Geometric polarimetry – part I: spinors and wave states

Article

Accepted Version

Bebbington, D. and Carrea, L. (2014) Geometric polarimetry – part I: spinors and wave states. IEEE Transactions on Geoscience and Remote Sensing, 52 (7). pp. 3908-3923. ISSN 0196-2892 doi: https://doi.org/10.1109/TGRS.2013.2278141 Available at http://centaur.reading.ac.uk/51711/

Geometric Polarimetry – Part I: Spinors and Wave States

David Bebbington and Laura Carrea

Abstract— **A new formal approach for representation of polarization states of coherent and partially coherent electromagnetic plane waves is presented. Its basis is a purely geometric construction for the normalised complex-analytic coherent wave as a generating line in the sphere of wave directions, and whose Stokes vector is determined by the intersection with the conjugate generating line. The Poincaré sphere is now located in physical space, simply a coordination of the wave sphere, its axis aligned with the wave vector. Algebraically, the generators representing coherent states are represented by spinors, and this is made consistent with the spinor-tensor representation of electromagnetic theory by means of an explicit reference spinor we call the *phase flag*. As a faithful unified geometric representation, the new model provides improved formal tools for resolving many of the geometric difficulties and ambiguities that arise in the traditional formalism.**

Index Terms— **state of polarization, geometry, covariant and contravariant spinors and tensors, bivectors, phase flag, Poincaré sphere.**

I. Introduction

OVER the last decade or so, considerable advances in digital signal processing capabilities, together with corresponding decreases in costs have meant that the cost premium on radars with full polarimetric capability has become hardly significant, and there is now little reason to forgo the valuable information that is available in vector scattering data. Both small scale systems as well as increasingly ambitious polarimetric radars have been developed for remote sensing, such as TerraSAR [1], which also incorporate multi-platform bistatic capabilities. Unsurprisingly perhaps, after a long period since the work of Davidowitz and Boerner [2] after which new theoretical developments for bistatic polarimetric radar were few [3] there has in recent years been some revival in work that attempts to determine what can be obtained from such measurements, and to understand the geometric factors [4], [5], [6]. Even in the context of monostatic polarimetry, the half century and more history of polarimetric radar has been punctuated by controversies over the interaction of geometry with polarization representations [7], [8], [9], [10], [11], [12]. In a recent paper [13] we showed that the widely accepted consimilarity concept [14] used to describe backscatter was not only in a strict sense unphysical but also masked a problem in the usual identification of antenna height with wave polarization state under general unitary basis transformation. In contrast with many other presentations on radar polarimetry, geometric polarimetry is not an attempt to fix any one problem, but a fundamental reformulation that may, by its generality, be expected to provide a unifying framework formalizing both traditional and modern representations in radar polarimetry. The phenomenon of electromagnetic polarization is self-evidently a geometric phenomenon, and polarimetry-based theses as well as primers on the subject invariably describe the polarization ellipse [15], [16]. It is when concepts of frequency domain analytic signal representation are imported that geometrically related difficulties appear to arise. Clearly, the field of the geometry needs to be expanded to embrace complex points, but from there on there are issues that until recently seem not to have been explicitly questioned. Perhaps the most obvious was the apparently widely held belief that wave-reversal implies the conjugation operation, which was finally disposed of in [13], where we asserted that conjugation always involves time-reversal; formally, a linear scattering operation cannot involve antilinear transition to a conjugate representation. More generally, in relation to the mathematical foundations of any theory, it is important when considering a space to understand the properties of that space. When considering a polarization ellipse, the vectors are commonly understood as elements of a Euclidean space. This is characterized by its metric (a quadratic form), which is invariant under the group of rotations. It seems natural enough, when considering a complex space that the obvious and appropriate generalization is to a Hermitian space. Hermitian products feature widely in polarimetry theory, but wherever they do, they are associated with time-averaged intensities. It is important to realize, however, that this does not imply that Hermiticity is a fundamental property in polarimetry; this is because it goes hand in hand with unitarity (energy conservation of the field), which by no means always holds. Whilst it is quite common for lossy, non-unitary processes to occur in radar polarimetry, it is always the case that coherent propagation preserves the quality of complete degree of polarization [17]. To express this another way, what is fundamental (and invariant) in polarimetry, indeed the only thing that truly is, is the Poincaré sphere. What should be carefully noted is that it is not possible to express this invariance in terms of Hermitian forms. In the framework we present here, Hermitian products do appear naturally in connection with time averaged intensities or covariances but are to be interpreted within a more general geometric framework. In [13] it was explained that a consistent theory of coherent polarization must require that wave states and antenna states must transformation contragradiently with respect to one another, which is to say that they have mutually

This work was supported by the Marie Curie Research Training Network AMPER (Contract number HPRN-CT-2002-00205).

D. Bebbington and L. Carrea are with the Centre for Remote Sensing & Environmetrics, School of Computer Science and Electronic Engineering, University of Essex, U.K. (email: david@essex.ac.uk; lcarrea@essex.ac.uk).

inverse transformations. That is, one must be covariant and the other contravariant. This distinction maps naturally into projective geometry, which provides much of the conceptual framework for geometric polarimetry since planes and points exhibit a precisely analogous duality in their homogeneous coordinate representations. The basic development of geometric polarimetry is divided between this and a companion paper. This one focuses on the development of wave-states as spinors starting from the tensor description of Maxwell's equations expressed in the frequency domain, while the next paper will develop antenna states as spinors from basic reciprocity principles and from there develops a scattering formalism valid for all geometries. The breakdown of this paper is as follows. In Sec. II, the background and the motivations for introducing a new approach to polarimetry based on geometry are given. Sec. III revises the algebra of spinors which are the foundation of geometric polarimetry. In Sec. IV the link between tensors, spinors and their geometric interpretation in projective spaces is explored. In particular, one-index spinors are interpreted as generators of a projective sphere. These are the fundamental ingredients to see how to derive the polarization state of a wave from the electromagnetic tensor in form of a one-index spinor (Sec. V) and to interpret it as one generator of a sphere. This sphere, the wave sphere, is then identified with Poincaré sphere in Sec. VI which is the most important finding of this part of the work. After some working examples in Sec. VII, Sec. VIII concludes the paper.

II. The background to geometric polarimetry

Geometric polarimetry (GP) represents a completely new approach to the foundations of polarimetry that achieves complete integration of analytic signal representation within a fully geometric model. The advantages of this should be clear, namely that this makes it possible to describe polarimetric relationships algebraically in ways that are independent of the basis and geometric frame. This goes further than using polar coordinates, for example, because these have singularities at the poles. Although in many practical cases such problems are minor, they can also be significant: for both forward and backscatter, a similar singularity arising from the parallelism or anti-parallelism of the incident and scattered wave vectors makes it impossible to define a unique scattering plane as is used in optical ellipsometry [16]. This has been implicitly significant in debates about the desirability of the backscatter alignment convention [10], and is an obstacle to developing a theory without exceptions. In developing GP we had a number of goals to fulfil. Firstly it should try as far as possible to maintain compatibility with existing polarimetry formalism and usages; secondly, it should be a comprehensive theory; thirdly, it should be developed from first principles. This last condition is most important, because perhaps the biggest problem with polarimetry has to do with its multidisciplinary character, of which a significant side effect is that methodologies have often been developed that are application specific, particularly with regard to restricted geometries. There have been attempts [18], [19], [7], [20], [10], [21], [22] to reformulate radar polarimetry but it seems difficult if not impossible to succeed in this if unnecessary assumptions are (even unwittingly) retained. The mathematical developments of the remainder of this paper as well as the motivations behind GP are perhaps best understood by considering the following observations. Firstly, as it is clear that unitary transformations do play a big role in polarimetry, spinors must implicitly be involved in the picture. Spinors are the carriers of unitary transformation [23], i.e. the things that are operated upon. They were first introduced by Cartan [24], but from the 1930's on [25], became important in physics for understanding relativistic quantum mechanics with spin. Secondly, it was established long ago that any relationship in tensor theory expressed in terms of relativistic 4-vectors can also, by straightforward means, be expressed in terms of spinors. Since electromagnetic theory as developed by Maxwell is inherently a relativistic theory expressible in terms of 4-vector tensors, there already appears to be a promising basis for reducing the more complex descriptors of fields to the desired simpler single linear spinor representation. It also turns out to be the case that it is via spinors that Hermitian products arise, in the context of the time-like (or dually, energy-like) components of 4-vectors, which accounts naturally for their importance in terms of energy invariance. The final element required to develop a basis for GP was a re-realization - of ideas developed in the 1930's by Veblen [26] and Ruse [27] - that the homogeneous coordinate representation of three-dimensional projective geometry provides a faithful representation for the algebra of 4-vectors. That is to say the tensor expressions translate to geometric constructions by interpreting the algebra as that of the relevant algebraic geometry. As much as anything, this is a visualization, or conceptualization, tool, since it is easier for us to understand constructions in three dimensions rather than four. More than that, however, is that this model allows us to draw on a very extensive literature on projective geometry which is in many ways richer than that of linear algebra. In this scheme of things, the metric tensor of special relativity is represented geometrically by an isotropic quadric surface, in other words a sphere, and is invariant under Lorentz transformations. We interpret this sphere both as the sphere of normalized wave-vectors in the analytical signal representation, and as the Poincaré sphere. As this paper will show, GP succeeds in relating fields, wave-vectors, vector potentials and elementary complex polarization states by geometrical constructions in one unified space. The validity of such constructions ensures the transformational and basis invariance that a formal theory should possess. In this sense, the wave-sphere and the Poincaré sphere are seen to be identified. This is very different from the usual picture, in which polarization states on the Poincaré sphere appear in an abstracted space via stereographic projection over the Argand plane of complex polarization ratios. In GP, coherent polarization states, represented algebraically as spinors, are geometrically neither points nor vectors, but rather, lines on the Poincaré sphere. To explain the emphasis on 'on' we should mention that a very important concept in complex projective geometry is that all quadric surfaces contain two families of lines - known as generators. No two generators of the same family intersect, and each member of one family intersects each of the other at a single point. To conceive of a sphere containing lines (each point of such a line being a point of the sphere) may seem at first bizarre, but this is because in that case only one point of any line is real. Architecturally, this general property of quadric surfaces has been exploited many times for constructing rigid curved surfaces comprising hyperboloids (such as in power station cooling towers and other structures [28]) which are constructed using intersecting families of beams. The complete amalgamation of analytic signal theory of coherent polarization states with geometry culminates in the recognition that polarization states with the usual exponential time signature are represented by one family of generators, while their conjugates are represented by the complementary family of generators. Coherent Stokes vectors are then defined by the unique and real intersection between a conjugate pair of generating lines on the Poincaré sphere. The central principle of GP, that the Poincaré sphere is an invariant, arises from the fact that any linear transformation on spinors corresponds to a Lorentz transformation on Stokes vectors [29]. Geometrically, such coherent transformations are Clifford translations of generators [30] on the sphere. Left and right translations represent coherent transformations within one or other family of generators, and there is no antilinear crossover from one family to the other. What GP boils down to, is that we consider operations on projective space in terms of actions on generators of an invariant sphere, and these rather than the spatial points, as the basic building blocks from which points and more complex objects are constructed. In this way, the minimum required specialization of the projective geometry is imposed. The interpretation of coherent states as generators of the Poincaré sphere is of course a radical departure from the conventional view of basic states as Jones vectors. The conventional interpretation would, however, not lead to the geometric construction of Stokes vectors. Although it has been necessary to focus on some of the difficulties that have arisen in traditional approaches, it should already be becoming clear that GP represents more than just a clean-up exercise involving traditional polarimetry. Given that the modern developments pioneered by Cloude and Pottier [31] are based on intrinsically spinorial representations, such as Pauli matrices, it can be anticipated that old and new representations in polarimetry can be much more effectively integrated within an overarching framework based on GP. Indeed, by means of a minor algebraic change, Cloude's target vectors [32] can be mapped in relation to the Poincaré sphere [33]. Fig. 1 provides a schematic overview illustrating the connections that can be expected to be made in such a unified framework, which we hope to present in further work.

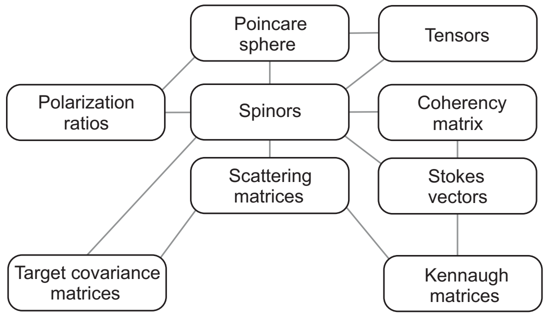


Fig. 1. Schematic relationships between polarimetric descriptors.

III. The algebra of spinors

This paper follows the notations and the conventions of the book by Penrose and Rindler[1] [35] which provides a very clear idea of what spinor representation signifies.

A. Spinor characters and inner products

The basic algebra of spinors [35] was reviewed in [13]. Fundamentally, a complex vector when treated as a spinor belongs to one of four distinct characters, namely contravariant ξ^A or covariant ξ_A, further delineated as either unprimed (ξ^A, ξ_A) or primed $(\bar{\xi}^{A'}, \bar{\xi}_{A'})$. Each character belongs to a distinct linear vector space, having a particular basis transformation rule. There are natural mappings between the different characters. Firstly, conjugation maps unprimed spinors into primed, and vice versa. Under the conjugation operation C, for example, overbar represents conjugation so

$$\xi^A \ \overset{C}{\longleftrightarrow} \ \bar{\xi}^{A'}, \tag{1}$$

where the component values are conjugated, but the index labels are also swapped reflecting the fact that the objects' transformation rule will also be conjugated. The rule works identically for covariant spinors. One of the most important aspects of spinor algebra is that there is an invariant, known as the *metric spinor* for each spinor character:

$$\epsilon_{AB} = \epsilon_{A'B'} = \epsilon^{AB} = \epsilon^{A'B'} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{2}$$

The equals $=$ sign here means numerical equality of the components in their matrix representation. These entities are invariant not only under unitary transformations but all complex unimodular transformations SU(2,C). This property is related to the invariance of the metric of special relativity. The metric spinor gets its name from the fact that the inner product of two spinors $\xi^A = \xi^0 o^A + \xi^1 \iota^A$ and $\eta^B = \eta^0 o^B + \eta^1 \iota^B$ is given by

$$\epsilon_{AB}\, \xi^A \eta^B = \xi^0 \eta^1 - \xi^1 \eta^0, \tag{3}$$

where $\{o^A, \iota^A\}$ is the basis of the space [13] and it is called *spin frame*. The inner product vanishes if the two spinors are the same up to scale factor. This is an affine measure of distance (linear) rather than the more usual quadratic measures of symmetric Euclidean metrics. When the spinors are normalised, the result is the sine of the half angle between the vectors the spinors represent. This inner product is invariant under basis transformation, and so the metric spinor induces a natural one to one mapping between covariant and contravariant spinors in the sense that there is a unique partner to any spinor,

$$\xi_A = \epsilon_{AB}\, \xi^B, \quad \xi^B = \xi_A\, \epsilon^{AB}. \tag{4}$$

It is for this reason that the spinor literature adopts the convention of using the same symbolic name after applying a metric spinor in this way to lower or raise an index. In

[1] which are the same as the conventions of the book of Misner [34].

physical applications this can be slightly confusing, and it is probably better to think of a physical spinor as inherently of the character that is appropriate to its physical role. For example, extensive variables like length and time are naturally associated with contravariant character, while their duals, like wavevectors and frequency are associated with covariant character. Implicity, the raising and lowering operations give rise to inner products between spinors of dual character, so that,

$$\xi_A \eta^A = \xi_A \, \eta_B \, \epsilon^{AB}. \tag{5}$$

Owing to the skew nature of the metric spinor this inner product is antisymmetric, in the sense that,

$$\xi_A \eta^A = -\eta_A \xi^A. \tag{6}$$

The elementary result that when spinors are linearly dependent their inner product is zero is frequently of use in simplifying and understanding spinor expressions. It should be noted that there is not a Hermitian inner product for spinors. Fundamentally, the reason for this is that such a product could not be generally invariant, but only unitary invariant. Physically, any such product has to be related to time or energy components of a $4-$vector, and is not a true scalar quantity. Linear transformations of spinors, including geometric and basis transformation have 2×2 matrix representations, for example, contravariant transformations take the form,

$$\eta^A = L^A{}_B \, \xi^B, \tag{7}$$

where $L^A{}_B$ is the spinor describing the transformation. When considering a basis transformation, covariant spinors have to transform by the inverse transformation, while priming introduces a conjugation.

B. Higher order spinors

For much of the remainder of this paper, we will be concerned not just with simple (rank one) spinors as discussed in this section, but with higher rank spinors, since vectors in spacetime are described by second rank spinors of mixed (primed and unprimed) character. Transformation rules for these follow naturally, because for consistency any higher rank spinor must transform in the same way as a simple product of rank one objects. So, for example, a second rank spinor, $X^{BB'}$ representing a vector, transforms into $Y^{AA'}$ as a kind of tensor product,

$$Y^{AA'} = L^A{}_B \, \bar{L}^{A'}{}_{B'} \, X^{BB'}, \tag{8}$$

where $\bar{L}^{A'}{}_{B'}$ is the transformation which acts on the conjugate space. If this were expressed in matrix algebra, the primed operator would be placed after the operand, and so would need to be written in transposed form. This then becomes the familiar rule for transformation of Hermitian matrices using the Hermitian transpose. A common example of this rule applies to rotation of a vector, using Cayley matrices:

$$\begin{pmatrix} t' + z' & x' - jy' \\ x' + jy' & t' - z' \end{pmatrix} = U \begin{pmatrix} t + z & x - jy \\ x + jy & t - z \end{pmatrix} U^{\dagger} = \\ = U^A{}_B X^{BB'} \bar{U}^{A'}_{B'} \tag{9}$$

where

$$U = \begin{pmatrix} e^{j\frac{\lambda}{2}} & 0 \\ 0 & e^{-j\frac{\lambda}{2}} \end{pmatrix} \begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} \begin{pmatrix} e^{j\frac{\chi}{2}} & 0 \\ 0 & e^{-j\frac{\chi}{2}} \end{pmatrix}, \tag{10}$$

and θ, λ, and χ are the Euler angles for the rotation. Each matrix contributes half of the overall rotation. For a unitary matrix, the Hermitian transpose is the inverse, so (9) is a similarity, but in the general case the spinor form of (9) shows that the operative rule in general is that the right hand side matrix is the Hermitian transpose. The fact that the matrix form of the equation involves a transpose is seen by the fact that the summation indices are not adjacent in the spinor form. It should be noted that the unitarity of (10) results in the invariance of the time-coordinate. This is in line with the deeper point that invariance of Hermitian inner products is conditional on unitarity. They are therefore not fundamental to a classical theory of polarimetry which includes dissipative processes.

C. Spin frame

An important concept in spinor geometry is that of a *spin frame*, in general determined by two spinors ξ^A and η^A such that their inner product (5) is one:

$$\xi_A \eta^A = 1. \tag{11}$$

For the standard spin frame, the spinors $\{o^A, \iota^A\}$

$$o^A = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \iota^A = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{12}$$

are defined. These are not only a linearly independent basis, but also define a spatial reference frame mapping to Hermitian matrices via (see [13]):

$$\begin{array}{lcl} \hat{\mathbf{t}} & \rightarrow & \frac{1}{2}\left(o^A o^{A'} + \iota^A \iota^{A'}\right) = \frac{1}{2}(\mathbf{l} + \mathbf{n}) \\ \hat{\mathbf{z}} & \rightarrow & \frac{1}{2}\left(o^A o^{A'} - \iota^A \iota^{A'}\right) = \frac{1}{2}(\mathbf{l} - \mathbf{n}) \\ \hat{\mathbf{x}} & \rightarrow & \frac{1}{2}\left(o^A \iota^{A'} + \iota^A o^{A'}\right) = \frac{1}{2}(\mathbf{m} + \bar{\mathbf{m}}) \\ \hat{\mathbf{y}} & \rightarrow & \frac{j}{2}\left(o^A \iota^{A'} - \iota^A o^{A'}\right) = \frac{j}{2}(\mathbf{m} - \bar{\mathbf{m}}) \end{array}, \tag{13}$$

where $(\mathbf{l}, \mathbf{n}, \mathbf{m}, \bar{\mathbf{m}})$ are null vectors (see [13]) and $(\hat{\mathbf{t}}, \hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}})$ the spatial reference frame. We have to notice that the fixed phase of (11) is a key element in fixing the phase of a wave relative to a spatial reference frame.

IV. Spinors and geometry

The aim of this paper is to perform a construction of the Poincaré sphere as a collection of complex lines. Each coherent state of polarization will be one element of the set of complex lines generating the sphere, obtained in form of a spinor from the full electromagnetic field. Such a line can be parameterized by a complex number equivalent to the polarization ratio. The first step is now to see how a spinor can be interpreted as a generating line of a sphere, showing the links between a sphere, a complex line and a spinor. We will attach the physical meaning to the spinor in the next sections to establish that this sphere is the Poincaré sphere.

A suitable starting point is the phase of the wave. The analytical representation of a plane wave is proportional to [36]

$$e^{j(\omega t-\mathbf{k}\cdot\mathbf{x})} \tag{14}$$

with ω the angular frequency and $\mathbf{k} = (k_x, k_y, k_z)$ the wavevector. According to relativity theory, the phase $\varphi = \omega t - \mathbf{k}\cdot\mathbf{x}$ is invariant [36]. This means that in two different reference frames (two observers in uniform relative motion) the plane wave would have different frequency ω and wavevector $\mathbf{k}$ but the phase φ would be the same. As a consequence, the invariance of the phase corresponds to the invariance of a sort of scalar product between two vectors with four components $(\frac{\omega}{c}, -k_x, -k_y, -k_z)$ and (ct, x, y, z):

$$\varphi = \left(\frac{\omega}{c}, -k_x, -k_y, -k_z\right) \cdot (ct, x, y, z), \tag{15}$$

where c is the speed of light in a vacuum. Because of this invariance, the frequency and the wavevector of any plane wave must form a $4-$vector. The tensor description elegantly expresses the invariance property and highlights the different behavior of the two $4-$vectors in case of change of reference frame. In fact, the $4-$vector $(ct, x, y, z) = x^a$ is a true vector since its dimension is a length, instead the wavevector is a $4-$gradient of a scalar invariant

$$\left(\frac{\omega}{c}, -k_x, -k_y, -k_z\right) = \left(\frac{1}{c}\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\varphi = \partial_a \varphi \tag{16}$$

which has the dimension of the inverse of a length. The $4-$vector x^a transforms contravariantly[2] with respect to the $4-$vector $k_a = (\frac{\omega}{c}, -k_x, -k_y, -k_z)$. The simplest demonstration of this is that if we change our unit of length from meters to centimeters, the numerical values of x^a would scale up while those of k_a would scale down. The tensor notation automatically takes account of this and tensors[3] like x^a are called *contravariant* while tensors like k_a are called *covariant*. The tensorial notation of the product (15) is:

$$\varphi = \sum_{a=0}^{3} k_a x^a \equiv k_a x^a, \tag{17}$$

where we have dropped summation sign adopting the Einstein summation convention where *upper* indices (x^a) are paired to *lower* (k_a)[4].

We shall try now to obtain representations for the complex lines generating the sphere. In order to do so, we consider now the contravariant $4-$vector $x^a = (\tau = ct, x, y, z)$. It is isomorphic[5] with a Hermitian matrix[6] which may be parameterized in the following form, via the mapping (13):

$$\begin{pmatrix} \tau+z & x+jy \\ x-jy & \tau-z \end{pmatrix} \qquad \tau, x, y, z \in \mathbb{R}. \tag{18}$$

[2]which means that the inverse transformation has to be used for k_a.

[3]A $4-$vector is a type of tensor.

[4]Inversely, if the distinction between contravariant and covariant were not made, scalar products would only be invariant under isometric transformation such as rotations or reflections, and not under unitary transformation in general.

[5]An isomorphism (from Greek: isos 'equal', and morphe 'shape') is a bijective map between two sets of elements.

[6]A matrix M is Hermitian if $M = M^\dagger$ where $M^\dagger$ denotes the conjugate transpose.

The condition that the matrix be singular may be expressed as

$$\tau^2 - x^2 - y^2 - z^2 = 0. \tag{19}$$

In special relativity this condition is satisfied by points lying on a light-cone through the origin [37]. An alternative interpretation is for (τ, x, y, z) to be considered as homogeneous projective coordinates [30]. For homogeneous coordinates scaling is unimportant,

$$(\tau, x, y, z) \equiv (\alpha\tau, \alpha x, \alpha y, \alpha z), \tag{20}$$

consequently in the three dimensional projective space $\mathbb{P}^3$, the equation (19) defines projectively a sphere (or more generally, a quadric surface):

$$1 - \left(\frac{x}{\tau}\right)^2 - \left(\frac{y}{\tau}\right)^2 - \left(\frac{z}{\tau}\right)^2 = 0. \tag{21}$$

This is reminiscent of the reduction of polarization states of arbitrary amplitude to a unit Poincaré sphere. We can notice that equation (19) is satisfied even if (τ, x, y, z) can be complex. That is, complex points lie on the sphere (19). On the other hand, the nullity of the determinant of the matrix (18) allows us to express it as a Kronecker product

$$\begin{pmatrix} \tau+z & x+jy \\ x-jy & \tau-z \end{pmatrix} = \tag{22}$$
$$= \begin{pmatrix} \xi^0 \\ \xi^1 \end{pmatrix} \otimes \begin{pmatrix} \bar{\eta}^{0'} & \bar{\eta}^{1'} \end{pmatrix} = \begin{pmatrix} \xi^0\bar{\eta}^{0'} & \xi^0\bar{\eta}^{1'} \\ \xi^1\bar{\eta}^{0'} & \xi^1\bar{\eta}^{1'} \end{pmatrix} \quad \tau, x, y, z \in \mathbb{C}.$$

where $\otimes$ denotes the Kronecker product. If the coordinates are real, the matrix is Hermitian and $\eta^A = \xi^A$. For general spinors ξ^A, η^A, the matrix is still singular but not Hermitian, so coordinates are complex. In spinor form (22) expresses the fact a singular $2-$indices spinor $X^{AB'}$ can be written as the outer product of two spinors:

$$X^{AB'} = \xi^A \bar{\eta}^{B'}. \tag{23}$$

Now if η^A be fixed but ξ^A variable, it can be seen that points (τ, x, y, z) form a linear one dimensional, complex projective subspace. The singularity of the matrix (22) implies the existence of a sphere in the projective space $\mathbb{P}^3$. Projectively, the spinor $\xi^A = (\xi^0, \xi^1)$ is a complex point and the collection of all the complex points, together with the infinity $(0, 1)$ is the complex projective unidimensional space $\mathbb{CP}^1$ [30] which geometrically is a complex line. The elements of $\mathbb{CP}^1$ are *spinors* and represented by a complex line. $\mathbb{CP}^1$ is a subspace of $\mathbb{CP}^3$, the natural complex extension of $\mathbb{P}^3$. The point P on the line is represented by the projective parameter $\zeta = \frac{\xi^1}{\xi^0}$ once two reference points R and Q have been chosen (see Fig. 2):

$$P = \xi^0 R + \xi^1 Q, \quad \Rightarrow \quad P = R + \zeta Q. \tag{24}$$

At this point we have a complex projective line and a

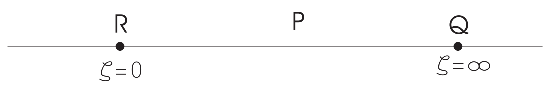


Fig. 2. A 'real' visualization of the complex line. $P = R + \zeta Q$ is any point on the line. For $\xi^1 = 0$ $(\zeta = 0)$, the point P coincides with R, and for $\xi^0 = 0$ $(\zeta = \infty)$ $P \equiv Q$.

projective sphere linked by the relation (22). The geometric significance of this link is an example in the theory of projective geometry that through any point on a quadric surface there pass two lines, each of which lies entirely within the surface [38]. Moreover, these two lines are generators of the quadric with the following properties: i) any point of the generating line is a point of the surface, ii) there are two families of generators, and through each point of the surface there pass two generators, one of each family, iii) generators of the same family do not intersect, iv) each line of one family intersects with every other line of the other family. A point, P, of the sphere uniquely determines both generators passing through it. An intuitive explanation of how complex lines can belong to a sphere and generate it, is given in Appendix I. The point of the preceding discursive outline is to emphasize the central principle of geometric polarimetry, which is to identify spinors

$$\xi^A = (\xi^0, \xi^1) \tag{25}$$

with one of the two sets of complex lines generating the sphere. We should stress at this stage that the sphere in question is to be thought of as the sphere of real unit vectors in three-dimensional space. The concept of Poincaré sphere will be derived in the next sections from this main result.

We have started from the phase of a plane wave built up with two 4−vectors, x^a and k_a. But so far we have considered only the contravariant coordinate 4−vector x^a in the physical space. Now since k_a is also a 4−vector we can think to make the same considerations we have done for x^a. However, there is a difference. The 4−vector k_a is covariant. Contravariant and covariant 4−vectors are of dual types. In particular, projectively contravariant 4−vectors, like x^a, are represented by points in the projective space $\mathbb{P}^3$ and covariant 4−vectors, like k_a are represented by planes. Points and planes form dual spaces. For details about points, planes and duality see Appendix II. Reconsidering the phase of the wave (16), the zero phase

$$k_a x^a = 0 \quad \Rightarrow \quad \begin{pmatrix} \frac{\omega}{c} \\ -k_x \\ -k_y \\ -k_z \end{pmatrix}^T \cdot \begin{pmatrix} \tau \\ x \\ y \\ z \end{pmatrix} = 0 \tag{26}$$

in the projective representation states that the plane k_a passes through the point x^a (see Appendix II for details) which belongs to the sphere $1 - \left(\frac{x}{\tau}\right)^2 - \left(\frac{y}{\tau}\right)^2 - \left(\frac{z}{\tau}\right)^2 = 0$. At this point, we can make the same consideration we have done for x^a keeping in mind that $k_a \equiv (\frac{\omega}{c}, -k_x, -k_y, -k_z)$ is a plane. The singularity of the Hermitian matrix, obtained via the mapping (13) in covariant form,

$$K_{AB'} \equiv \begin{pmatrix} K_{00'} & K_{01'} \\ K_{10'} & K_{11'} \end{pmatrix} \equiv \begin{pmatrix} k_0 - k_z & -k_x + jk_y \\ -k_x - jk_y & k_0 + k_z \end{pmatrix}. \tag{27}$$

defines projectively a sphere

$$1 - \left(\frac{k_x}{k_0}\right)^2 - \left(\frac{k_y}{k_0}\right)^2 - \left(\frac{k_z}{k_0}\right)^2 = 0 \tag{28}$$

with $k_0 = \frac{\omega}{c}$. The sphere this time is an envelope of tangent planes k_a. The singularity of the matrix (27) can be expressed as:

$$k_a \Omega^{ab} k_b = 0 \mapsto \begin{pmatrix} k_0 \\ k_1 \\ k_2 \\ k_3 \end{pmatrix}^T \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} k_0 \\ k_1 \\ k_2 \\ k_3 \end{pmatrix} = 0 \tag{29}$$

where $k_1 = -k_x$, $k_2 = -k_y$, $k_3 = -k_z$, so that from here on numerical as well as symbolic index positions are to be explicitly interpreted as contravariant x^a or covariant k_a according to the position. We shall refer to Ω^{ab} as the wave sphere, since (29) expresses the Fourier transform of the wave equation in free space[7]. Now we can also give a projective meaning to $k^a = \Omega^{ab} k_b = (k_0, k^1 = k_x, k^2 = k_y, k^3 = k_z)$ as the point of tangency of the plane k_b on the sphere Ω^{ab}. In the Fourier domain, the interpretation of the 4−vectors k_a as homogeneous coordinates in three-dimensional projective space implies normalization of the frequency

$$k_a \rightarrow \left(1, \frac{k_1}{k_0}, \frac{k_2}{k_0}, \frac{k_3}{k_0}\right). \tag{30}$$

As we consider harmonic waves individually, and in many applications a quasi monochromatic assumption is justified this creates few problems. In return, the interpretation of the linear algebra as three-, rather then four-dimensional is beneficial from the point of view of visualization. As for $X^{AB'}$ in (23), the singularity of the matrix (27), allow us to express it as the Kronecker product

$$K_{AB'} = \begin{pmatrix} K_{00'} & K_{01'} \\ K_{10'} & K_{11'} \end{pmatrix} = \begin{pmatrix} \kappa_0 \\ \kappa_1 \end{pmatrix} \otimes \begin{pmatrix} \bar{\kappa}_{0'} & \bar{\kappa}_{1'} \end{pmatrix} \tag{31}$$

which is the outer product, this time, of the two covariant spinors, the wave spinor and its conjugate:

$$\kappa_A = (\kappa_0, \kappa_1) \quad \text{and} \quad \bar{\kappa}_{A'} = (\bar{\kappa}_{0'}, \bar{\kappa}_{1'}). \tag{32}$$

Like the contravariant spinor (25) they also represent complex lines on the sphere. The concept of duality is still valid for spinors, namely covariant and contravariant spinors are still duals of each other[8]. However, in the three dimensional projective space $\mathbb{P}^3$, the dual of a line is a line or lines are self-dual. The projective description of a line is given in Appendix II.

V. MAXWELL'S EQUATIONS AND THE WAVE SPINOR

In the polarimetric literature the state of polarization is customarily described with reference to the electric field vector, although in earlier literature it was the magnetic field vector. Notwithstanding, to divorce one from another in an electromagnetic wave is artificial since one can never propagate without the other. We will start from the electromagnetic field tensor which contains both the components of the electric and magnetic fields. We will extract from it the direction of propagation to obtain the electromagnetic potential. This will be the right quantity to use to derive the polarization state since we will be able to establish from it the form of spinor with one index ψ^A that is usually designated as the coherent

[7]As we will see in the section V-B in the Fourier domain $\partial_a \rightarrow jk_a$.

[8]It means that they transform covariantly to one another (see footnote 2).

polarization state without recourse to the complefixication of the Euclidean electric field $\mathbf{E}$. In this way, the nature of complex unitary transformation for rotation in space will become very clear.

In the next section we derive the spinor form of the electromagnetic field and in the following one we project out from the electromagnetic potential a spinor containing the polarization information.

A. Maxwell's equations and the tensor and spinor forms of fields

Whilst in engineering and applied science, the vector calculus form of Maxwell's equations is prevalent, they are more concisely formulated in tensor form. The electromagnetic field tensor F_{ab} contains the components (E_x, E_y, E_z) of the electric field $\mathbf{E}$ and (B_x, B_y, B_z) of the magnetic induction $\mathbf{B}$ [36]:

$$F_{ab} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -cB_z & cB_y \\ -E_y & cB_z & 0 & -cB_x \\ -E_z & -cB_y & cB_x & 0 \end{pmatrix}. \quad (33)$$

Together with the Hodge dual of F_{ab} [39],

$$^*F_{ab} = \begin{pmatrix} 0 & -cB_x & -cB_y & -cB_z \\ cB_x & 0 & -E_z & E_y \\ cB_y & E_z & 0 & -E_x \\ cB_z & -E_y & E_x & 0 \end{pmatrix}, \quad (34)$$

and a further field tensor G_{ab} for linear media containing the components (D_x, D_y, D_z) of the electric displacement $\mathbf{D}$ and (H_x, H_y, H_z) of the magnetic field $\mathbf{H}$,

$$G_{ab} = \begin{pmatrix} 0 & cD_x & cD_y & cD_z \\ -cD_x & 0 & -H_z & H_y \\ -cD_y & H_z & 0 & -H_x \\ -cD_z & -H_y & H_x & 0 \end{pmatrix}, \quad (35)$$

the Maxwell equations in S.I. units can be written as:

$$\partial_a {}^*F^{ab} = 0 \quad \Rightarrow \quad \begin{cases} \nabla \cdot \mathbf{B} = 0 \\ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \end{cases} \quad (36)$$

$$\partial_a G^{ab} = J^b \quad \Rightarrow \quad \begin{cases} \nabla \cdot \mathbf{D} = \rho \\ \nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \end{cases} \quad (37)$$

where $J^b = (c\rho, J_x, J_y, J_z)$ is the current 4-vector with ρ the charge density and $\mathbf{J}$ the current density and $\partial_a = \frac{\partial}{\partial x^a} = (\frac{1}{c}\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$ as in (16). It is clear that the field F_{ab} is a skew symmetric tensor, namely $F_{ab} = -F_{ba}$, containing six independent real components, the $\mathbf{E}$ and $\mathbf{B}$ components. We can then interpret it as a projective line (see Appendix II), since the condition (111) is satisfied by plane wave radiating fields. In fact, this condition can be expressed as ${}^*F^{ab}F_{ab} = 0$ and it corresponds to $\mathbf{E} \cdot c\mathbf{B} = 0$. Comparing the tensor F^{ab}

$$F^{ab} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -cB_z & cB_y \\ E_y & cB_z & 0 & -cB_x \\ E_z & -cB_y & cB_x & 0 \end{pmatrix}. \quad (38)$$

with (108) and (109) we can write the Pluecker coordinates of the line (110) (see Appendix II):

$$(E_x, E_y, E_z, -cB_x, -cB_y, -cB_z) \quad (39)$$

which are the $\mathbf{E}$ and $\mathbf{B}$ cartesian components. In spinor form, a real electromagnetic field may be represented by a mixed spinor [35],

$$F_{ABA'B'} = \varphi_{AB}\, \epsilon_{A'B'} + \epsilon_{AB}\, \bar{\varphi}_{A'B'}. \quad (40)$$

where ϵ_{AB} is the spinor metric defined in (2) and φ_{AB} is called the electromagnetic spinor. Because $\epsilon_{A'B'}$ and ϵ_{AB} are constant spinors and $\bar{\varphi}_{A'B'}$ is the conjugate of φ_{AB}, this explains why this φ_{AB} as symmetric spinor encodes all the information. Since $\varphi_{AB} = \varphi_{BA}$ is symmetric, it has three independent complex components which are related to [35]

$$\mathbf{E} - jc\mathbf{B} \quad (41)$$

for real field-vectors $\mathbf{E}$, $\mathbf{B}$. In matrix form, φ_{AB}, a spinor with two indices like ϵ_{AB} is a two-by-two matrix.

The information of the field is bundled in a two-index entity, the electromagnetic spinor φ_{AB}. From the quantum physical standpoint, the electromagnetic field is carried by photons, particles of spin equal to one. If we want to represent a quantum field with a spinor, a simple rule for the number of spinor indices is that the number of indices of like type is twice the quantum spin. So φ_{AB} automatically represents a spin 1 boson such as a photon. It is also the case that a spinor with primed indices $\bar{\varphi}_{A'B'}$ represents the antiparticle field, and in case of a photon the opposite helicity.

Our aim is to derive the polarization state in the form of spinor with one index ψ^A that is usually designated as the coherent polarization state. However, a spinor with one index, like ψ^A should represent quantum fields for fermions with spin equal to $\frac{1}{2}$, e.g. the massless neutrino. Therefore from the physicist's point of view it may appear manifestly incorrect to represent an electromagnetic field by a spinor with one index[9]. The solution to this problem, which explains why the polarimetric notation of transverse wave states and Stokes vectors has not to our knowledge been formalized in spinor form, is that there is missing structure. Absence of this missing structure from the representation means that polarimetric tensor and spinor expressions do not appear to transform correctly geometrically. From the polarimetrist's point of view, absence of the missing structure leaves room for ambiguity in the way Jones vectors should be transformed when the direction of propagation is variable, something already highlighted in the work of Ludwig [41]. In order to obtain the polarization state in a one index spinor and see where the missing structure is hidden we need to consider the electromagnetic potential as the primary object rather than the field tensor.

B. The electromagnetic potential and the wave spinor

The expression (40) is not suitable to derive a one-index spinor to represent a harmonic polarization state since the polarization information is equally contained in φ_{AB} and

[9]except in a two dimensional representation [40].

$\bar{\varphi}_{A'B'}$. For this representation, a more convenient spinor to use, which carries all the required information can be derived from the electromagnetic potential[10]. It is a $4-$vector whose components are the electrostatic potential ϕ and the magnetic vector potential $\mathbf{A}$:

$$\Phi^a \equiv (\phi, \mathbf{A}). \quad (42)$$

The electromagnetic tensor F_{ab} can be expressed in terms of Φ_a as [36], [42]:

$$F_{ab} = \partial_b \Phi_a - \partial_a \Phi_b \quad \Rightarrow \quad \begin{cases} \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} - \nabla\phi \\ \mathbf{B} = \nabla \times \mathbf{A} \end{cases} \quad (43)$$

where $\Phi_a \equiv (\phi, -\mathbf{A})$. If we restrict attention to the Fourier domain then the derivative becomes simply an algebraic operation. In fact,

$$\partial_a \;\rightarrow\; -jk_a \quad (44)$$

and the (43) becomes the skew symmetrized outer product,

$$F_{ab} = j(k_a \Phi_b - k_b \Phi_a). \quad (45)$$

Note that, from now on, since we are working in the Fourier domain, the harmonic analytical signal representation is implicit. The fields and the derived objects become complex.

At this point we can notice that, given the potential, in order to derive the field, F_{ab}, it is necessary to assume the wave vector, k_a. However, in polarimetry, this is the goal: to strip off assumed or known quantities: frequency, direction of propagation, even amplitude and phase, to arrive at the *polarization state*. Now, in spinor form, the vector potential Φ_a is a $4-$vector isomorphic to an Hermitian matrix, $\Phi_{AB'}$ like any covariant $4-$vector (27)

$$\Phi_{AB'} \equiv \begin{pmatrix} \phi - A_z & -A_x + jA_y \\ -A_x - jA_y & \phi + A_z \end{pmatrix}. \quad (46)$$

Moreover, as is well known, the vector potential has gauge freedom, so is not uniquely specified for a given field. In fact, the field F_{ab}[11] is not altered if the potential Φ_a is changed subtracting the $4-$gradient of some arbitrary function χ:

$$\tilde{\Phi}_a = \Phi_a - \partial_a \chi \quad \Rightarrow \quad \begin{cases} \tilde{\phi} = \phi - \frac{\partial \chi}{\partial t} \\ \tilde{\mathbf{A}} = \mathbf{A} + \nabla\chi \end{cases}. \quad (47)$$

In relativity texts, a particular gauge, namely the Lorenz gauge, is typically singled out, as it is a unique choice that is invariant under Lorentz transformations. However, if this generality is not required other choices are possible:

- the radiation gauge, also known as the transverse gauge, has the consequence that

$$k^a F_{ab} = 0, \quad (48)$$

- the Coulomb gauge can be expressed as

$$\omega_a \Phi^a = 0, \quad (49)$$

  with $\omega_a \equiv (1, 0, 0, 0)$, and implies $\phi = 0$.

These two choices are not mutually exclusive and can be simultaneously satisfied for a radiating plane wave. Then we have for a wave propagating in the $z-$direction that the radiation gauge condition implies that $A_z = 0$ and together with the Coulomb gauge we obtain

$$\Phi_a \;\equiv\; (0, -A_x, -A_y, 0) \;\rightarrow\; \Phi_{AB'} = \begin{pmatrix} 0 & \Phi_{01'} \\ \Phi_{10'} & 0 \end{pmatrix} = \\ = \begin{pmatrix} 0 & -A_x + jA_y \\ -A_x - jA_y & 0 \end{pmatrix}. \quad (50)$$

This contains two generally non-vanishing components that transform conjugately with respect to one another, and can be identified with components of opposite helicity (the two circular polarization components), since the $\mathbf{E}$ vector is algebraically proportional to the vector potential in the Fourier domain (43). Formally (50) is the complex Jones vector, in spinor form. It is because (50) anticommutes with rotations about the wavevector that such rotations can be performed as one-sided full angle rotations in SU(2). In order to obtain the traditional 'Jones vector' representation as the one-index spinor ψ_A, it is necessary to find a form of projection of $\Phi_{AB'}$ onto a spinor with one index. The polarization information contained in (50) can be amalgamated into a single spinor simply by contracting with a *constant* spinor $\bar{\theta}^{B'}$:

$$\psi_A = jk_0\, \Phi_{AB'}\bar{\theta}^{B'} = jk_0 \begin{pmatrix} 0 & \Phi_{01'} \\ \Phi_{10'} & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} jk_0\, \Phi_{01'} \\ jk_0\, \Phi_{10'} \end{pmatrix}, \quad (51)$$

with $k_0 = \frac{\omega}{c}$. This achieves the stated goal and $jk_0\, \Phi_{01'}$ and $jk_0\, \Phi_{10'}$ are the circular polarization components[12]. It has been necessary to introduce extra structure, namely the contraction with $\bar{\theta}^{B'}$ to obtain the one-index representation ψ_A. This explains the apparent physical inappropriateness of the one-index spinor representation. While the extra structure may for obvious reasons be unattractive to the theoretical physicists, it is by contrast of value to the practical polarimetrists because it can be inserted when needed to resolve questions relating to the geometry of scattering. The choice of the fixed spinor

$$\bar{\theta}^{B'} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (52)$$

may at first appear an arbitrary one. The component amplitudes must be equal to give equal weighting to each handedness of polarization. Introduction of a differential phase between components would be equivalent to a rotation of the spin reference frame. To avoid introduction of an unnecessary extra arbitrary parameter, we fix a convention that

$$\bar{\theta}^{B'} = \bar{\kappa}^{B'} + \bar{\lambda}^{B'}, \quad (53)$$

where (κ^A, λ^A) is the *spin frame* for the wave. In standard coordinates,

$$\kappa^A = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \lambda^A = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (54)$$

and the wave travels in the $z-$direction, while its circular polarization components are referenced to the $xz-$plane. The spinor $\bar{\theta}^{B'}$ fixes the phases of both components of the resulting

[10]The vector potential is often used in antenna theory, where the potential in the far field can be simply related to each current element in the source.

[11]and consequently $\mathbf{E}$ and $\mathbf{B}$.

[12]In fact in the Fourier representation, the $\mathbf{E}$ vector is algebraically proportional to the vector potential $\mathbf{A}$. Using (43) and (44) we obtain $\mathbf{E} = -jk_0\mathbf{A}$. Then, $jk_0\, \Phi_{01'} = E_x - jE_y$ and $jk_0\, \Phi_{10'} = E_x + jE_y$ are the left and right circular polarization components.

wave spinor ψ_A. For this reason, we name it the *phase flag*. To conclude this section, we have been able to define the spinor containing the polarization information from the electromagnetic field using a constant spinor we named phase flag. This definition is valid in any reference frame, namely it is valid for any direction of propagation considered. Table I summaries the main steps to follow to extract the polarization information from the electromagnetic tensor.

VI. Spinors, phase and the Poincaré sphere

So far we have deliberately avoided more than scant reference to the Poincaré sphere. The reason for this is that by integrating the material from the previous section with the projective interpretation of Sections III and IV we are now able to arrive at the first remarkable consequence of this work, namely that the Poincaré sphere may be *identified* with the wave sphere by a direct geometrical construction. In order to see this, we have simply to interpret geometrically the algebraic relations of the previous section. The vector potential in its covariant form Φ_a may be represented projectively as a plane as we have seen in the section IV. If no gauge condition is specified, then Φ_a would represent a general plane in projective space. However, the radiation gauge condition $k^a F_{ab} = 0$ (48) implies that

$$k^a \Phi_a = 0 \quad \Rightarrow \quad \frac{\omega}{c}\phi - k_x A_x - k_y A_y - k_z A_z = 0 \quad (55)$$

namely that Φ_a is any plane that passes through the point of tangency of the wave plane k_a with the wave sphere Ω^{ab} as shown in Fig. 3 (see (26) and following). The Coulomb gauge condition $\omega_a \Phi^a = 0$ (49), however, imposes the condition that Φ_a passes through the center of the wave sphere $\omega_a \equiv (1, 0, 0, 0)$. Taken together, these conditions imply that the plane Φ_a passes through the axis of the sphere normal to the wave plane k_a (Fig. 4). The appropriateness of the projective

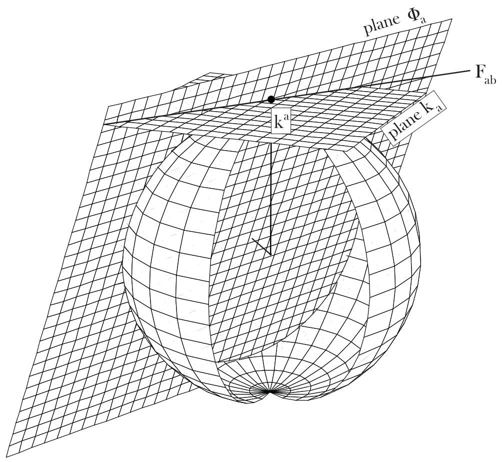


Fig. 3. Relation between wave vector, electromagnetic field and the potential for the radiation gauge. In this gauge, the plane Φ_a is any plane intersecting the point k^a on the wave sphere Ω^{ab}, hinging around the line F_{ab} which is the intersection between the plane Φ_a and the wave plane k_a.

interpretation is seen by noting that the significance of the line intersection of the planes Φ_a and k_a is that it represents projectively the electromagnetic field tensor F_{ab}.

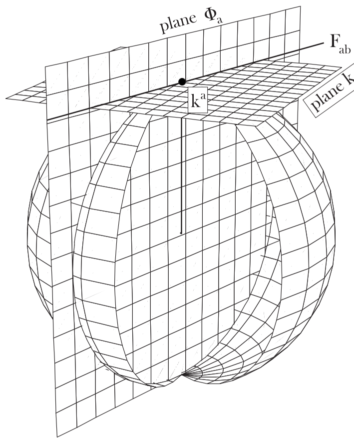


Fig. 4. Relation between wave vector, electromagnetic field and the potential for the Coulomb gauge. In this gauge, Φ_a is the plane through the point k^a and the center of the wave sphere Ω^{ab}. The electromagnetic field is the same line of Fig. 3, the intersection between the plane Φ_a and the wave plane k_a.

Let us suppose that the field tensor F_{ab} is given. Algebraically, using homogeneous coordinates, F_{ab} (45) in the Fourier domain is proportional to

$$F_{ab} \propto k_a \Phi_b - k_b \Phi_a \quad (56)$$

and it expresses the dual homogeneous coordinates of a line (115)-(117), called Pluecker coordinates[13] in the projective space $\mathbb{P}^3$ [30].

Now, given the line F_{ab} and the wavevector k_a we have that Φ_a is a general plane in $\mathbb{P}^3$. Imposing the radiation gauge condition (48) we have that the plane Φ_a passes through the point k^a but it has still the freedom to pivot about the line F_{ab} (see Fig. 3). The unique representation is then obtained by choosing the one element of the family of planes Φ_a that passes through the center of the sphere, namely the plane which satisfy the Coulomb condition (49). Very nicely the geometry neatly illustrates the gauge freedom.

We now come to our main objective: to express the one index spinor ψ_A geometrically. In the geometric interpretation, the totality of wave states with wave vector k^a are obtained by the lines F_{ab} that pass through the k^a axis in the plane k_a. Note that we may admit complex lines as the geometry is generally valid over complex coordinates. For every line F_{ab}, the plane Φ_a is thus uniquely determined given the gauge fixing. Now, we consider the spinor equivalent of the plane Φ_a which is $\Phi_{AB'}$ as in (46) still geometrically represented as a plane. As seen in the section IV, spinors can be interpreted as complex lines generating the wave sphere (see equations (27)-(31)). We suppose that one of the two generators of the wave sphere $\bar{\theta}^{B'}$ is chosen as a fixed reference generator. Then provided $\bar{\theta}^{B'}$ does not lie in the plane $\Phi_{AB'}$ (which is guaranteed by the gauge choice), there is a unique point of the generator $\bar{\theta}^{B'}$

[13]In general F_{ab} is a bivector [43] because of its antisymmetry property. For a plane wave radiating field the bivector becomes simple and it is representable as the line (56). The condition for the bivector F_{ab} to be simple is $\tilde{F}_{ab}F^{ab} = 0$ which is equivalent to $\mathbf{E} \cdot c\mathbf{B} = 0$ and to (111). The condition for the bivector to be null is $F_{ab}F^{ab} = 0$ which is equivalent to $|\mathbf{E}|^2 = |c\mathbf{B}|^2$. This condition states that the bivector intersects the sphere Ω^{ab}. Both conditions are met by a plane wave radiating field.

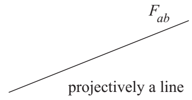
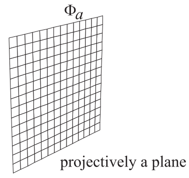
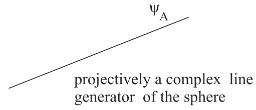
	The electromagnetic field	
$F_{ab} = j(k_a\Phi_b - k_b\Phi_a)$	$F_{ab} = \begin{pmatrix} 0 & E_x & E_y & 0 \\ -E_x & 0 & 0 & cB_y \\ -E_y & 0 & 0 & -cB_x \\ 0 & -cB_y & cB_x & 0 \end{pmatrix}$	
$\downarrow$ dropping the wave vector $k_a = (1,0,0,-1)$ information		
	The electromagnetic potential	
$\Phi_a = \frac{j}{k_0} F_{ab} t^b$	$\Phi_a = \frac{j}{k_0}(0, -E_x, -E_y, 0)$ $\Phi_{AB'} = \frac{j}{k_0} \begin{pmatrix} 0 & -E_x + jE_y \\ -E_x - jE_y & 0 \end{pmatrix}$	
$\downarrow$ dropping the reference $\bar{\theta}^{B'} = \binom{1}{1}$ information		
	The polarization spinor	
$\psi_A = jk_0 \Phi_{AB'} \bar{\theta}^{B'}$	$\psi_A = \begin{pmatrix} E_x - jE_y \\ E_x + jE_y \end{pmatrix}$	

TABLE I
FROM THE ELECTROMAGNETIC FIELD TO THE POLARIZATION SPINOR.

that intersects $\Phi_{AB'}$ for any wave state and thus we can write

$$\psi_A = \Phi_{AB'}\bar{\theta}^{B'}. \tag{57}$$

The spinor ψ_A represents the generator of the complementary family to $\bar{\theta}^{B'}$ of the sphere that intersect $\Phi_{AB'}$ in the same point as the $\bar{\theta}^{B'}-$generator, since

- a linear relation must exist between these objects, due to the fact that generators of opposite type have a unique intersection. In fact, reconsidering the equation (57) we can rewrite it as

$$\Phi_{AB'}\psi^A\bar{\theta}^{B'} = 0, \tag{58}$$

  having contracted both sides of equation (57) with ψ^A and using $\psi_A\psi^A = 0$. Now, if $\Phi_{AB'}$ represents a plane, then $\psi^A\bar{\theta}^{B'}$ represents a point p^a. This point is in general complex since the matrix obtained $P^{AB'} = \psi^A\bar{\theta}^{B'}$ is only singular but not necessarily Hermitian as in (23) and it is the intersection of the two generators (two lines) of the wave sphere of opposite families, ψ^A and $\bar{\theta}^{B'}$. Moreover, this point lies on the wave sphere since the matrix $P^{AB'}$ is singular. Finally, the relation (58) tells us that that this null point belongs to the plane $\Phi_{AB'}$.
- (57) is the only such relation that transforms covariantly and homogeneously.

Now, the reason we claim that we can identify the wave sphere with the Poincaré sphere is found by considering the effects of pure rotations about the k^a axis for any wave state: it is seen that keeping $\bar{\theta}^{B'}$ fixed when the plane Φ_a rotates by 180°, the same point of intersection arises. This explains the double rotation phenomenon of Stokes vectors, when rotation takes place around the wave vector. In fact we can construct the Stokes vector noting that only one point of any generator is real. It is clear that

$$\psi_A\bar{\psi}_{A'} = \Psi_{AA'} = \begin{pmatrix} \psi_0\bar{\psi}_{0'} & \psi_0\bar{\psi}_{1'} \\ \psi_1\bar{\psi}_{0'} & \psi_1\bar{\psi}_{1'} \end{pmatrix} \rightarrow S_a \tag{59}$$

is a singular and Hermitian matrix representing a coherency matrix for a single wave-state. It is isomorphic to a real vector S_a, the Stokes vector[14] via the mapping (13)

$$\begin{aligned} S &= (S_0, S_1, S_2, S_3) = \\ &= (|\psi_0|^2 + |\psi_1|^2, 2\,\mathrm{Re}[\psi_0\bar{\psi}_{1'}], -2\,\mathrm{Im}[\psi_0\bar{\psi}_{1'}], |\psi_0|^2 - |\psi_1|^2) \end{aligned} \tag{60}$$

whose discriminating condition

$$S_0^2 - S_1^2 - S_2^2 - S_3^2 = 0 \tag{61}$$

recognizably represents a purely polarized state. Geometrically the vanishing of the determinant of $\Psi_{AA'}$ expresses the condition that a plane should contain two conjugate generators (ψ_A and $\bar{\psi}_{A'}$) of the polarization sphere, so that the covariant Stokes vector S_a represents a real tangent plane of the Poincaré sphere which now is identified with the wave sphere. In fact the condition (61) can be expressed as

$$S_a\Omega^{ab}S_b = 0 \tag{62}$$

which is the equation of a sphere in homogeneous coordinates (as in (29) and (28)).

We finally remark that S is not a true tensor object because, like ψ_A it omits structure (the phase flag essentially) that also has to transform geometrically.

[14]The components of the Stokes vector appear to be different from the usual ones, since we have used throughout this paper circular polarization basis instead of the usual linear basis.

The introduction of the phase flag is the key element to identify the polarization state for a wave vector in any direction with a spinor. Any transformation of geometric orientation does not change the spinor algebra. The spinor expression (57) transforms covariantly so that the relation will remain valid in any rotated frame if all the elements composing the relation are transformed with the appropriate rule. The essential distinction between geometric and polarimetric basis transformations is that in the former case the phase flag $\bar{\theta}^{B'}$ must be transformed for consistency, while in the latter $\bar{\theta}^{B'}$ must be regarded as fixed. Fixing $\bar{\theta}^{B'}$ and varying $\Phi_{AB'}$ we obtain all the polarization states for one direction of propagation.

VII. Illustrative examples

We will now report some numerical examples in order to illustrate how all this really works.

We start first to choose one direction of propagation, which is as usual the $z-$direction. The covariant $4-$vector k_a is then

$$k_a = k_0(1, 0, 0, -1) \tag{63}$$

with $k_0 = \frac{\omega}{c}$. We choose now a linear polarized wave at $60°$ to the $x-$axis propagating along z. The electromagnetic field tensor will be

$$F_{ab} = E\,\mathrm{e}^{j(\omega t - \mathbf{k}\cdot\mathbf{x}+\alpha)} \begin{pmatrix} 0 & \cos\frac{\pi}{3} & \sin\frac{\pi}{3} & 0 \\ -\cos\frac{\pi}{3} & 0 & 0 & \cos\frac{\pi}{3} \\ -\sin\frac{\pi}{3} & 0 & 0 & \sin\frac{\pi}{3} \\ 0 & -\cos\frac{\pi}{3} & -\sin\frac{\pi}{3} & 0 \end{pmatrix} \cdot \tag{64}$$

where E is the amplitude of the wave, $\omega t - \mathbf{k}\cdot\mathbf{x} = \varphi$ is the phase (15) and α is an initial phase.

In order to isolate Φ_a from F_{ab}, we use the (45) contracting F_{ab} with $t^b = (1, 0, 0, 0)$. We obtain

$$F_{ab}t^b = j(k_a\Phi_b - k_b\Phi_a)t^b \quad \Rightarrow \quad F_{a0} = -j\,k_0\,\Phi_a \tag{65}$$

and Φ_a has components

$$\Phi_a = \frac{j}{k_0}\,E\,\mathrm{e}^{j(\varphi+\alpha)}\left(0, -\cos\frac{\pi}{3}, -\sin\frac{\pi}{3}, 0\right). \tag{66}$$

Using (46), the corresponding $\Phi_{AB'}$ is

$$\Phi_{AB'} = \frac{j}{k_0}\,E\,\mathrm{e}^{j(\varphi+\alpha)}\begin{pmatrix} 0 & -\cos\frac{\pi}{3} + j\sin\frac{\pi}{3} \\ -\cos\frac{\pi}{3} - j\sin\frac{\pi}{3} & 0 \end{pmatrix}. \tag{67}$$

Now after selecting the spin frame $\{o^A, \iota^A\}$ (12), the fixed generator $\bar{\theta}^{B'}$ according to (53) is

$$\bar{\theta}^{B'} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}. \tag{68}$$

We are now ready to compute the polarization spinor ψ_A with the (51):

$$\begin{aligned}\psi_A &= jk_0\,\Phi_{AB'}\bar{\theta}^{B'} = jk_0\,(\Phi_{A0'}\bar{\theta}^{0'} + \Phi_{A1'}\bar{\theta}^{1'}) = \\ &= E\,\mathrm{e}^{j(\varphi+\alpha)}\frac{1}{\sqrt{2}}\begin{pmatrix}\mathrm{e}^{-j\frac{\pi}{3}}\\ \mathrm{e}^{+j\frac{\pi}{3}}\end{pmatrix},\end{aligned} \tag{69}$$

which represents the Jones vector in circular polarization basis for a $60°$ linear polarization. The corresponding polarization ratio is:

$$\mu = \mathrm{e}^{j\frac{2\pi}{3}}. \tag{70}$$

We can calculate the corresponding Stokes vector using (60) and we obtain:

$$S_a = E^2\left(1, \cos\frac{2\pi}{3}, \sin\frac{2\pi}{3}, 0\right). \tag{71}$$

which is the Stokes vector for a linear polarization with orientation angle $\psi = 60°$. Now we can do the same calculation using the tensors instead of the spinors and for brevity we omit the multiplication constants. We have the plane Φ_a. We calculate the Pluecker coordinates θ^{ab} of the line corresponding to the generator $\bar{\theta}^{B'}$ and then the intersection of the plane potential with the line generator, in order to find the polarization state, the generator of the other type. We can use (31) where $\bar{\lambda}_{B'} = \begin{pmatrix}1\\ \lambda\end{pmatrix}$ and $\kappa_A = \begin{pmatrix}1\\ \mu\end{pmatrix}$. We obtain:

$$\kappa_A\bar{\lambda}_{B'} = \begin{pmatrix}1 & \lambda \\ \mu & \mu\lambda\end{pmatrix} = \begin{pmatrix}\tau - z & -x + jy \\ -x - jy & \tau + z\end{pmatrix}. \tag{72}$$

Since we want to compute the generator through $\bar{\theta}^{B'}$ then using (68) $\lambda = -1$. Using the mapping (13), the corresponding one index tensor is:

$$t^a = (1 - \mu, 1 - \mu, j + j\mu, -1 - \mu). \tag{73}$$

In order to compute the projective Pluecker coordinates of this line, we consider two points on this line for example $p^a = (1, 1, 0, 0)$ and $q^a = (0, 0, j, -1)$, and we use the relations (108)-(110):

$$\theta^{\,ab} = \begin{pmatrix} 0 & 0 & j & -1 \\ 0 & 0 & j & -1 \\ -j & -j & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}, \tag{74}$$

$$(0, -j, 1, 0, 1, j). \tag{75}$$

In order to find the generator of the other type, the polarization state, we compute the intersection of the line $\theta^{\,ab}$ with the plane Φ_b:

$$\psi^a = \theta^{\,ab}\Phi_b = \left(\sin\frac{\pi}{3}, \sin\frac{\pi}{3}, -\cos\frac{\pi}{3}, -j\cos\frac{\pi}{3}\right). \tag{76}$$

The spinor corresponding to this point will be computed using (72):

$$\mu = \frac{\tau + z}{-x + jy} = \mathrm{e}^{2j\frac{\pi}{3}}, \tag{77}$$

with of course $\tau = \sin\frac{\pi}{3}$, $x = \sin\frac{\pi}{3}$, $y = -\cos\frac{\pi}{3}$, $z = -j\cos\frac{\pi}{3}$. The spinor can be expressed as:

$$\psi_A \propto \begin{pmatrix}1\\ \mathrm{e}^{j\frac{2\pi}{3}}\end{pmatrix} \tag{78}$$

which is projectively the same as (69).

Now we apply a rotation in space to the bivector F^{ab}, for example a rotation of $45°$ about the $y-$axis. The rotation matrix turns out to be:

$$R^a{}_b = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\frac{\pi}{4} & 0 & -\sin\frac{\pi}{4} \\ 0 & 0 & 1 & 0 \\ 0 & \sin\frac{\pi}{4} & 0 & \cos\frac{\pi}{4} \end{pmatrix}. \tag{79}$$

The new electromagnetic field tensor will be:

$$\tilde{F}^{cd} = R^{c}_{\ a} R^{d}_{\ b} F^{ab} = \tag{80}$$

$$\begin{pmatrix} 0 & -\cos\frac{\pi}{4}\cos\frac{\pi}{3} & -\sin\frac{\pi}{3} & -\sin\frac{\pi}{4}\cos\frac{\pi}{3} \\ \cos\frac{\pi}{4}\cos\frac{\pi}{3} & 0 & \sin\frac{\pi}{4}\sin\frac{\pi}{3} & \cos\frac{\pi}{3} \\ \sin\frac{\pi}{3} & -\sin\frac{\pi}{4}\sin\frac{\pi}{3} & 0 & \cos\frac{\pi}{4}\sin\frac{\pi}{3} \\ \sin\frac{\pi}{4}\cos\frac{\pi}{3} & -\cos\frac{\pi}{3} & -\cos\frac{\pi}{4}\sin\frac{\pi}{3} & 0 \end{pmatrix}.$$

We can notice that the equivalent matrix calculation to (80) would be:

$$\tilde{F} = RFR^{T} \tag{81}$$

where F, $\tilde{F}$ and R indicate the corresponding matrices and R^{T} is the transpose of the rotation matrix R.

The new wave vector will be:

$$\tilde{k}^{d} = R^{d}_{\ a} k^{a} = \left(1, -\sin\frac{\pi}{4}, 0, \cos\frac{\pi}{4}\right) \tag{82}$$

and

$$\tilde{k}_{a} = \left(1, \sin\frac{\pi}{4}, 0, -\cos\frac{\pi}{4}\right). \tag{83}$$

The new potential can be computed like in (65) and we obtain:

$$\tilde{\Phi}_{a} = \left(0, -j\cos\frac{\pi}{4}\cos\frac{\pi}{3}, -j\sin\frac{\pi}{3}, -j\sin\frac{\pi}{4}\cos\frac{\pi}{3}\right). \tag{84}$$

We can easily verify that the potential is still a plane through the origin ω^{a} and through the point $\tilde{k}^{a}$:

$$\tilde{\Phi}_{a}\omega^{a} = 0, \quad \tilde{\Phi}_{a}\tilde{k}^{a} = 0 \tag{85}$$

In this calculation we are changing the orientation and calculating the corresponding new polarization state. In order to compute the new polarization state we have to rotate the generator θ^{ab} as well, to obtain:

$$\tilde{\theta}^{cd} = R^{c}_{\ a} R^{d}_{\ b} \theta^{ab} = \tag{86}$$

$$= \begin{pmatrix} 0 & \sin\frac{\pi}{4} & j & -\cos\frac{\pi}{4} \\ -\sin\frac{\pi}{4} & 0 & j\cos\frac{\pi}{4} & -1 \\ -j & -j\cos\frac{\pi}{4} & 0 & -j\sin\frac{\pi}{4} \\ \cos\frac{\pi}{4} & 1 & j\sin\frac{\pi}{4} & 0 \end{pmatrix}.$$

In order to find the polarization state we need to find the point where the line $\tilde{\theta}^{cd}$ intersects the plane $\tilde{\Phi}_{a}$ as before[15].

$$\tilde{\psi}^{a} = \tilde{\theta}^{ab}\tilde{\Phi}_{b} = \left(\sin\frac{\pi}{3}, j\cos\frac{\pi}{3}\sin\frac{\pi}{4} + \sin\frac{\pi}{3}\cos\frac{\pi}{4}, -\cos\frac{\pi}{3}, -j\cos\frac{\pi}{3}\cos\frac{\pi}{4} + \sin\frac{\pi}{3}\sin\frac{\pi}{4}\right). \tag{87}$$

Using the definition (72) we can calculate the polarization ratios of the two generators:

$$\mu = \frac{\tau + z}{-x + jy} = \frac{e^{j\frac{2\pi}{3}}\cos\frac{\pi}{8} - \sin\frac{\pi}{8}}{\cos\frac{\pi}{8} + e^{j\frac{2\pi}{3}}\sin\frac{\pi}{8}}, \tag{88}$$

$$\lambda = \frac{\tau + z}{-x - jy} = \frac{\cos\frac{\pi}{8} + \sin\frac{\pi}{8}}{-\cos\frac{\pi}{8} + \sin\frac{\pi}{8}}. \tag{89}$$

[15]Of course $\tilde{\psi}^{a} = R^{a}_{\ d}\psi^{d}$ but we have gone through the full calculation again in order to make clear how tensors and spinors work.

Now we perform the same calculation using spinors. We want to calculate the new polarization state for a new orientation $\tilde{k}^{a}$. It will be much faster and simpler. The unitary spinor corresponding to $R^{a}_{\ b}$ is $U^{A}_{\ B}$:

$$U^{A}_{\ B} = -\cos\frac{\pi}{8}\,\epsilon^{A}_{\ B} - j\sin\frac{\pi}{8}(\tau^{A}\omega_{B} + \omega^{A}\tau_{B}) = \\ = \begin{pmatrix} \cos\frac{\pi}{8} & \sin\frac{\pi}{8} \\ -\sin\frac{\pi}{8} & \cos\frac{\pi}{8} \end{pmatrix} \tag{90}$$

where

$$\tau^{A} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ j \end{pmatrix} \quad \omega^{A} = \frac{1}{\sqrt{2}}\begin{pmatrix} j \\ 1 \end{pmatrix} \tag{91}$$

are the spinor corresponding to the axis of rotation, the $y-$axis $T_{a} = (1, 0, 1, 0)$ and the orthogonal one $O_{a} = (1, 0, -1, 0)$. $U^{A}_{\ B}$ represents the rotation in space the generates the new orientation $\tilde{k}^{a}$.

The new polarization spinor can be simply computed as

$$\tilde{\psi}^{A} = U^{A}_{\ B}\psi^{B} = \frac{j\,e^{-j\frac{\pi}{3}}}{\sqrt{2}}\begin{pmatrix} -e^{j\frac{2\pi}{3}}\cos\frac{\pi}{8} + \sin\frac{\pi}{8} \\ \cos\frac{\pi}{8} + e^{j\frac{2\pi}{3}}\sin\frac{\pi}{8} \end{pmatrix}. \tag{92}$$

The corresponding covariant form of the spinor is:

$$\tilde{\psi}_{A} = \frac{j\,e^{-j\frac{\pi}{3}}}{\sqrt{2}}\begin{pmatrix} -\cos\frac{\pi}{8} - e^{j\frac{2\pi}{3}}\sin\frac{\pi}{8} \\ -e^{j\frac{2\pi}{3}}\cos\frac{\pi}{8} + \sin\frac{\pi}{8} \end{pmatrix} \tag{93}$$

and the corresponding polarization ratio is exactly (88)

$$\mu = \frac{e^{j\frac{2\pi}{3}}\cos\frac{\pi}{8} - \sin\frac{\pi}{8}}{\cos\frac{\pi}{8} + e^{j\frac{2\pi}{3}}\sin\frac{\pi}{8}}. \tag{94}$$

The new phase flag $\tilde{\bar{\theta}}^{B'}$ can be simply computed as

$$\tilde{\bar{\theta}}^{A'} = U^{A'}_{\ B'}\bar{\theta}^{B'} = \frac{1}{\sqrt{2}}\begin{pmatrix} \cos\frac{\pi}{8} + \sin\frac{\pi}{8} \\ \cos\frac{\pi}{8} - \sin\frac{\pi}{8} \end{pmatrix}. \tag{95}$$

Using the conjugate version of (4), the corresponding covariant phase flag is

$$\tilde{\bar{\theta}}_{B'} = \epsilon_{A'B'}\tilde{\bar{\theta}}^{A'} = \frac{1}{\sqrt{2}}\begin{pmatrix} -\cos\frac{\pi}{8} + \sin\frac{\pi}{8} \\ +\cos\frac{\pi}{8} + \sin\frac{\pi}{8} \end{pmatrix} \tag{96}$$

and the corresponding polarization ratio is (89).

The corresponding Stokes vectors are

$$\tilde{\psi}^{A} \rightarrow \tilde{S}_{a} = \left(1, \cos\frac{2\pi}{3}\cos\frac{\pi}{4}, \sin\frac{2\pi}{3}, \cos\frac{2\pi}{3}\sin\frac{\pi}{4}\right), \tag{97}$$

$$\tilde{\bar{\theta}}^{B'} \rightarrow \Theta_{b} = \left(1, -\cos\frac{\pi}{4}, 0, -\sin\frac{\pi}{4}\right). \tag{98}$$

$\tilde{\psi}^{A}$ is the new polarization state for a new orientation $\tilde{k}^{a}$ which corresponds to a new phase flag $\tilde{\bar{\theta}}^{B'}$.

VIII. Discussion and Conclusions

The representation that has been arrived at allows a spinor to describe a polarization state for a wave vector in *any* direction. In any fixed direction, we extend to Jones vector calculus in any chosen basis by applying unitary transformation to the polarization spinor alone, and not to the phase flag. The natural representation turns out, as might be expected, in terms of a circular polarization basis. If we change the direction of the wave vector we have also to apply the unitary transformation to the phase flag.

Geometrically this representation can be determined since two generators of the sphere ψ_A and $\bar{\theta}^{B'}$ (one of each kind) pass through any point of it, and then through every tangential plane of the wave sphere there pass two generators ψ_A and $\bar{\psi}_{A'}$ which are complex conjugates.

For the coherent field, one represents the polarization state of the propagating wave as

$$\psi_A \propto \mathrm{e}^{jk_a x^a} = \mathrm{e}^{j(\omega t - \mathbf{k}\cdot\mathbf{x})} \tag{99}$$

and the other, the conjugate field as

$$\bar{\psi}_{A'} \propto \mathrm{e}^{-jk_a x^a} = \mathrm{e}^{j(-\omega t + \mathbf{k}\cdot\mathbf{x})}. \tag{100}$$

These fields are conjugate solutions, but *both* propagate in the same direction, as the equations for constant phase surface are identical. The use of strict spinor algebra prevents the often-committed error of associating a conjugated Jones vector with a backward propagating field. In each wave plane, therefore, there is one generator for the unconjugated Fourier component, and it is possible to show that the entire collection of these forms a ruled surface (regulus) on the wave sphere. The generator lying in the plane $(k_0, \mathbf{k})$ corresponds to the wave state of -1 helicity (LHC), while the generator in the plane corresponding to the 'backward' direction $(k_0, -\mathbf{k})$ is that of $+1$ helicity (RHC).

This is the origin of the apparent conjugation/reversal symmetry. The lines on the regulus form a one-dimensional linear space, and in this sense we can say

$$\psi_A \equiv \begin{pmatrix}1\\0\end{pmatrix} + \psi\begin{pmatrix}0\\1\end{pmatrix} \quad \text{or} \quad \psi_A \equiv \psi_0\begin{pmatrix}1\\0\end{pmatrix} + \psi_1\begin{pmatrix}0\\1\end{pmatrix} \tag{101}$$

for a homogeneous projective spinor in which the complex polarization ratio $\psi = \frac{\psi_1}{\psi_0}$ runs from 0 to ∞. As we have shown there is an isomorphism between the generators of the sphere and the set of the spinors ψ_A, and this can be effected in an invariant way with respect to any linear change of basis. Thus we have arrived at a unified polarimetric description in which both the geometry of 'world space' and the abstract mapping of the generators of the sphere are handled consistently using spinors. The geometric interpretation we have introduced explains the fundamental place of the Poincaré sphere in polarimetry as an invariant object under linear transformations with its reguli whose generators are wave spinors, constituting its invariant subspaces. Identifying generators as states of polarization, the structure of the Poincaré sphere is preserved under all linear processes. The sphere is considered an invariant of the theory, the *absolute quadric* [30]. The Poincaré sphere and the wave sphere are hereby unified. The well-known phenomenon of 'double rotation' of Stokes vectors with respect to rotation of world coordinates is down to the fact that $\bar{\theta}^{B'}$ is not rotated when a basis transformation is made while for geometric transformations the phase flag is included. In other words, geometric transformations are two-sided. Confusion as to the nature of the Jones vector as a true vector or unitary spinor can be attributed to the fact that commutation rules allow rotations in the polarization plane to be expressed as one-sided.

Although spinor rotation is largely unfamiliar to polarimetrists, it involves nothing intrinsically complicated. The careful distinctions it makes between conjugate spaces, and covariant and contravariant forms are precisely what is required to make sense of unitary concepts in polarimetry. In the sequel to this paper we propose to address wider questions of antennas and scattering.

Appendix I
The generating lines of a quadric surface

To those unfamiliar with complex geometry it may appear surprising that a sphere *contains* straight lines: this is a fact that escapes us because on a sphere or any quadric with positive curvature only one point on each such line is real; all other points are complex. A simple way to "see" how a complex line can belong to the sphere is to consider planes intersecting the sphere. We consider for simplicity a sequence of planes parallel to the $xy-$plane, cutting the $z-$axis at z_c. The equation in the plane of intersection

$$x^2 + y^2 = 1 - z_c^2 \tag{102}$$

is the equation of a circle which degenerates in a point of tangency ($z_c = \pm 1$). In this case, the point of tangency is not the whole solution. In fact, we have also two intersecting complex lines with gradient $\pm j$

$$x^2 + y^2 = 0 \quad \Rightarrow \quad x = \pm jy \tag{103}$$

having one real point $(1, 0, 0, 1)$ at their intersection. What is surprising is that the two lines lie completely on the sphere surface. In fact, each point on these lines has coordinate $(1, \pm jy, y, 1)$ which belongs to the sphere. This is a simple example to illustrate that a sphere contains complex points and in particular lines built up with complex points! Now, the next step is to see how such lines can generate the sphere surface. Since it is difficult to imagine this we can consider instead a quadric surface with negative curvature[16]. For such quadric surfaces, the generating lines can be wholly real, a fact that is exploited architecturally, e.g. in the design of cooling towers as cylindrical hyperboloids [28]. In Fig. 5 we show how a line can generate an hyperboloid. We can easily see that i) any point of the generating line is a point of the surface, ii) there are two families of generators, and through each point of the surface there pass two generators, one of each family, iii) generators of the same family do not intersect, iv) each line of one family intersects with every other line of the other family.

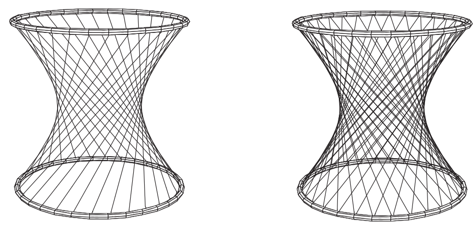


Fig. 5. A quadric surface in the form of a hyperboloid. On the left we can see the hyperboloid generated by one line rotating around one axis. In the picture on the right we show the two family of generators.

[16]In reality, in the complex projective space all the non-degenerate quadrics are indistinguishable from each other [30].

APPENDIX II
DUALITY AND THE PROJECTIVE LINE, POINT AND PLANE

We want to explore the meaning of duality in the projective context. First, we consider the equation of a plane in the three dimensional Euclidean space:

$$f(x,y,z) = ux + vy + wz + q = 0. \tag{104}$$

The normal vector to the plane is given by

$$\mathbf{N} = \nabla f = (u,v,w). \tag{105}$$

The plane may be characterized via its normal vector from the origin, so a set of three coordinates (u,v,w) can refer to a plane rather than a point, giving rise to a 'dual' interpretation. If we consider the homogeneous projective space, we add a fourth initial coordinate and we have that the condition:

$$\begin{pmatrix} \sigma \\ u \\ v \\ w \end{pmatrix}^T \cdot \begin{pmatrix} \tau \\ x \\ y \\ z \end{pmatrix} = \sigma\tau + ux + vy + wz = 0 \tag{106}$$

may be seen as the condition for a plane (σ,u,v,w) to pass through the point (τ,x,y,z). Comparing (106) with (104) we have that $q = \sigma\tau$. Since the set of homogeneous coordinates (σ,u,v,w) are defined as a gradient (105) we associate it with a covariant $4-$vector u_a and we associate the set of coordinates (τ,x,y,z) interpreted as a point with a contravariant $4-$vector x^a. The linear equation of a plane (106) can be written as $u_a x^a = 0$, with $u_a = (\sigma,u,v,w)$. Hence the components of any covariant vector u_a are to be regarded as the coordinates of a plane.

Instead, a line can be built linking two points but also it is the intersection of two planes, duals of points. Given two points $p^a = (p^0,p^1,p^2,p^3)$ and $q^a = (q^0,q^1,q^2,q^3)$ in homogeneous coordinates, the projective description of the line passing through the two points are given by the numbers:

$$p^i q^j - p^j q^i \tag{107}$$

which build the tensor

$$l^{ab} = p^a q^b - q^a p^b. \tag{108}$$

Since $l^{ab} = -l^{ba}$ the tensor is clearly skew-symmetric:

$$l^{ab} = \begin{pmatrix} 0 & -l^{10} & -l^{20} & -l^{30} \\ l^{10} & 0 & l^{12} & -l^{31} \\ l^{20} & -l^{12} & 0 & l^{23} \\ l^{30} & l^{31} & -l^{23} & 0 \end{pmatrix} \tag{109}$$

and therefore the distinct elements are reduced to six

$$\{l^{10}, l^{20}, l^{30}, l^{23}, l^{31}, l^{12}\}. \tag{110}$$

However, they are not independent since they always satisfy

$$l^{10} l^{23} + l^{20} l^{31} + l^{30} l^{12} = 0 \tag{111}$$

which is the determinant of a 4×4 matrix (p^a, q^a, p^a, q^a) that is identically zero. The coordinates (110) connected by the relation (111) are called Pluecker (or Grassmann) coordinates of a line. Again overall scaling is unimportant, namely the set

$$\{\alpha l^{10}, \alpha l^{20}, \alpha l^{30}, \alpha l^{23}, \alpha l^{31}, \alpha l^{12}\} \tag{112}$$

represents the same line as (110) does. If the first coordinate of the points is not zero, it is easy to show that the coordinates have a nice Euclidean interpretation, namely

$$\begin{aligned} (l^{10}, l^{20}, l^{30}) &= \mathbf{p} - \mathbf{q} \\ (l^{23}, l^{31}, l^{12}) &= \mathbf{p} \times \mathbf{q} \end{aligned} \tag{113}$$

with $\mathbf{p} = (\frac{p^1}{p^0}, \frac{p^2}{p^0}, \frac{p^3}{p^0})$, $\mathbf{q} = (\frac{q^1}{q^0}, \frac{q^2}{q^0}, \frac{q^3}{q^0})$ and where $\times$ denotes the cross product. The first set of coordinates describes the direction of the line from $\mathbf{q}$ to $\mathbf{p}$ and the second describes the plane containing the line and the origin. The condition (111) is equivalent to the identically null product

$$(\mathbf{p} - \mathbf{q}) \cdot (\mathbf{p} \times \mathbf{q}) \tag{114}$$

where $\cdot$ denotes the scalar product.

Alternatively, we consider the planes $p_a = \Omega_{ab} p^b$ and $q_a = \Omega_{ab} q^b$. We define the skew-symmetric tensor l_{ab}

$$l_{ab} = p_a q_b - q_a p_b \tag{115}$$

whose components are related to the components of l^{ab} in (108)

$$l_{ab} = \Omega_{ac}\Omega_{bd} l^{cd} = \begin{pmatrix} 0 & l^{10} & l^{20} & l^{30} \\ -l^{10} & 0 & l^{12} & -l^{31} \\ -l^{20} & -l^{12} & 0 & l^{23} \\ -l^{30} & l^{31} & -l^{23} & 0 \end{pmatrix} \tag{116}$$

which is the dual of the tensor ${}^*r_{ab}$ representing the line intersection of the planes p_a and q_a:

$${}^*r_{ab} = l_{ab}. \tag{117}$$

The dual of a tensor is defined through the full antisymmetric Levi Civita symbol ε^{abcd} and it is related to the tensor r^{ab} :

$$r^{ab} = -\frac{1}{2}\varepsilon^{abcd}\,{}^*r_{cd} \tag{118}$$

which components can be written in function of the components of l^{ab}:

$$r^{ab} = \begin{pmatrix} 0 & -l^{23} & -l^{31} & -l^{12} \\ l^{23} & 0 & -l^{30} & l^{20} \\ l^{31} & l^{30} & 0 & -l^{10} \\ l^{12} & -l^{20} & l^{10} & 0 \end{pmatrix}. \tag{119}$$

The Pluecker coordinates of the line intersection of p_a and q_a will be the set:

$$\{l_{23}, l_{31}, l_{12}, -l_{10}, -l_{20}, -l_{30}\}. \tag{120}$$

Again considering the Euclidean interpretation as in (121), the vector $\mathbf{p}$ and $\mathbf{q}$ now represents the normal to the planes. For this reason

$$\begin{aligned} (l^{23}, l^{31}, l^{12}) &= \mathbf{p} \times \mathbf{q} \\ (-l^{10}, -l^{20}, -l^{30}) &= \mathbf{q} - \mathbf{p} \end{aligned} \tag{121}$$

this time the first set of coordinates namely the direction of the line is described by $\mathbf{p} \times \mathbf{q}$ and the second set namely the plane containing the line and the origin is described by $\mathbf{q} - \mathbf{p}$. We emphasize that this is only an Euclidean interpretation that can help us to visualize things but the Pluecker coordinates are coordinates in the projective space and not in the Euclidean space.

ACKNOWLEDGMENT

This work was supported by the Marie Curie Research Training Network (RTN) AMPER (Contract number HPRN-CT-2002-00205). The authors are indebted to Prof. W-M. Boerner and Prof. M. Chandra for their support and fruitful discussions. L.C. is also deeply indebted to Prof. G. Wanielik for his inexaustible trust and encouragement. The authors would also like to acknowledge the diverse and invaluable discussions and debates with the late Dr. Ernst Lüneburg. Finally, the authors would like to thank the anonymous reviewers for their valuable comments and suggestions to improve the quality of the paper.

REFERENCES

[1] "http://www.dlr.de/tsx/start_en.htm."

[2] M. Davidovitz and W.-M. Boerner, "Extension of Kennaugh's optimal polarization concept to the asymmetric scattering," *IEEE Trans. on Antenna and Propagation*, vol. 34, no.4, pp. 569–574, 1986.

[3] S. Cloude, "On the status of bistatic polarimetry theory," in *IEEE International Conference on Geoscience and Remote Sensing, IGARSS 2005*, Seul, Korea, Nov. 2005.

[4] C. Titin-Schnaider, "Power optimization for polarimetric bistatic random mechanisms," *IEEE Trans. on Geoscience and Remote Sensing*, vol. 45, no.11, pp. 3646–3660, 2007.

[5] D. Bebbington, L. Carrea, and G. Wanielik, "Applications of geometric polarization to invariance properties in bistatic scattering," *Advances in Radio Science*, vol. 3, pp. 421–425, 2005.

[6] L. Bombrun, "Extension of the target scattering vector model to the bistatic case," in *IEEE International Conference on Geoscience and Remote Sensing, IGARSS 2010*, Honolulu, Hawaii, July 2010.

[7] A. Kostinski and W.-M. Boerner, "On the foundations of radar polarimetry," *IEEE Trans. on Antenna and Propagation*, vol. 34, pp. 1395–1404, 1986.

[8] H. Mieras, "Comments on 'On the foundations of radar polarimetry', by A.B. Kostinski and W-M. Boerner," *IEEE Trans. on Antenna and Propagation*, vol. 34, pp. 1470–1471, 1986.

[9] A. Kostinski and W.-M. Boerner, "Reply to 'Comments on On the foundations of radar polarimetry' by H. Mieras," *IEEE Trans. on Antenna and Propagation*, vol. 34, pp. 1471–1473, 1986.

[10] J. Hubbert, "A comparison of radar, optics, and specular null polarization theory," *IEEE Trans. on Geoscience and Remote Sensing*, vol. 32, no.3, pp. 658–671, 1994.

[11] E. Lüneburg, "Comments on the 'Specular null polarization theory', by J.C. Hubbert," *IEEE Trans. on Geoscience and Remote Sensing*, vol. 35, no. 4, pp. 1070–1071, 1997.

[12] J. Hubbert, "Reply to 'Comments on The specular null polarization theory', by E. Lüneburg," *IEEE Trans. on Geoscience and Remote Sensing*, vol. 35, no.4, pp. 1071–1072, 1997.

[13] D. Bebbington and L. Carrea, "On mathematical and physical principles of transformations of the coherent radar backscatter matrix," *IEEE Trans. Geoscience and Remote Sensing*, 2012, accepted for publication.

[14] Z. Czyż, "Lüneburg's concepts of time and frame reversal resulting in his voltage equation as a polarimetric form of the Lorentz reciprocity theorem and its further representation using the Poincaré sphere tangential phasors," in *URSI Commission F Symposium*, Ispra, Italy, 2005.

[15] H. Mott, *Polarization in Antennas and Radar*. John Wiley and Sons, 1986.

[16] R. Azzam and N. Bashara, *Ellipsometry and Polarized Light*. Elsevier Science Pub Co, 1987.

[17] M. Galletti, D. Bebbington, M. Chandra, and T. Borner, "Measurement and characterization of entropy and degree of polarization of weather radar targets," *IEEE Trans. on Geoscience and Remote Sensing*, vol. 46, no. 10, pp. 3196–3207, 2008.

[18] E. Lüneburg, "Aspects of radar polarimetry," *Elektrik-Turkish Journal of Electrical Engineering and Computer Science*, vol. 10, no.2, pp. 219–243, 2002.

[19] C. Graves, "Radar polarization power scattering matrix," *Proceedings of I.R.E.*, vol. 44, pp. 248–252, February 1956.

[20] A. Agrawal and W.-M. Boerner, "Redevelopment of Kennaugh's target characteristic polarization state theory using polarization transformation ratio formalism for the coherent case," *IEEE Trans. Geoscience and Remote Sensing*, vol. 27, no.1, pp. 2–14, 1989.

[21] J. Hubbert and V. Bringi, "Specular null polarization theory: Applications to radar meteorology," *IEEE Trans. on Geoscience and Remote Sensing*, vol. 34, no.4, pp. 859–873, 1996.

[22] Z. Czyż, "Polarization and phase sphere of tangential phasors and its applications," in *Proc. SPIE Wideband Interferometric Sensing and Imaging Polarimetry*, H. Mott and W. Boerner, Eds., vol. 3120, 1997, pp. 268–294.

[23] H. Jones, *Groups, Representations and Physics*. Taylor & Francis, 1998.

[24] E. Cartan, *The Theory of Spinors*. Paris: Hermann, 1966.

[25] O. Laporte and G. E. Uhlenbeck, "Application of spinor analysis to the Maxwell and Dirac equations," *Phys. Rev.*, vol. 37, pp. 1380–1397, 1931.

[26] O. Veblen, *Projektive Relativitaetstheorie*. Berlin: Springer Verlag, 1933.

[27] H. Ruse, "On the geometry of the electromagnetic field in general relativity," *Proc. London Math. Soc.*, vol. 41, no. 4, pp. 302–322, 1936.

[28] M. Berger, *Geometry II*. Berlin, Heidelberg, New York: Springer Verlag, 1987.

[29] C. Brown, "Unified formalism for polarization phenomena using Stokes vectors and the Lorentz group," in *SPIE Conf. on Polarization Analysis and Measurements*, vol. 1746, San Diego, CA, July 1992.

[30] J. Semple and G. Kneebone, *Algebraic Projective Geometry*. Oxford: Clarendon Press, 1952.

[31] S. Cloude and E. Pottier, "A review of target decomposition theorems in radar polarimetry," *IEEE Trans. Geoscience and Remote Sensing*, vol. 34, no.2, pp. 498–518, 1996.

[32] S. Cloude, "Target decomposition theorems in radar scattering," *IEE Electronics Letters*, vol. 21(1), pp. 22–24, 1985.

[33] D. Bebbington, E. Krogager, and M. Hellmann, "Vectorial generalization of target helicity," in 3^{rd} *European Conference on Synthetic Aperture Radar (EuSAR)*, May 2000, pp. 531–534.

[34] C. Misner, K. Thorne, and J. Wheeler, *Gravitation*. New York: W.H. Freeman and Company, 1973.

[35] R. Penrose and W. Rindler, *Spinors and Space-Time*. Cambridge University Press, 1984, vol. 1.

[36] J. Jackson, *Classic Electrodynamics*. John Wiley and Sons, 1999.

[37] M. Tonnelat, *The Principle of Electromagnetic Theory and of Relativity*. Dordrecht, Holland: D. Reidel Publishing Company, 1966.

[38] J. Todd, *Projective and Analytical Geometry*. London: Sir Isaac Pitman & Sons, LTD, 1960.

[39] E. Post, *Formal Structure of Electromagnetics - General Covariance and Electromagnetics*. Dover Publications Inc., 1998.

[40] M. Berry and S. Klein, "Geometric phases from stacks of crystal plates," *J. Mod. Optics*, vol. 43, no. 1, pp. 165–180, 1996.

[41] A. Ludwig, "The definition of cross polarization," *IEEE Trans. on Antenna and Propagation*, vol. 21, pp. 116–119, 1973.

[42] A. Barut, *Electrodynamics and Classical Theory of Fields and Particles*. Dover Publications, 1964.

[43] P. Boulanger and M. Hayes, *Bivectors and Waves in Mechanics and Optics*. Chapman and Hall, 1993.

David Bebbington Received the B.A. degree in experimental and theoretical physics and the Ph.D. degree in radio astronomy from Cambridge University, Cambridge, U.K., in 1977 and 1986, respectively. From 1981 to 1984, he worked on millimeter wave propagation research at the Rutherford Appleton Laboratory. Since 1984, he has been at the University of Essex, Essex, U.K., and currently holds the post of Senior Lecturer in the School of Computer Science and Electronic Engineering. His interests are polarimetry, applications of wave propagation in remote sensing, and weather radars.

Laura Carrea received a Laurea degree in physics from the University of Turin, Italy in 1997 and a PhD in Computer Science and Electronic Engineering from the University of Essex, UK in 2013. From 1999 to 2004 she worked at the Faculty of Electrical Engineering and Information Technology at the Chemnitz University of Technology, Germany. Since 2004 she is with Centre for Remote Sensing & Environmetrics, School of Computer Science and Electronic Engineering, University of Essex, U.K. Her interests are centered on theoretical aspects of polarization and scattering for radar remote sensing.