GRADIENT ESTIMATES FOR MEAN CURVATURE FLOW WITH NEUMANN BOUNDARY CONDITIONS

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ABSTRACT. We study the mean curvature flow of graphs both with Neumann boundary conditions and transport terms. We derive boundary gradient estimates for the mean curvature flow. As an application, the existence of the mean curvature flow of graphs is presented. A key argument is a boundary monotonicity formula of a Huisken type derived using reflected backward heat kernels. Furthermore, we provide regularity conditions for the transport terms.

1. Introduction

We consider the mean curvature flow of graphs with transport terms and Neumann boundary conditions:

$$\begin{cases} \dfrac{\partial_t u}{\sqrt{1+|du|^2}} = \operatorname{div}\left(\dfrac{du}{\sqrt{1+|du|^2}}\right) + \boldsymbol{f}(x,u,t)\cdot\dfrac{(-du,1)}{\sqrt{1+|du|^2}}, & x\in\Omega,\ t>0,\\ du\cdot\nu\big|_{\partial\Omega}=0, & t>0,\\ u(x,0)=u_0(x), & x\in\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with a smooth boundary, ν is an outer unit normal vector on $\partial\Omega$, $u = u(x,t) : \Omega\times[0,\infty) \to \mathbb{R}$ is an unknown function, $du := (\partial_{x_1}u, \dots, \partial_{x_n}u)$ is the tangential gradient of u, $u_0 = u_0(x) : \Omega \to \mathbb{R}$ is given initial data, and $\boldsymbol{f} : \Omega\times\mathbb{R}\times[0,\infty) \to \mathbb{R}^{n+1}$ is a given transport term. For a solution u of (1.1) and $t > 0$, the graph of $u(x,t)$, which is

$$\Gamma_t := \{(x, u(x,t)) : x \in \Omega\}, \tag{1.2}$$

satisfies the mean curvature flow with the transport term, which is subjected to right angle boundary conditions given by

$$\begin{cases} \boldsymbol{V} = \boldsymbol{H} + (\boldsymbol{f}\cdot\boldsymbol{n})\boldsymbol{n}, & \text{on } \Gamma_t,\ t>0,\\ \Gamma_t \perp \partial(\Omega\times\mathbb{R}), & t>0, \end{cases} \tag{1.3}$$

where $\boldsymbol{n} := \frac{1}{\sqrt{1+|du|^2}}(-du,1)$ is the unit normal vector of Γ_t, $\boldsymbol{V} := \frac{\partial_t u}{\sqrt{1+|du|^2}}\boldsymbol{n}$ is the normal velocity vector of Γ_t, and $\boldsymbol{H} := \operatorname{div}(\frac{du}{\sqrt{1+|du|^2}})\boldsymbol{n}$ is the mean curvature vector of Γ_t. To study the behavior of Γ_t, we need to investigate $v := \sqrt{1+|du|^2}$, which is the volume element of Γ_t. Thus, it is important to derive gradient estimates for (1.1). Interior gradient estimates for (1.1) were studied in [3, 13], and sharp gradient estimates for (1.1)

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with the assumption $\boldsymbol{f} \equiv 0$ were obtained in [2] via the maximum principle. For the mean curvature flow with Neumann boundary conditions and without transport terms, several up-to-the-boundary regularity results were provided in [1, 4, 5, 8, 11, 12]. Takasao provided a sufficient condition for the transport terms to obtain interior gradient estimates for (1.1) in [13], but the condition was not optimal. However, reasonable conditions for the transport terms for the regularity of weak mean curvature flow were obtained in [7, 14]. In this paper, we obtain an a priori gradient estimate with reasonable conditions for the transport terms.

Our problem (1.1) imposes Neumann boundary conditions; thus, up-to-the-boundary gradient estimates are also important. The first author studied weak mean curvature flow with Neumann boundary conditions via phase field methods in [10]. To study boundary behavior, it was important to derive an ε-diffused boundary monotonicity formula of a Huisken type via a reflected backward heat kernel (cf. [5], [6]). Thus, it is also important to derive a boundary monotonicity formula for (1.1) and determine the optimal regularity condition for the transport terms. In this paper, we derive a boundary monotonicity formula for (1.1) and as an application, we derive an a priori boundary gradient estimate and prove the existence of a classical solution of (1.1).

This paper is organized as follows. In section 2, we present basic notation and the main results. In section 3, we derive a boundary monotonicity formula for (1.1). In section 4, we derive the boundary gradient estimates for (1.1) and some integral estimates for the transport terms. In section 5, we prove the existence of the classical solution of (1.1).

2. Preliminaries and main results

2.1. **Notation.** Let $\boldsymbol{\nu}$ be an outer unit normal vector on $\partial(\Omega \times \mathbb{R})$; $\boldsymbol{\nu} = (\nu, 0)$. For n-dimensional symmetric matrices A and B, define the inner product $A : B$ as $A : B = \operatorname{tr}(AB)$. Set $Q_T := \Omega \times (0, T)$ and $Q_T^{\varepsilon} := \Omega \times (\varepsilon, T)$ for $0 < \varepsilon < T$. Let d and D be the gradients of Ω and $\Omega \times \mathbb{R}$, respectively. Let D_{Γ_t} and Δ_{Γ_t} be the gradient and Laplacian of Γ_t, respectively. For a solution u of (1.1), let $h := -\operatorname{div}\left(\frac{du}{\sqrt{1+|du|^2}}\right)$, $v := \sqrt{1+|du|^2}$. Then, equation (1.1) becomes

$$\partial_t u = -vh + (\boldsymbol{f} \cdot \boldsymbol{n})v. \tag{2.1}$$

2.2. **Main results.** Let $T_0 > 0$ be fixed. We impose a regularity assumption on the transport term such that

$$\|\boldsymbol{f}\|_{L_t^q L_x^p(\Gamma_t)} := \left(\int_0^{T_0} \left(\int_{\Gamma_t} |\boldsymbol{f}(X,t)|^p \, d\mathscr{H}^n\right)^{\frac{q}{p}} dt\right)^{\frac{1}{q}} < \infty, \ \frac{n}{p} + \frac{2}{q} < 1, \ p, q \geq 1. \tag{2.2}$$

Remark 2.1. Using the Meyer-Ziemer inequality (cf. [15, p. 266, Theorem 5.12.4]),

$$\int_{\Gamma_t} |\boldsymbol{f}(X,t)|^p \, d\mathscr{H}^n \leq C \|\boldsymbol{f}(\cdot, t)\|^p_{W^{1,p}(\Omega\times\mathbb{R})};$$

hence, our assumption (2.2) is fulfilled if $\boldsymbol{f} \in L^q([0, T_0] : W^{1,p}(\Omega \times \mathbb{R}))$.

First we derive an a priori gradient estimate for (1.1).

Theorem 2.2 (A priori estimate for the gradient). *Let u be a classical solution of* (1.1) *on $\Omega \times (0, T_0)$. Assume Ω is convex, $u_0 \in W^{1,\infty}(\Omega)$, and the transport term $\boldsymbol{f}$ satisfies* (2.2). *Then there exist $T > 0$ and $C_1 > 0$ such that*

$$\sup_{0<t<T, x\in\overline{\Omega}} \sqrt{1 + |du(x,t)|^2} \le C_1(1 + \|du_0\|_\infty^2). \tag{2.3}$$

The regularity assumption (2.2) is reasonable. Moreover, using the scale transform

$$x = \lambda y, \quad t = \lambda^2 s, \quad w(y,s) = \frac{1}{\lambda} u(x,t),$$

we obtain

$$\begin{cases} \dfrac{\partial_s w}{\sqrt{1+|dw|^2}} = \operatorname{div}\left(\dfrac{dw}{\sqrt{1+|dw|^2}}\right) + \lambda \boldsymbol{f}(\lambda y, \lambda w, \lambda^2 s) \cdot \dfrac{(-dw, 1)}{\sqrt{1+|dw|^2}}, \\ dw = du. \end{cases}$$

Then

$$\|\lambda \boldsymbol{f}(\lambda y, \lambda w, \lambda^2 s)\|_{L^q_s L^p_y} = \lambda^{1-\frac{n}{p}-\frac{2}{q}} \|\boldsymbol{f}\|_{L^q_t L^p_x}$$

and $\|\lambda \boldsymbol{f}(\lambda y, \lambda w, \lambda^2 s)\|_{L^q_s L^p_y} \to 0$ as $\lambda \to 0$ if (2.2) is fulfilled; that is, the transport is a small perturbation for blow-up arguments. Note that the regularity assumption (2.2) is the same as the assumption for the parabolic Allard's regularity theory developed by Kasai-Tonegawa [7, 14]. Furthermore, our results include results from the study by the second author [13] because our argument also applies to interior gradient estimates.

From the regularity estimate (2.3), the graph Γ_t subjected to (1.2) is C^1-Riemannian manifold up to the boundary. Furthermore, the graph Γ_t is perpendicular to $\partial\Omega \times \mathbb{R}$, which is the boundary of a cylinder $\Omega \times \mathbb{R}$. In terms of partial differential equations, Theorem 2.2 can be regarded as an up-to-the-boundary parabolic smoothing effect for $\partial_t u - \sqrt{1+|du|^2} \operatorname{div}\Big(\frac{du}{\sqrt{1+|du|^2}}\Big)$. The non-divergence elliptic differential operator $-\sqrt{1+|du|^2} \operatorname{div}\Big(\frac{du}{\sqrt{1+|du|^2}}\Big)$ is degenerate; however, using the gradient estimates, we may consider the elliptic operator to be uniformly elliptic. Theorem 2.2 also can be regarded as a parabolic smoothing effect for the mean curvature operator. To summarize, (2.3) determines how we obtain regularity of the mean curvature flow.

To prove Theorem 2.2, we derive a boundary monotonicity formula of a Huisken type for (1.1). We introduce reflected backward heat kernels to compute the boundary integrals and derive integral estimates for the transport terms under assumption (2.2). Theorem 2.2 is obtained using upper Gaussian density bounds for the volume element $\sqrt{1+|du|^2}\, dx$. Note that our argument does not rely on the maximum principle for (1.1).

Next, we demonstrate the existence of a classical solution of (1.1). We assume parabolic Hölder continuity for $\boldsymbol{f}$; that is, there is $\alpha \in (0,1]$ such that

$$K := \sup_{(X,t),(Y,s)\in(\Omega\times\mathbb{R})\times(0,T_0)} \frac{|\boldsymbol{f}(X,t) - \boldsymbol{f}(Y,s)|}{|X-Y|^\alpha + |t-s|^{\alpha/2}} < \infty \tag{2.4}$$

Theorem 2.3 (Existence of a classical solution). *Assume Ω is convex, $u_0 \in W^{1,\infty}(\Omega)$ with $du_0 \cdot \nu = 0$ on $\partial\Omega$ and the transport term $\boldsymbol{f}$ satisfies* (2.4) *with some $\alpha \in (0,1]$. Then, there exist a constant $T > 0$ and a unique solution $u \in C(\overline{Q_T}) \cap C^{2,\alpha}(Q_T^\varepsilon)$ (for all*

$\varepsilon \in (0,T)$) of (1.1) with $u(0) = u_0$. Furthermore, for any $\varepsilon > 0$ there exists $C_2 > 0$ such that

$$\|u\|_{C^{2,\alpha}(Q_T^\varepsilon)} \le C_2. \tag{2.5}$$

Theorem 2.3 provides a sufficient condition for studying the relationship between the boundary of the mean curvature flow and transport terms in the C^2 sense. Brakke flow without transport was constructed by the first author and Tonegawa [10]. According to the results of Theorems 2.2 and 2.3, we can demonstrate the existence and regularity of Brakke flow with Neumann boundary conditions and transport terms with assumption (2.3) or (2.4).

Theorem 2.3 is deduced from the Schauder fixed point theorem for the linearized problem of (1.1). Theorem 2.2 is employed as an a priori gradient estimate for the Schauder fixed point theorem. As a result of the gradient bounds, the mean curvature operator can be computed in the same class as the uniformly elliptic operator; hence, we can derive the Schauder estimates for (1.1) and apply the Schauder fixed point theorem.

3. Monotonicity of the metric

Our first task is to establish an up-to-the-boundary monotonicity formula of a Huisken type.

Lemma 3.1.

$$\partial_t v - \Delta_{\Gamma_t} v - \left(\frac{du}{v}\cdot dv\right)\frac{\partial_t u}{v} = -|A_t|^2 v - \frac{2|D_{\Gamma_t} v|^2}{v} + du\cdot d(\boldsymbol{f}\cdot\boldsymbol{n}), \tag{3.1}$$

where A_t is the second fundamental form of Γ_t.

Proof. According to Ecker-Huisken [3],

$$-\Delta_{\Gamma_t} v + |A_t|^2 v + \frac{2|D_{\Gamma_t} v|^2}{v} - v^2(D_{\Gamma_t} h\cdot \boldsymbol{e}_{n+1}) = 0$$

where $\boldsymbol{e}_{n+1} = (0,\dots,0,1)$. Because

$$\begin{aligned} v^2(D_{\Gamma_t} h\cdot\boldsymbol{e}_{n+1}) &= v^2(Dh\cdot\boldsymbol{e}_{n+1} - (Dh\cdot\boldsymbol{n})(\boldsymbol{n}\cdot\boldsymbol{e}_{n+1}))\\ &= dh\cdot du\\ &= -d\left(\frac{\partial_t u}{v}\right)\cdot du + d(\boldsymbol{f}\cdot\boldsymbol{n})\cdot du \quad (\because (1.1))\\ &= -\partial_t v + \left(\frac{du}{v}\cdot dv\right)\frac{\partial_t u}{v} + d(\boldsymbol{f}\cdot\boldsymbol{n})\cdot du, \end{aligned}$$

we obtain (3.1). $\square$

Let

$$R := \frac{1}{\|\text{principal curvature of } \partial\Omega\|_{L^\infty(\partial\Omega)}}.$$

Because $\partial\Omega$ is smooth and compact, $0 < R \le \infty$. For $r < R$, let N_r denote the interior tubular neighborhood of $\partial\Omega$;

$$N_r := \{x\in\Omega : \mathrm{dist}(x,\partial\Omega) < r\}.$$

For $x \in N_r$, there uniquely exists $\zeta(x) \in \partial\Omega$ such that $\operatorname{dist}(x, \partial\Omega) = |x - \zeta(x)|$. Thus, we define the reflection point x with respect to $\partial\Omega$ as $\tilde{x} = 2\zeta(x) - x$. We fix a radially symmetric cut-off function $\eta = \eta(|X|) \in C^\infty(\mathbb{R}^{n+1})$ such that

$$0 \le \eta \le 1, \quad \frac{\partial \eta}{\partial r} \le 0, \quad \operatorname{spt} \eta \subset B_{R/2}, \quad \eta = 1 \text{ on } B_{R/4}.$$

For $0 < t < s$ and $X = (x, x_{n+1})$, $Y = (y, y_{n+1}) \in N_R \times \mathbb{R}$, we define the n-dimensional backward heat kernel $\rho_{(Y,s)}(X,t)$ and reflected backward heat kernel $\tilde{\rho}_{(Y,s)}(X,t)$ as

$$\begin{aligned} \rho_{(Y,s)}(X,t) &:= \frac{1}{(4\pi(s-t))^{\frac{n}{2}}} \exp\left(-\frac{|X-Y|^2}{4(s-t)}\right), \\ \tilde{\rho}_{(Y,s)}(X,t) &:= \frac{1}{(4\pi(s-t))^{\frac{n}{2}}} \exp\left(-\frac{|\tilde{X}-Y|^2}{4(s-t)}\right), \end{aligned} \tag{3.2}$$

where $\tilde{X} = (\tilde{x}, x_{n+1})$. For fixed $0 < t < s$ and $X, Y \in N_R \times \mathbb{R}$, we define a truncated version of ρ and $\tilde{\rho}$ as

$$\begin{aligned} \rho_1 = \rho_1(X,t) &:= \eta(X-Y)\rho_{(Y,s)}(X,t), \\ \rho_2 = \rho_2(X,t) &:= \eta(\tilde{X}-Y)\tilde{\rho}_{(Y,s)}(X,t). \end{aligned} \tag{3.3}$$

To derive Huisken's monotonicity formula,

$$\frac{(\boldsymbol{w} \cdot D\rho)^2}{\rho} + ((I - \boldsymbol{w} \otimes \boldsymbol{w}) : D^2\rho) + \partial_t \rho = 0 \tag{3.4}$$

is the crucial identity, where $\rho = \rho_{(Y,s)}(X,t)$ and $\boldsymbol{w} \in \mathbb{R}^{n+1}$ is any unit vector. In [10], a similar identity for the reflected backward heat kernel $\tilde{\rho}_{(Y,s)}$ was obtained.

Lemma 3.2 ([10]). *For $\boldsymbol{w} = (w_i) \in \mathbb{R}^{n+1}$ with $|\boldsymbol{w}| = 1$ and $\tilde{\rho} = \tilde{\rho}_{(Y,s)}(X,t)$,*

$$\begin{aligned} &\frac{(\boldsymbol{w} \cdot D\tilde{\rho})^2}{\tilde{\rho}} + ((I - \boldsymbol{w} \otimes \boldsymbol{w}) : D^2\tilde{\rho}) + \partial_t \tilde{\rho} \\ &\qquad = \sum_{i,j,k=1}^{n+1} \left(\frac{(\delta_{ij} - w_i w_j) D_{X_j}(\nu_i \nu_k)(\tilde{X}_k - Y_k)}{s-t} \right) \tilde{\rho} \end{aligned} \tag{3.5}$$

for $0 < t < s$ and $X, Y \in N_R \times \mathbb{R}$, where $\boldsymbol{\nu} = (\nu_i) = (\nu_i(\zeta(X)))$ is the unit outer-pointing normal to $\partial\Omega \times \mathbb{R}$ and $(\delta_{ij}) = I$.

Proof. In the following, we prove for self-containedness. Because $D\zeta(X) = I - \boldsymbol{\nu} \otimes \boldsymbol{\nu}$ and $\tilde{X} = 2\zeta(X) - X$,

$$\begin{aligned} D|\tilde{X} - Y|^2 &= 2(I - \boldsymbol{\nu} \otimes \boldsymbol{\nu})(\tilde{X} - Y), \\ \partial_{X_i}\partial_{X_j}|\tilde{X} - Y|^2 &= 2\delta_{ij} - 4\sum_k \partial_{X_k}(\nu_i \nu_k)(\tilde{X}_k - Y_k). \end{aligned} \tag{3.6}$$

Using (3.6), we obtain

$$
\begin{aligned}
\partial_t \tilde{\rho} &= \left(\frac{n}{2(s-t)} - \frac{|\tilde{X}-Y|^2}{4(s-t)^2} \tilde{\rho} \right), \\
D\tilde{\rho} &= -\frac{D|\tilde{X}-Y|^2}{4(s-t)} \tilde{\rho}, \\
D^2\tilde{\rho} &= \left(\frac{D|\tilde{X}-Y|^2 \otimes D|\tilde{X}-Y|^2}{16(s-t)^2} - \frac{1}{2(s-t)} \right. \\
&\qquad \left. + \left(\sum_k \frac{\partial_{X_k}(\nu_i \nu_k)(\tilde{X}_k - Y_k)}{s-t} \right)_{i,j} \right) \tilde{\rho}.
\end{aligned} \tag{3.7}
$$

Using (3.7) and noting that $|D|\tilde{X}-Y|^2|^2 = 4|\tilde{X}-Y|^4$, we obtain (3.5). □

We next prove a local boundary monotonicity inequality.

Lemma 3.3. *Let $\phi \in C^1([0,\infty) : C^2(\Omega))$ be a non-negative function and $0 < \alpha < \frac{1}{2}$. Then there exist positive numbers C_3, C_4 and $C_5 > 0$ such that*

$$
\begin{aligned}
&\frac{d}{dt} \int_{\Gamma_t} \phi(\rho_1 + \rho_2)\, d\mathscr{H}^n \\
&\le \int_{\Gamma_t} (\rho_1 + \rho_2) \left(\partial_t \phi - \Delta_{\Gamma_t} \phi - \left(d\phi \cdot \frac{du}{v} \right) \frac{\partial_t u}{v} \right) d\mathscr{H}^n \\
&\quad + \frac{1}{4} \int_{\Gamma_t} \phi(\rho_1 + \rho_2)(\boldsymbol{f} \cdot \boldsymbol{n})^2\, d\mathscr{H}^n \\
&\quad + C_3 \mathscr{H}^n(\Gamma_t) + C_4 (s-t)^{-\alpha} \int_{\Gamma_t} \phi(\rho_1 + \rho_2)\, d\mathscr{H}^n \\
&\quad + C_5 \int_{\Gamma_t \cap \operatorname{spt} \rho_2} |\tilde{X} - Y| \phi\, d\mathscr{H}^n \\
&\quad + \int_{\partial \Gamma_t} (\rho_1 + \rho_2)(D_{\Gamma_t} \phi \cdot \boldsymbol{\nu})\, d\mathscr{H}^{n-1}.
\end{aligned} \tag{3.8}
$$

Proof. **1.** For $i = 1, 2$

$$
\begin{aligned}
\frac{d}{dt}\int_{\Gamma_t} \phi\rho_i \, d\mathscr{H}^n &= \frac{d}{dt}\int_\Omega \phi(x,t)\rho_i(x,u(x,t),t)v(x,t)\,dx \\
&= \int_\Omega \partial_t\phi(x,t)\rho_i(x,u(x,t),t)v(x,t)\,dx \\
&\quad + \int_\Omega \phi(x,t)\partial_{x_{n+1}}\rho_i(x,u(x,t),t)\partial_t u(x,t)v(x,t)\,dx \\
&\quad + \int_\Omega \phi(x,t)\partial_t\rho_i(x,u(x,t),t)v(x,t)\,dx \qquad (3.9)\\
&\quad + \int_\Omega \phi(x,t)\rho_i(x,u(x,t),t)\partial_t v(x,t)\,dx \\
&= \int_{\Gamma_t} (\partial_t\phi\rho_i + \phi\partial_t\rho_i + \phi\partial_{x_{n+1}}\rho_i\partial_t u)\,d\mathscr{H}^n \\
&\quad + \int_\Omega \phi(x,t)\rho_i(x,u(x,t),t)\frac{du(x,t)\cdot d(\partial_t u(x,t))}{v(x,t)}\,dx.
\end{aligned}
$$

2. We consider the last term of equation (3.9). Using integration by parts, we obtain

$$
\begin{aligned}
&\int_\Omega \phi(x,t)\rho_i(x,u(x,t),t)\frac{du(x,t)\cdot d(\partial_t u(x,t))}{v(x,t)}\,dx \\
&= -\int_\Omega \operatorname{div}\left(\phi(x,t)\rho_i(x,u(x,t),t)\frac{du(x,t)}{v(x,t)}\right)\partial_t u(x,t)\,dx \\
&= -\int_\Omega d\phi(x,t)\cdot\frac{du(x,t)}{v(x,t)}\rho_i(x,u(x,t),t)\frac{\partial_t u(x,t)}{v(x,t)}v(x,t)\,dx \\
&\quad - \int_\Omega \phi(x,t)\big(d\rho_i(x,u(x,t),t) \\
&\quad + \partial_{x_{n+1}}\rho_i(x,u(x,t),t)du(x,t)\big)\cdot\frac{du(x,t)}{v(x,t)}\frac{\partial_t u(x,t)}{v(x,t)}v(x,t)\,dx \\
&\quad - \int_\Omega \phi(x,t)\rho_i(x,u(x,t),t)\operatorname{div}\left(\frac{du(x,t)}{v(x,t)}\right)\frac{\partial_t u(x,t)}{v(x,t)}v(x,t)\,dx \\
&= -\int_{\Gamma_t}\left(d\phi\cdot\frac{du}{v}\right)\rho_i\frac{\partial_t u}{v}\,d\mathscr{H}^n - \int_{\Gamma_t}\phi\left((d\rho_i + \partial_{x_{n+1}}\rho_i du)\cdot\frac{du}{v}\right)\frac{\partial_t u}{v}\,d\mathscr{H}^n \\
&\quad + \int_{\Gamma_t}\phi\rho_i h\frac{\partial_t u}{v}\,d\mathscr{H}^n.
\end{aligned}
$$

We note that

$$
\begin{aligned}
\partial_{x_{n+1}}\rho_i v - \partial_{x_{n+1}}\rho_i\frac{|du|^2}{v} - d\rho_i\cdot\frac{du}{v} &= \partial_{x_{n+1}}\rho_i\left(v - \frac{|du|^2}{v}\right) - d\rho_i\cdot\frac{du}{v} \\
&= \partial_{x_{n+1}}\rho_i\frac{1}{v} - d\rho_i\cdot\frac{du}{v} = (D\rho\cdot\boldsymbol{n}).
\end{aligned}
$$

Hence, we obtain

$$\frac{d}{dt}\int_{\Gamma_t}\phi\rho_i\,d\mathscr{H}^n=\int_{\Gamma_t}\partial_t\phi\rho_i\,d\mathscr{H}^n+\int_{\Gamma_t}\phi\partial_t\rho_i\,d\mathscr{H}^n+\int_{\Gamma_t}\phi(D\rho_i\cdot\boldsymbol{n})\frac{\partial_t u}{v}\,d\mathscr{H}^n$$
$$-\int_{\Gamma_t}\Big(d\phi\cdot\frac{du}{v}\Big)\rho_i\frac{\partial_t u}{v}\,d\mathscr{H}^n+\int_{\Gamma_t}\phi\rho_i h\frac{\partial_t u}{v}\,d\mathscr{H}^n.$$

3. Using (1.1) or (2.1),

$$\begin{aligned}
&(\phi(D\rho_i\cdot\boldsymbol{n})+\phi\rho_i h)\,\frac{\partial_t u}{v}\\
&=(\phi(D\rho_i\cdot\boldsymbol{n})+\phi\rho_i h)\,(-h+\boldsymbol{f}\cdot\boldsymbol{n})\\
&=\phi(D\rho_i\cdot\boldsymbol{H})-\phi\rho_i|\boldsymbol{H}|^2+\phi\rho_i\left(\left(\frac{D^\perp\rho_i}{\rho_i}-\boldsymbol{H}\right)\cdot\boldsymbol{n}\right)(\boldsymbol{f}\cdot\boldsymbol{n})\\
&=-\phi\rho_i\left|\boldsymbol{H}-\frac{D^\perp\rho_i}{\rho_i}\right|^2+\phi\frac{|D^\perp\rho_i|^2}{\rho_i}-\phi(D^\perp\rho_i\cdot\boldsymbol{H})\\
&\quad+\phi\rho_i\left(\left(\frac{D^\perp\rho_i}{\rho_i}-\boldsymbol{H}\right)\cdot\boldsymbol{n}\right)(\boldsymbol{f}\cdot\boldsymbol{n})\\
&\le\phi\frac{|D^\perp\rho_i|^2}{\rho_i}-\phi(D^\perp\rho_i\cdot\boldsymbol{H})+\frac{1}{4}\phi\rho_i(\boldsymbol{f}\cdot\boldsymbol{n})^2,
\end{aligned}$$

where $\boldsymbol{H}=-h\boldsymbol{n}$ and $D^\perp\rho_i=(D\rho_i\cdot\boldsymbol{n})\boldsymbol{n}$ are used. Therefore,

$$\begin{aligned}
\frac{d}{dt}\int_{\Gamma_t}\phi\rho_i\,d\mathscr{H}^n\le&\int_{\Gamma_t}\partial_t\phi\rho_i\,d\mathscr{H}^n-\int_{\Gamma_t}\Big(d\phi\cdot\frac{du}{v}\Big)\rho_i\frac{\partial_t u}{v}\,d\mathscr{H}^n\\
&+\int_{\Gamma_t}\phi\left(\partial_t\rho_i+\frac{|D^\perp\rho_i|^2}{\rho_i}-(D^\perp\rho_i\cdot\boldsymbol{H})\right)d\mathscr{H}^n\\
&+\frac{1}{4}\int_{\Gamma_t}\phi\rho_i(\boldsymbol{f}\cdot\boldsymbol{n})^2\,d\mathscr{H}^n.
\end{aligned}$$

According to the divergence theorem on Γ_t,

$$\begin{aligned}
&-\int_{\Gamma_t}\phi(D^\perp\rho_i\cdot\boldsymbol{H})\,d\mathscr{H}^n\\
&=-\int_{\Gamma_t}\phi(D\rho_i\cdot\boldsymbol{H})\,d\mathscr{H}^n\\
&=\int_{\Gamma_t}\mathrm{div}_{\Gamma_t}(\phi D\rho_i)\,d\mathscr{H}^n-\int_{\partial\Gamma_t}\phi(D\rho_i\cdot\boldsymbol{\nu})\,d\mathscr{H}^{n-1}\\
&=\int_{\Gamma_t}D_{\Gamma_t}\phi\cdot D\rho_i\,d\mathscr{H}^n+\int_{\Gamma_t}\phi((I-\boldsymbol{n}\otimes\boldsymbol{n}):D^2\rho_i)\,d\mathscr{H}^n\\
&\quad-\int_{\partial\Gamma_t}\phi(D\rho_i\cdot\boldsymbol{\nu})\,d\mathscr{H}^{n-1}\\
&=-\int_{\Gamma_t}\rho_i\Delta_{\Gamma_t}\phi\,d\mathscr{H}^n+\int_{\Gamma_t}\phi((I-\boldsymbol{n}\otimes\boldsymbol{n}):D^2\rho_i)\,d\mathscr{H}^n\\
&\quad+\int_{\partial\Gamma_t}(\rho_i(D_{\Gamma_t}\phi\cdot\boldsymbol{\nu})-\phi(D\rho_i\cdot\boldsymbol{\nu}))\,d\mathscr{H}^{n-1}.
\end{aligned}$$

Using (3.4) and (3.5), we obtain

$$\frac{|D^{\perp}\rho_1|^2}{\rho_1} + ((I - \boldsymbol{n}\otimes\boldsymbol{n}) : D^2\rho_1) + \partial_t \rho_1 \leq C_6 \tag{3.10}$$

and

$$\begin{aligned} &\frac{|D^{\perp}\rho_2|^2}{\rho_2} + ((I - \boldsymbol{n}\otimes\boldsymbol{n}) : D^2\rho_2) + \partial_t \rho_2 \\ &\leq \sum_{i,j,k=1}^{n+1} \left(\frac{(\delta_{ij} - n_i n_j) D_{X_j}(\nu_i \nu_k)(\tilde{X}_k - Y_k)}{s-t} \right) \tilde{\rho} + C_8 \leq \frac{C_7|\tilde{X} - Y|}{s-t}\rho_2 + C_8 \end{aligned} \tag{3.11}$$

for some constants C_6, C_7, $C_8 > 0$.

To compute the integration of (3.11), we decompose the integration as

$$\begin{aligned} &\int_{\Gamma_t} \phi \frac{C_7|\tilde{X} - Y|}{s-t}\rho_2 \, d\mathscr{H}^n \\ &\leq \int_{\Gamma_t \cap \{|\tilde{X} - Y| \leq (s-t)^{\frac{1}{4}}\}} \phi \frac{C_7|\tilde{X} - Y|}{s-t}\rho_2 \, d\mathscr{H}^n \\ &\quad + \int_{\Gamma_t \cap \{|\tilde{X} - Y| \geq (s-t)^{\frac{1}{4}}\}} \phi \frac{C_7|\tilde{X} - Y|}{s-t}\rho_2 \, d\mathscr{H}^n \\ &=: I_1 + I_2. \end{aligned}$$

I_1 is estimated as

$$I_1 \leq C_7 (s-t)^{-\frac{3}{4}} \int_{\Gamma_t \cap \{|\tilde{X} - Y| \leq (s-t)^{\frac{1}{4}}\}} \phi\rho_2 \, d\mathscr{H}^n \leq C_7 (s-t)^{-\frac{3}{4}} \int_{\Gamma_t} \phi\rho_2 \, d\mathscr{H}^n. \tag{3.12}$$

I_2 is estimated by

$$I_2 \leq \frac{C_7}{(s-t)^{1+\frac{n}{2}}} e^{-\frac{1}{4\sqrt{s-t}}} \int_{\Gamma_t \cap \operatorname{spt}\rho_2} \phi|\tilde{X} - Y| \, d\mathscr{H}^n \leq C_9 \int_{\Gamma_t \cap \operatorname{spt}\rho_2} \phi|\tilde{X} - Y| \, d\mathscr{H}^n \tag{3.13}$$

for some constant $C_9 > 0$.

Using (3.12), (3.13), and $D(\rho_1 + \rho_2)\cdot\boldsymbol{\nu}\big|_{\partial\Omega} \equiv 0$, we compute

$$\begin{aligned} &\frac{d}{dt}\int_{\Gamma_t} \phi(\rho_1 + \rho_2) \, d\mathscr{H}^n \\ &\leq \int_{\Gamma_t} (\rho_1 + \rho_2) \left(\partial_t \phi - \Delta_{\Gamma_t}\phi - \left(d\phi \cdot \frac{du}{v} \right) \frac{\partial_t u}{v} \right) d\mathscr{H}^n \\ &\quad + \frac{1}{4}\int_{\Gamma_t} (\rho_1 + \rho_2)\phi(\boldsymbol{f}\cdot\boldsymbol{n})^2 \, d\mathscr{H}^n \\ &\quad + (C_6 + C_8)\mathscr{H}^n(\Gamma_t) + C_7(s-t)^{-\frac{3}{4}} \int_{\Gamma_t} \phi\rho_2 \, d\mathscr{H}^n \\ &\quad + C_9 \int_{\Gamma_t \cap \operatorname{spt}\rho_2} \phi|\tilde{X} - Y| \, d\mathscr{H}^n + \int_{\partial\Gamma_t} (\rho_1 + \rho_2)(D_{\Gamma_t}\phi\cdot\boldsymbol{\nu}) \, d\mathscr{H}^{n-1}. \end{aligned}$$

For $C_3 = \frac{C_6 + C_8}{4}$, $C_4 = C_7$, and $C_5 = C_9$, we obtain (3.8). □

We use the following lemma to handle the boundary integral.

Lemma 3.4. *If Ω is convex, then*

$$(D_{\Gamma_t} v \cdot \boldsymbol{\nu})|_{\partial(\Omega\times\mathbb{R})} \le 0 \tag{3.14}$$

for all $t > 0$.

Proof. Because

$$\begin{aligned} D_{\Gamma_t} v &= Dv - (Dv \cdot \boldsymbol{n})\boldsymbol{n} \\ &= (dv, 0) + \frac{1}{v^2}(dv \cdot du)(-du, 1), \end{aligned}$$

using the boundary condition of u

$$\begin{aligned} (D_{\Gamma_t} v \cdot \boldsymbol{\nu})|_{\partial(\Omega\times\mathbb{R})} &= \left((dv \cdot \nu) + \frac{1}{v^2}(dv \cdot du)(-du \cdot \nu) \right) \Big|_{\partial\Omega} \\ &= \frac{1}{2v} d|du|^2 \cdot \nu|_{\partial\Omega} \\ &= \frac{1}{v} A_t(du, du)|_{\partial\Omega}, \end{aligned}$$

where A_t is the second fundamental form of Γ_t. Because Ω is convex, $A_t(du, du) \le 0$. $\square$

Using (3.1), (3.8), and (3.14), monotonicity of the metric is obtained as follows:

Proposition 3.5. *For $0 < \alpha < \frac{1}{2}$,*

$$\begin{aligned} &\frac{d}{dt} \int_{\Gamma_t} v(\rho_1 + \rho_2)\, d\mathscr{H}^n \\ &\le -\int_{\Gamma_t} (\rho_1 + \rho_2) \left(|A_t|^2 v + \frac{2|D_{\Gamma_t} v|^2}{v} - du \cdot d(\boldsymbol{f} \cdot \boldsymbol{n}) \right) d\mathscr{H}^n \\ &\quad + \frac{1}{4} \int_{\Gamma_t} v(\rho_1 + \rho_2)(\boldsymbol{f} \cdot \boldsymbol{n})^2\, d\mathscr{H}^n \\ &\quad + C_3 \mathscr{H}^n(\Gamma_t) + C_4 (s-t)^{-\alpha} \int_{\Gamma_t} v(\rho_1 + \rho_2)\, d\mathscr{H}^n \\ &\quad + C_5 \int_{\Gamma_t \cap \operatorname{spt} \rho_2} |\tilde{X} - Y| v\, d\mathscr{H}^n. \end{aligned} \tag{3.15}$$

4. Gradient estimates

We deduce the integral estimates for the transport terms.

Lemma 4.1. *Let $\boldsymbol{f} \in L^p_x L^q_t(\Gamma_t)$ with $1 - \frac{n}{p} - \frac{2}{q} > 0$. Then there is a constant $C_{10} > 0$ depending only on n, p, and q such that*

$$\begin{aligned} \int_0^s dt \int_{\Gamma_t} (\rho_1 + \rho_2) du \cdot d(\boldsymbol{f} \cdot \boldsymbol{n})\, d\mathscr{H}^n &\le \frac{1}{2} \int_0^s dt \int_{\Gamma_t} (\rho_1 + \rho_2)|A_t|^2 v\, d\mathscr{H}^n \\ &\quad + \int_0^s dt \int_{\Gamma_t} (\rho_1 + \rho_2) \frac{|D_{\Gamma_t} v|^2}{v}\, d\mathscr{H}^n \\ &\quad + \|v\|_\infty^3 \|\boldsymbol{f}\|_{L^p_x L^q_t(\Gamma_t)} (1 + \|\boldsymbol{f}\|_{L^p_x L^q_t(\Gamma_t)}). \end{aligned} \tag{4.1}$$

Proof. **1.** For simplicity, set $\bar{\rho} := \rho_1 + \rho_2$. Then

$$\begin{aligned}
&\int_{\Gamma_t} (\rho_1 + \rho_2)(du \cdot d(\boldsymbol{f} \cdot \boldsymbol{n}))\, d\mathscr{H}^n \\
&= \int_{\Omega} \bar{\rho}(du \cdot d(\boldsymbol{f} \cdot \boldsymbol{n}))v\, dx \\
&= -\int_{\Omega} (\bar{\rho}\Delta u v + (du \cdot d(\bar{\rho}(x,u,t)))v + \bar{\rho}(du \cdot dv))(\boldsymbol{f} \cdot \boldsymbol{n})\, dx \\
&= -\int_{\Gamma_t} \left(\bar{\rho}\Delta u + (du \cdot d(\bar{\rho}(x,u,t))) + \bar{\rho}\Big(\frac{du}{v} \cdot dv\Big)\right)(\boldsymbol{f} \cdot \boldsymbol{n})\, d\mathscr{H}^n.
\end{aligned}$$

Here

$$h = -\operatorname{div}\left(\frac{du}{v}\right) = -\frac{1}{v}\Delta u + \frac{1}{v^2}(du \cdot dv);$$

hence,

$$\begin{aligned}
\int_{\Gamma_t} \bar{\rho} du \cdot d(\boldsymbol{f} \cdot \boldsymbol{n})\, d\mathscr{H}^n = &\int_{\Gamma_t} \bar{\rho} v h(\boldsymbol{f} \cdot \boldsymbol{n})\, d\mathscr{H}^n \\
&- 2\int_{\Gamma_t} \bar{\rho}\Big(\frac{du}{v} \cdot dv\Big)(\boldsymbol{f} \cdot \boldsymbol{n})\, d\mathscr{H}^n \\
&- \int_{\Gamma_t} (du \cdot d(\bar{\rho}(x,u,t)))(\boldsymbol{f} \cdot \boldsymbol{n})\, d\mathscr{H}^n \\
=:& \; I_1 + I_2 + I_3.
\end{aligned}$$

2. I_1 is estimated as

$$\begin{aligned}
|I_1| &\le \frac{1}{2n}\int_{\Gamma_t} \bar{\rho} h^2 v\, d\mathscr{H}^n + \frac{n}{2}\int_{\Gamma_t} \bar{\rho} v(\boldsymbol{f} \cdot \boldsymbol{n})^2\, d\mathscr{H}^n \\
&\le \frac{1}{2}\int_{\Gamma_t} \bar{\rho}|A_t|^2 v\, d\mathscr{H}^n + \frac{n}{2}\int_{\Gamma_t} \bar{\rho} v(\boldsymbol{f} \cdot \boldsymbol{n})^2\, d\mathscr{H}^n
\end{aligned} \tag{4.2}$$

because $h^2 \le n|A_t|^2$.

3. Note that $D_{\Gamma_t} v = Dv - (Dv \cdot \boldsymbol{n})\boldsymbol{n}$,

$$\begin{aligned}
|D_{\Gamma_t} v|^2 &= |Dv|^2 - (Dv \cdot \boldsymbol{n})^2 \\
&= |dv|^2 - \frac{1}{v^2}(du \cdot dv)^2 \quad (\because Dv = dv) \\
&\ge |dv|^2 - \frac{1}{v^2}|du|^2|dv|^2 \\
&= |dv|^2\left(1 - \frac{1}{v^2}(v^2 - 1)\right) \\
&= \frac{1}{v^2}|dv|^2.
\end{aligned}$$

Therefore,

$$
\begin{aligned}
|I_2| &\le \int_{\Gamma_t} \bar{\rho}\frac{(du\cdot dv)^2}{v^5}\,d\mathscr{H}^n + \int_{\Gamma_t} \bar{\rho}v^3(\boldsymbol{f}\cdot\boldsymbol{n})^2\,d\mathscr{H}^n \\
&\le \int_{\Gamma_t} \bar{\rho}\frac{|du|^2|dv|^2}{v^5}\,d\mathscr{H}^n + \int_{\Gamma_t} \bar{\rho}v^3(\boldsymbol{f}\cdot\boldsymbol{n})^2\,d\mathscr{H}^n \\
&= \int_{\Gamma_t} \bar{\rho}\left(\frac{1}{v^3}-\frac{1}{v^5}\right)|dv|^2\,d\mathscr{H}^n + \int_{\Gamma_t} \bar{\rho}v^3(\boldsymbol{f}\cdot\boldsymbol{n})^2\,d\mathscr{H}^n \\
&\le \int_{\Gamma_t} \bar{\rho}\frac{|D_{\Gamma_t}v|^2}{v}\,d\mathscr{H}^n + \int_{\Gamma_t} \bar{\rho}v^3(\boldsymbol{f}\cdot\boldsymbol{n})^2\,d\mathscr{H}^n.
\end{aligned}
\tag{4.3}
$$

4. In the following, we derive the integral estimates for the transport terms. Using the Hölder inequality,

$$
\begin{aligned}
&\left|\int_0^s dt\int_{\Gamma_t}\bar{\rho}(\boldsymbol{f}\cdot\boldsymbol{n})^2v^3\,d\mathscr{H}^n\right| \\
&\le \|v\|_\infty^3\left(\int_0^s dt\left(\int_{\Gamma_t}\bar{\rho}^{p'}\,d\mathscr{H}^n\right)^{\frac{q'}{p'}}\right)^{\frac{1}{q'}}\|\boldsymbol{f}\|^2_{L^p_xL^q_t(\Gamma_t)},
\end{aligned}
$$

where $\frac{2}{p}+\frac{1}{p'}=1$ and $\frac{2}{q}+\frac{1}{q'}=1$. Using the convexity of Ω, $|\tilde{X}-Y|\ge|X-Y|$; hence,

$$
\begin{aligned}
\int_{\Gamma_t}\bar{\rho}^{p'}\,d\mathscr{H}^n &\le \frac{2}{(4\pi(s-t))^{\frac{np'}{2}}}\int_{\Gamma_t}\exp\left(-\frac{p'|X-Y|^2}{4(s-t)}\right)d\mathscr{H}^n \\
&\le C_{11}(s-t)^{-\frac{np'}{2}+\frac{n}{2}},
\end{aligned}
$$

where $C_{11}>0$ is some constant. Therefore,

$$
\left(\int_0^s dt\left(\int_{\Gamma_t}\bar{\rho}^{p'}\,d\mathscr{H}^n\right)^{\frac{q'}{p'}}\right)^{\frac{1}{q'}}<\infty
$$

if $-\frac{nq'}{2}+\frac{nq'}{2p'}>-1$, which provides $1-\frac{n}{p}-\frac{2}{q}>0$. Using (4.2) and (4.3), we obtain

$$
\begin{aligned}
\int_0^s(|I_1|+|I_2|)\,dt \le{}& \frac{1}{2}\int_0^s dt\int_{\Gamma_t}\bar{\rho}|A_t|^2v\,d\mathscr{H}^n + \int_0^s dt\int_{\Gamma_t}\bar{\rho}\frac{|D_{\Gamma_t}v|^2}{v}\,d\mathscr{H}^n \\
&+ C_{12}\|v\|_\infty^3\|\boldsymbol{f}\|^2_{L^p_xL^q_t(\Gamma_t)}
\end{aligned}
\tag{4.4}
$$

for a positive constant $C_{12}>0$.

5. Because

$$
|du\cdot d(\bar{\rho}(x,u,t))| = |du\cdot d\bar{\rho}+|du|^2\bar{\rho}_{x_{n+1}}|\le v^2|D\bar{\rho}|,
$$

we obtain

$$
|I_3|\le\int_{\Gamma_t}v^2|D\bar{\rho}||\boldsymbol{f}\cdot\boldsymbol{n}|\,d\mathscr{H}^n.
$$

Then using the Hölder inequality,

$$
\int_0^s|I_3|\,dt\le\|v\|_\infty^2\left(\int_0^s dt\left(\int_{\Gamma_t}|D\bar{\rho}|^{p'}\,d\mathscr{H}^n\right)^{\frac{q'}{p'}}\right)^{\frac{1}{q'}}\|\boldsymbol{f}\|_{L^p_xL^q_t(\Gamma_t)},
\tag{4.5}
$$

where $\frac{1}{p} + \frac{1}{p'} = 1$ and $\frac{1}{q} + \frac{1}{q'} = 1$. Using the convexity of Ω,

$$|D\bar{\rho}| \leq C_{13} \frac{1}{(s-t)^{\frac{1}{2}+\frac{n}{2}}} \exp\left(-\frac{|X-Y|^2}{8(s-t)}\right),$$

where $C_{13} > 0$ is some constant. Therefore,

$$\int_{\Gamma_t} |D\bar{\rho}|^{p'} \, d\mathscr{H}^n \leq C_{13}(s-t)^{-\frac{p'}{2}-\frac{np'}{2}+\frac{n}{2}};$$

hence,

$$\left(\int_0^s dt \left(\int_{\Gamma_t} |D\bar{\rho}|^{p'} \, d\mathscr{H}^n\right)^{\frac{q'}{p'}}\right)^{\frac{1}{q'}} < \infty$$

if $-\frac{q'}{2} - \frac{nq'}{2} + \frac{nq'}{2p'} > -1$, which provides $1 - \frac{n}{p} - \frac{2}{q} > 0$. Therefore, using (4.5) we obtain

$$\int_0^s |I_3| \, dt \leq C_{14} \|v\|_\infty^2 \|\boldsymbol{f}\|_{L_x^p L_t^q(\Gamma_t)} \tag{4.6}$$

for some constant $C_{14} > 0$. Combining (4.4) and (4.6), we obtain (4.1). □

Proposition 4.2. *Let u be a classical solution of* (1.1) *on $\Omega \times (0, T_0)$. Assume $u_0 \in W^{1,\infty}(\Omega)$ and the transport term $\boldsymbol{f}$ satisfies* (2.2). *Then, for $\Omega' \Subset \Omega$, there exist $T > 0$ and $C_{15} > 0$ such that*

$$\sup_{0<t<T,\ x\in\Omega'} \sqrt{1 + |du(x,t)|^2} \leq C_{15}(1 + \|du\|_\infty^2).$$

Proposition 4.2 is deduced by a similar argument to that presented in [13]. Using Lemma 4.1, we may treat the transport terms and the regularity assumption can be relaxed as (2.2).

Proof of Theorem 2.2. Fix $Y \in \Omega \times \mathbb{R}$ and $T \in (0,1)$. For $s < T$ and $0 < t' < s$, using (4.1) and $\mathscr{H}^n(\Gamma_t) \leq \|v\|_\infty |\Omega|$

$$\begin{aligned}
&\int_0^{t'} \int_{\Gamma_t} (\rho_1 + \rho_2) \left(-|A_t|^2 v - \frac{2|D_{\Gamma_t} v|^2}{v} + du \cdot d(\boldsymbol{f} \cdot \boldsymbol{n})\right) d\mathscr{H}^n \\
&\quad + \frac{1}{4} \int_{\Gamma_t} v(\rho_1 + \rho_2)(\boldsymbol{f} \cdot \boldsymbol{n})^2 \, d\mathscr{H}^n + C_3 \mathscr{H}^n(\Gamma_t) \\
&\quad + C_5 \int_{\Gamma_t \cap \operatorname{spt} \rho_2} |\tilde{X} - Y| v \, d\mathscr{H}^n dt \\
\leq & C_{16} \int_0^{t'} \|v(\cdot, t)\|_\infty^3 \, dt,
\end{aligned}$$

where $C_{16} > 0$ is a positive constant. Let $M_T = \sup_{0<t<T} \|v(\cdot,t)\|_\infty$. Using (3.15) and the Gronwall inequality, for $0 < t < s$ and $0 < \alpha < \frac{1}{2}$,

$$
\begin{aligned}
&\exp\left(-C_4 \int_0^t (s-\tau)^{-\alpha}\, d\tau\right) \int_{\Gamma_t} v(\rho_1(X,t) + \rho_2(X,t))\, d\mathscr{H}^n \\
\leq &\int_{\Gamma_0} v(x,0)(\rho_1(X,0) + \rho_2(X,0))\, d\mathscr{H}^n \\
&+ C_{16} \int_0^t \exp\left(\frac{C_4}{1-\alpha}((s-\tau)^{1-\alpha} - s^{1-\alpha})\right) \|v(\cdot,\tau)\|_\infty^3\, d\tau \\
\leq &\; 2\|v(\cdot,0)\|_\infty^2 + C_{16} M_T^3 t.
\end{aligned}
$$

For $t \to s$,

$$v(y,s) \leq C_{17}(\|v(\cdot,0)\|_\infty^2 + s M_T^3),$$

where C_{17} is a positive constant.

Now, select (y,s) such that $M_T = v(y,s)$ and $Y = (y, u(y,s))$. Then,

$$C_{17} T M_T^3 - M_T + C_{17}\|v(\cdot,0)\|_\infty^2 \geq 0. \tag{4.7}$$

Suppose that there exists $C > 1 + C_{17}$ such that

$$M_T > C\|v(\cdot,0)\|_\infty^2$$

for all $T \in (0,1)$. Then (4.7) implies that

$$C_{17} \frac{T}{\|v(\cdot,0)\|_\infty^2} M_T^3 \geq \frac{M_T}{\|v(\cdot,0)\|_\infty^2} - C_{17} > C - C_{17} > 1,$$

which is contradiction as taking $T \downarrow 0$. □

5. Existence of classical solutions

Finally, we prove Theorem 2.3. To use the Schauder estimates, we provide the following:

Lemma 5.1. *Let $T > 0$ and $u \in C^{2,1}(Q_T)$ be a solution of* (1.1). *Then*

$$\sup_{Q_T} |u| \leq \sup_{\Omega\times\mathbb{R}\times[0,1]} |\boldsymbol{f}|\, T + \sup_{\Omega} |u_0|. \tag{5.1}$$

Proof. We set $w(x,t) = \sup_{\Omega\times\mathbb{R}\times[0,1]} |\boldsymbol{f}|\, t + \sup_\Omega |u_0|$. We note that

$$\partial_t w \geq \sqrt{1+|dw|^2}\, \mathrm{div}\left(\frac{dw}{\sqrt{1+|dw|^2}}\right) + \boldsymbol{f}(x,w,t)\cdot(-dw,1).$$

Using the maximum principle, we determine that

$$w \geq u, \qquad (x,t) \in Q_T.$$

Similarly to the above argument,

$$u \geq -w, \qquad (x,t) \in Q_T.$$

Hence, we obtain (5.1). □

Proof of Theorem 2.3. Fix $\alpha \in (0,1)$. We assume that $u_0 \in C^{2,\alpha}(\Omega)$ and let $T \in (0,1)$, which is given by Theorem 2.2. Let $\beta \in (0,\alpha]$ and we set $X := C^{1,\beta}(Q_T)$. We consider the following linear parabolic type equation:

$$\tag{5.2} \begin{cases} \partial_t u = \displaystyle\sum_{i,j=1}^{n} a_{ij}(dw)\partial_{x_i x_j} u + \boldsymbol{f}(x,w,t)\cdot(-du,1), & \text{in } Q_T, \\ du\cdot\nu\Big|_{\partial\Omega} = 0, \\ u\Big|_{t=0} = u_0, & \text{on } \Omega, \end{cases}$$

where $w \in X$ and $a_{ij}(r) = \left(\delta_{ij} - \dfrac{r_i r_j}{1+|r|^2}\right)$ for $r = (r_1,\dots,r_n)$. Because

$$\tag{5.3} \begin{aligned} \|a_{ij}(dw)\|_{C^{\alpha\beta}(Q_T)} &\le \|a_{ij}(dw)\|_{C^{\beta}(Q_T)} \\ &\le \|a_{ij}\|_{C^1(\mathbb{R}^n)}\|dw\|_{C^{\beta}(Q_T)} \le \|a_{ij}\|_{C^1(\mathbb{R}^n)}\|w\|_X \end{aligned}$$

for any $w \in X$, (5.2) is uniformly parabolic in Q_T. Note that $\|a_{ij}\|_{C^1(\mathbb{R}^n)} < \infty$. Using (2.4), we obtain

$$\tag{5.4} \|\boldsymbol{f}(\cdot,w,\cdot)\|_{C^{\alpha\beta}(Q_T)} \le K\|w\|_{C^{\beta}(Q_T)} \le K\|w\|_X$$

for any $w \in X$. Hence, for any $w \in X$ there exists a unique solution $u_w \in C^{2,\alpha\beta}(Q_T) \subset X$ of (5.2) such that

$$\tag{5.5} \|u_w\|_{C^{2,\alpha\beta}(Q_T)} \le C_{18},$$

where $C_{18} > 0$ depends only on $n, \alpha, \beta, \|w\|_X, \|u_0\|_{C^{2,\alpha}(\Omega)}$ and K (see [9, Theorem 4.5.3]).

We define $A : X \to X$ as $Aw = u_w$. Note that A is continuous and compact. We show that

$$S := \{u \mid u = \sigma A u \text{ in } X, \text{ for some } \sigma \in [0,1]\}$$

is bounded in X. If $u \in S$, then

$$\tag{5.6} \begin{cases} \partial_t u = \displaystyle\sum_{i,j=1}^{n} a_{ij}(du)\partial_{x_i x_j} u + \boldsymbol{f}(x,u,t)\cdot(-du,\sigma), & \text{in } Q_T, \\ du\cdot\nu\Big|_{\partial\Omega} = 0, \\ u\Big|_{t=0} = \sigma u_0, & \text{on } \Omega. \end{cases}$$

According to Theorem 2.2,

$$\tag{5.7} \sup_{Q_T}|du| \le C_{19},$$

where $C_{19} = C_{19}(\sup_{\Omega}|du_0|, \|\boldsymbol{f}\|_{C^0(\Omega\times\mathbb{R}\times[0,T])}) > 0$. Because $du\cdot\nu = 0$ on $\partial\Omega$, we can use similar arguments to the interior Schauder estimates (cf. [9, Theorem 6.2.1]); hence,

$$\tag{5.8} \|du\|_{C^{\beta}(Q_T)} \le C_{20},$$

where $C_{20} = C_{20}(n, \sup_{Q_T}|u|, \sup_{Q_T}|du|, \|du_0\|_{C^{\alpha}(\Omega)}, \sup_{\Omega\times\mathbb{R}\times[0,T]}|\boldsymbol{f}|, \partial\Omega) > 0$.

Using the same argument as (5.5),

$$\tag{5.9} \|u\|_X \le \|u\|_{C^{2,\alpha\beta}(Q_T)} \le C_{21},$$

where $C_{21} = C_{21}(n, \alpha, \|u_0\|_{C^{2,\alpha}(\Omega)}, C_{20}, K) > 0$ (see [9]). According to (5.7), (5.8), and (5.9), C_{21} depends only on $n, \alpha, \|u_0\|_{C^{2,\alpha}(\Omega)}, \sup_\Omega |du_0|$ and K. Thus, S is bounded in X. According to Schauder's fixed point theorem, there exists a solution $u \in C^{2,\alpha}(Q_T)$ of (1.1).

We return to the assumption that u_0 is a Lipschitz function with Lipschitz constant $L > 0$. Set $\varepsilon > 0$. We choose smooth functions u_0^k converging uniformly to u_0 on Ω. We note that according to Theorem 2.2,

$$\sup_{Q_T} |du^k| \leq C_1(1 + L^2)$$

for all $k \geq 1$. Using an argument similar to (5.8), (5.9) and the interior Schauder estimates, there exists $C_7 = C_7(n, \alpha, L, \varepsilon, K) > 0$ such that

$$\sup_k \|u^k\|_{C^{2,\alpha}(Q_T^\varepsilon)} \leq C_7,$$

where u^k is the solution of (1.1) with $u^k(x, 0) = u_0^k(x)$ in Ω. Note that $\varepsilon = \mathrm{dist}(Q_T^\varepsilon, \partial Q_T)$.

Hence, for any $\varepsilon > 0$, passing to a subsequence if necessary, $\{u^k\}_{k=1}^\infty$ converges to a classical solution u in Q_T^ε and we obtain (2.5). Therefore, by diagonal arguments, we obtain the solution $u \in C(\overline{Q_T}) \cap C^{2,1}(Q_T^\varepsilon)$. The maximum principle implies the uniqueness of u. Thus, we have proved Theorem 2.3. $\square$

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