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# Classifying the Clique-Width
# of *H*-Free Bipartite Graphs[⋆]

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**Abstract.** Let $G$ be a bipartite graph, and let $H$ be a bipartite graph
with a fixed bipartition $(B_H, W_H)$. We consider three different, natural
ways of forbidding $H$ as an induced subgraph in $G$. First, $G$ is $H$-free
if it does not contain $H$ as an induced subgraph. Second, $G$ is strongly
$H$-free if no bipartition of $G$ contains an induced copy of $H$ in a way
that respects the bipartition of $H$. Third, $G$ is weakly $H$-free if $G$ has
at least one bipartition that does not contain an induced copy of $H$ in a
way that respects the bipartition of $H$. Lozin and Volz characterized all
bipartite graphs $H$ for which the class of strongly $H$-free bipartite graphs
has bounded clique-width. We extend their result by giving complete
classifications for the other two variants of $H$-freeness.

**Keywords:** clique-width; bipartite graph; graph class

## 1 Introduction

The *clique-width* of a graph $G$ is a well-known graph parameter that has been
studied both in a structural and in an algorithmic context. It is the minimum
number of labels needed to construct $G$ by using the following four operations:

(i) creating a new graph consisting of a single vertex $v$ with label $i$;
(ii) taking the disjoint union of two labelled graphs $G_1$ and $G_2$;
(iii) joining each vertex with label $i$ to each vertex with label $j$ ($i \neq j$);
(iv) renaming label $i$ to $j$.

We refer to the surveys of Gurski [19] and Kamiński, Lozin and Milanič [21] for
an in-depth study of the properties of clique-width.

We say that a class of graphs has *bounded* clique-width if every graph from
the class has clique-width at most $c$ for some constant $c$. As many NP-hard graph
problems can be solved in polynomial time on graph classes of bounded clique-
width [13,22,27,28], it is natural to determine whether a certain graph class has

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bounded clique-width and to find new graph classes of bounded clique-width. In particular, many papers have determined the clique-width of graph classes characterized by one or more forbidden induced subgraphs [1,2,5–12,15,20,23–26].

In this paper we focus on classes of bipartite graphs characterized by a forbidden induced subgraph $H$. A graph $G$ is $H$-free if it does not contain $H$ as an induced subgraph. If $G$ is bipartite, then when considering notions for $H$-freeness, we may assume without loss of generality that $H$ is bipartite as well. For bipartite graphs, the situation is more subtle as one can define the notion of freeness with respect to a fixed ordered bipartition $(B_H, W_H)$ of $H$. This leads to two other notions (see also Section 2 for formal definitions). We say that a bipartite graph $G$ is strongly $H$-free if no bipartition of $G$ contains an induced copy of $H$ in a way that respects the bipartition of $H$. Strongly $H$-free graphs have been studied with respect to their clique-width, although under less explicit terminology (see e.g. [21,24,25]). In particular, Lozin and Volz [25] completely determined those bipartite graphs $H$, for which the class of strongly $H$-free graphs has bounded clique-width (we give an exact statement of their result in Section 3). If $G$ has at least one bipartition that does not contain an induced copy of $H$ in a way that respects the bipartition of $H$ then $G$ is said to be weakly $H$-free. As we shall see, any $H$-free graph is strongly $H$-free, and any strongly $H$-free graph is weakly $H$-free, whereas the two reverse statements do not always hold. Moreover, as far as we are aware, the notion of being weakly $H$-free has not been studied with respect to the clique-width of bipartite graphs.

**Our Results:** We completely classify the classes of $H$-free bipartite and weakly $H$-free bipartite graphs of bounded clique-width. In this way, we have identified a number of new graph classes of bounded clique-width. Before stating our classification results precisely in Section 3, we first give some terminology and examples in Section 2. In Section 4 we give the proofs of our results.

## 2 Preliminaries

We first give some terminology on general graphs and notation to denote various well-known graphs. In Section 2.1 we introduce labelled bipartite graphs. We illustrate the definitions of $H$-freeness, strong $H$-freeness and weak $H$-freeness of bipartite graphs with some examples. As we will explain, these examples also make clear that all three notions are different from each other.

**General graphs:** Let $G$ and $H$ be graphs. We write $H \subseteq_i G$ to indicate that $H$ is an induced subgraph of $G$. A bijection $f : V_G \rightarrow V_H$ is called a (*graph*) *isomorphism* when $uv \in E_G$ if and only if $f(u)f(v) \in E_H$. If such a bijection exists then $G$ and $H$ are *isomorphic*. Let $\{H_1, \ldots, H_p\}$ be a set of graphs. A graph $G$ is $(H_1, \ldots, H_p)$-*free* if no $H_i$ is an induced subgraph of $G$. If $p = 1$ we may write $H_1$-free instead of $(H_1)$-free. The *disjoint union* $G + H$ of two vertex-disjoint graphs $G$ and $H$ is the graph with vertex set $V_G \cup V_H$ and edge set $E_G \cup E_H$. We denote the disjoint union of $r$ vertex-disjoint copies of $G$ by $rG$.

**Special Graphs:** For $r \geq 1$, the graphs $C_r, K_r, P_r$ denote the cycle, complete graph and path on $r$ vertices, respectively, and the graph $K_{1,r}$ denotes the star on $r+1$ vertices. If $r = 3$, the graph $K_{1,r}$ is also called the *claw*. For $1 \leq h \leq i \leq j$, let $S_{h,i,j}$ denote the tree that has only one vertex $x$ of degree 3 and that has exactly three leaves, which are of distance $h, i$ and $j$ from $x$, respectively. Observe that $S_{1,1,1} = K_{1,3}$. A graph $S_{h,i,j}$ is said to be a *subdivided claw*. A graph $G$ is a *linear forest* if every connected component of $G$ is a path.

## 2.1 Labelled Bipartite Graphs

A graph $G$ is *bipartite* if its vertex set can be partitioned into two (possibly empty) independent sets. Let $H$ be a bipartite graph. We say that $H$ is a *labelled* bipartite graph if we are also given a *black-and-white labelling* $\ell$, which is a labelling that assigns either the colour "black" or the colour "white" to each vertex of $H$ in such a way that the two resulting monochromatic colour classes $B_H^\ell$ and $W_H^\ell$ form a *bipartition* of $V_H$ into two (possibly empty) independent sets. From now on we denote a graph $H$ with such a labelling $\ell$ by $H^\ell = (B_H^\ell, W_H^\ell, E_H)$. Here the pair $(B_H^\ell, W_H^\ell)$ is *ordered*, that is, $(B_H^\ell, W_H^\ell, E_H)$ and $(W_H^\ell, B_H^\ell, E_H)$ are different labelled bipartite graphs.

We say that two labelled bipartite graphs $H_1^\ell$ and $H_2^{\ell^*}$ are *isomorphic* if the (unlabelled) graphs $H_1$ and $H_2$ are isomorphic, and if in addition there exists an isomorphism $f : V_{H_1} \to V_{H_2}$ such that for all $u \in V_{H_1}$, $u \in W_{H_1}^\ell$ if and only if $f(u) \in W_{H_2}^{\ell^*}$. Moreover, if $H_1 = H_2$, then $\ell$ and $\ell^*$ are said to be *isomorphic* labellings. For example, the bipartite graphs $(\{u, v\}, \emptyset)$ and $(\{x, y\}, \emptyset)$ are isomorphic, and the labelled bipartite graph $(\{u, v\}, \emptyset, \emptyset)$ is isomorphic to the labelled bipartite graph $(\{x, y\}, \emptyset, \emptyset)$. However, $(\{x, y\}, \emptyset, \emptyset)$ is neither isomorphic to $(\emptyset, \{x, y\}, \emptyset)$ nor to $(\{x\}, \{y\}, \emptyset)$ (also see Fig. 1).

We write $H_1^\ell \subseteq_{li} H_2^{\ell^*}$ if $H_1 \subseteq_i H_2$, $B_{H_1}^\ell \subseteq B_{H_2}^{\ell^*}$ and $W_{H_1}^\ell \subseteq W_{H_2}^{\ell^*}$. In this case we say that $H_1^\ell$ is a *labelled* induced subgraph of $H_2^{\ell^*}$. Note that the two labelled bipartite graphs $H_1^{\ell_1}$ and $H_2^{\ell_2}$ are isomorphic if and only if $H_1^{\ell_1}$ is a labelled induced subgraph of $H_2^{\ell_2}$, and vice versa.

● ●                ○ ○                ● ○

(a) $(2P_1)^b$         (b) $(2P_1)^{\overline{b}}$         (c) $(P_1)^b + (P_1)^{\overline{b}}$
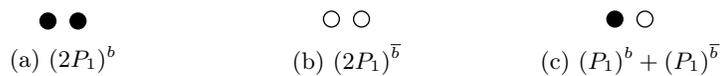
Fig. 1: The three pairwise non-isomorphic labellings of $2P_1$. The labellings $b$ and $\overline{b}$ will be formally defined later.

Let $G$ be an (unlabelled) bipartite graph, and let $H^\ell$ be a labelled bipartite graph. The graph $G$ is *strongly* $H^\ell$-*free* if for every labelling $\ell^*$ of $G$, $G^{\ell^*}$ does not contain $H^\ell$ as a labelled induced subgraph. The graph $G$ is *weakly* $H^\ell$-*free* if there is a labelling $\ell^*$ of $G$ such that $G^{\ell^*}$ does not contain $H^\ell$ as a labelled induced subgraph. Note that these two notions of freeness are only defined for

(unlabelled) bipartite graphs. Let $\{H_1^{\ell_1}, \ldots, H_p^{\ell_p}\}$ be a set of labelled bipartite graphs. Then a graph $G$ is *strongly (weakly)* $(H_1^{\ell_1}, \ldots, H_p^{\ell_p})$-*free* if $G$ is strongly (weakly) $H_i^{\ell_i}$-free for $i = 1, \ldots, p$.

The following lemma shows that for all labelled bipartite graphs $H^\ell$, the class of $H$-free graphs is a (possibly proper) subclass of the class of strongly $H^\ell$-free bipartite graphs and that the latter graph class is a (possibly proper) subclass of the class of weakly $H^\ell$-free bipartite graphs.

**Lemma 1.** *Let $G$ be a bipartite graph and $H^\ell$ be a labelled bipartite graph. The following two statements hold:*

*(i) If $G$ is $H$-free, then $G$ is strongly $H^\ell$-free.*
*(ii) If $G$ is strongly $H^\ell$-free, then $G$ is weakly $H^\ell$-free.*

*Moreover, the two reverse statements are not necessarily true.*

*Proof.* Statements (i) and (ii) follow by definition.

The following examples, which are also depicted in Fig. 2, show that the reverse statements may not necessarily be true. Let $G$ be isomorphic to $P_3$ with $V_G = \{u_1, u_2, u_3\}$ and $E_G = \{u_1u_2, u_2u_3\}$. Let $H = 2P_1$. We denote the vertex set and edge set of $H$ by $V_H = \{x_1, x_2\}$ and $E_H = \emptyset$, respectively.

Let $H^\ell = (P_1)^b + (P_1)^{\bar{b}} = (\{x_1\}, \{x_2\}, \emptyset)$ (see also Fig. 1). We first notice that $G$ is not $H$-free, because $G[\{u_1, u_3\}]$ is isomorphic to $2P_1$. However, we do have that $G$ is strongly $H^\ell$-free, because $H^\ell$ is neither a labelled induced subgraph of $G^b = (\{u_1, u_3\}, \{u_2\}, E_G)$ nor of $G^{\bar{b}} = (\{u_2\}, \{u_1, u_3\}, E_G)$.

Let $H^{\ell^*} = (2P_1)^b = (\{x_1, x_2\}, \emptyset, E_H)$ (see also Fig. 1). Then $G$ is not strongly $H^{\ell^*}$-free, because $(\{u_1, u_3\}, \emptyset, \emptyset)$ is isomorphic to $H^{\ell^*}$. However, $G$ is weakly $H^{\ell^*}$-free, because $H^{\ell^*}$ is not a labelled induced subgraph of $G^{\bar{b}} = (\{u_2\}, \{u_1, u_3\}, E_G)$. □

$$u_1 \;\text{—}\; u_2 \;\text{—}\; u_3$$

(a) $G$          (b) $G^b$          (c) $G^{\bar{b}}$
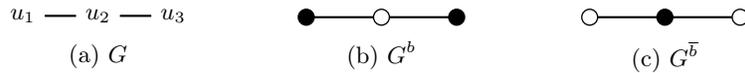
Fig. 2: The graph $G = P_3$ and its two labellings.

Next, we prove a lemma which is related to Lemma 1 and which follows immediately from the corresponding definitions.

**Lemma 2.** *Let $G$ and $H$ be bipartite graphs. Then $G$ is $H$-free if and only if $G$ is strongly $H^\ell$-free for all black-and-white labellings $\ell$ of $H$.*

A graph $G$ that contains a graph $H$ as an induced subgraph may be weakly $H^\ell$-free for all black-and-white labellings $\ell$ of $H$; take for instance the graphs

$G = P_3$ and $H = 2P_1$ as in the proof of Lemma 1. However, we can make the following observation, which also follows directly from the corresponding definitions.

**Lemma 3.** *Let $H$ be a bipartite graph with a unique black-and-white labelling $\ell$ (up to isomorphism). Then every bipartite graph $G$ is $H$-free if and only if it is weakly $H^\ell$-free.*

Note that there exist both connected bipartite graphs (for example $H = P_6$) and disconnected bipartite graphs (for example $H = 2P_2$) that satisfy the condition of Lemma 3.

Let $H^\ell = (B_H^\ell, W_H^\ell, E_H)$ be a labelled bipartite graph. The *opposite* of $H^\ell$ is defined as the labelled bipartite graph $H^{\bar{\ell}} = (W_H^\ell, B_H^\ell, E_H)$; in other words it is the labelled bipartite graph obtained from $H^\ell$ by recolouring the black vertices to be white and vice versa. We say that $\bar{\ell}$ is the *opposite* black-and-white labelling of $\ell$. Suppose that $H$ is a bipartite graph such that among all its black-and-white labellings, all those that maximize the number of black vertices are isomorphic. In this case we pick one of such labelling and call it $b$ (see also Fig. 1). Note that there are graphs for which such a labelling does not exist. For example, the graph $S_{1,2,2}$ has two non-isomorphic labellings and both of them have the same number of black vertices (see also Fig. 3). If such a unique labelling $b$ does exist, we let $\bar{b}$ denote the opposite labelling to $b$.

Fig. 3: The two labellings of $S_{1,2,2}$.

Two black-and-white labellings of a bipartite graph $H$ are said to be *equivalent* if they are isomorphic or opposite to each other; otherwise they are said to be *non-equivalent*. Note that if a linear forest has two non-equivalent labellings then it must contain at least two components with an odd number of vertices. The following lemma follows directly from the definitions.

**Lemma 4.** *Let $\ell$ and $\ell^*$ be two equivalent black-and-white labellings of a bipartite graph $H$. Then the class of strongly (weakly) $H^\ell$-free graphs is equal to the class of strongly (weakly) $H^{\ell^*}$-free graphs.*

We will also need the following two lemmas.

**Lemma 5.** *Let $H^\ell$ be a labelled bipartite graph. Then $H \subseteq_i P_2 + P_4$ or $H \subseteq_i P_6$ if and only if $H^\ell \subseteq_{li} (P_2 + P_4)^b$ or $H^\ell \subseteq_{li} (P_6)^b$.*

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*Proof.* Clearly, if $H^\ell \subseteq_{li} (P_2 + P_4)^b$ or $H^\ell \subseteq_{li} (P_6)^b$ then $H \subseteq_i P_2 + P_4$ or $H \subseteq_i P_6$.

Now suppose $H \subseteq_i P_2 + P_4$ or $H \subseteq_i P_6$ and let $\ell$ be a labelling of $H$. We will show that $H^\ell \subseteq_{li} (P_2 + P_4)^b$ or $H^\ell \subseteq_{li} (P_6)^b$. Note that $P_2 + P_4$ and $P_6$ have a unique labelling $b$ (up to isomorphism). We may therefore assume that $H \notin \{P_2 + P_4, P_6\}$. If $H^\ell$ is not a labelled induced subgraph of one of $\{(P_2 + P_4)^b, (P_6)^b\}$ then $H$ must have two non-equivalent black-and-white labellings. Since $H$ is a linear forest, it must have at least two components with an odd number of vertices. Therefore $H \in \{2P_1, 3P_1, P_1 + P_3, 2P_1 + P_2\}$. However, in all these cases, for every labelling $\ell$ of $H$, $H^\ell \subseteq_{li} (P_6)^b$ or $H^\ell \subseteq_{li} (P_2 + P_4)^b$. This completes the proof. □

**Lemma 6.** *Let $H \in \mathcal{S}$. Then $H$ is $(2P_1 + 2P_2, 2P_1 + P_4, 4P_1 + P_2, 3P_2, 2P_3)$-free if and only if $H = sP_1$ for some integer $s \geq 1$ or $H$ is an induced subgraph of one of the graphs in $\{K_{1,3} + 3P_1, K_{1,3} + P_2, P_1 + S_{1,1,3}, S_{1,2,3}\}$.*

*Proof.* Let $H \in \mathcal{S}$. First suppose that $H = sP_1$ for some integer $s \geq 1$ or that $H$ is an induced subgraph of one of the graphs in $\{K_{1,3} + 3P_1, K_{1,3} + P_2, P_1 + S_{1,1,3}, S_{1,2,3}\}$. It is readily seen that $H$ is $(2P_1 + 2P_2, 2P_1 + P_4, 4P_1 + P_2, 3P_2, 2P_3)$-free. We may therefore assume that $H$ contains at least one edge.

Now suppose that $H$ is $(2P_1 + 2P_2, 2P_1 + P_4, 4P_1 + P_2, 3P_2, 2P_3)$-free. Let $D_1, \ldots D_r$ be the connected components of $H$, where $|V_{D_1}| \leq \cdots \leq |V_{D_r}|$. Since $H$ contains an edge, we find that $|V_{D_r}| \geq 2$. Since $H$ is $(4P_1 + P_2)$-free, it follows that $r \leq 4$.

Suppose that $r = 4$. Because $H$ is $(2P_1 + 2P_2)$-free, it follows that $D_3 = P_1$, so $H = 3P_1 + D_3$. Because $H$ is $(2P_1 + P_4)$-free, $D_3$ must be $P_4$-free. As $H \in \mathcal{S}$, this means that $D_3$ is isomorphic to one of $\{K_{1,3}, P_2, P_3\}$. Hence, $H$ is an induced subgraph of $K_{1,3} + 3P_1$.

Now suppose that $r = 3$ Because $H$ is $3P_2$-free, it follows that $D_1 = P_1$. Since $H$ is $2P_3$-free and $H \in \mathcal{S}$, it follows that $D_2$ is $P_3$-free, so $D_2 \in \{P_1, P_2\}$. Because $H$ is $(2P_1 + P_4)$-free, $D_3$ must be $P_4$-free. As $H \in \mathcal{S}$, this means that $D_3 \in \{K_{1,3}, P_2, P_3\}$. Because $H$ is $(4P_1 + P_2)$-free, the combination $D_2 = P_2$ and $D_3 = K_{1,3}$ is not possible. Hence, if $D_2 = P_1$ then $H$ is an induced subgraph of $K_{1,3} + 2P_1$ and if $D_2 = P_2$ then $H$ is an induced subgraph of $P_1 + P_2 + P_3$. This means that $H$ is an induced subgraph of $K_{1,3} + 3P_1$ or of $S_{1,2,3}$, respectively.

Now suppose that $r = 2$. Because $H$ is $2P_3$-free and $H \in \mathcal{S}$, we find that $D_1 \in \{P_1, P_2\}$ and that $D_2$ is either a path or a subdivided claw. Because $H$ is $(2P_1 + P_4)$-free, $D_2$ is $P_6$-free. Suppose that $D_2$ is a path. Then $D_2 \subseteq_i P_5$. If $D_2 = P_5$ then $D_1 = P_1$, as $H$ is $3P_2$-free. Hence we find that $H$ is an induced subgraph of $P_1 + P_5$ or $P_2 + P_4$, which are induced subgraphs of $P_1 + S_{1,1,3}$ and $S_{1,2,3}$, respectively. Suppose that $D_2$ is a subdivided claw, say $D_2 = S_{a,b,c}$ for some $1 \leq a \leq b \leq c$. Then, because $H$ is $(2P_1 + 2P_2)$-free, $a = b = 1$. Because $H$ is $(2P_1 + P_4)$-free, $c \leq 3$. Moreover, if $2 \leq c \leq 3$ then $D_1 = P_1$, as $H = (2P_1 + 2P_2)$-free. Hence, we find that $H$ is an induced subgraph of $K_{1,3} + P_2$ or $P_1 + S_{1,1,3}$.

Now suppose that $r = 1$, in which case $H$ is connected. As $H \in \mathcal{S}$, we find that $H$ is either a path or a subdivided claw. If $H$ is a path then, as $H$ is $2P_3$-free,

$H$ is an induced subgraph of $P_6$, which means that $H \subseteq_i S_{1,2,3}$. Suppose that $H$ is a subdivided claw, say $H = S_{a,b,c}$ for some $1 \leq a \leq b \leq c$. Because $H$ is $3P_2$-free, we find that $a = 1$. Because $H$ is $2P_3$-free, we find that $b \leq 2$ and that $c \leq 3$. Hence, $H$ is an induced subgraph of $S_{1,2,3}$. This completes the proof. $\square$

## 3    The Classifications

A full classification of the boundedness of the clique-width of strongly $H^\ell$-free bipartite graphs was given by Lozin and Volz [25], except that in their result the trivial case when $H^\ell$ or $H^{\overline{\ell}} = (sP_1)^b$ for some $s \geq 1$ was missing. Their proof is correct except that it overlooked this case, which occurs when one of the colour classes of the labelled graph $H^\ell$ is empty. However, strongly $(sP_1)^b$-free bipartite graphs can have at most $2s - 2$ vertices, and as such form a class of bounded clique-width. Below we state their result after incorporating this small correction, followed by our results for the other two variants of freeness. We refer to Fig. 4 for pictures of the labelled bipartite graphs used in Theorems 1 and 3.

**Theorem 1 ([25]).** *Let $H^\ell$ be a labelled bipartite graph. The class of strongly $H^\ell$-free bipartite graphs has bounded clique-width if and only if one of the following cases holds:*

- *$H^\ell$ or $H^{\overline{\ell}} = (sP_1)^b$ for some $s \geq 1$*
- *$H^\ell$ or $H^{\overline{\ell}} \subseteq_{li} (K_{1,3} + 3P_1)^b$*
- *$H^\ell$ or $H^{\overline{\ell}} \subseteq_{li} (K_{1,3} + P_2)^b$*
- *$H^\ell$ or $H^{\overline{\ell}} \subseteq_{li} (P_1 + S_{1,1,3})^b$*
- *$H^\ell$ or $H^{\overline{\ell}} \subseteq_{li} (S_{1,2,3})^b$*

**Theorem 2.** *Let $H$ be a graph. The class of $H$-free bipartite graphs has bounded clique-width if and only if one of the following cases holds:*

- *$H = sP_1$ for some $s \geq 1$*
- *$H \subseteq_i K_{1,3} + 3P_1$*
- *$H \subseteq_i K_{1,3} + P_2$*
- *$H \subseteq_i P_1 + S_{1,1,3}$*
- *$H \subseteq_i S_{1,2,3}$.*

**Theorem 3.** *Let $H^\ell$ be a labelled bipartite graph. The class of weakly $H^\ell$-free bipartite graphs has bounded clique-width if and only if one of the following cases holds:*

- *$H^\ell$ or $H^{\overline{\ell}} = (sP_1)^b$ for some $s \geq 1$*
- *$H^\ell$ or $H^{\overline{\ell}} \subseteq_{li} (P_1 + P_5)^b$*
- *$H^\ell \subseteq_i (P_2 + P_4)^b$*
- *$H^\ell \subseteq_i (P_6)^b$.*

Note that Theorem 2 is exactly the unlabelled variant of Theorem 1. Indeed, if $H^\ell$ is a labelled bipartite graph then the class of $H$-free bipartite graphs is contained in the class of strongly $H^\ell$-free bipartite graphs (by Lemma 1), so all the bounded cases carry over. However, we need to do some more work to deal with the unbounded cases, so Theorem 2 does not follow from Theorem 1 as a direct corollary.

Also note that by Lemma 5, we can also state Theorem 3 as follows. (We originally stated the theorem in this form in the extended abstract of this paper [17].)

**Theorem 3 (equivalent formulation).** *Let $H^\ell$ be a labelled bipartite graph. The class of weakly $H^\ell$-free bipartite graphs has bounded clique-width if and only if one of the following cases holds:*

- $H^\ell$ *or* $H^{\bar\ell} = (sP_1)^b$ *for some* $s \geq 1$
- $H^\ell$ *or* $H^{\bar\ell} \subseteq_{li} (P_1 + P_5)^b$
- $H \subseteq_i P_2 + P_4$
- $H \subseteq_i P_6$.



(a) $(sP_1)^b$ for $s = 5$

(b) $(K_{1,3} + 3P_1)^b$

(c) $(K_{1,3} + P_2)^b$

(d) $(P_1 + P_5)^b$
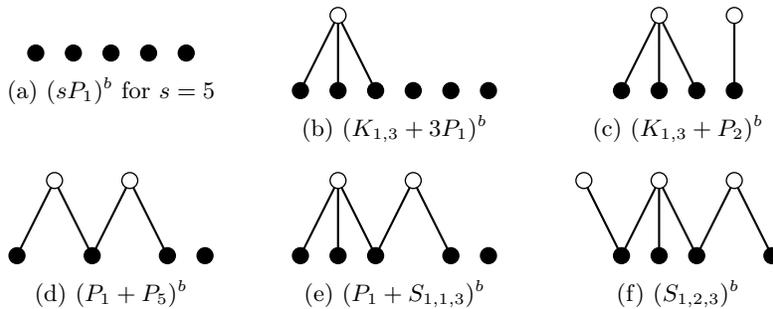
(e) $(P_1 + S_{1,1,3})^b$

(f) $(S_{1,2,3})^b$

Fig. 4: The labelled bipartite graphs used in Theorems 1 and 3.

## 4 The Proofs of Our Results

We first recall a number of basic facts on clique-width known from the literature. We then state a number of other lemmas which we use to prove Theorems 2 and 3.

### 4.1 Facts about Clique-width

For two disjoint vertex subsets $X$ and $Y$ in a (not necessarily bipartite) graph $G$, the *bipartite complementation* operation with respect to $X$ and $Y$ acts on $G$ by replacing every edge with one end-vertex in $X$ and the other one in $Y$ by a non-edge and vice versa. The *bipartite complement* of a bipartite graph *with respect*

*to* a bipartition $(B, W)$ is the bipartite graph with bipartition $(B, W)$ obtained from $G$ by applying a bipartite complementation between $B$ and $W$. For instance, the graph $2P_2$ has a unique bipartition (up to isomorphism) and it therefore has only one bipartite complement, namely $2P_2$. On the other hand $2P_1$ does not have a unique bipartition, and both $2P_1$ and $P_2$ can be obtained as bipartite complements of it, depending on the choice of partition. The *edge subdivision* operation replaces an edge $vw$ in a graph by a new vertex $u$ with edges $uv$ and $uw$.

Let $k \geq 0$ be a constant and let $\gamma$ be some graph operation. We say that a graph class $\mathcal{G}'$ is $(k, \gamma)$-*obtained* from a graph class $\mathcal{G}$ if the following two conditions hold:

(i) every graph in $\mathcal{G}'$ is obtained from a graph in $\mathcal{G}$ by performing $\gamma$ at most $k$ times, and

(ii) for every $G \in \mathcal{G}$ there exists at least one graph in $\mathcal{G}'$ obtained from $G$ by performing $\gamma$ at most $k$ times.

If we allow arbitrarily many applications of $\gamma$ then we write that $\mathcal{G}'$ is $(\infty, \gamma)$-obtained from $\mathcal{G}$.

We say that $\gamma$ *preserves* boundedness of clique-width if for any finite constant $k$ and any graph class $\mathcal{G}$, any graph class $\mathcal{G}'$ that is $(k, \gamma)$-obtained from $\mathcal{G}$ has bounded clique-width if and only if $\mathcal{G}$ has bounded clique-width.

**Fact 1.** Vertex deletion preserves boundedness of clique-width [14,23].

**Fact 2.** Bipartite complementation preserves boundedness of clique-width [21].

**Fact 3.** For a class of graphs $\mathcal{G}$ of *bounded* maximum degree, let $\mathcal{G}'$ be a class of graphs that is $(\infty, \mathtt{es})$-obtained from $\mathcal{G}$, where $\mathtt{es}$ is the edge subdivision operation. Then $\mathcal{G}$ has bounded clique-width if and only if $\mathcal{G}'$ has bounded clique-width [21].

We also use some other elementary results on the clique-width of graphs. In order to do so we need the notion of a *wall*. We do not formally define this notion, but instead refer to Fig. 5, in which three examples of walls of different height are depicted. A *k-subdivided wall* is a graph obtained from a wall after subdividing each edge exactly $k$ times for some constant $k \geq 0$. The next well-known lemma follows from combining Fact 3 with the fact that walls have maximum degree 3 and unbounded clique-width (see e.g. [21]).

**Lemma 7.** *For every constant $k$, the class of $k$-subdivided walls has unbounded clique-width.*

We let $\mathcal{S}$ be the class of graphs each connected component of which is either a subdivided claw $S_{h,i,j}$ for some $1 \leq h \leq i \leq j$ or a path $P_r$ for some $r \geq 1$. Note that every graph in $\mathcal{S}$ is of maximum degree at most 3 and every connected component of a graph in $\mathcal{S}$ has at most one vertex of degree 3. This leads to the following lemma, which is well known and follows from the fact that walls have maximum degree at most 3 and from Lemma 7 by choosing an appropriate value for $k$ (also note that $k$-subdivided walls are bipartite for all $k \geq 0$).
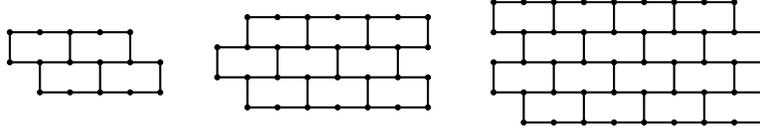
Fig. 5: Walls of height 2, 3, and 4, respectively.

**Lemma 8.** *Let $\{H_1, \ldots, H_p\}$ be a finite set of graphs. If $H_i \notin \mathcal{S}$ for $i = 1, \ldots, p$ then the class of $(H_1, \ldots, H_p)$-free bipartite graphs contains all $\left(\max\{|V_{H_1}|, \ldots, |V_{H_p}|\}\right)$-subdivided walls, and thus has unbounded clique-width.*

The following lemma is due to Lozin and Rautenbach [24].

**Lemma 9 ([24]).** *Let $\{H_1^{\ell_1}, \ldots, H_p^{\ell_p}\}$ be a finite set of labelled bipartite graphs. For $i = 1, \ldots, p$, let $F_i$ denote the bipartite complement of $H_i$ with respect to $(B_{H_i}^{\ell_i}, W_{H_i}^{\ell_i})$. If $H_i \notin \mathcal{S}$ for all $1 \le i \le p$ or $F_i \notin \mathcal{S}$ for all $1 \le i \le p$, then the class of strongly $(H_1^{\ell_1}, \ldots, H_p^{\ell_p})$-free bipartite graphs has unbounded clique-width.*

In the next two lemmas we list a number of classes of $H$-free bipartite graphs that have unbounded clique-width. The first of these is due to Lozin and Volz.

**Lemma 10 ([25]).** *The class of $2P_3$-free graphs is unbounded.*

**Lemma 11.** *The class of $H$-free bipartite graphs has unbounded clique-width if $H \in \{2P_1 + 2P_2, 2P_1 + P_4, 4P_1 + P_2, 3P_2\}$.*

*Proof.* Let $H \in \{2P_1 + 2P_2, 2P_1 + P_4, 4P_1 + P_2, 3P_2\}$, and let $\{H^{\ell_1}, \ldots, H^{\ell_p}\}$ be the set of all non-equivalent labelled bipartite graphs isomorphic to $H$. For $i = 1, \ldots, p$, let $F_i$ denote the bipartite complement of $H$ with respect to $(B_H^{\ell_i}, W_H^{\ell_i})$. We will show that every $F_i$ does not belong to $\mathcal{S}$. Then, by Lemma 9 the class of strongly $(H_1^{\ell_1}, \ldots, H_p^{\ell_p})$-free bipartite graphs has unbounded clique-width. Because a bipartite graph is $H$-free if and only if it is strongly $(H_1^{\ell_1}, \ldots, H_p^{\ell_p})$-free (by Lemmas 2 and 4), this means that the class of $H$-free bipartite graphs has unbounded clique-width.

Suppose $H \in \{2P_1 + 2P_2, 2P_1 + P_4\}$. Let $V_H = \{x_1, \ldots, x_6\}$ with $E_H = \{x_1x_2, x_3x_4\}$ if $H = 2P_1 + 2P_2$ and $E_H = \{x_1x_2, x_2x_3, x_3x_4\}$ if $H = 2P_1 + P_4$. Then $H$ has only two non-equivalent black-and-white labellings. We may assume without loss of generality that one of these two labellings colours $x_1, x_3, x_5, x_6$ black and $x_2, x_4$ white, whereas the other one colours $x_1, x_3, x_5$ black and $x_2, x_4, x_6$ white. Let $F_1$ and $F_2$ be the bipartite complements corresponding to the first and second labellings, respectively. The vertices $x_2, x_4, x_5, x_6$ induce a $C_4$ in $F_1$, whereas the vertices $x_1, x_4, x_5, x_6$ induce a $C_4$ in $F_2$. Hence, $F_1$ and $F_2$ do not belong to $\mathcal{S}$.

Suppose $H = 4P_1 + P_2$. Let $V_H = \{x_1, \ldots, x_6\}$ and $E_H = \{x_1x_2\}$. Then $H$ has three non-equivalent black-and-white labellings. We may assume without loss

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of generality that the first one colours $x_1, x_3, x_4, x_5, x_6$ black and $x_2$ white, the second one colours $x_1, x_3, x_4, x_5$ black and $x_2, x_6$ white, and the third one colours $x_1, x_3, x_4$ black and $x_2, x_5, x_6$ white. Let $F_1, F_2, F_3$ denote the corresponding bipartite complements. The vertices $x_2, \ldots, x_6$ induce a $K_{1,4}$ in $F_1$. The vertices $x_2, x_3, x_4, x_6$ induce a $C_4$ in $F_2$ and $F_3$. Hence, none of $F_1, F_2, F_3$ belongs to $\mathcal{S}$.

Suppose $H = 3P_2$. Let $V_H = \{x_1, \ldots, x_6\}$ and $E_H = \{x_1x_2, x_3x_4, x_5x_6\}$. Let $\ell$ be a black-and-white labelling of $H$ that colours $x_1, x_3, x_5$ black and $x_2, x_4, x_6$ white. Then every other labelling $\ell^*$ of $H$ is isomorphic to $\ell$. The bipartite complement of $H$ with respect to $(B_H^\ell, W_H^\ell)$ is isomorphic to $C_6$, which does not belong to $\mathcal{S}$. □

The last lemma we need before proving the main results of this paper is the following one (we use it several times in the proof of Theorem 3).

**Lemma 12.** *Let $H^\ell$ be a labelled bipartite graph. The class of weakly $H^\ell$-free bipartite graphs has unbounded clique-width in both of the following cases:*

*(i) $H^\ell$ contains a vertex of degree at least 3, or*
*(ii) $H^\ell$ contains four independent vertices, not all of the same colour.*

*Proof.* Let $b_1$ be a black-and-white labelling of $4P_1$ that colours three vertices black and one vertex white. Let $b_2$ be a black-and-white labelling of $4P_1$ that colours two vertices black and two vertices white. We show below that the class of weakly $H^\ell$-free bipartite graphs has unbounded clique-width if $H^\ell \in \{(K_{1,3})^b, (4P_1)^{b_1}, (4P_1)^{b_2}\}$. Then we are done by Lemma 4.

Consider a 1-subdivided wall $G'$ obtained from a wall $G$. Recall that 1-subdivided walls are bipartite. Moreover, the vertices that were introduced when subdividing every edge of $G$ all have degree 2 and the set of these vertices forms one class of a bipartition $(B, W)$ of $G'$. Let this class be $B$. Then $(K_{1,3})^b$ is not a labelled induced subgraph of $(B, W, E_{G'})$. Hence, $G'$ is weakly $(K_{1,3})^b$-free. This means that the class of weakly $(K_{1,3})^b$-free graphs contains the class of 1-subdivided walls. As such, it has unbounded clique-width by Lemma 7. The bipartite complement $G''$ of $G'$ with respect to $(B, W)$ is weakly $(4P_1)^{b_1}$-free, as $(K_{1,3})^b$ is the bipartite complement of $(4P_1)^{b_1}$ and $(K_{1,3})^b$ is not a labelled induced subgraph of $(B, W, E_{G'})$. Hence, the class of weakly $(4P_1)^{b_1}$-free graphs has unbounded clique-width by Fact 2. The class of weakly $(4P_1)^{b_2}$-free bipartite graphs has unbounded clique-width by Lemma 1 and Theorem 1. □

### 4.2 The Proof of Theorem 2

*Proof.* First suppose that $H$ does not contain an edge, so $H = sP_1$ for some $s \geq 1$. Then every $H$-free bipartite graph $G$ has at most $s - 1$ vertices in each partition class for every bipartition. This means that the clique-width of $G$ is at most $2s - 2$. For the remainder of the proof we therefore assume that $H$ contains at least one edge.

If $H \in \{K_{1,3} + 3P_1, K_{1,3} + P_2, P_1 + S_{1,1,3}, S_{1,2,3}\}$ then the claim follows from combining Lemma 1 with Theorem 1. Now suppose that $H$ is not an induced

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subgraph of one of the graphs in $\{K_{1,3}+3P_1, K_{1,3}+P_2, P_1+S_{1,1,3}, S_{1,2,3}\}$. Then by Lemma 6, either $H \notin \mathcal{S}$ or, $H$ is not $(2P_1+2P_2, 2P_1+P_4, 4P_1+P_2, 3P_2, 2P_3)$-free. Hence, the clique-width of the class of $H$-free bipartite graphs is unbounded by Lemmas 8, 10 and 11. □

### 4.3  The Proof of Theorem 3

*Proof.* We first consider the bounded cases. First suppose $H^\ell = (sP_1)^b$ for some $s \geq 1$ (the $H^{\bar{\ell}} = (sP_1)^b$ case is equivalent). Then every weakly $H^\ell$-free bipartite graph has a bipartition $(B, W)$ with $|B| \leq s-1$. The clique-width of such graphs is at most $s + 1$: first introduce the vertices of $B$ using distinct labels and then use two more labels for the vertices of $W$, introducing them one-by-one.

Suppose $H^\ell = (P_2 + P_4)^b$ or $(P_6)^b$. Then $H \subseteq_i S_{1,2,3}$, which implies that that the class of $H$-free bipartite graphs has bounded clique-width by Theorem 2. All black-and-white labellings of $P_2 + P_4$ are isomorphic. Similarly, all black-and-white labellings of $P_6$ are isomorphic. Hence, the class of $H$-free bipartite graphs coincides with the class of weakly $H^\ell$-free graphs by Lemma 3. We therefore conclude that the latter class also has bounded clique-width.

Finally, let $H^\ell = (P_1 + P_5)^b$. Note that in a $(P_1 + P_5)^b$, the set of four black vertices have every possible neighbourhood among the two white vertices. Therefore, if we apply a bipartite complementation in a labelled bipartite graph between a subset of the white vertices and the set of *all* black vertices, then any six vertices that form a $(P_1+P_5)^b$ in the original graph will still form a $(P_1+P_5)^b$ in the obtained graph and vice versa. Suppose $G$ is a weakly $(P_1 + P_5)^b$-free bipartite graph. Then $G$ has a labelling $\ell^*$ such that $(P_1 + P_5)^b$ is not a labelled induced subgraph of $(B_G^{\ell^*}, W_G^{\ell^*}, E_G)$. If $|B_G^{\ell^*}|$ is even, then we delete a vertex of $B_G^{\ell^*}$. We may do this by Fact 1. Hence $|B_G^{\ell^*}|$ may be assumed to be odd. Let $X$ be the subset of $W_G^{\ell^*}$ that consists of all vertices that are adjacent to less than half of the vertices of $B_G^{\ell^*}$. We apply a bipartite complementation between $X$ and $B_G^{\ell^*}$. We may do this by Fact 2. Let $G_1$ be the resulting bipartite graph, with bipartition classes $B_{G_1}^{\ell^*} = B_G^{\ell^*}$ and $W_{G_1}^{\ell^*} = W_G^{\ell^*}$. Since $|B_{G_1}^{\ell^*}| = |B_G^{\ell^*}|$ is odd, in the graph $G_1$ every vertex of $W_{G_1}^{\ell^*}$ is adjacent to more than half of the vertices in $B_{G_1}^{\ell^*}$.

Suppose $B_{G_1}^{\ell^*}$ contains three vertices $b_1, b_2, b_3$ and $W_{G_1}^{\ell^*}$ contains two vertices $w_1, w_2$ such that $G_1^{\ell^*}[b_1, b_2, b_3, w_1, w_2]$ is isomorphic to $(P_1+2P_2)^b$. Because every vertex of $W_{G_1}^{\ell^*}$ is adjacent to more than half of the vertices in $B_{G_1}^{\ell^*}$, $w_1$ and $w_2$ have at least one common neighbour $b_4 \in B_{G_1}^{\ell^*}$. Then $G_1^{\ell^*}[b_1, b_2, b_3, b_4, w_1, w_2]$ is isomorphic to $(P_1 + P_5)^b$. However, then $G^{\ell^*}[b_1, b_2, b_3, b_4, w_1, w_2]$ is also isomorphic to $(P_1 + P_5)^b$ (irrespective of whether $w_1$ or $w_2$ belong to $X$), which is a contradiction. We conclude that $G_1$ is weakly $(P_1 + 2P_2)^b$-free. As observed above, this means that $G_1$ has bounded clique-width. Hence $G$ has bounded clique-width.

We now consider the unbounded cases. Let $H^\ell$ be a labelled bipartite graph that is not isomorphic to one of the (bounded) cases considered already. Suppose

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that $H$ contains a cycle or an induced subgraph isomorphic to $2P_3$. Then the class of weakly $H^\ell$-free graphs has unbounded clique-width by combining Lemma 1 with Theorem 2. Suppose that $H$ contains a vertex of degree at least 3. Then the class of weakly $H^\ell$-free bipartite graphs has unbounded clique-width by Lemma 12(i). It remains to consider the case when $H = sP_1 + tP_2 + P_r$ for some constants $1 \le r \le 6$, $s \ge 0$ and $t \ge 0$, where $\max\{s, t\} \ge 1$ (as $H^\ell$ is not a labelled induced subgraph of $P_6$). We will show that in most of the cases we need to consider, we can find four pairwise disjoint vertices in $H^\ell$ that are not all of the same colour, in which case we apply Lemma 12(ii).

Suppose $5 \le r \le 6$. Assume without loss of generality that three vertices of the copy of $P_r$ in $H^\ell$ are coloured black. If $r = 6$ or $t \ge 1$ or some copy $P_1$ in $H^\ell$ is coloured white, or two copies of $P_1$ in $H^\ell$ are coloured black, then we apply Lemma 12(ii). Hence, $H^\ell = (P_1 + P_5)^b$, which is not possible by assumption.

Suppose $r = 4$. If two vertices in the induced subgraph of $H^\ell$ isomorphic to $sP_1 + tP_2$ have the same colour then we apply Lemma 12(ii). Hence we may assume that $s \le 2$ and $t \le 1$, and moreover that $s = 0$ if $t = 1$. Also we would have $H^\ell \subseteq_{li} (P_2 + P_4)^b$ if $s = 0$ and $t = 1$ or if $s = 1$ and $t = 0$. Hence, it remains to consider the case $s = 2$ and $t = 0$, such that one copy of $P_1$ is coloured black and the other one white. In that case, we apply Lemma 12(ii).

Suppose $r = 3$. Assume without loss of generality that the two vertices of the copy of $P_3$ in $H^\ell$ are coloured black. Recall that $s \ge 1$ or $t \ge 1$. If $t \ge 2$, then we apply Lemma 12(ii). Suppose $t = 1$. Then $s = 0$ otherwise $H^\ell$ would contain an induced $4P_1$ in which not all the vertices are the same colour, in which case we would apply Lemma 12(ii). However, this means that $H^\ell \subseteq_{li} (P_2 + P_4)^b$. Now suppose $t = 0$. Then $s \ge 2$, as otherwise $H^\ell \subseteq_{li} (P_2 + P_4)^b$. If $s \ge 3$ then $H^\ell$ contains an induced $4P_1$ in which not all the vertices are the same colour, in which case we apply Lemma 12(ii). Hence, $s = 2$ and both copies are coloured black (otherwise we apply Lemma 12(ii)). However, in this case $H^\ell$ is a labelled induced subgraph of $(P_1 + P_5)^b$, which is not possible by assumption.

Finally suppose that $r \le 2$. Then we may write $H = sP_1 + tP_2$ instead. We must have $s + t \ge 4$ or $t \ge 3$, otherwise $H^\ell \subseteq_{li} (P_2 + P_4)^b$. If $t = 0$ then since $H^\ell \ne (sP_1)^b$ and $H^\ell \ne (sP_1)^{\overline{b}}$ we can find four copies of $P_1$ in $H$ that are not all of the same colour and apply Lemma 12(ii). If $t \ge 1, s + t \ge 4$, we can also find four copies of $P_1$ that are not all of the same colour and apply Lemma 12(ii). Finally, suppose $s = 0, t = 3$. In this case we combine Lemmas 1 and 11. This completes the proof. $\qquad\square$

## 5 Conclusions

We have completely determined those bipartite graphs $H$ for which the class of $H$-free bipartite graphs has bounded clique-width. We also characterized exactly those labelled bipartite graphs $H$ for which the class of weakly $H$-free bipartite graphs has bounded clique-width. These results complement the known characterization of Lozin and Volz [25] for strongly $H$-free bipartite graphs. A natural

direction for further research would be to characterize, for each of the three notions of $H$-freeness, the clique-width of classes of $\mathcal{H}$-free bipartite graphs when $\mathcal{H}$ is an arbitrary set containing at least two graphs. Here, the underlying research question is to determine what kinds of properties ensure that a graph class has bounded clique-width. As mentioned in Section 1, many results exist in the literature. In a series of follow-up papers [3,4,16,18] we have tried to address this question by determining classes of $(H_1, H_2)$-free (general) graphs, $H$-free split graphs, $H$-free chordal graphs and $H$-free weakly chordal graphs of bounded and unbounded clique-width. In each of these papers, we have applied our results for $H$-free bipartite graphs as useful lemmas.

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