good in sector II requires 10% less of the means of production produced by sector I. Similarly, $\eta_{l1} = 5\%$ implies that the production of each unit of the means of production in sector I requires 5% less of its workers' labor time. When the growth rate is negative, factor productivity declines and production becomes more intensive in that input. A negative productivity growth such as $\eta_{a11} = -15\%$ implies, for example, that the production of means of production in sector I requires 15% more means of production per unit of output.

Given the total amount of monetary capital to be invested and its distribution between sectors at the beginning of each period (M_t and $f_{i,t}$ respectively), and given a certain productive structure ($a_{11}^o, a_{12}^o, l_1^o, l_2^o$), we can calculate sector supplies $x_{i,t}$ via equation (6) and then market prices $p_{i,t}$ via equation (7) once we have determined sector expenditures $M'_{i,t}$. Because firms in sector II produce and sell only one type of final consumption good, and because firms in sector I produce and sell only one type of means of production, there is only one market price prevailing within each sector at any time t. With the technical coefficient set (a_{11}^o, a_{12}^o, l_1^o, l_2^o) given and with an exogenous money wage w, the determination of market prices allows for the subsequent determination of profit rates per unit of output in each sector using the fact that $[p_{1,t-1}a_{1i}^o + wl_i^o](1 + r_{i,t}) = p_{i,t}$. Rearranging it yields the equation for the profit rate per unit produced using current technology, $r_{i,t}^o$:

$$r_{i,t}^o = \frac{p_{i,t}}{[p_{1,t-1}a_{1i}^o + wl_i^o]} - 1 \tag{13}$$

After the random creation of a new technique (a_{11}^n, a_{12}^n, l_1^n, l_2^n), capitalists will adopt it if doing so increases their individual profits at the current money wage and prevailing market prices. The profit rate associated with the new technology is $r_{i,t}^n$:

$$r_{i,t}^n = \frac{p_{i,t}}{\left[p_{1,t-1}a_{1i}^n + wl_i^n\right]} - 1 \tag{14}$$

Firms calculate the expected profit rate $r_{i,t}^n$ associated with the new technology based on a given money wage and current market prices because the individual firm cannot anticipate how the uncoordinated adoption of the new technique will affect market prices. As long as $r_{i,t}^n > r_{i,t}^o$ firms will keep adopting the new technology.

The evolutionary diffusion of a new technique (a_{11}^n, a_{12}^n, l_1^n, l_2^n) can then be formalized with the dynamics of replication. The variable $\upsilon_{i,t} \in [0,1]$ indicates the share of firms in sector i that adopt the new technique at time t, while $(1 - \upsilon_{i,t})$ indicates the share that does not update and remains with the older technique. Because each sector has a large collection of firms and supposing that they interact through random pairwise matching, we can use a simple replicator equation for the diffusion of innovations. Normalizing population sizes to unity allows us to work with population shares in each sector as follows:

$$\begin{gathered} \upsilon_{i,t+1} = \upsilon_{i,t} + \upsilon_{i,t}\left(1 - \upsilon_{i,t}\right)[r_{i,t}^n - r_{i,t}^o] \\ \Delta\upsilon_{i,t+1} = \upsilon_{i,t}\left(1 - \upsilon_{i,t}\right)[r_{i,t}^n - r_{i,t}^o] \\ \Delta\upsilon_{i,t+1} = \upsilon_{i,t}[r_{i,t}^n - \overline{r_{i,t}}] \end{gathered} \tag{15}$$

The term $\upsilon_{i,t}\left(1 - \upsilon_{i,t}\right)$ is the variance of the firms within each sector and the term $\left[r_{i,t}^n - r_{i,t}^o\right]$ is the differential replication selection, so that the updating process is payoff monotonic. The third line in equation (15) follows from the fact that the average profit rate in each sector is: $\overline{r_{i,t}} = \left(\upsilon_{i,t}\right)\left[r_{i,t}^n\right] + \left(1 - \upsilon_{i,t}\right)\left[r_{i,t}^o\right]$. In terms of growth rates the updating process is simply $\hat{\upsilon}_{i,t+1} = r_{i,t}^n - \overline{r_{i,t}}$.

During the gradual adoption of an innovation, each sector will consist of a combination of firms producing with the older and newer technologies side by side. Technical change and its evolutionary diffusion imply that older and newer cost structures coexist until the newer technique completely replaces the older one. Given the investment of monetary capital $M_{i,t}$ in each sector, the new quantities supplied can be found by dividing the capital invested by the cost structure:

$$x_{i,t} = \frac{M_{i,t}}{(\upsilon_{i,t})[p_{1,t-1}a_{1i}^{n} + wl_{i}^{n}] + (1 - \upsilon_{i,t})[p_{1,t-1}a_{1i}^{o} + wl_{i}^{o}]} \qquad (16)$$

The denominator in equation (16) implies that the cost structure of each sector is a linear combination of the firms operating with the new technique and the firms operating with the old technique. The endogenous term $\upsilon_{i,t}$ in this linear combination of cost structures comes from the replicator dynamic in equation (15). As firms gradually update to the new technique, the share $\upsilon_{i,t}$ rises and displaces the older productive structure $[p_{1,t-1}a_{1i}^{o} + wl_{i}^{o}]$ in favor of the newer cost structure $[p_{1,t-1}a_{1i}^{n} + wl_{i}^{n}]$. The supply equation in (16) thus replaces the previous supply equation in (6) which only works with a single technology.

Given the newer cost structure and sector supplies, market prices must be recalculated jointly with the total end-of-period expenditures. When the diffusion of the new technology begins to affect market prices, it will immediately affect profit rates associated with the old ($r_{i,t}^{o}$) and new ($r_{i,t}^{n}$) techniques. As long as $r_{i,t}^{n} > r_{i,t}^{o}$ holds true, firms will keep updating and driving $\upsilon_{i,t}$ up through the replication process. Changes in $\upsilon_{i,t}$ then have further feedback effects on the quantities produced through the supply equations in each sector. New supplies then continue to affect market prices. New market prices for both means of production and consumption goods, in turn, will affect firms' profitability once again, which will then trigger further adoption of the newer technique. Once the intra-sector replicator equation produces a new value for $\upsilon_{i,t}$ the system keeps looping until a stationary state is reached for the share of firms using the new technology in each sector. In the limiting case of $\upsilon_{i,t} = 1$, all firms adopt the new technology and the old technique vanishes completely.

The transition from $\upsilon_{i,t} = 0$ towards $\upsilon_{i,t} = 1$ formalizes the mechanism of innovation diffusion through the replication of successful strategies. Intra-sector competition for greater profits through cost reduction drives the gradual adoption of new technologies that save on input usage. Each single firm behaves without coordinating with any other firm. Atomized agents only respond to profitability signals and do not, and actually cannot care about aggregate outcomes when deciding upon the adoption of a new technique of production. The dynamic interplay between individual actions and aggregate outcomes produces a chain of events that escapes from the control of any single entity.

Average rates of profit in each sector depend on the prevailing market prices and on the linear combination between older and newer production techniques:

$$\overline{r_{i,t}} = \frac{p_{i,t}}{(\upsilon_{i,t})\left[p_{1,t-1}a_{1i}^{n} + wl_{i}^{n}\right] + \left(1 - \upsilon_{i,t}\right)\left[p_{1,t-1}a_{1i}^{o} + wl_{i}^{o}\right]} - 1 = \frac{\Delta M_{i,t}}{M_{i,t}} \qquad (17)$$

As soon as average profit rates in each sector ($\overline{r_{1,t}}$ and $\overline{r_{2,t}}$) change from their previous position they then affect intra-sector competition via the micro replicator dynamic in equation (15) as well as it affects inter-sector competition via the macro replicator dynamic in equation (11). Because of profitability differentials between sectors, in each period a fraction of the capitalists will flow into the sector with the higher average profit rate $\overline{r_{i,t}}$. When capitals flow into a different sector they bring in more investments while withdrawing their monetary capitals from the sector with lower profitability. More investment in one sector via changes in $f_{i,t}$ means additional supply in the sector with capital inflows and lower supply in the sector with capital outflows. Inter-sector adjustments in equation (11) then affect intra-sector dynamics via equation (15), which then feeds back again into equation (11) and so forth.

The evolutionary, nonlinear, and complex features of this model appear in the permanent interaction among the macro and micro replication dynamics in equations (11) and (15). The out-of-equilibrium adjustments and the evolution of the system over time explicitly reflect the interplay of unintended social consequences of uncoordinated individual actions. This interplay generates multiple long-run equilibria, each with varying properties in terms of asymptotic stability and instability. In the next section I analyze the stationary states and the evolutionary strategies that might prevail in the long run.

4. Long-Run Equilibria and Evolutionary Stability

In an evolutionary setting, differential replication offers a behavioral foundation for spontaneous and path-dependent interactions of multiple uncoordinated agents. The replicator dynamic also provides a behavioral foundation for the Nash equilibrium by way of the concept of *evolutionary stability* or asymptotic self-correction. Individuals are adaptive agents that are not always forward-looking inter-temporal maximizers, but are most often bearers of behavioral rules. For this reason, an evolutionary model of social dynamics emphasizes not individuals but behavioral rules. What individuals do with their local knowledge is then crucial in determining which behavioral rules succeed or perish over time. Hence equilibrium is no longer static but instead becomes a temporal concept. Some equilibria, for example, might be evolutionarily irrelevant because they do not have evolutionary stability (Bowles, 2006; Gintis, 2009).

This evolutionary model has multiple dimensions and is not linear. Its nonlinearity in multiple dimensions makes its behavior substantially more complex, at the cost of preventing us from having an analytical solution to the evolution of the system. But the model becomes more intuitive if we focus on the trajectories of the three replicator equations $(f_{1,t}, v_{1,t}, v_{2,t})$ towards their long-run stationary states. Stationary states are those states at which the replicator reaches a fixed point with no further changes in the replication process($\Delta f_{1,t} = 0, \Delta v_{1,t} = 0, \Delta v_{2,t} = 0$). The crucial procedure is to know which strategies are going to prevail asymptotically (when $t \rightarrow \infty$).

In an evolutionary game with replicator dynamics we know that the *evolutionary stable strategies* prevail over the long run. An *evolutionary stable strategy* (ESS) is a best response to itself and hence it is a symmetric Nash equilibrium that is also asymptotically stable in its respective replicator equation. Shocks are self-correcting around a steady state that has evolutionary stability. *Evolutionary stability* implies both self-correction and asymptotic attractiveness, hence the system converges over time to a stationary point with evolutionary stability. The concept of evolutionary stability is therefore a refinement of the concept of Nash equilibrium (Bowles, 2006; Elaydi, 2005; Gintis, 2009).

In Table 1 I summarize the stationary states and asymptotic properties of each replicator equation. The long-run equilibrium for the whole model can be any combination out of the four cases (but with only three possible stationary states) from each of the three replicator equations. Because the overall long-run equilibrium requires Nash equilibria in every subgame, the long-run equilibrium is also a subgame perfect equilibrium.

[Table 1 about here]

For the macro inter-sector dynamic there are four possible cases: (i) Investment in sector I and investment in sector II are both ESS so capitalists will allocate all investment to either of these sectors depending on the initial conditions, parameters, and the evolution of the system, such that any interior solution is unstable; (ii) Investment in sector I is the only ESS hence the system converges to all investment being allocated to sector I and none to sector II, which happens because the average profit rate in sector I is consistently above the average profit rate in sector II; (iii) Investment in sector II is the only ESS so the system converges to all investment being allocated to sector II and none to sector I, which happens because the average profit rate in sector II is consistently above the average profit rate in sector I; (iv) When there is no ESS the system converges to an interior stable solution in which a share of the investment goes to sector I and the other share goes to sector II, such that average profit rates are equalized asymptotically across sectors. In this last case, profit rates are not just *equal* across sectors but truly *equalized* in the sense that the equality in sector profitability has evolutionary stability and hence is self-correcting over the long run.

For the micro intra-sector dynamic in sector I there are also four possible cases: (i) Adoption and non-adoption of the new technique of production are both ESS, hence the sector converges to either full adoption or zero adoption depending on the initial conditions, parameters, and the evolution of the system, and any interior solution is unstable; (ii) Adoption of the new technique is the only ESS hence the sector converges to full adoption, a case in which the profit rate associated with the newer technique is systematically greater than the profit rate associated with the older technique; (iii) Non-adoption is the only ESS hence no firm will innovate, a case in which the profit rate associated with the newer technique is systematically lower than the profit rate associated with the older technique; (iv) There is no ESS and hence the sector converges to a stable interior solution with a share of the capitalists adopting the innovation while the other share does not innovate, such that the profit rates associated with the older and newer techniques equalize over time within sector I. For the micro intra-sector dynamic in sector II there are four possible cases and these are analogous to the dynamic in sector I.

An extra layer of complexity exists in this evolutionary model because the technical coefficients in the input-output matrix are *exogenous but not constant*. Technological progress means that the production coefficients can change over time, which implies that a strategy that was an ESS *before* the technical change might not be an ESS *after* the innovation is introduced. As long as we have exogenous innovations brought into the system, the ESSs themselves will change over time. For example, investment in sector I might be the only ESS before an innovation is introduced, but after the innovation is introduced the new technical coefficients might make investment in sector II become the only ESS. The asymptotic properties and the existence of ESSs are therefore sensitive to the changes in the technical coefficients that the exogenous innovations generate.

When the interior solution in the replicator dynamic is asymptotically stable the system tends to converge to it since the fixed points at the edges of the system will be unstable. When the interior solution is unstable then the system will tend to converge to the stable solutions at the edges of the system where the populations of firms become homogenous. In the macro inter-sector replicator, when the interior solution $0 < f_{i,t}^{*} < 1$ such that $\Delta f_{i,t+1} = 0$ is stable, the system will tend to converge to it because the solutions $f_{1,t} = 1$ and $f_{1,t} = 0$ such that $\Delta f_{i,t+1} = 0$ at the edges of the system will be unstable. But if the interior solution becomes unstable then the system will tend to converge to either edge at $f_{1,t} = 1$ or at $f_{1,t} = 0$ since these will be the stable stationary states. The same reasoning applies to the micro intra-sector replicators. When the interior solution $0 < \upsilon_{i,t}^{*} < 1$ such that $\Delta \upsilon_{i,t+1} = 0$ is stable then the other fixed points $\upsilon_{i,t} = 0$ and $\upsilon_{i,t} = 1$ at the edges are unstable. But if the interior solution is unstable then the fixed points $\upsilon_{i,t} = 0$ and $\upsilon_{i,t} = 1$ at the edges are stable.

Stability requires the payoff of a strategy to increase less than the competing payoff when the agents adopting that strategy increase their share in the population. Given the nonlinear and multi-dimensional properties of the model, the stability analysis of this dynamic system of difference (or recurrence) equations would require us: (i) to find every possible fixed point in terms of the exogenous parameters; (ii) linearize the system around each fixed point; (iii) compute the Jacobian matrix of partial derivatives at every fixed point; (iv) compute the eigenvalues and eigenvectors of the Jacobian matrix at every fixed point; (v) check whether the absolute value of every eigenvalue lies within the unit circle (Scheinerman, 2000; Elaydi, 2005). This mathematical exercise would give us a precise diagnosis of the stability of every possible stationary state. Stability analysis must be carried out for every possible fixed point under both closures (Say's law and Keynes' effective demand), but such procedure is extremely cumbersome and little would be gained in terms of economic theory. Fortunately we can have a more intuitive understanding of the evolution of the system by simulating the model and observing its behavior over time.

Which long-run equilibrium will prevail, and whether or not the stationary state will be stable, depends on which closure we impose on the system. In the next section I analyze the implications of choosing an exogenous rate of exploitation as a first closure, which in fact implies adopting Say's law. In the subsequent section I then analyze an alternative closure for the model that employs Keynes' principle of effective demand and which automatically makes the rate of exploitation dependent upon it.

5. Say's Law and the Exogenous Rate of Exploitation

The first closure that I present for the evolutionary model posits an exogenous rate of exploitation. An exogenous rate of exploitation means that the rate of exploitation must be the *same* both in the production sphere and in the circulation sphere, or that there is no difference between the realized rate of exploitation and the rate of exploitation in production, which in fact implies the deployment of Say's law.

When the rate of exploitation is predetermined in the model then all the value (including surplus value and profits) produced must be realized in sales, otherwise the rate of exploitation would not be held constant throughout the circuit of capital. Since all surplus value must be realized, and given that in the circuit of capital the initial investment in labor power and means of production take place at the beginning of each period, an exogenous rate of exploitation is equivalent to an exogenous profit-wage ratio and hence an exogenous functional distribution of income between wages and profits. In the Marxist circuit of capital, since the production of value already presupposes an initial investment in wages and machines, an exogenous rate of exploitation predetermines the amount of profits even before they are actually realized, and hence it corresponds to an also exogenous distribution of value added between wages and profits. If we were to consider realization problems associated with the lack of effective demand, the rate of exploitation (or the distribution of income between wages and profits) could not be predetermined exogenously at any given level.

Once the rate of exploitation (or the distribution of income) is given exogenously, whatever its level may be, this exogenous rate of exploitation predetermines the amount of profits to be realized in each sector and hence prevents profit rates from equalizing throughout the economy. Under Say's law, under an exogenous rate of exploitation, or under a given distribution of income between wages and profits, the economy has no self-correction mechanism that would make profit rates sensitive to the amount invested in each sector.

Profit rate equalization necessitates profit rates to be equal across sectors and also necessitates that this equality in profitability be self-correcting and hence to have evolutionary stability. If profit rates are equal across sectors but this equality is not stable asymptotically then profit rates have not been equalized. Profit rates can indeed be numerically equal across sectors when the rate of exploitation is exogenous, but this equality will be transient and not stable asymptotically.

The long-run market prices that prevail in a competitive economy under profit rate equalization across sectors are what Marx called *prices of production* and Adam Smith called *natural prices*. These prices of production are the prices that have evolutionary stability and therefore are the ones that operate as gravitational centers in a competitive environment. Under an exogenous rate of exploitation or under Say's law, profit rates do not equalize in the long run and hence prices of production cannot function as gravitational centers for market prices. The idea of a *gravitational center* in classical Political Economy is thus what we now name *evolutionary stability* or asymptotic self-correction in evolutionary game theory. In the lines that follow I develop this reasoning in more detail.

The l_i hours worked per unit of output produce the value added that corresponds to the summation of the value of labor power and the surplus value, or value added as the summation of wages and profits. The value of labor power is simply the wage bill. The paid labor time wl_i generates a surplus value of ewl_i, in which e indicates the rate of exploitation or the ratio of the surplus to the value of labor power. Workers in sector i produce $wl_i(1+e)$ of value added per unit of output but only get back the value of their labor power corresponding to wl_i, thus leaving the surplus or unpaid labor time ewl_i to the capitalists hiring them. Workers get their money wages and spend it as they like, not bound to any real wage specified in terms of a bundle of goods.

In this closure of the model I suppose the rate of exploitation e to be exogenous and equal for every firm in either sector. For e to be predetermined we must impose Say's law so that all value and surplus value produced are realized. Total sales in each sector ($M'_{i,t}$) correspond to the monetary capital realized and hence comprise the summation of the constant and variable capitals invested plus all of the surplus value generated in the production phase. Profits in each sector $\left(\Delta M_{i,t}\right)$ originate from the share of unpaid labor time $\left(e\, wl_i\, x_{i,t}\right)$. Hence:

$$M'_{i,t} = \left[p_{1,t-1}a_{1i} + wl_i(1+e)\right]x_{i,t} \tag{20}$$

$$\Delta M_{i,t} = M'_{i,t} - M_{i,t} = e\, wl_i\, x_{i,t} \tag{21}$$

In qualitative terms, profits originate from surplus value. Under Say's law, the causality in quantitative terms then runs from exploitation to profits ($e \rightarrow \Delta M_{i,t}$). Profits originate *qualitatively* from surplus value, and under Say's law the *quantity* of surplus value produced determines the amount of surplus value realized and hence determines the amount of profits in the system. This particular relation between profitability and exploitation derives from the fact that the price system is such that $p_{i,t} = p_{1,t-1}a_{1\mathrm{i}} + wl_i + ewl_i = \left[p_{1,t-1}a_{1\mathrm{i}} + wl_i\right]\left(1 + r_{i,t}\right)$. Rearranging terms and solving for the profit rate gives us that:

$$r_{i,t} = \frac{e}{1 + \left(\frac{p_{1,t-1}}{w}\right)\left(\frac{a_{1i}}{l_i}\right)} \tag{22}$$

Equation (22) is the usual Marxist relation in which the profit rate is the rate of exploitation divided by one plus the organic composition of capital. The organic composition is, in turn, the relative price $\left(\frac{p_{1,t-1}}{w}\right)$ times the technical composition $\left(\frac{a_{1i}}{l_i}\right)$ between constant and variable capital. When e is predetermined, the long-run $r_{i,t}$ will also be predetermined in both sectors I and II. The exogenous rate of exploitation predetermines profitability because in equation (22) we have that e, w, a_{1i}, and l_i are all parameters and Say's law in sector I also predetermines the path of $p_{1,t}$: because $0 \leq a_{11} < 1$, there is a stationary state such that $p_{1,t} \rightarrow \frac{wl_1(1+e)}{1-a_{11}}$ as $t \rightarrow \infty$. However, even though market prices converge to a stationary state, this stationary state is *not* a price of production as profit rates are *not* equalized across sectors!

The wage share in value added is $\frac{V}{V+S} = \frac{1}{1+e}$, where V is the value of labor power (wage bill), S is surplus value or profits, and hence V+S is the flow of value added in the economy. Given that in the circuit of capital the expenditure with wages (or V) happens at the start of the circuit, as in equation (1), when the rate of exploitation (e) is exogenous it will automatically predetermine the wage and profit shares of national income. The functional distribution of income becomes exogenous and thus unresponsive to capital accumulation.

Once firms begin to adopt technological innovations, the productive structure in each sector becomes a linear combination of the firms using the older and newer technologies side by side. The recalculation of profitability then requires weighting the surplus value produced by the respective shares of firms employing the newer and older techniques. Equations (23) and (24) below replace equations (20) and (21) as soon as a new technique is introduced:

$$M'_{i,t} = \{(v_{i,t})[p_{1,t-1}a^n_{1i} + wl^n_i(1+e)] + (1-v_{i,t})[p_{1,t-1}a^o_{1i} + wl^o_i(1+e)]\}x_{i,t} \tag{23}$$

$$\Delta M_{i,t} = M'_{i,t} - M_{i,t} = \{(v_{i,t})[wl^n_i e] + (1-v_{i,t})[wl^o_i e]\}x_{i,t} \tag{24}$$

For a given rate of exploitation, increments in the share of firms adopting the new technology $(v_{i,t})$ *might* reduce the average profit rate prevailing in the sector if the new technology employs relatively less labor power. I stress the conditional clause 'might' since the final effect on profitability can only be known after the repricing of both the means of production and the final consumption good.

A full analytical solution to this system is too cumbersome and not illuminating enough to be worthwhile, but simulations are feasible and useful and provide some intuitive sense of how the system works. To simulate the model in its first closure it is necessary to fix eleven parameters and three initial conditions. The initial technical coefficients are set to (a_{11}^{o}, a_{12}^{o}, l_{1}^{o}, l_{2}^{o}) = (0.2, 0.1, 0.7, 0.7) for the old technology. The nominal wage w is set to 10 dollars per work hour, the rate of exploitation e is set to 110%, and only μ=20% of the firms migrate to another sector according to inter-sector average profitability differentials. The initial aggregate monetary capital $M_{t=1}$ is set to 100 dollars, and is initially distributed as 60% to the sector manufacturing means of production ($f_{1,t=1} = 0.6$) and 40% to the sector producing the final consumption good ($f_{2,t=1} = 0.4$). The market price for the means of production produced in sector I starts from 50 dollars per unit ($p_{1,t=0} = 50$).

The model is set to run for 400 periods. For the first 49 rounds the trajectories evolve without any kind of innovation or technical change. At period t=50 I introduce an innovation in sector II that increases labor productivity by 100% while increasing the use of machines by 100% per unit of output such that productivity gains are $\eta_{a12} = -50\%$ and $\eta_{l2} = 100\%$ and hence (a_{11}^{o}, a_{12}^{n}, l_{1}^{o}, l_{2}^{n}) = (0.2, 0.2, 0.7, 0.35). This machine-intensive labor-saving innovation generates a strong increase in the technical composition of capital in the sector producing the consumption good. At time t=100 I introduce an innovation in sector I that increases labor productivity by 150% and the use of the machines by 100% per unit of output such that productivity gains are $\eta_{a11} = -50\%$ and $\eta_{l1} = 150\%$ and hence (a_{11}^{n}, a_{12}^{n}, l_{1}^{n}, l_{2}^{n}) = (0.4, 0.2, 0.28, 0.35). This innovation also implies a strong machine-intensive labor-saving technical change in the sector producing the means of production.

[Figure 1 about here]

In Figure 1 I report the evolution of key variables from the first closure of the model with Say's law and an exogenous rate of exploitation. The uncoordinated implementation of the new technologies increases the profit rate only for those firms initially adopting the innovation, but the gradual diffusion of the new production technologies results in lower levels of profitability for all capitalists over time. The uncoordinated struggle for greater individual profits via technical change can indeed lead to a reduction in the average profitability for all competitors over time. As expected, even though the real wage is determined endogenously, the exogenous rate of exploitation predetermines constant profit and wage shares of value added.

Profit rates do not equalize under Say's law, even though they are temporarily equal across sectors for two specific moments in time during the implementation of the new techniques of production. With an exogenous rate of exploitation the model has no self-correction mechanism to equalize profit rates over the long run. This happens because the exogenous rate of exploitation turns the macro replicator dynamic unresponsive to the amount invested in each sector.

Say's law implies that the interior solution $f_{1,t}^{*}$ in the macro replicator dynamic is asymptotically unstable. When the model is closed with Say's law, the exogenous and therefore predetermined rate of exploitation makes profit rates in each sector insensitive to changes in $f_{1,t}$. Thus the interior solution $f_{1,t}^{*}$, under which profit rates equalize across sectors, will not have evolutionary stability. Because $f_{1,t}^{*}$ does not have evolutionary stability, production prices do not exist and hence cannot function as gravitational centers for market prices. In this case $f_{1,t}$ will tend to move to the stable stationary points at either $f_{1,t} = 0$ or $f_{1,t} = 1$. Asymptotically the total monetary capital under Say's law is invested either 100% in sector I or 100% in sector II, depending on which sector has the permanently higher average profit rate.

6. Effective Demand and the Endogenous Rate of Exploitation

In this section I offer a second closure for the evolutionary model in which the rate of exploitation is endogenous and dependent on the level of effective demand. Under Keynes' principle of effective demand the rate of exploitation necessarily becomes endogenous to aggregate expenditures and is thus no longer solely determined in the production sphere. Once effective demand is brought into the model, the rate of exploitation, and thus the distribution of income between wages and profits automatically become dependent on the level of aggregate expenditures. We can either have an exogenous rate of exploitation or the principle of effective demand at play, but not both simultaneously.

The principle of effective demand implies that value originates in the production phase but is only validated in circulation via consumption and investment expenditures. Expenditures determine incomes. And because surplus value is a type of income, both the amount of surplus value realized and the level of exploitation can only be known *after* the end-of-period consumption and investment expenditures have been carried out. Marx himself was well aware of the role of effective demand in the realization of value. For example:

> "The general possibility of crisis is given in the process of metamorphosis of capital itself, and in two ways: in so far as money functions as means of circulation, [the possibility of crisis lies in] the separation of purchase and sale; and in so far as money functions as means of payment, it has two different aspects, it acts as measure of value and as realisation of value. These two aspects [may] become separated. If in the interval between them the value has changed, if the commodity at the moment of its sale is not worth what it was worth at the moment when money was acting as a measure of value and therefore as a measure of the reciprocal obligations, then the obligation cannot be met from the proceeds of the sale of the commodity, and therefore the whole series of transactions which retrogressively depend on this one transaction, cannot be settled." (Marx, 1861-1863, Theories of Surplus Value, volume II, chapter 17, section 11).

When effective demand determines the amount of surplus value realized and hence the level of exploitation, sector profitability becomes sensitive to the amount of investment expenditures in each sector. An endogenous rate of exploitation that is dependent upon effective demand confers evolutionary stability to the equalization of profitability, and therefore allows for prices of production to operate as gravitational centers for market prices. Profit rates equalize over the long run as long as sector profitability responds negatively to the amount invested in each sector, such that short-run deviations are self-correcting in the long run.

Supposing that workers do not save and that there is no consumption credit, the total demand for the consumption good produced in sector II (denoted by $M'_{2,t}$) is simply the total wage bill in the economy. Given that the wage bills in each sector must be weighted by the shares of firms using the old and the new technologies, we have that the total wage bill and hence total consumption expenditures are:

$$M'_{2,t} = \{(v_{1,t})[wl_1^n] + (1 - v_{1,t})[wl_1^0]\}\, x_{1,t} + \{(v_{2,t})[wl_2^n] + (1 - v_{2,t})[wl_2^0]\}\, x_{2,t} \tag{25}$$

The investment function is surely the more sensitive part of the model. In this closure I opt for the neo-Keynesian investment function à la Joan Robinson (1962; see also Dutt, 2011 and 1990; Marglin, 1984). It assumes that firms operate at full capacity utilization and that the desired amount of investment is a function of the observed profit rate. Given that there are firms operating with the newer and older technologies simultaneously in each sector, the aggregate demand for the means of production produced in sector 1 (denoted by $M'_{1,t}$) is:

$$M'_{1,t} = M'_{1,t-1} + \gamma_1\{(v_{1,t})[r_{1,t-1}^n] + (1 - v_{1,t})[r_{1,t-1}^0]\}\, M_{1,t-1} + \gamma_2\{(v_{2,t})[r_{2,t-1}^n] + (1 - v_{2,t})[r_{2,t-1}^0]\}\, M_{2,t-1} \tag{26}$$

The parameters γ_i indicate the sensitivity of investment expenditures to the observed profit rates in each sector, and the autonomous component of investment expenditures is simply the investment carried out in the previous period $(M'_{1,t-1})$. The end-of-period expenditures with consumption and investment then connect the circuit of capital in period t with the circuit in period $t + 1$. Aggregate demand at the end of period t is $M_t' = M_{1,t}' + M_{2,t}'$, which is identical to the aggregate monetary capital at the beginning of period $t+1$, such that $M_{t+1} = M_t'$ with causality running from M_{t+1} to M_t'.

The endogenous rates of exploitation within each sector (denoted by $e_{i,t}$) are the sector surplus value *realized* over the nominal wage bill:

$$e_{i,t} = \frac{M_{i,t}' - M_{i,t}}{\{(v_{i,t})[wl_i^n] + (1 - v_{i,t})[wl_i^0]\}\, x_{i,t}} \qquad (27)$$

In equation (27) we can observe that the rates of exploitation $e_{i,t}$ in each sector depend directly on the level of effective demand $M_{i,t}'$ from equations (25) and (26). As under Say's law, in *qualitative* terms profits originate from surplus value. But contrary to Say's law, under which in *quantitative* terms the causality runs from exploitation to profits ($e \rightarrow \Delta M_{i,t}$), the principle of effective demand implies that in *quantitative* terms the causality now runs from profits to exploitation ($e \leftarrow \Delta M_{i,t}$). Even though profits originate qualitatively from surplus value, under Keynes' principle of effective demand the amount of profit determines the quantity of surplus value realized.

This evolutionary model is nonlinear, has no closed form solution, and cannot be solved analytically. A way to circumvent this analytical shortcoming is to simulate the model numerically many times over and check under which parameter values and initial conditions the equilibrium with equalized profit rates is either evolutionarily stable or unstable. Simulations indicate that initial conditions have solely negligible temporary effects over the first time periods. But the input-output coefficients and the sensitivity of investment to changes in profit rates do have large and long-lasting effects on the dynamics of the system. The result obtained from several rounds of simulations is that as long as the economy is viable, such that it does not collapse because of permanently negative profit rates, the system features a long-run equilibrium with equalized profit rates as an evolutionary stable stationary state.

I simulate the model in its second closure to illustrate these points. To facilitate comparison with the model in its first closure I keep the same parameter values and innovation patterns as in the previous simulation. For the investment function I set $\gamma_1 = \gamma_2 = 0.5$, and investment demand begins at 50 dollars ($M'_{1,t=1} = 50$) as an initial condition. Simulation results for key variables are reported in Figure 2. Similarly to the first simulation, all firms in both sectors adopt the new technologies over the long run (such that $\upsilon_{1,t} \to 1$ and $\upsilon_{2,t} \to 1$ when $t \to \infty$).

[Figure 2 about here]

Contrary to Say's law, Keynes' principle of effective demand makes profit rate equalization an evolutionary stable long-run equilibrium. Unlike the model in its first closure under an exogenous rate of exploitation, when the macro inter-sector replicator had an unstable interior solution that forced the system to move to its edges in the long run, the macro inter-sector replicator under effective demand now reaches an interior stationary state $f^*_{1,t}$ with equalized profitability that is stable asymptotically.

With the explicit incorporation of effective demand, profit rates do equalize and this equality has evolutionary stability, thus prices of production can operate as gravitational centers for market prices. Unlike the first closure under Say's law, and because the rates of exploitation in each sector are now endogenous, the functional distribution of income between wages and profits is no longer predetermined. The level of exploitation, the real wage, as well as the wage and profit shares of value added now all respond to the trajectory of aggregate expenditures.

Under Keynes' principle of effective demand the economy-wide average profit rate functions as a gravitational center for the sector average profit rates, hence any short-run deviations from the economy-wide average are soon self-corrected. With the introduction of effective demand the system displays negative feedback effects that make profitability equalization self-correcting, so that the total monetary capital in the economy will not all flow into a single sector over the long run. Unlike the first closure when all investment did migrate to a single sector, the macro inter-sector replicator dynamic now reaches an interior stationary state that is stable asymptotically and in which a fraction of the total monetary capital flows to sector 1 and the other fraction to sector 2.

7. Conclusion and Implications

In this paper I have claimed and demonstrated the following points regarding the Marxist theory of exploitation and accumulation. First, an exogenous rate of exploitation implies Say's law since under an exogenous rate of exploitation the volume of profits is predetermined even before it is realized in the market. Because the rate of exploitation is equivalent to the profit-wage ratio, an exogenous rate of exploitation implies an exogenous functional distribution of income between wages and profits. An exogenous distribution of income between wages and profits thus also implies Say's law. Second, an exogenous rate of exploitation, or equivalently an exogenous functional distribution of income, prevents profit rates from equalizing across sectors. Third, Marx himself was logically inconsistent in *Capital III* when supposing an exogenous rate of exploitation together with equalizing profit rates. Fourth, because profit rates cannot equalize, an exogenous rate of exploitation or Say's law prevent prices of production from operating as gravitational centers to market prices. Fifth, once Keynes' principle of effective demand is introduced, the rate of exploitation and the wage and profit shares become endogenous and dependent on the level of aggregate expenditures. Sixth, when the rate of exploitation is dependent on effective demand, profit rates will equalize across sectors and hence prices of production can operate as gravitational centers for market prices. Seventh, the idea of a gravitational center in classical Political Economy is what we now name a stationary state with evolutionary stability (or asymptotic self-correction) in evolutionary game theory. Eighth, profit rate equalization requires profit rates to be equal and this equality to have evolutionary stability, such that short-run deviations from equilibrium are self-corrected over time.

The main implication of these results is that we can *either* have: (a) an exogenous rate of exploitation, an exogenous profit-wage ratio, an exogenous functional distribution of income between wages and profits, Say's law, non-equalizing profit rates, and no prices of production as gravitational centers for market prices; *or* (b) an endogenous rate of exploitation, an endogenous profit-wage ratio, an endogenous functional distribution of income between wages and profits, Keynes' principle of effective demand, equalizing profit rates across sectors, and prices of production as gravitational centers for market prices.

If we aim at developing a theory that is both empirically relevant and logically consistent, Marxist scholars must therefore drop Say's law and incorporate Keynes' principle of effective demand for a proper understanding of how capital accumulation determines the rate of exploitation, the functional distribution of income, and the equalization of profit rates.

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Tables and Figures

Table 1: Stationary States and Asymptotic Properties of the Model

	Stationary States		
(a) Macro Inter-Sector Replicator			
Investment in sector I is ESS Investment in sector II is ESS	$\Delta f_{i,t+1} = 0$	$\overline{r_{1,t}} \gtreqless \overline{r_{2,t}}$	$f_{1,t} = 1$ is stable $f_{1,t} = 0$ is stable $0 < f_{i,t}^{*} < 1$ is unstable
Investment in sector I is ESS Investment in sector II is not ESS	$\Delta f_{i,t+1} = 0$	$\overline{r_{1,t}} > \overline{r_{2,t}}$	$f_{1,t} = 1$ is stable $f_{1,t} = 0$ is unstable
Investment in sector I is not ESS Investment in sector II is ESS	$\Delta f_{i,t+1} = 0$	$\overline{r_{1,t}} < \overline{r_{2,t}}$	$f_{1,t} = 1$ is unstable $f_{1,t} = 0$ is stable
No ESS	$\Delta f_{i,t+1} = 0$	$\overline{r_{1,t}} = \overline{r_{2,t}}$	$0 < f_{i,t}^{*} < 1$ is stable $f_{1,t} = 1$ is unstable $f_{1,t} = 0$ is unstable
(b) Micro Intra-Sector Replicator in Sector I			
Innovate is ESS Not innovate is ESS	$\Delta v_{1,t+1} = 0$	$r_{1,t}^{n} \gtreqless r_{1,t}^{o}$	$v_{1,t} = 1$ is stable $v_{1,t} = 0$ is stable $0 < v_{1,t}^{*} < 1$ is unstable
Innovate is ESS Not innovate is not ESS	$\Delta v_{1,t+1} = 0$	$r_{1,t}^{n} > r_{1,t}^{o}$	$v_{1,t} = 1$ is stable $v_{1,t} = 0$ is unstable
Innovate is not ESS Not innovate is ESS	$\Delta v_{1,t+1} = 0$	$r_{1,t}^{n} < r_{1,t}^{o}$	$v_{1,t} = 1$ is unstable $v_{1,t} = 0$ is stable
No ESS	$\Delta v_{1,t+1} = 0$	$r_{1,t}^{n} = r_{1,t}^{o}$	$0 < v_{1,t}^{*} < 1$ is stable $v_{1,t} = 1$ is unstable $v_{1,t} = 0$ is unstable
(c) Micro Intra-Sector Replicator in Sector II			
Innovate is ESS Not innovate is ESS	$\Delta v_{2,t+1} = 0$	$r_{2,t}^{n} \gtreqless r_{2,t}^{o}$	$v_{2,t} = 1$ is stable $v_{2,t} = 0$ is stable $0 < v_{2,t}^{*} < 1$ is unstable
Innovate is ESS Not innovate is not ESS	$\Delta v_{2,t+1} = 0$	$r_{2,t}^{n} > r_{2,t}^{o}$	$v_{2,t} = 1$ is stable $v_{2,t} = 0$ is unstable
Innovate is not ESS Not innovate is ESS	$\Delta v_{2,t+1} = 0$	$r_{2,t}^{n} < r_{2,t}^{o}$	$v_{2,t} = 1$ is unstable $v_{2,t} = 0$ is stable
No ESS	$\Delta v_{2,t+1} = 0$	$r_{2,t}^{n} = r_{2,t}^{o}$	$0 < v_{2,t}^{*} < 1$ is stable $v_{2,t} = 1$ is unstable $v_{1,t} = 0$ is unstable

Figure 1: Simulation of the Evolutionary Model with Say's Law
and an Exogenous Rate of Exploitation

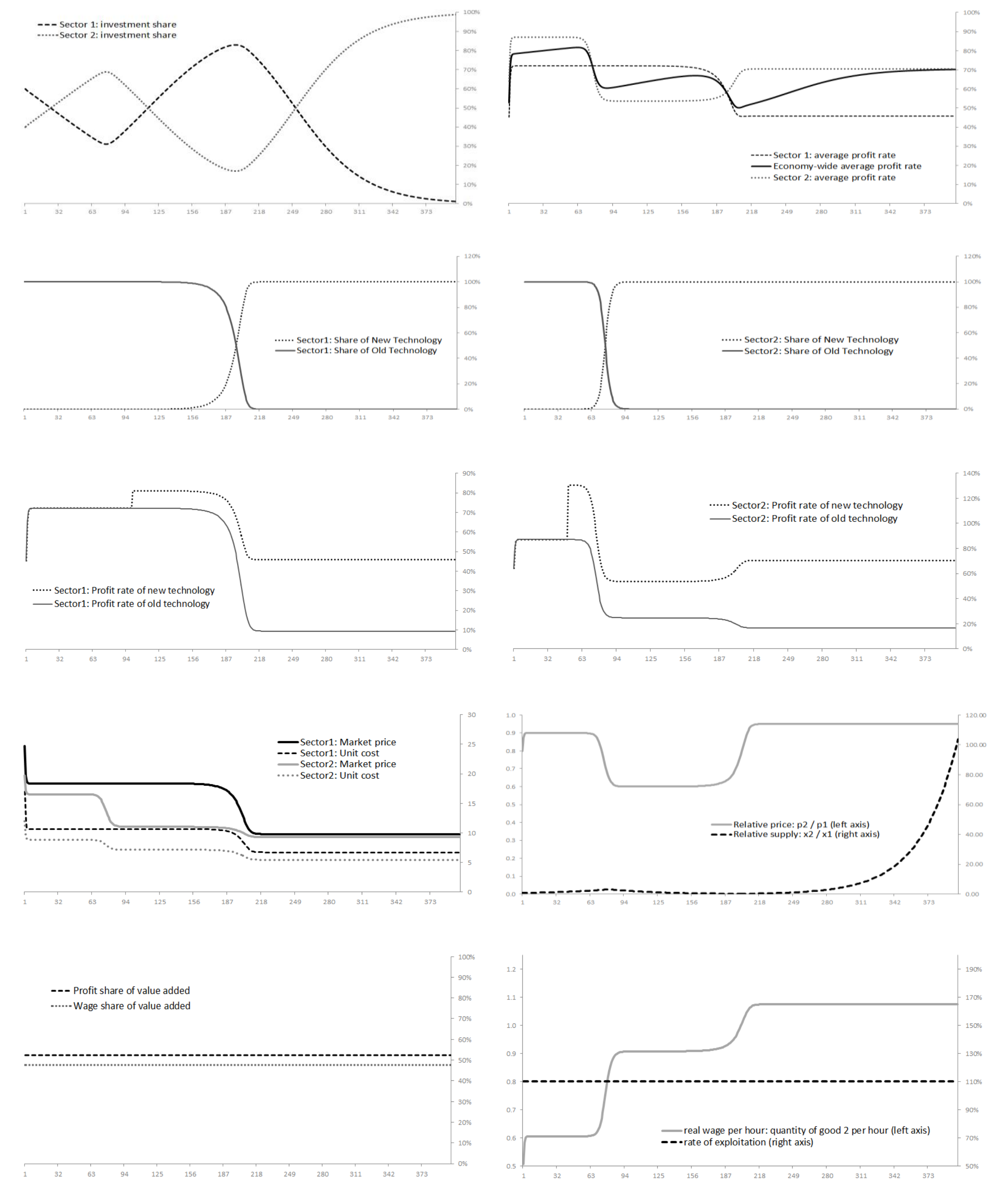

Figure 2: Simulation of the Evolutionary Model with Effective Demand and Endogenous Rates of Exploitation