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Sapporo, 2015

PREFACE

This volume is intended as the proceedings of Sapporo Symposium on Partial Differential Equations, held on August 19 through August 21 in 2015 at Faculty of Science, Hokkaido University.

Sapporo Symposium on PDE has been held annually to present the latest developments on PDE with a broad spectrum of interests not limited to the methods of a particular school. Professor Taira Shirota started the symposium more than 35 years ago. Professor Kôji Kubota and late Professor Rentaro Agemi made a large contribution to its organization for many years.

We always thank their significant contribution to the progress of the Sapporo Symposium on PDE.

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S. Ei, Y. Giga, S. Jimbo, H. Kubo, T. Ozawa, T. Sakajo
H. Takaoka, Y. Tonegawa, and K. Tsutaya

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The 40th Sapporo Symposium
on Partial Differential Equations
（第４０回偏微分方程式論札幌シンポジウム）

Period（期間）　August 19, 2015 - August 21, 2015

Venue（場所）　Room 310, Faculty of Science Building #7, Hokkaido University
北海道大学 理学部 7 号館講義室 7 - 310 室

URL　　　http://www.math.sci.hokudai.ac.jp/sympo/sapporo/program150819.html

Organizers（組織委員）　Shinichiro Ei（栄伸一郎）
Hideo Takaoka（高岡秀夫）

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An integration by parts formula for Feynman path integrals — Time slicing approximation method —*†

Daisuke Fujiwara ‡

August, 2015

Abstract

We are concerned with rigorously defined, by time slicing method, Feynman path integral $\int_{\Omega_{x,y}} F(\gamma)e^{i\nu S(\gamma)}\mathcal{D}(\gamma)$ of a functional $F(\gamma)$, cf. [6]. Here $\Omega_{x,y}$ is the set of paths $\gamma(t)$ in $\mathbf{R}^d$ starting from a point $y \in \mathbf{R}^d$ at time s and arriving at $x \in \mathbf{R}^d$ at time s', $S(\gamma)$ is the action of γ and $\nu = 2\pi h^{-1}$, with Planck's constant h. If $p(\gamma)$ is a vector field on the path space with suitable property, we prove the following integration by parts formula for Feynman path integral:

$$\int_{\Omega_{x,y}} DF(\gamma)[p(\gamma)]e^{i\nu S(\gamma)}\mathcal{D}(\gamma) = -\int_{\Omega_{x,y}} F(\gamma)\mathrm{Div}\, p(\gamma)e^{i\nu S(\gamma)}\mathcal{D}(\gamma) - i\nu \int_{\Omega_{x,y}} F(\gamma)DS(\gamma)[p(\gamma)]e^{i\nu S(\gamma)}\mathcal{D}(\gamma).$$

Here $DF(\gamma)[p(\gamma)]$ and $DS(\gamma)[p(\gamma)]$ are differentials of $F(\gamma)$ and $S(\gamma)$ evaluated in the direction of $p(\gamma)$, respectively, and $\mathrm{Div}\, p(\gamma)$ is divergence of vector fields $p(\gamma)$. This formula is an analogy to Elworthy's integration by parts formula for Wiener integrals, cf. [1]. As an application, we prove a semiclassical asymptotic formula of the Feynman path integrals which gives us a sharp information in the case $F(\gamma^*) = 0$. Here γ^* is the stationary point of the phase $S(\gamma)$.

1 Feynman path integrals

Feynman's path integral introduced by [2] is a method to construct the fundamental solution $k(s', s;, x, y)$ of Schrödinger equation using Lagrangian of classical mechanics

$$L(t, \dot{x}, x) = \frac{1}{2}|\dot{x}|^2 - V(t, x).$$

Here $V(t.x)$ is the potential field and x is the position of the particle and $\dot{x}$ is the velocity.

Let $[s, s']$ be a time interval. Action of a path, $\gamma : [s, s'] \ni t \to \gamma(t) \in \mathbf{R}^d$ is

$$S(\gamma) = \int_s^{s'} L(t, \dot{\gamma}(t), \gamma(t))dt.$$

Let x, y be arbitrary points of $\mathbf{R}^d$. Let $\Omega_{xy} = \{\gamma; [s, s'] \to \mathbf{R}^d, \gamma(s) = y, \gamma(s') = x\}$ be the path space starting from y at time s and arriving at x at time s'. Feynman's path integral is the following quite formal formula using $S(\gamma)$.

$$k(s', s; x, y) = \frac{1}{N} \sum_{\gamma \in \Omega_{xy}} exp\left(\frac{i}{\hbar}S(\gamma)\right), \quad N \text{ is the normalizing factor .} \tag{1}$$

Here summation $\sum_{\gamma \in \Omega}$ is summation over all paths in Ω. Since Ω is a continuum, it is customary to use symbol of integration,i.e.,

$$k(s', s; x, y) = \int_{\Omega_{xy}} \exp\left(\frac{i}{\hbar}S(\gamma)\right)\mathcal{D}[\gamma]. \tag{2}$$

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‡Gakushuin university

This is called Feynman path integral. More generally, one can discuss integration of the form

$$\int_{\Omega_{xy}} F(\gamma) \exp\left(\frac{i}{\hbar} S(\gamma)\right) \mathcal{D}[\gamma]$$

for functional $F(\gamma)$ of γ.

1.1 Time slicing approximation for Feynman path integral

The formula (1) or (2) is quite formal. Feynman gave more solid formulation in [2] and we follow him. We assume that $d = 1$ for simplicity.

Classical path is the solution of the variational problem,

$$\delta S(\gamma_0) = 0, \quad \gamma_0(s) = y, \quad \gamma_0(s') = x.$$

Let Δ be an arbitrary division of the interval $[s, s']$

$$\Delta : s = T_0 < T_1 < \cdots < T_J < T_{J+1} = s'. \tag{3}$$

We set $\tau_j = T_j - T_{j-1}$, ($j = 1, 2, \ldots, J+1$) and $|\Delta| = max_{1 \le j \le J+1} \tau_j$.

For $j = 1, 2, \ldots, J$, choose an arbitrary point $x_j \in \mathbf{R}$. We set $x_0 = y, x_{J+1} = x$. Consider classical path γ_1 starting from (T_0, x_0) and ending at (T_1, x_1). Connect (T_1, x_1) and (T_2, x_2), by classical path. Similarly, connect all pairs (T_j, x_j) and (T_{j+1}, x_{j+1}), $j = 0, 1, \ldots, J$ by classical paths. Thus we get a long path $\gamma_\Delta(x_{J+1}, x_J, \ldots, x_1, x_0)$ connecting (T_0, x_0) and (T_{J+1}, x_{J+1}) that may not be smooth. It may have edge at (T_j, x_j) ,$j = 1, 2, \ldots, J$. Some time we abbreviate $\gamma_\Delta(x_{J+1}, x_J, \ldots, x_1, x_0)$ to γ_Δ.

The action $S(\gamma_\Delta)$ of $\gamma_\Delta(x_{J+1}, x_J, \ldots, x_0)$ is a function of $(x_{J+1}, x_J, \ldots, x_1, x_0)$ if Δ is fixed.

$$S(\gamma_\Delta)(x_{J+1}, x_J, \ldots, x_0) = \int_s^{s'} L(t, \dot{\gamma}_\Delta(t), \gamma_\Delta(t)) dt = \sum_{j=1}^{J+1} \int_{T_{j-1}}^{T_j} L(t, \dot{\gamma}_j(t), \gamma_j(t)) dt.$$

Similarly if a functional $F(\gamma)$ of γ is given, $F(\gamma_\Delta)$ is a function of $(x_{J+1}, x_J, \ldots, x_1, x_0)$. For the sake of brevity we often abbreviate $F(\gamma_\Delta)$ to F_Δ and $S(\gamma_\Delta)$ to S_Δ.

Feynman formulated:

$$\int_{\Omega_{xy}} F(\gamma) \exp\left(i\nu S(\gamma)\right) \mathcal{D}[\gamma] = \lim_{|\Delta| \to 0} I[F_\Delta](\Delta; \nu, b, a, x, y), \tag{4}$$

where

$$I[F_\Delta](\Delta; \nu, s', s, x, y) \tag{5}$$
$$= \prod_{j=1}^{J+1} \left(\frac{\nu}{2\pi i \tau_j}\right)^{1/2} \int_{\mathbf{R}^J} F(\gamma_\Delta)(x_{J+1}, x_J, \ldots, x_1, x_0) \exp\left(i\nu S(\gamma_\Delta)(x_{J+1}, x_J, \ldots, x_1, x_0)\right) \prod_{j=1}^{J} dx_j.$$

Sice the integral on the right hand side of this does not converge absolutely, we regard this as an oscillatory integral. We call $I[F_\Delta](\Delta; \nu, s', s, x, y)$ time slicing approximation of Feynman path integral.

1.2 Our assumptions for Potential

For simplicity, we assume that the configuration space is $\mathbf{R}^1$.

Our assumption for potential $V(t, x)$ is the following. cf. W.Pauli [7].

Assumption 1.1 *1. $V(t, x)$ is a real continuous function of (t, x) and of class C^∞ in x.*

2. $\forall m \ge 0 \ \exists v_m \ge 0$ such that

$$\max_{|\alpha|=m} \sup_{(t,x) \in [s,s'] \times \mathbf{R}} \left|\partial_x^\alpha V(t,x)\right| \le v_m (1 + |x|)^{\max\{2-m, 0\}}.$$

One can prove the following

Proposition 1.2 *Let $\delta_0 > 0$ be so small that $\frac{\delta_0^2 dv_2}{8} < 1$.*

If $|s' - s| \le \delta_0$, then $\forall x, y \in \mathbf{R}$ there exists a unique classical path γ such that $\gamma(s) = y$ and $\gamma(s') = x$.

Let Δ be an arbitrary division of $[s, s']$ as (3).

Assume that $|\Delta| \leq \delta_0$. Let $\forall x_j \in \mathbf{R}$, $j = 1, 2, \ldots, J$. We set $x_0 = y, x_{J+1} = x$. We can uniquely define the piecewise classical path $\gamma_\Delta(x_{J+1}, \ldots, x_0)$ by Proposition 1.2. Let $\Gamma_{x,y}(\Delta)$ denote the space of all piecewise classical paths associated with division Δ. Then $\Gamma_{x,y}(\Delta)$ is identified with $\mathbf{R}^J$, if $|\Delta| \leq \delta_0$.

2 Property of classical action

From now we always assume $|s' - s| \leq \delta_0$.

For any $x, y \in \mathbf{R}$ the classical path γ with $\gamma(s) = y, \gamma(s') = x$ is unique. We write

$$S(s', s, x, y) = S(\gamma). \tag{6}$$

Calculation shows:

Proposition 2.1 *If $|s' - s| \leq \delta_0$, $S(s', s, x, y)$ is of the following form:*

$$S(s', s, x, y) = \frac{|x - y|^2}{2(s' - s)} + (s' - s)\phi(s', s, x, y).$$

The function $\phi(s', s, x, y)$ is a function of (s', s, x, y) of class C^1 and $\exists C > 0$ such that

$$|\phi(s', s, x, y)| \leq C(1 + |x|^2 + |y|^2). \tag{7}$$

Moreover, $\phi(s', s, x, y)$ is a C^∞ function of (x, y) and $\forall m \geq 2$ we have

$$\max_{2 \leq |\alpha| + |\beta| \leq m} \sup_{(x,y) \in \mathbf{R}^2} \left| \partial_x^\alpha \partial_y^\beta \phi(s', s, x, y) \right| = \kappa_m < \infty.$$

In particular,

$$\kappa_2 \leq \frac{v_2}{2} \left(1 - \frac{v_2 \delta_0^2}{8} \right)^{-1}.$$

Let Δ be the division (3) of the interval $[s, s']$

We discuss the time slicing approximation of path integral.

$$I[F_\Delta](\Delta; \nu, s', s, x, y) = \prod_{j=1}^{J+1} \left(\frac{\nu}{2\pi i \tau_j} \right)^{1/2} \int_{\mathbf{R}^J} F_\Delta(x_{J+1}, x_J, \ldots, x_1, x_0) e^{i\nu S_\Delta(x_{J+1}, x_J, \ldots, x_1, x_0)} \prod_{j=1}^J dx_j.$$

Consider $J \times J$ matrix Ψ whose (j, k) element is

$$\Psi_{jk} = \partial_{x_j} \partial_{x_k} S_\Delta(x_{J+1}, x_J, \ldots, x_1, x_0).$$

Then we divide the matrix Ψ into two parts.

$$\Psi = H_\Delta + W_\Delta,$$

where

$$H_\Delta = \begin{pmatrix} \frac{1}{\tau_1} + \frac{1}{\tau_2} & -\frac{1}{\tau_2} & 0 & 0 & \cdots & 0 & 0 \\ -\frac{1}{\tau_2} & \frac{1}{\tau_2} + \frac{1}{\tau_3} & -\frac{1}{\tau_3} & 0 & \cdots & 0 & 0 \\ 0 & -\frac{1}{\tau_3} & \cdots & \cdots & \cdots & \cdots & 0 \\ \vdots & \vdots & \cdots & \cdots & \cdots & \vdots & -\frac{1}{\tau_J} \\ 0 & 0 & 0 & \cdots & 0 & -\frac{1}{\tau_J} & \frac{1}{\tau_J} + \frac{1}{\tau_{J+1}} \end{pmatrix}$$

and

$$W_\Delta \tag{8}$$
$$= \begin{pmatrix} \partial_{x_1}^2(\tau_1\phi_1 + \tau_2\phi_2) & \partial_{x_2}\partial_{x_1}\tau_2\phi_2 & 0 & 0 & \cdots & 0 & 0 \\ \partial_{x_1}\partial_{x_2}\tau_2\phi_1 & \partial_{x_2}^2(\tau_2\phi_2 + \tau_3\phi_3) & \partial_{x_3}\partial_{x_2}\tau_3\phi_3 & 0 & \cdots & 0 & 0 \\ 0 & \partial_{x_2}\partial_{x_3}\tau_3\phi_3 & \cdots & \cdots & \cdots & \cdots & \cdots \\ \vdots & \vdots & \cdots & \cdots & \cdots & \vdots & \partial_{x_J}\partial_{x_{J-1}}\tau_J\phi_J \\ 0 & 0 & 0 & \cdots & 0 & \partial_{x_{J-1}}\partial_{x_J}\tau_J\phi_J & \partial_{x_J}^2(\tau_J\phi_J + \tau_{J+1}\phi_{J+1}) \end{pmatrix}$$

The matrix H_Δ is a constant matrix with determinant

$$\det H_\Delta = \frac{\tau_1 + \tau_2 + \cdots + \tau_{J+1}}{\tau_1 \tau_2 \ldots \tau_{J+1}} = \frac{(s' - s)}{\tau_1 \tau_2 \ldots \tau_{J+1}}.$$

It has it inverse H_Δ^{-1}. Regarding W_Δ as an perturbation, we write

$$\Psi = H_\Delta (I + H_\Delta^{-1} W_\Delta).$$

Proposition 2.2 *Let $0 < \delta_1$ be so small that $\delta_1 \leq \delta_0$ and $\kappa_2 \delta_1^2 < 1$. Let $|s' - s| \leq \delta_1$. Then $\forall (x_{J+1}, x_J, \ldots, x_1, x_0) \in \mathbf{R}^{J+2}$*

$$(1 - \kappa_2 \delta_1^2)^J \leq \det(I + H_\Delta^{-1} W_\Delta) \leq (1 + \kappa_2 \delta_1^2)^J.$$

Assume $|s' - s| \leq \delta_1$. Let γ^* be the unique classical path starting from y at time s and reaching x at time s'. Let $x_j^* = \gamma^*(T_j)$ for $j = 0, 1, 2, \ldots, J+1$ and $W_\Delta^* = W_\Delta \big|_{x_j = x_j^*, 1 \leq j \leq J}$. We set

$$D(\Delta; s', s, x, y) = \det(I + H_\Delta^{-1} W_\Delta^*) = \left(\frac{\tau_1 \tau_2 \ldots \tau_{J+1}}{(s' - s)} \right) \det Hess_{x_J^*, x_{J-1}^*, \ldots x_1^*} S_\Delta (x_{J+1}, x_J, \ldots, x_1, x_0).$$

Here $Hess_{x_j^*, \ldots x_1^*} S_\Delta (x_{J+1}, x_J, \ldots, x_1, x_0)$ is the Hessian matrix at $(x_J^*, \ldots x_1^*)$ of $S_\Delta (x_{J+1}, x_J, \ldots, x_1, x_0)$.

Proposition 2.3 *If $|t - s| \leq \delta_1$, then there exists the limit*

$$\lim_{|\Delta| \to 0} D(\Delta; t, s, x, y) = D(t, s, x, y) \tag{9}$$

and $(\frac{\nu}{2\pi i (t - s)})^{1/2} D(t, s, x, y)^{-1/2}$ satisfies the transport equation. Let $(-\frac{d^2}{dt^2})^{-1}$ be the green operator of Dirichlet boundary problem. Then $D(t, s, x, y)$ is the infinite dimensional determinant

$$D(t, s, x, y) = \det \left(-\frac{d^2}{dt^2} - \partial_x^2 V(t, \gamma^*) \right) \left(-\frac{d^2}{dt^2} \right)^{-1}. \tag{10}$$

3 Stationary phase method for integrals over a space of large dimension

Let Δ be a division of interval $[s, s']$ as (3). We call a division Δ' of the interval $[s, s']$ coarser than the division Δ if Δ is a refinement of Δ'. Let $0 = j_0 < j_1 < \cdots < j_p < j_{p+1} = J + 1$ be any subsequence of $\{0, 1, \ldots, J, J+1\}$. Then

$$\Delta' : a = T_{j_0} < T_{j_1} < \cdots < T_{j_p} < T_{j_{p+1}} = b \tag{11}$$

is a division of the interval $[s, s']$ coarser than Δ. There exists a natural embedding map $\Gamma_{x,y}(\Delta') \subset \Gamma_{x,y}(\Delta)$. We shall write $\iota_\Delta^\Delta f : \Gamma_{x,y}(\Delta') \to \mathbf{C}$ for the pull back of a function $f : \Gamma_{x,y}(\Delta) \to \mathbf{C}$ by this embedding.

It is clear that

$$\iota_{\Delta'}^\Delta S(x_{J+1}, x_{j_p}, \ldots, x_{j_1}, x_0) = \sum_{n=1}^{p+1} S(T_{j_n}, T_{j_{n-1}}, x_{j_n}, x_{j_{n-1}}). \tag{12}$$

The interval $[a, b]$ itself is a particular division of $[s, s']$, which we write $\Delta(J+1)$. Then $\iota_{\Delta(J+1)}^\Delta S_\Delta (x_{J+1}, x_0) = S(b, a, x, y)$.

For fixed Δ we discuss

$$I(\Delta, S, a_\lambda, \nu)(x_{J+1}, x_0) \tag{13}$$

$$= \prod_{j=1}^{J+1} \left(\frac{\nu}{2\pi i \tau_j} \right)^{1/2} \int_{\mathbf{R}^J} a_\lambda (x_{J+1}, x_J, \ldots, x_1, x_0) e^{(i\nu S_\Delta (x_{J+1}, x_J, \ldots, x_1, x_0))} \prod_{j=1}^J dx_j.$$

Assumption for the amplitude $a_\lambda (x_{J+1}, x_J, \ldots, x_1, x_0)$ is the following:

Assumption 3.1 *Let m be a nonnegative number. The amplitude function $a_\lambda = a_\lambda(x_{J+1}, x_J, \ldots, x_1, x_0)$ may depend on a parmeter λ. For any negative integer K there exists a positive constant A_K and X_K such that $X_K \geq 1$ such that*

1.

$$\left| \left(\prod_{j=0}^{J+1} \partial_{x_j}^{\alpha_j} \right) a_\lambda(x_{J+1}, x_J, \ldots, x_1, x_0) \right| \leq A_K X_K^{J+2} (1 + |\lambda| + |x_{J+1}| + |x_J| + \ldots + |x_1| + |x_0|)^m.$$

   if $|\alpha_j| \leq K$ for all $j = 0, 1, \ldots, J + 1$.

2. *Let Δ' be*

$$\Delta' : s = T_{j_0} < T_{j_1} < \cdots < T_{j_p} < T_{j_{p+1}} = s' \tag{14}$$

   any division coarser than Δ. Let $0 = j_0 < j_1 - 1 < j_1 < j_2 - 1 < \cdots < j_n < j_{n+1} - 1 = J + 1$, $n = 1, \ldots, J$ be any sequence of indices. Let $\{\alpha_j\}$ be a sequence of indices which satisfies $|\alpha_j| \leq K$. Then for any $(x_0, x_{j_1-1}, x_{j_1}, \ldots, x_{j_{n-1}}, x_{j_n-1}, x_{j_n}, x_{J+1}) \in \mathbf{R}^{2n+2}$

$$\left| \partial_{x_0}^{\alpha_0} \partial_{x_{J+1}}^{\alpha_{J+1}} \left(\prod_{k=0}^{p+1} \partial_{x_{j_k}}^{\alpha_{j_k}} \right) (\iota_{\Delta'}^\Delta a_\lambda)(x_{J+1}, x_{j_p}, \ldots, x_{j_1}, x_0) \right|$$

$$\leq A_K X_K^{2p+2} (1 + |\lambda| + |x_{J+1}| + |x_{j_p}| + \ldots + |x_{j_1}| + |x_0|)^m.$$

Set $T = |s' - s|$. For integers $1 \leq k < l \leq J + 1$ we set

$$S_{l,j}(x_l, \ldots, x_{j-1}) = \sum_{k=j}^{l} S_k(x_k, x_{k-1})$$

and for any fixed $(x_l, x_{j-1}) \in \mathbf{R}^2$ let $S_{l,j}^*(x_l, x_{j-1})$ be the stationary value of

$$(x_{l-1}, \ldots, x_j) \to S_l(x_l, x_{l-1}) + S_{l-1}(x_{l-1}, x_{l-2}) + \cdots + S_j(x_j, x_{j-1})$$

at the stationary point $(x_{l-1}^*, x_{l-2}^*, \ldots, x_j^*)$. Since $x_k^*, j \leq k < l$ is a function of (x_l, x_{j-1}), we denote it by $x_k^*(x_l, x_{j-1})$. i.e.,

$$S_{l,j}^*(x_l, x_{j-1}) = S_l(x_l, x_{l-1}^*) + \sum_{k=j+1}^{l-1} S_k(x_k^*(x_l, x_{j-1}), x_{k-1}^*(x_l, x_{j-1})) + S_j(x_j^*, x_{j-1}). \tag{15}$$

We understand that $S_{j,j}(x_j, x_{j-1}) = S_j(x_j, x_{j-1})$. Note that $S_{J+1,1}(x_{J+1}, \ldots, x_0) = S(x_{J+1}, \ldots, x_0)$. We use also the notation

$$D_{x_{l-1}^*, \ldots, x_j^*}(S_{l,j}; x_l, x_{j-1}) = \left(\frac{\tau_l + \cdots + \tau_j}{\tau_l \cdots \tau_j} \right) \det Hess(\sum_{k=j}^{l} S_k(x_k, x_{k-1}))\Big|_{x_k = x_k^*, j \leq k \leq l-1},$$

Here $Hess$ means the Hessian at the critical point $(x_{l-1}^*, \ldots, x_j^*)$. For any $k = 1, 2, \ldots, J + 1$ we define the division

$$\Delta(k) : s = T_0 < T_k < T_{k+1} < \cdots < T_{J+1} = s'. \tag{16}$$

$\Delta(1) = \Delta$ and $\Delta(J+1)$ is $\Delta(J+1); s = T_0 < T_{J+1}$, i.e.,the interval itself without any intermediate dividing poit. Then

Theorem 3.2 *Suppose that $T \leq \delta$ and $a_\lambda(x_{J+1}, x_J, \ldots, x_1, x_0)$ satisfies Assumption **??**. We further assume that $|\Delta|T \leq 1$. Then*

$$I(\Delta; S, a_\lambda, \nu)(x_{J+1}, x_0) \tag{17}$$

$$= \left(\frac{\nu}{2\pi i T} \right)^{1/2} e^{i\nu S_{J+1,1}^*(x_{J+1}, x_0)} D_{x_J^*, \ldots, x_1^*}(S_{J+1,1}; x_{J+1}, x_0)^{-1/2} \left(\iota_{\Delta(J+1)}^\Delta a_\lambda(x_{J+1}, x_0) + \nu^{-1} T p(\Delta, x_{J+1}, x_0) \right)$$

$$+ \nu^{-1} T^2 |\Delta| \left(\frac{\nu}{2\pi i T} \right)^{1/2} e^{i\nu S_{J+1,1}^*(x_{J+1}, x_0)} q(\Delta, x_{J+1}, x_0)$$

$$+ \nu^{-2} T^2 \left(\frac{\nu}{2\pi i T} \right)^{1/2} e^{i\nu S_{J+1,1}^*(x_{J+1}, x_0)} r(\Delta, \nu, x_{J+1}, x_0).$$

Here

$$p(\Delta, x_{J+1}, x_0) \tag{18}$$

$$= -\frac{i}{2T}\sum_{j=1}^{J}\frac{T_j\tau_{j+1}}{T_{j+1}}(\iota_{\Delta(J+1)}^{\Delta(j)}D_{x_{j-1}^*,\dots,x_1^*}(S_{j,1};x_j,x_0)^{1/2}\partial_{x_j}^2(D_{x_{j-1}^*,\dots,x_1^*}(S_{j,1},x_j,x_0)^{-1/2}\iota_{\Delta(j)}^{\Delta}a_\lambda))(x_{J+1},x_0).$$

$q(\Delta, x_{J+1}, x_0)$ *is independent of* ν. *And functions* $q(\Delta, x_{J+1}, x_0)$ *and* $r(\Delta, \nu, x_{J+1}, x_0)$ *satifies the following estimate. For any* $K \geq 0$ *there exists an integer* $M(K) \geq 0$ *and a constant* $C_K > 0$ *independent of* Δ *such that*

$$(1 + |\lambda| + |x_{J+1}| + |x_0|)^{-m}|\partial_{x_{J+1}}^{\alpha_{J+1}}\partial_{x_0}^{\alpha_0}q(\Delta, x_{J+1}, x_0)| \leq C_K A_{M(K)} \tag{19}$$

$$(1 + |\lambda| + |x_{J+1}| + |x_0|)^{-m}|\partial_{x_{J+1}}^{\alpha_{J+1}}\partial_{x_0}^{\alpha_0}r(\Delta, \nu, x_{J+1}, x_0)| \leq C_K A_{M(K)}, \tag{20}$$

if multi-indices α_0, α_{J+1} *satisfies* $|\alpha_0| \leq K$ *and* $|\alpha_{J+1}| \leq K$. *[3], [5].*

Remark 1 *Tsuchida [8] treated the case of non-zero vector potential.*

4 Convergence of Feynman integral

We discuss convergence of Feynman path integral. Our discussion is valid only for those $F(\gamma)$ that have rather restrictive properties.

Assumption 4.1 (N.Kumano-go's condition) *Let* m *be a non-negative constant. There exists a bounded Borel measure* $\rho \geq 0$ *on* $[s, s']$. *For any non-negative integer* K *there exist positive constants* A_K *and* X_K *such that for any division* Δ *and for any indices* α_j, $j = 0, 1, 2, \dots, J+1$ *satisfying* $|\alpha_j| \leq K$ *there holds the following inequlities.*

$$\left|\left(\prod_{k=0}^{J+1}\partial_{x_j}^{\alpha_j}\right)F(\gamma_\Delta(x_{J+1}, x_J, \dots, x_1, x_0))\right| \leq A_K X_K^{J+2}(1 + |x_{J+1}| + |x_J| + \dots + |x_1| + |x_0|)^m.$$

$$\left|\left(\prod_{n=0}^{p+1}\partial_{x_j}^{\alpha_j}\right)(\partial_{x_k}^\beta F_\Delta)(x_{J+1}, \dots, x_{k+1}, x_k, x_{k-1}, \dots, x_0)\right|$$

$$\leq A_K X_K^{p+2}\rho([T_{k-1}, T_{k+1}])(1 + |x_{J+1}| + |x_J| + \dots + |x_1| + |x_0|)^m \qquad if \ |\beta| \geq 1. \tag{21}$$

Remark 2 $F(\gamma) \equiv 1$ *clearly satisfies this assumption.*

Example 4.2 *Let* $\rho(t)$ *be a function of bounded-variation on* $[s, s']$ *and* $f(t, x)$ *be a continuous function of* $(t, x) \in [s, s'] \times \mathbf{R}$ *and infinitely differentiable in* x. *Suppose that for any* α *there exists a positive constant* C_α *such that*

$$|\partial_x^\alpha f(t, x)| \leq C_\alpha(1 + |x|)^m, \tag{22}$$

with some $m \geq 0$ *independent of* α *and* (t, x). *Then the following functional satisfies Assumptions (4.1).*

$$F(\gamma) = \int_s^{s'} f(t, \gamma(t))\, d\rho(t). \tag{23}$$

The next theorem was proved by Kumano-go [6], while the case $F(\gamma) \equiv 1$ had been known [4].

Theorem 4.3 *Assume that the integrand* $F(\gamma)$ *satisfies Assumption 4.1 and Assumption* **??** *above and* $|s' - s|$ *is so small that* $|s' - s| \leq \delta_0$. *Then the limit*

$$K[F](\nu, s', s, x, y) = \lim_{|\Delta|\to 0} I[F_\Delta](\Delta; \nu, s', s, x, y) \tag{24}$$

exists in $\mathcal{B}_{(m)}(\mathbf{R}^2)$. $K[F](\nu, s', s, x, y)$ *is of the form*

$$K[F](\nu, s', s, x, y) = \left(\frac{\nu}{2\pi i(s' - s)}\right)^{1/2}e^{i\nu S(\gamma^*)}k(F; \nu, s', s, x, y). \tag{25}$$

$k(F; \nu, s', s, x, y)$ can be written as

$$k(F; \nu, s', s, x, y) = D(s', s, x, y)^{-1/2} \left(F(\gamma^*) + \nu^{-1} R[F](\nu, s', s, x, y) \right). \tag{26}$$

More precisely, for any non negative K there exist a positive constant $C_{\alpha\beta}$ and an integer $M(K)$ such that

$$|\partial_x^\alpha \partial_y^\beta R[F](\nu, s', s, x, y)| \le C_K A_{M(K)} |s - s'|(|s - s'| + \rho([s, s']))(1 + |x| + |y|)^m,$$

Remark 3 *We may write*

$$\int_\Omega e^{i\nu S(\gamma)} F(\gamma) \mathcal{D}[\gamma] = K[F](\nu, s', s, x, y). \tag{27}$$

Remark 4 *Equality (26) together with (??) imply semi-classical asymptotic formula.*

Theorem 4.4 *In the case $F(\gamma) \equiv 1$, $K[1](\nu, s', s, x, y) = \int_{\Omega_{x,y}} e^{i\nu S(\gamma)} F(\gamma) \mathcal{D}[\gamma]$ is infact the fundamental solution of Schrödinger equation.*

5 An integration by parts formula

[1] We set $s = 0$ and $s' = T$ for simplicity. Let $\mathcal{X} = L^2([0, T])$ and $\mathcal{H} = H^1([0, T])$ be the real L^2-Sobolev space of order 1. For any $x, y \in \mathbf{R}$, we write $\mathcal{H}_{x,y} = \{\gamma \in \mathcal{H} : \gamma(0) = y, \gamma(T) = x\}$. $\mathcal{H}_{x,y}$ is an infinite dimensional differentiable manifold. Its tangent space at $\gamma \in \mathcal{H}_{x,y}$ is identified with the Hilbert space $\mathcal{H}_0 = H_0^1([0, T]) = \{\gamma \in \mathcal{H}; \gamma(0) = \gamma(T) = 0\}$ equipped with the inner product

$$(h_1, h_2)_{\mathcal{H}_0} = \int_0^T \frac{d}{dt} h_1(t) \frac{d}{dt} h_2(t) \, dt.$$

We denote the norm in $\mathcal{H}_0$ by $\|h\|_{\mathcal{H}_0}$ for $h \in \mathcal{H}_0$.

Let $\tilde{\rho} : \mathcal{H} \to \mathcal{X}$ be the canonical embedding and $\rho : \mathcal{H}_0 \to L^2(0, T)$ be its restriction to $\mathcal{H}_0$ and $\rho^* : L^2(0, T) \to \mathcal{H}_0$ be its adjoint. We use the symbol $\mathcal{L}(\mathcal{X})$ for the Banach space of all bounded linear operators in $L^2(0, T)$ equipped with operator norm.

5.1 m-smooth functional

We use the following notation : Let $\mathcal{Y}$ be a Banach space with norm $\| \ \|_{\mathcal{Y}}$. Let Δ be a division of $[0, T]$, γ_Δ and $\{x_{J+1}, x_J, \ldots, x_1, x_0\}$ be as before. Assume that $F(\gamma_\Delta)$ is a map $F : \Gamma(\Delta) \ni \gamma_\Delta \to F(\gamma_\Delta) \in \mathcal{Y}$ and is infinitely differentiable with respect to $(x_{J+1}, \ldots, x_0)$. Let K be a nonnegative integer, m be a nonnegative constant and $X \ge 1$ be a constant. Then we define a norm of $F(\gamma_\Delta)$ defined on $\Gamma(\Delta)$:

$$\|F(\gamma_\Delta)\|_{\{\mathcal{Y}; \Delta, m, K, X,\}} \tag{28}$$

$$= \max_{\alpha_0 \le K, \ldots \alpha_{J+1} \le K} \quad \sup_{x_j \in \mathbf{R}, \text{for } 0 \le j \le J+1} (1 + |x_{J+1}| + \cdots + |x_0|)^{-m} \left\| \prod_{j=0}^{J+1} X^{-|\alpha_j|} \partial_{x_j}^{\alpha_j} F(\gamma_\Delta) \right\|_{\mathcal{Y}},$$

where max is taken over all multi-indices α with $m(\alpha) \le K$ and sup is taken over all $(x_{J+1}, \ldots, x_0) \in \mathbf{R}^{J+2}$. Moreover if $F(\gamma)$ is defined on $\mathcal{H}$, then we define

$$\|F\|_{\{\mathcal{Y}; m, K, X\}} = \sup_\Delta \|F\|_{\{\mathcal{Y}; \Delta, m, K, X\}}, \tag{29}$$

where sup is taken over all divisions Δ of $[0, T]$. If $\mathcal{Y} = \mathbf{R}$ or $\mathbf{C}$, we simply write $\|F\|_{\{\Delta, m, K, X,\}}$ or $\|F\|_{\{m, K, X\}}$.

Suppose that a functional $F(\gamma); \mathcal{H}_{x,y} \to \mathbf{C}$ is Fréchet differentiable at γ. Then $DF(\gamma)$ denotes its differential. For $h \in \mathcal{H}_0$,

$$DF(\gamma)[h] = (DF(\gamma), h)_{\mathcal{H}_0} \quad \forall h \in \mathcal{H}_0.$$

---

[1] cf. [9]

Moreover, if there exists a density $f_\gamma(s) \in L^2(0, T)$ such that

$$DF(\gamma)[h] = \int_0^T f_\gamma(s)\rho h(s)\, ds, \quad \text{for} \quad \forall h \in \mathcal{H}_0, \tag{30}$$

then we often denote $f_\gamma(s)$ by $\dfrac{\delta F(\gamma)}{\delta \gamma(s)}$ or $\dfrac{\delta}{\delta \gamma(s)} F(\gamma)$.

Definition 5.1 *Let $m \geq 0$ be a constant. We call $F(\gamma)$ a m-smooth functional if $F(\gamma)$ satisfies the following conditions.*

F-I *$F(\gamma)$ is an infinitely differentiable map from $\mathcal{H}_{xy}$ to $\mathbf{C}$.*

F-2 *$\forall x, \forall y \in \mathbf{R}$ and $\gamma \in \mathcal{H}_{xy}$ the differential $DF(\gamma)$ has its density $\dfrac{\delta F(\gamma)}{\delta \gamma(s)}$, that is, $\forall \gamma \in \mathcal{H}_{x,y} \forall h \in \mathcal{H}_0$*

$$DF(\gamma)[h] = \int_0^T \frac{\delta F(\gamma)}{\delta \gamma(s)} \rho h(s) ds,$$

F-3 *Functional $\dfrac{\delta F(\gamma)}{\delta \gamma(s)}$ is a continous functional of $\mathcal{H}_{x,y} \times [0, T] \ni (\gamma, s) \longrightarrow \mathbf{C}$. It is infinitely differentiable with respect to $\gamma \in \mathcal{H}_{x,y}$ if s is fixed.*

F-4 *For any integer $K \geq 0$ there are constants $A_K > 0$ and $X_K \geq 1$ such that $\forall K = 0, 1, 2, \ldots,$,*

$$A_K = \|F(\gamma)\|_{\{m, K, X_K\}} + \sup_{s \in [0, T]} \left\| \frac{\delta F(\gamma)}{\delta \gamma(s)} \right\|_{\{m, K, X_K\}} < \infty. \tag{31}$$

Remark 5 *Let δ be so small that $v_2 \delta^2 < 4$ and $v_2 \delta < 1$. If $T \leq \delta$, a m-smooth functional satisfies condition of N. Kumano-go 4.1 and its Feynman path integral converges.*

5.2   Some Operators of trace class

Let $B : \mathcal{X} \to \mathcal{X}$ be a bounded linear operator with operator norm $\|B\|_{\mathcal{L}(\mathcal{X})}$.

Proposition 5.2 *Linear operators $\rho^* B \rho : \mathcal{H}_0 \to \mathcal{H}_0$ and $\rho \rho^* B : \mathcal{X} \to \mathcal{X}$ are of trace class. Their traces are equal.*

Let $\exists k(s, t) \in L^2([0, T] \times [0, T])$ be such that $\forall f \in L^2(0, T)$

$$\rho \rho^* B f(s) = \int_0^T k(s, t) f(t) dt. \tag{32}$$

In particular, the kernel function of $\rho \rho^*$ is the Green operator for the Dirichlet boundary value problem.

Proposition 5.3 *$k(s, t)$ has the properties*

1. *For $\forall s \in [0, T]$ $k(s, t)$ is well defined function of t in $L^2(0, T)$.*
2. *$[0, T] \ni s \to k(s, *)$ is a strongly continuous mapping from $[0, T]$ to $L^2([0, T])$.*
3. *For almost every $t \in [0, T]$ $k(s, t)$ is in the image of ρ as a function of s.*

Proposition 5.4 *For $\epsilon > 0$ define*

$$k_\epsilon(s, t) = \epsilon^{-1} \int_{t-\epsilon}^{t+\epsilon} k(s, t) dt. \tag{33}$$

Then $k_\epsilon(s, t)$ is continuous on $[0, T] \times [0, T]$. Moreover,

$$Trace(\rho \rho^* B) = \lim_{\epsilon \to 0} \int_a^b k_\epsilon(s, s)\, ds. \tag{34}$$

6 admissible vector field

Let $p : \mathcal{H}_{xy} \ni \gamma \to p(\gamma) \in \mathcal{H}_0$. Then $p(\gamma)$ is a tangent vector field on $\mathcal{H}_{xy}$. We write as usual $p(\gamma, s) = \rho p(\gamma)(s)$. We have $\partial_s p(\gamma, s) \in L^2(0, T)$.

Definition 6.1 (Admissible vector field) *We say that $p(\gamma)$ is an admissible vector field if $p(\gamma)$ has the following properties:*

1. *There exits a C^1 map $q : \mathcal{H} \to L^2(0, T)$ such that*

$$p(\gamma) = \rho^* q(\gamma), \quad (\gamma \in \mathcal{H}_{x,y}).$$

2. *When we restrict $q(\gamma)$ to $\mathcal{H}_{x,y}$, the Fréchet differential $Dq(\gamma) : \mathcal{H}_0 \ni h \to Dq(\gamma)[h] \in L^2(0, T)$ can be boudedly extended to a bounded linear map $B(\gamma)$ in $L^2(0, T)$, that is, for any $h \in \mathcal{H}_0$,*

$$Dq(\gamma)[h] = B(\gamma)\rho h.$$

We often write $\dfrac{\delta q(\gamma)}{\delta \gamma}$ for $B(\gamma)$.

Let $Dp(\gamma) : \mathcal{H}_0 \to \mathcal{H}_0$ be Fréchet differential of $p(\gamma)$ restricted to $\mathcal{H}_{x,y}$ at $\gamma \in \mathcal{H}_{x,y}$. Then it is clear that for all $h \in \mathcal{H}_0$,

$$Dp(\gamma)[h] = \rho^* B(\gamma)\rho h.$$

That is, for all $h_1, h_2 \in \mathcal{H}_0$,

$$(Dp(\gamma)[h_1], h_2)_{\mathcal{H}_0} = \Big(B(\gamma)\rho h_1, \rho h_2 \Big)_{L^2(0,T)}.$$

Definition 6.2 (Divergence of a vecor field) *Suppose that $p(\gamma)$ is an admissible vector field. We define its divergence $\operatorname{Div} p(\gamma)$ at $\gamma \in \mathcal{H}_{x,y}$ by the follwing equality:*

$$\operatorname{Div} p(\gamma) = \operatorname{tr}\rho^* B(\gamma)\rho = \operatorname{tr}\rho^* \frac{\delta q(\gamma)}{\delta \gamma}\rho.$$

Definition 6.3 *Let $p(\gamma)$ be an admissible vector field. The map $\rho\rho^* B(\gamma)$ is an operator of trace class. We denote its kernel function by $\dfrac{\delta p(\gamma, s)}{\delta \gamma(t)}$, i.e.,*

$$\rho\left(Dp(\gamma)[h]\right)(s) = \int_0^T \frac{\delta p(\gamma, s)}{\delta \gamma(t)}\rho h(t)\, dt.$$

It is clear that for any $h \in \mathcal{H}_0$,

$$\int_0^T \frac{\delta p(\gamma, s)}{\delta \gamma(t)}\rho h(t)\, dt = Dp(\gamma, s)[h].$$

As a result of discussions in the previous subsection we have the following proposition.

Proposition 6.4 *Assume $p(\gamma)$ is an admissible vectorfield. Then*

$$\operatorname{Div} p(\gamma) = \int_0^T \frac{\delta p(\gamma, t)}{\delta \gamma(t)}dt.$$

The notion of admissible vector field defined above is an analogy to infinitesimal version of " admissible transformation" in the case of Wiener integral.

6.1 Integration by parts formula

Definition 6.5 *Let m' be a nonnegative number. We say that the vector field $p(\gamma)$ is an m'-admissible vector field if it has all the following properties:*

P1 *p is an infinitely differentiable map $p : \mathcal{H} \ni \gamma \to p(\gamma) \in \mathcal{H}_0$ of which the restriction to $\mathcal{H}_{xy}$ is admissible for any fixed $x, y \in \mathbf{R}$, that is, there is a C^∞ map $q : \mathcal{H} \to \mathcal{X}$ such that $p(\gamma) = \rho^* q(\gamma)$ for $\gamma \in \mathcal{H}_{x,y}$ and that for all $h \in \mathcal{H}_0$, $Dq(\gamma)[h] = B(\gamma)\rho h$, where $B(\gamma) \in \mathcal{L}(\mathcal{X})$.*

P2 *The map $\mathcal{H} \ni \gamma \to B(\gamma) \in \mathcal{L}(\mathcal{X})$ is infinitely differentiable. For any non-negative integer K there exists a positive constant Y_K such that*

$$B_K = \|q(\gamma)\|_{\{\mathcal{X}, m', K, Y_K\}} + \|B(\gamma)\|_{\{\mathcal{L}(\mathcal{X}); m', K, Y_K\}} < \infty. \tag{35}$$

We often write $\dfrac{\delta q(\gamma)}{\delta \gamma}$ for $B(\gamma)$.

Let δ_0 be as in (1.2). Our main theorem is the following [9]:

Theorem 6.6 (Integration by parts) *Let $T \le \delta$. Suppose that $F(\gamma)$ is an m-smooth functional and that $p(\gamma)$ is an m'-admissible vector field. We further assume that two of $DF(\gamma)[p(\gamma)]$, $F(\gamma)\mathrm{Div}\,p(\gamma)$ and $F(\gamma)DS(\gamma)[p(\gamma)]$ are F-integrable. Then the rest is also F-integrable and the following equality holds.*

$$\int_{\Omega_{xy}} DF(\gamma)[p(\gamma)]e^{i\nu S(\gamma)}\mathcal{D}(\gamma) \tag{36}$$

$$= -\int_{\Omega_{xy}} F(\gamma)\mathrm{Div}\,p(\gamma)e^{i\nu S(\gamma)}\mathcal{D}(\gamma) - i\nu \int_{\Omega_{xy}} F(\gamma)DS(\gamma)[p(\gamma)]e^{i\nu S(\gamma)}\mathcal{D}(\gamma).$$

Remark 6 cf. N.Kumano-go [6]. *If $p(\gamma, s)$ is independent of γ, i.e., $p(\gamma, s) = h(s)$ then $\mathrm{Div}p(\gamma) = 0$ and the above formula reduces to*

$$\int_{\Omega_{x,y}} DF(\gamma)[h]e^{i\nu S(\gamma)}\mathcal{D}(\gamma) = -i\nu \int_{\Omega_{x,y}} F(\gamma)DS(\gamma)[h]e^{i\nu S(\gamma)}\mathcal{D}(\gamma). \tag{37}$$

7 Application to semiclassical asymtotic behaviour of Feynman path integrals

7.1 A sharper asymptotic formula.

We always assume $T < \delta_0$. Let $F(\gamma)$ be an m-smooth functional. Then semiclassical asymptotic formula was proved by Kumano-go [6].

What happens if $F(\gamma^*) = 0$? Integration by parts formula enables us to get a sharper informations even in this case.

Assumption 7.1 *1. $F(\gamma)$ is a real valued m-smooth functional. For fixed $\gamma \in \mathcal{H}_{x,y}$, $\dfrac{\delta F(\gamma)}{\delta \gamma(s)}$ is a function in $L^2(0, T)$, which we write $\dfrac{\delta F(\gamma)}{\delta \gamma}$. The map $\mathcal{H}_{x,y} \ni \gamma \to \dfrac{\delta F(\gamma)}{\delta \gamma} \in L^2(0, T)$ is a C^∞ map. There exists a C^∞ map $\mathcal{H}_{x,y} \ni \gamma \to A(\gamma) \in B(L^2)$ such that for any $h \in \mathcal{H}_0$,*

$$D\frac{\delta F(\gamma)}{\delta \gamma}[h] = A(\gamma)\rho h. \tag{38}$$

2. Linear ator $A(\gamma)$ has integral kernel $k_\gamma(s, t)$ which is continous in $(s, t) \in [0, T] \times [0, T]$ and we have for any $K = 0, 1, 2, \ldots$

$$\sup_{(s,t)} \|k_\gamma(s, t)\|_{\{m, K, X_K\}} < \infty. \tag{39}$$

Suppose $F(\gamma)$ satifies above conditions and $F(\gamma^*) = 0$. Let $\gamma_\theta = \theta\gamma + (1-\theta)\gamma^*$ for $0 \le \theta \le 1$. We define an element $\zeta(\gamma) \in L^2(0,T)$ by

$$\zeta(\gamma, t) = \int_0^1 \left.\frac{\delta F(\gamma)}{\delta \gamma(t)}\right|_{\gamma=\gamma_\theta} d\theta. \tag{40}$$

Define a vector field

$$p(\gamma) = \rho^*(I - \tilde{W}(\gamma)\rho\rho^*)^{-1}\zeta(\gamma). \tag{41}$$

Theorem 7.2 *Suppose $F(\gamma)$ is an m-smooth functional with some $m \ge 0$ and it satisfies the additional assumption 7.1. Asume further that $F(\gamma^*) = 0$. Define $\zeta(\gamma,t)$ and $p(\gamma)$ as above. Then we have*

$$\int_{\Omega_{xy}} F(\gamma) e^{i\nu S(\gamma)} \mathcal{D}[\gamma] = -(i\nu)^{-1} \int_{\Omega_{xy}} \mathrm{Div}p(\gamma) e^{i\nu S(\gamma)} \mathcal{D}[\gamma]. \tag{42}$$

Apply Kumano-go's theorem of semiclassical asymptotics, c.f. [6], to (42), we have the following theorem.

Theorem 7.3 *Under the same assumption as in Theorem7.2 the following asymptotic formula holds:*

$$\int_{\Omega_{xy}} F(\gamma) e^{i\nu S(\gamma)} \mathcal{D}[\gamma] = \left(\frac{-i\nu}{2\pi T}\right)^{1/2} D(T,0,x,y)^{-1/2} e^{i\nu S(\gamma^*)} \big(-(i\nu)^{-1}\mathrm{Div}p(\gamma^*) + \nu^{-2}r(\nu,T,0,x,y)\big).$$

where the remainder term $r(\nu,T,0,x,y)$ has the property such that for $\forall \alpha, \beta$ there exists a positive constant $C_{\alpha\beta}$

$$\left|\partial_x^\alpha \partial_y^\beta r(\nu,T,0,x,y)\right| \le C_{\alpha\beta}(1 + |x| + |y|)^m. \tag{43}$$

Let $G_{\gamma^*}(t,s)$ be the Green function of differential equation of Jacobi field:

$$-\left(\frac{d^2}{dt^2} + \partial_x^2 V(t, \gamma^*(t))\right) u(t) = f(t), \qquad u(0) = 0 = u(T). \tag{44}$$

Calculation shows:

Theorem 7.4 *Under the same assumption as in Theorem7.3*

$$\mathrm{Div}p(\gamma^*) = \frac{1}{2}\int_0^T\int_0^T \frac{\delta}{\delta\gamma(t)}\left(G_{\gamma^*}(t,s)\frac{\delta F(\gamma^*)}{\delta\gamma(s)}\right) ds dt$$

$$= \frac{1}{2}\int_0^T\int_0^T \frac{\delta G_{\gamma^*}(t,s)}{\delta\gamma(t)}\frac{\delta F(\gamma^*)}{\delta\gamma(s)} ds dt + \frac{1}{2}\int_0^T\int_0^T G_{\gamma^*}(t,s)\frac{\delta^2 F(\gamma^*)}{\delta\gamma(s)\delta\gamma(t)} ds dt.$$

Example: − the moment of degree 2−.
Let $a(s,t)$ be a smooth function uniformly bouded with its all derivatives. Consider the functional

$$F(\gamma) = \int_0^T\int_0^T (\gamma(s) - \gamma^*(s))(\gamma(t) - \gamma^*(t))a(s,t)\, ds dt$$

and its integral

$$\int_{\Omega_{x,y}} F(\gamma) e^{i\nu S(\gamma)}\, \mathcal{D}[\gamma]. \tag{45}$$

This is an analogue of "the moment of degree 2" of Wiener integral:

$$\int_\Omega (\gamma(s) - (\frac{s}{T}x + (1 - \frac{s}{T})y))(\gamma(t) - (\frac{t}{T}x + (1 - \frac{t}{T})y))a(s,t)\, \mathcal{W}[\gamma].$$

Notice that $F(\gamma^*) = 0$. We can apply our method. The semi-classical asymptotic formula of (45) is

$$\int_{\Omega_{x,y}} F(\gamma) e^{i\nu S(\gamma)}\, \mathcal{D}[\gamma] =$$

$$= \left(\frac{\nu}{2\pi i T}\right)^{1/2} D(T, 0; x, y)^{-1/2} e^{i\nu S^*(\gamma)}$$

$$\times \left(-(i\nu)^{-1} \int_0^T \int_0^T G_{\gamma^*}(s, t) a(s, t)\, ds dt + \nu^{-2} R(\nu, T, 0, x, y)\right).$$

Here $G_{\gamma^*}(s, t)$ is the Green function for the Jacobi differential equation of Euler equation.

This means that the semiclassical asymptotic of the "covariance matrix" equals $G_{\gamma^*}(s, t)$.

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Geometric interfacial motion:
Coupling surface diffusion and mean curvature motion

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Abstract

Mean curvature motion as well as surface diffusion both constitute geometric interfacial motions which have received considerable attention. However in many applications a complex coupling of surfaces occurs whose evolution may be described using both these types of motion. In my lecture, a variety of such physical problems will be described. While sometimes an anisotropic formulation might seem to be preferable, often an isotropic formulation is helpful to consider. Some analytic and numerical results will be presented, in addition to some supporting experimental evidence.

I shall discuss some problems in which various evolving interfaces appear, with some interfaces evolving by mean curvature motion and some by surface diffusion. My interest in this direction is reflected collaborations with J. Cahn, Y. Kanel, L. Peres-Hari, R. Dal Passo, L. Giacomelli, Vilenkin, [9, 8, 27, 10, 22, 18, 19, 20, 21, 29, 28] with special thanks my students V. Derkach, O. Zelekman-Smirin, and A. Zigelman [30, 13, 35, 11, 12, 31, 34].

Below is a partial list of some of the problems which we have in mind.

1. Polycrystalline films are material layers composed of a number of individual crystals, or grains. When these films are thin, the effects of their exterior surface can affect the stability of the film. Since thin films are in common use in numerous technological applications, the interest in their stability is considerable.

2. Sintering is a technology with a long history, but which still in active use to this day. In this process, a collection of metallic particles, produced previously by evaporation and condensation, are placed into a mold. Under pressure, the collections of particles combines to yield one solid metallic block with some desired shape.

3. Under appropriate compositional conditions, ordering can been induced in a previously homogenous material. If the composition differs slightly from these conditions, the excess composition can emerge

as droplets along the boundaries between the ordered regions. This phenomena can be modelled with some success by a coupled Allen-Cahn/Cahn-Hilliard system of equations with degenerate mobilities.

In all of the above applications, the coupled motion of surface diffusion with motion by mean curvature appears quite naturally. In each of these applications, there are additional effects have been neglected and which arguably should be included. However, the coupled motion, in and of itself, is not overly well understood or studied, so it is reasonable to isolate it and study it, even given its limitations. So let us define motion by mean curvature and motion by surface diffusion, then to return to see how we should like to coupled the motions.

1 Motion by mean curvature

For simplicity, let us consider

$$\Gamma(t) := \{X(\alpha, t) : [0,1] \times [0,T] \to R^2\}, \tag{1}$$

to be a smooth (and smoothly parameterized) embedded curve in R^2. For $\alpha \in (0,1)$, we may define $\hat{n}$, the unit normal to $\Gamma(t)$ at α, and we may define κ, the mean curvature of $\Gamma(t)$ at α via the inscribed circle at α. Then, setting $\hat{V} := \frac{\partial}{\partial t} X$ for $(\alpha, t) \in (0,1) \times (0,T)$, the motion known as motion by mean curvature is given by

$$V_n = \mathcal{A}\kappa, \tag{2}$$

where $V_n := \hat{n} \cdot \hat{V}$ is the normal velocity, and $\mathcal{A}$ is a dimensional parameter, known as the reduced mobility whose dimensions are given by $[\mathcal{A}] = L^2/T$. It is straightforward to extended the above definition to smooth (and smoothly parameterized) embedded hypersurfaces in R^3, taking now κ to be the sum of the principle curvatures.

An early reference, if not the earliest reference, to motion by mean curvature is Mullins 1956 [24]. In the mathematical literature, early references are Brakke 1978 [7], Huisken 1984 [16], Grayson 1987 [14]. For a survey, see Bellettini 2013 [6].

2 Motion by surface diffusion

To define motion by surface diffusion, we may proceed as above and consider first, for simplicity, $\Gamma(t) := X(\alpha, t)$ to be a smooth and smoothly parame-

terized embedded curve in R^2 prescribed as in (1), with $\hat{n}$, κ, $\hat{V}$, V_n defined as above for $(\alpha, t) \in (0, 1) \times (0, T)$. Taking $s(= s(t))$ to be an arc-length parametrization of $\Gamma(t)$, e.g. $s := \int_0^\alpha |\frac{\partial}{\partial \alpha} X(\tilde{\alpha}, t)| \, d\tilde{\alpha}$, the motion known as motion by surface diffusion is given by

$$V_n = -\mathcal{B} \frac{\partial^2 \kappa}{\partial s^2}, \tag{3}$$

where $\mathcal{B}$ is a dimensional kinetic coefficient with dimensions $[\mathcal{B}] = L^4/T$. The above definition may be extended to smooth (and smoothly parameterized) embedded hypersurfaces in R^3, by defining κ as the sum of the principle curvatures and by replacing the operator $\frac{\partial^2}{\partial s^2}$ with the Laplace-Beltrami operator $\triangle_s$. The Laplace-Beltrami operator on a smooth hypersurface may be defined as $\triangle_s := \sum_{i=1}^2 \frac{\partial^2}{\partial s_i^2}$, where s_i $i = 1, 2$ are locally defined arc-length parameterizations in the directions of the principle curvatures, so $\triangle_s$ indeed generalizes $\frac{\partial^2}{\partial s^2}$.

An early reference, if not the earliest reference, to surface diffusion in the literature is Mullins 1957 [25]. Some more recent references include [23, 3].

3   Coupling the two motions

Why should one wish to couple the two motions discussed above? Let us focus on the first problem in the list given earlier, that of polycrystalline films. Because the microstructure of polycrystalline materials, namely the way in which the individual crystals or grains are arranged within the specimen, strongly influences the material properties, it has attracted many studies since Mullins' 1956 [24] work on the dynamics of grain boundaries, and earlier static discussions by Smith [33] and Herring [15]. Much emphasis has been on considering grain boundaries within the material, with much emphasis on larger systems and and grain boundary distributions. Additional effects have been included, such anisotropy, elastic effects, and defects.

When a polycrystalline specimen is thin enough to be considered a film, or more specifically a thin film, the exterior surface is nowhere too far from the grain boundaries within, and the effects of the dynamics of the exterior surface on the structural stability of the specimen needs to be taken into account. A classical model though somewhat simplistic model for the evolution of the exterior surface, due to Mullins 1957 [25], predicts that it evolves by surface diffusion. While many of the grain boundaries lie totally within the interior of the material, some continue on until they intersect the exterior

surface. This gives rise a system in which numerous grain boundaries, which for simplicity are being taken to evolve by mean curvature, are coupled with the exterior surface, which for simplicity is being taken to evolve by surface diffusion. As there is much to be studied in regard to these coupled motion, a possible simplification is to consider some special simple geometries.

4   Some special geometries

In polycrystalline specimens with bamboo structure, the specimen is one grain thick throughout. Accordingly all of the grains intersect the exterior surface at least twice, above and below. Within the interior of the specimen, triple junctions form where three grain boundaries intersect. On the exterior surface, where three intersecting grain boundaries intersect with the exterior surface, quadruple junctions form. So although though this situation is simpler, it still is quite complicated. A further simplification is to consider the 2D symmetric thermal grooving geometry proposed by Mullins [25], described below. Other special geometries are possible to consider, some of which will be described a bit further on.

5   A phase field approach

One approach to modeling this type of coupled motions is to embed the problem description within a phase field approach, and one way to accomplish this is to consider some type of Allen-Cahn/Cahn-Hilliard system with degenerate mobility. Such a system was derived to describe the dynamics of simultaneous order-disorder and phase transitions, as the formal continuum limit of discrete lattice dynamics, in the context of the Krzanowski instability described earlier [9]. For the resultant system, it was possible to formally derive the coupled motion by mean curvature and surface diffusion by considering an appropriately defined long time asymptotic limit [28, 29].

6   The Mullins' sharp interface approach

Mullins considered a geometrically simple exterior surface-grain boundary configuration with important experimental implications and outlined a dynamic problem for its evolution [25], following the general thermodynamic framework outlined by Gibbs. The geometry which he considered consisted of an initially flat exterior surface of infinite extent, coupled with an initially

flat grain boundary of semi-infinite extent which intersected the exterior surface perpendicularly. The exterior surface was assumed to evolve by surface diffusion, and the grain boundary, which in theory could evolve by motion by mean curvature, could be expected to remain flat due to symmetry considerations. Although this reflects a 3D problem because of the uniform cross-section of the its geometry, its dynamics reduce to a 2D problem. To obtain a 2D problem formulation, boundary conditions are required at the triple junction, in addition to far-field and initial conditions. The following conditions were proposed by Mullins: 1) persistence, meaning that the grain boundary remained attached to the exterior surface, 2) balance of mechanical forces, known also as Herring's law or Young's law for isotropic systems, 3) continuity of the chemical potential, which, based on thermodynamic considerations of Gibbs, implies continuity of the mean curvature, 4) balance of mass flux along the exterior surface, assuming that there is no mass flux along the grain boundary. Following the problem description above, assuming asymptotic planarity, and fixing the initial location of the exterior surface along the $x-$axis and the initial location of the grain boundary along the negative $y-$axis, yields the following problem formulation for the evolution of the exterior surface,

$$
\begin{cases}
y_t = -\mathcal{B}\Big(\frac{\kappa_x}{\sqrt{1+y_x^2}}\Big)_x, \quad \kappa = \frac{y_{xx}}{(1+y_x^2)^{3/2}}, \\
\qquad\qquad x \in (-\infty,0)\cup(0,\infty), \ t \in (0,\infty), \\
y(0^+,t) = y(0^-,t), \quad t \in (0,\infty), \\
\arctan(y_x(0^+,t)) = \arctan(y_x(0^-,t)) + 2\arcsin(\frac{m}{2}), \quad t \in (0,\infty), \\
\kappa(0^+,t) = \kappa(0^-,t), \quad t \in (0,\infty), \\
\frac{\kappa_x}{\sqrt{1+y_x^2}}\big|_{(0^+,t)} = \frac{\kappa_x}{\sqrt{1+y_x^2}}\big|_{(0^-,t)}, \quad t \in (0,\infty), \\
y(\pm\infty,t) = y_x(\pm\infty,t) = 0, \quad t \in (0,\infty), \\
y(x,0) = 0, \quad x \in (-\infty,0)\cap(0,\infty),
\end{cases}
\tag{4}
$$

where $m = \gamma_{exterior\ surface}/\gamma_{grain\ boundary}$ and $\gamma_{exterior\ surface}$, $\gamma_{grain\ boundary}$ are, respectively, the surface free energies of the grain boundary and the exterior surface.

Mullins assumed that $0 < m \ll 1$, which is physically reasonable, and linearized the system in (4) about $y \equiv 0$. For the resultant linear system he found self-similar solutions of the form $y(x,t) = m(\mathcal{B}t)^{1/4}Y(\xi)$ where $\xi = x/(\mathcal{B}t)^{1/4}$, and derived a series formula for $Y(\xi)$, which allowed him to

predict that $y(0,t) = a_0\,m(\mathcal{B}t)^{1/4}$ with $a_0 \approx -.78$. The quantity $y(0,t)$ is known as the depth of the *thermal groove*, since $\mathcal{B}$ increases with temperature. Based on the linearization of the equation in (4), he also identified a traveling wave solution [26] describing a groove which steadily translates along the exterior surface with speed $V_0 = \mathcal{B}(m/d)^3$, where d is the groove depth. In this solution, no attachment of the groove to a grain boundary is prescribed and d is not fixed.

Despite the years which have passed, many question remain regarding the nonlinear problem formulation given in (4). For example, it is not known whether or not it possesses self-similar solutions, although a positive answer has been demonstrated for (4) with partially linearized boundary conditions. In the next two sections we will outline some of what is known regarding traveling wave solutions for problem formulations related to (4).

7 Travelling wave solutions

In [22, 18, 19, 20] we considered a configuration in which a grain boundary is attached at a thermal groove to an exterior surface in *quarter loop geometry*. The quarter loop geometry refers to a two grain system which may be assumed to be of semi-infinite extent, in which the exterior surface is asymptotically flat, with one internal grain boundary, which intersects the exterior surface and partitions the specimen into two grains, then asymptotes to one side, say to the right, to some specific depth below the exterior surface, say $-H$. If we assume that the profiles of both the exterior surface and the grain boundary can be described by graphs of functions, say by $h = h(x,t)$ and $u = u(x,t)$, then we may obtain a problem formulation, much as in (4) above, except that now we need to consider $x \in (-\infty, s(t)) \cup (s(t), \infty)$, where $s = s(t)$ with $s(0) = 0$ since the groove can move, and we need to amplify the resultant equations with the equations:

$$
\begin{cases}
u_t = \mathcal{A}\kappa, \quad \kappa = \frac{u_{xx}}{(1+u_x^2)^{3/2}}, \quad x \in (-\infty, s(t)) \cup (s(t), \infty),\ t \in (0,\infty), \\[2mm]
y(s(t)^-, t) = u(s(t)^+, t), \quad t \in (0,\infty), \\[2mm]
\arctan(u_x(0^+, t)) = -\frac{\pi}{2} + \frac{\arctan(y_x(0^+,t)) + \arctan y_x(0^-,t))}{2}, \quad t \in (0,\infty), \\[2mm]
u(\infty, t) = -H, \quad t \in (0,\infty), \\[2mm]
u(x, 0) = u_0(x), \quad x \in (0,\infty),
\end{cases}
$$

$$(5)$$

where $u_0(x)$ needs to be prescribed. In [19] it was demonstrated that there indeed exists a travelling wave solution to the resultant system if $0 < m$ is sufficiently small. Energetic considerations imply that at least in theory, m may assume values throughout the range $0 \leq m < 2$.

It turns out that the restriction on m resulted from looking for travelling wave solutions which could be described as graphs of functions. The restriction could be removed by formulating the travelling wave problem in terms of the angles, $\Phi(s_1)$ and $\Psi(s_2)$, where $\Phi(s_1)$ and $\Psi(s_2)$ denote respective the angles between the tangent to the grain boundary and the exterior surface and the $x-$axis, and where s_1, s_2 are respective arc-length parameterizations for $\Phi(s_1)$ and $\Psi(s_2)$, with $s_1 \in (0, \infty)$, $s_2 \in (-\infty, 0) \cup (0, \infty)$, and with the thermal groove is taken to be located at $s_1 = s_2 = 0$. Setting $s = (V/\mathcal{B})^{1/3} s_2$, the problem to be satisfied by Ψ could be formulated as:

$$\begin{cases} \Psi_{sss} = \sin \Psi, \quad s \in (-\infty, 0) \cup (0, \infty), \\[2mm] \Psi(0^+) = \Psi(0^-), \\[2mm] \Psi_s(0^+) = \Psi_s(0^-) + 2\arcsin(m/2), \\[2mm] \Psi_{ss}(0^+) = \Psi_{ss}(0^-), \\[2mm] \Psi(\pm\infty) = 0. \end{cases} \tag{6}$$

Based on this formulation and using a shooting argument, the following could be proven:

Theorem [22] For all $m \in [0, 2)$, there exists a travelling wave solution to the coupled exterior surface and grain boundary problem for the quarter loop geometry. For $0 < m \ll 1$, there exists a travelling wave solution which can be described via graphs of functions.

For all $0 \leq m < 2$, given a solution Ψ to (6), Φ can be prescribed in term of Ψ, and the wave speed is given by $V = \mathcal{B}(\lambda/\mathcal{B}))^3$ where λ is the unique real solution to the following algebraic equation

$$\lambda^3 + \lambda^2 \Psi_{ss}(0+) - \frac{\mathcal{A}H^2}{\mathcal{B}}\Big[\frac{\pi}{2} - \frac{1}{2}(\Psi(0^+) + \Psi(0^-)\Big] = 0. \tag{7}$$

The existence of nonclassical travelling wave solutions could be demonstrated numerically for $m \in (0, 2)$ sufficiently large. For the case of classical travelling wave solutions for $0 < m \ll 1$ sufficiently small, advective stability of the travelling wave solution could be demonstrated [5]. Further stability results are possible [17].

8 Estimating the reduced mobility

The equation for motion by mean curvature, (2), contains a kinetic parameter, $\mathcal{A}$, the reduced mobility. In the context of evolution of microstructure, it is not so obvious how to estimate this parameter. Shvindlerman and colleagues developed a methodology based on the half-loop geometry [2, 4]. The half loop geometry refers to a bi-crystal with bamboo structure. It consists of a layer with finite height and with infinite lateral extent, containing two crystals. One crystal is embedded within the other, and the grain boundary which separates them is U-shaped, with ends which are asymptotically parallel. They found grain boundaries in the laboratory, which fitted the described above and which appeared to move with constant speed. Neglecting the effects of the exterior surface and assuming the grain boundary within its interior to move by mean curvature motion, it was reasonable to look for travelling wave solutions with this shape. Such solutions, which they referred to as U-shape solutions, were already known to exist [24] and could be prescribed explicitly by $Y_\pm(x - V_0 t)$, where

$$V_0 = \frac{A\pi}{4Q}, \quad Y_\pm(x) = \pm\Big(Q - (2Q/\pi)\arcsin\big(e^{-\frac{\pi}{2Q}x}\big)\Big), \quad x \in \mathrm{R}^+, \quad (8)$$

and $2Q$ is the asymptotic width of the embedded crystal, far from the "nose" of the U-shaped profile. Later in the mathematical literature, these solutions came to known as grim reaper solutions [1]. These solutions could be used to estimate the reduced mobility $\mathcal{A}$, using the formula for V_0 in (8), [2, 4, 32].

Based on our discussions up to now, it is clear that if the exterior surfaces evolve by surface diffusion, then they should have some effect of the motion, and hence on the process of estimation or measurement of $\mathcal{A}$. Accordingly, Anna Zigelman set out to estimate this effect in her MSc and PhD theses [31, 34, 35]. To simplify the analysis, she considered the effect of surface diffusion only in the neighborhood of the thermal groove. She derived equations for the leading order corrections to the shape of the exterior surface, the grain boundary profile, the shape of the thermal groove, as well as to the traveling wave velocity, using asymptotic analysis based on the assumption that $0 < m \ll 1$ in two distinguished limits:

$$(i) \quad L/Q = m^{1/3}, \qquad (ii) \quad L/Q = O(1). \quad (9)$$

Limit (i) can be considered as a thin film (thin specimen) limit [31, 34], and limit (ii) better models the conditions of Shvindlerman's original experiments [34]. In both cases, the exterior surface could be shown at leading

order to satisfy a fourth order parabolic problem which decouples from the rest of the system describing the motion. The leading order perturbation to the shape of the thermal groove could be described by a second order parabolic equation with forcing which depended on the angle of inclination of the exterior surface along the thermal groove, as well as on V_1, the perturbation to the velocity, V_0 at the nose. After some analysis, the perturbation to the velocity, V_1, could be prescribed in the form:

$$V_1(t) = \frac{d}{dt}(r * \mathrm{B})(t), \quad \mathrm{r(t)} = \mathcal{L}^{-1}\Big[\frac{1}{\omega\,\mathcal{L}[\mathcal{G}](\omega)}\Big](\mathrm{t}), \tag{10}$$

where $\mathcal{L}$, $\mathcal{L}^{-1}$ denote, respectively, the Laplace transform and its inverse, $*$ denotes the convolution operator, B depends on the inclination of the exterior surface along the thermal groove, and $\mathcal{G}$ is a weighted integral of an appropriately defined heat kernel.

Since in experiments the motion was viewed from above, measurement were made not of V_n and κ, but rather of $(V_n)_{\text{effective}}$, the projection of the normal velocity at the nose on the plane of observation, and $\kappa_{\text{effective}}$, the curvature at the nose of the projection of the shape of the groove root on the plane of observation. Assuming that the measurements were made based on the formula

$$(V_n)_{\text{effective}} = A_{\text{effective}}\,\kappa_{\text{effective}}, \tag{11}$$

our results could prescribe a correction to the reduced mobility measurement due the influence of motion of the exterior surface as follows:

$$\frac{A_{\text{effective}}}{A} = \frac{(V_n)_{\text{effective}}}{V_n} \cdot \frac{\kappa}{\kappa_{\text{effective}}}, \tag{12}$$

with V_n, κ being approximated via their zeroth order description and the leading order corrections.

9   Parametric surfaces

Vadim Derkach, in his MSc. and PhD Theses considered some special geometries, containing a small number of grains, and undertook numerical analysis based on finite differences, parametric surfaces, and staggered grids, studying pitting, annihilation, dewetting, and void formation [11, 12, 13].

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Interpolation error estimates on triangles and its application to the finite element method

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Abstract

We propose a simple method to obtain sharp upper bounds for the interpolation error constants over the given triangular elements. These constants are important for analysis of interpolation error and especially for the error analysis in the finite element method. In our method, interpolation constants are bounded by the product of the solution of corresponding finite dimensional eigenvalue problems and constant which is slightly larger than one. Guaranteed upper bounds for these constants are obtained via the numerical verification method. Furthermore, we introduce remarkable formulas for the upper bounds of these constants.

Keywords: Interpolation error constant, Numerical verification method, finite element method

1 Introduction

The analysis of interpolation error is important in a lot of applications such as the approximate theory and the error estimation for the solution of finite element method. In order to estimate the interpolation errors, we have to obtain the upper bounds of the constants which appear in some kinds of norm inequalities. These are called interpolation error constants.

Let T be given triangle in $\mathbb{R}^2$ and define function spaces $V^{1,1}(T), V^{1,2}(T), V^2(T)$ as follows:

$$V^{1,1}(T) = \left\{ \varphi \in H^1(T) \;\middle|\; \int_T \varphi \, dxdy = 0 \right\},$$

$$V^{1,2}(T) = \left\{ \varphi \in H^1(T) \;\middle|\; \int_{\gamma_k} \varphi \, ds = 0, \quad \forall k = 1,2,3 \right\},$$

$$V^2(T) = \left\{ \varphi \in H^2(T) \;\middle|\; \varphi(p_k) = 0, \quad \forall k = 1,2,3 \right\},$$

where p_1, p_2, p_3 and $\gamma_1, \gamma_2, \gamma_3$ are vertices and edges of T, respectively. Under these settings, it is known that the following interpolation error constants $C_1(T), C_2(T), C_3(T)$ and $C_4(T)$ exist:

$$C_1(T) = \sup_{u \in V^{1,1}(T) \backslash 0} \frac{\|u\|_{L^2(T)}}{\|\nabla u\|_{L^2(T)}}, \qquad C_2(T) = \sup_{u \in V^{1,2}(T) \backslash 0} \frac{\|u\|_{L^2(T)}}{\|\nabla u\|_{L^2(T)}},$$

$$C_3(T) = \sup_{u \in V^2(T) \backslash 0} \frac{\|u\|_{L^2(T)}}{|u|_{H^2(T)}}, \qquad C_4(T) = \sup_{u \in V^2(T) \backslash 0} \frac{\|\nabla u\|_{L^2(T)}}{|u|_{H^2(T)}}.$$

where $|\cdot|_{H^k(\Omega)}$ means H^k semi-norm defined later. Here, $C_1(T)$ is known as the Poincaré constant whose existence is assured by Poincaré-Friedrichs inequality [20]. The existence proof of $C_3(T)$ and $C_4(T)$ are found in [3] and the existence of $C_2(T)$ is proved by the same way as [3, Lemma2.1]. Besides that, there is a plenty of literature on these constants, such as [7, 12] on $C_1(T)$, [14] on $C_2(T)$, [1] on $C_3(T)$ and [1, 3, 4, 5, 7, 8, 9, 10, 13, 14, 15, 17, 18, 22] on $C_4(T)$ and its application to the finite element method.

Nevertheless of the importance for the practical applications, it is very difficult to obtain sharp upper bounds for these constants. In this paper, we present a simple method to obtain explicit and sharp upper bounds for them. Furthermore, we obtained the following remarkable formulas for the upper bounds:

$$C_1(T) < K_1(T) = \sqrt{\frac{A^2 + B^2 + C^2}{28} - \frac{S^4}{A^2 B^2 C^2}},$$

$$C_2(T) < K_2(T) = \sqrt{\frac{A^2 + B^2 + C^2}{54} - \frac{S^4}{2A^2 B^2 C^2}},$$

$$C_3(T) < K_3(T) = \sqrt{\frac{A^2 B^2 + B^2 C^2 + C^2 A^2}{83} - \frac{1}{24}\left(\frac{A^2 B^2 C^2}{A^2 + B^2 + C^2} + S^2\right)},$$

$$C_4(T) < K_4(T) = \sqrt{\frac{A^2 B^2 C^2}{16 S^2} - \frac{A^2 + B^2 + C^2}{30} - \frac{S^2}{5}\left(\frac{1}{A^2} + \frac{1}{B^2} + \frac{1}{C^2}\right)},$$

where A, B, C are the edge lengths of triangle T and S is the area of T.

Numerical results shows that the upper bounds obtained by these formulas are sharp enough for the practical applications. Moreover, $K_1(T) \sim K_4(T)$ is convenient for practical calculations since these formulas consists of just four arithmetic operations and the square root. We have to note that, by our method, we can only prove these formulas for the "given" triangles. To prove the formulas for "any" triangle, we need some continuation techniques and asymptotic analysis. We indeed succeeded to prove these formulas for "any" triangle, but we will show it in another paper because of the space limit.

2 Application to the finite element method

Interpolation constants can be applied to the error estimates for the finite element method. There are two kinds of error estimates: *a priori* error estimate

and *a posteriori* error estimate. In this section, we only introduce the simple case, that is, *a priori* error estimate for Poisson's equation with homogeneous Dirichlet boundary conditions. For more general case, see [4, 5, 3, 13, 17, 14] for *a priori* error estimate and [4, 8, 14] for *a posteriori* error estimate.

Let Ω be convex polygonal domain in $\mathbf{R}^2$. Then we consider the following Poisson's equation:

$$\begin{cases} -\Delta\varphi = f & \text{in} & \Omega, \\ \varphi = 0 & \text{on} & \partial\Omega, \end{cases} \tag{1}$$

We divides Ω into finite triangular subregions and denote them by $\tau_1, \tau_2, \cdots, \tau_n$. Let S_h be the set of continuous functions on Ω which is linear in each τ_k and vanishes at $\partial\Omega$. Then we obtain the finite element solution $\varphi_h \in S_h$ by

$$(\nabla\varphi_h, \nabla\eta)_{L^2(\Omega)} = (f, \eta)_{L^2(\Omega)}, \qquad \forall \eta \in S_h.$$

For the finite element solution, the following error estimates holds:

Theorem 1. *Assume $f \in L^2(\Omega)$. Then, for φ the exact solution of (1) and the finite element solution φ_h,*

$$\|\nabla(\varphi_h - \varphi)\|_{L^2(\Omega)} \leq \max_{1 \leq k \leq n} C_4(\tau_k)\|f\|_{L^2(\Omega)},$$

$$\|\varphi_h - \varphi\|_{L^2(\Omega)} \leq \max_{1 \leq k \leq n} C_4(\tau_k)^2\|f\|_{L^2(\Omega)},$$

holds.

Proof. For φ, we define interpolation function $\varphi_h^* \in S_h$ by

$$\varphi_h^*|_{\tau_k} = \Pi_{\tau_k}^{(P1)}\varphi,$$

where $\Pi_{\tau_k}^{(P1)}$ is a linear interpolation which coincides with φ at the vartices of τ_k. Then, from Céa's lemma [4, 5],

$$\|\nabla(\varphi_h - \varphi)\|_{L^2(\Omega)} = \min_{w \in S_h} \|\nabla(w - \varphi)\|_{L^2(\Omega)} \leq \|\nabla(\varphi_h^* - \varphi)\|_{L^2(\Omega)}$$

holds.

It is known that φ the solution of (1) belongs to $H^2(\Omega)$ when Ω is bounded, convex and polygonal domain [6]. Therefore, from the definition of C_4, we have

$$\|\nabla(\varphi_h - \varphi)\|_{L^2(\Omega)} \leq \|\nabla(\varphi_h^* - \varphi)\|_{L^2(\Omega)} = \sqrt{\sum_{k=1}^{n} \|\nabla(\varphi_h^* - \varphi)\|_{L^2(\tau_k)}^2}$$

$$\leq \sqrt{\sum_{k=1}^{n} C_4(\tau_k)^2 |\varphi|_{H^2(\tau_k)}^2} \leq \max_{1 \leq k \leq n} C_4(\tau_k) \sqrt{\sum_{k=1}^{n} |\varphi|_{H^2(\tau_k)}^2}$$

$$= \max_{1 \leq k \leq n} C_4(\tau_k) \, |\varphi|_{H^2(\Omega)}.$$

It is also known that if Ω is bounded polygonal domain,

$$|\varphi|_{H^2} = \|\Delta\varphi\|_{L^2}$$

holds [6, 11]. Therefore, we have the following *a priori* error estimate:

$$\|\nabla(\varphi_h - \varphi)\|_{L^2(\Omega)} \leq \max_{1 \leq k \leq n} C_4(\tau_k)\,|\varphi|_{H^2(\Omega)} \leq \max_{1 \leq k \leq n} C_4(\tau_k)\,\|f\|_{L^2(\Omega)}.$$

We can also obtain L^2 error estimate

$$\|\varphi_h - \varphi\|_{L^2(\Omega)} \leq \max_{1 \leq k \leq n} C_4(\tau_k)^2 \|f\|_{L^2(\Omega)}$$

by Aubin-Nitsche's technique [5, 2, 19]. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

3 Definitions and preliminaries

For given triangle T, let $p_1(T), p_2(T), p_3(T)$ be vertices of T and $\gamma_1(T), \gamma_2(T), \gamma_3(T)$ be edge $p_2(T)p_3(T)$, $p_3(T)p_1(T)$, $p_1(T)p_2(T)$, respectively. Let $n(T)$ be the outer normal unit vector on ∂T, $\nu(T)$ be the direction vector which takes counterclockwise direction through ∂T and $ds(T)$ be the line element on ∂T. We omit "(T)" if there is no possibility of confusion. We use Cartesian coordinates (x, y) and use the usual notation for L^2 norm and define H^k semi-norm $|\cdot|_{H^k(T)}$ by $|u|_{H^k(\Omega)}^2 = \sum_{j=0}^k \binom{k}{j} \left\| \frac{\partial^k u}{\partial x^j \partial y^{k-j}} \right\|_{L^2(\Omega)}^2$. $T_{a,b}$ denotes triangle whose vertices are $(0,0)$, $(1,0)$ and (a,b). We use subscripts to indicate partial derivatives.

Let Q_α and Q_β denote the following polynomial spaces:

$$Q_\alpha = \left\{ a_1(x^2 + y^2) + a_2 x + a_3 y + a_4 \ \middle| \ a_1, \cdots, a_4 \in \mathbb{R} \right\},$$

$$Q_\beta = \left\{ a_1 x^2 + a_2 xy + a_3 y^2 + a_4 x + a_5 y + a_6 \ \middle| \ a_1, \cdots, a_6 \in \mathbb{R} \right\}.$$

Note that both Q_α and Q_β are invariant under constant shifts and rotations and thus they are independent of the choice of the coordinates. Let τ be the given triangle and we define two kinds of second order interpolation $\Pi_\tau^{(\alpha)}\varphi$ for $\varphi \in H^1(\tau)$ and $\Pi_\tau^{(\beta)}\varphi$ for $\varphi \in H^2(\tau)$ on triangle τ as follows:

$$\begin{cases} \Pi_\tau^{(\alpha)}\varphi \in Q_\alpha \\ \displaystyle\int_{\gamma_k} \Pi_\tau^{(\alpha)}\varphi\,ds = \int_{\gamma_k} \varphi\,ds, \qquad k = 1, 2, 3, \\ \displaystyle\iint_\tau \Pi_\tau^{(\alpha)}\varphi\,dxdy = \iint_\tau \varphi\,dxdy, \end{cases}$$

$$\begin{cases} \Pi_\tau^{(\beta)}\varphi \in Q_\beta \\ \Pi_\tau^{(\beta)}\varphi(p_k) = \varphi(p_k), \qquad k = 1, 2, 3, \\ \displaystyle\int_{\gamma_k} \nabla\Pi_\tau^{(\beta)}\varphi \cdot n\,ds = \int_{\gamma_k} \nabla\varphi \cdot n\,ds, \qquad k = 1, 2, 3. \end{cases}$$

In the rest of this section, we prepare some preliminary lemmas.

Lemma 1. *If $\varphi \in V^2(\tau)$ satisfies*

$$\int_{\gamma_k} \nabla\varphi \cdot n \, ds = 0, \quad k = 1, 2, 3,$$

then

$$\varphi_x, \ \varphi_y \in V^{1,2}(\tau)$$

holds.

Proof. From $\varphi(p_1) = \varphi(p_2) = \varphi(p_3) = 0$, we have

$$\int_{\gamma_k} \nabla\varphi \cdot \nu \, ds = 0, \qquad k = 1, 2, 3.$$

Then, together with the assumption,

$$\int_{\gamma_k} \nabla\varphi \cdot w \, ds = 0, \qquad k = 1, 2, 3,$$

holds for any fixed vector w, which proves the lemma. $\qquad\qquad\square$

On the interpolations $\Pi_\tau^{(\alpha)}$ and $\Pi_\tau^{(\beta)}$, the following orthogonal properties hold:

Lemma 2. *For $\varphi \in H^1(\tau)$,*

$$\|\nabla\Pi_\tau^{(\alpha)}\varphi\|_{L^2(\tau)}^2 + \|\nabla(\varphi - \Pi_\tau^{(\alpha)}\varphi)\|_{L^2(\tau)}^2 = \|\nabla\varphi\|_{L^2(\tau)}^2.$$

Lemma 3. *For $\varphi \in H^2(\tau)$,*

$$|\Pi_\tau^{(\beta)}\varphi|_{H^2(\tau)}^2 + |\varphi - \Pi_\tau^{(\beta)}\varphi|_{H^2(\tau)}^2 = |\varphi|_{H^2(\tau)}^2.$$

Proof of Lemma 2. Since $\Pi_\tau^{(\alpha)}\varphi$ does not depend on the choice of the coordinates, we consider the x-axis to be aligned with the edge γ_1 and take $p_1 = (0,0)$, $p_2 = (h, 0)$, $p_3 = (ah, bh)$ and

$$\Pi_\tau^{(\alpha)}\varphi = a_1(x^2 + y^2) + a_2 x + a_3 y + a_4.$$

Then, the divergence theorem yields

$$\|\nabla\varphi\|_{L^2(\tau)}^2 - \|\nabla\Pi_\tau^{(\alpha)}\varphi\|_{L^2(\tau)}^2 - \|\nabla(\varphi - \Pi_\tau^{(\alpha)}\varphi)\|_{L^2(\tau)}^2$$

$$= 2\iint_\tau \nabla(\varphi - \Pi_\tau^{(\alpha)}\varphi) \cdot \nabla\Pi_\tau^{(\alpha)}\varphi \, dxdy$$

$$= 2\iint_\tau \mathrm{div}\Big((\varphi - \Pi_\tau^{(\alpha)}\varphi)\nabla\Pi_\tau^{(\alpha)}\varphi\Big) dxdy - 2\iint_\tau (\varphi - \Pi_\tau^{(\alpha)}\varphi)\Delta\Pi_\tau^{(\alpha)}\varphi \, dxdy$$

$$= 2 \oint_{\partial \tau} (\varphi - \Pi_\tau^{(\alpha)} \varphi) \nabla \Pi_\tau^{(\alpha)} \varphi \cdot n \, ds - 8 a_1 \iint_\tau (\varphi - \Pi_\tau^{(\alpha)} \varphi) \, dx dy$$

$$= 2 \oint_{\partial \tau} (\varphi - \Pi_\tau^{(\alpha)} \varphi) \begin{pmatrix} 2a_1 x + a_2 \\ 2a_1 y + a_3 \end{pmatrix} \cdot n \, ds$$

$$= 4 a_1 \oint_{\partial \tau} (\varphi - \Pi_\tau^{(\alpha)} \varphi) \begin{pmatrix} x \\ y \end{pmatrix} \cdot n \, ds$$

$$= 4 a_1 \int_{\gamma_1} (\varphi - \Pi_\tau^{(\alpha)} \varphi) \begin{pmatrix} x - h \\ y \end{pmatrix} \cdot n \, ds + 4 a_1 \int_{\gamma_2} (\varphi - \Pi_\tau^{(\alpha)} \varphi) \begin{pmatrix} x - ah \\ y - bh \end{pmatrix} \cdot n \, ds$$

$$+ 4 a_1 \int_{\gamma_3} (\varphi - \Pi_\tau^{(\alpha)} \varphi) \begin{pmatrix} x \\ y \end{pmatrix} \cdot n \, ds$$

$$= 4 a_1 \int_{\gamma_1} \sqrt{(x-h)^2 + y^2} \, (\varphi - \Pi_\tau^{(\alpha)} \varphi) \, \nu \cdot n \, ds$$

$$+ 4 a_1 \int_{\gamma_2} \sqrt{(x-ah)^2 + (y-bh)^2} \, (\varphi - \Pi_\tau^{(\alpha)} \varphi) \, \nu \cdot n \, ds$$

$$+ 4 a_1 \int_{\gamma_3} \sqrt{x^2 + y^2} \, (\varphi - \Pi_\tau^{(\alpha)} \varphi) \, \nu \cdot n \, ds = 0$$

$\square$

Proof of Lemma 3. Same as previous lemma, we take $p_1 = (0,0)$, $p_2 = (h,0)$, $p_3 = (ah, bh)$ and
$$\Pi_\tau^{(\beta)} = a_1 x^2 + a_2 xy + a_3 y^2 + a_4 x + a_5 y + a_6.$$

Then, the divergence theorem yields

$$|\varphi|_{H^2(\tau)}^2 - |\Pi_\tau^{(\beta)} \varphi|_{H^2(\tau)}^2 - |\varphi - \Pi_\tau^{(\beta)} \varphi|_{H^2(\tau)}^2$$

$$= 2 \iint_\tau \Big((\varphi - \Pi_\tau^{(\beta)} \varphi)_{xx} (\Pi_\tau^{(\beta)} \varphi)_{xx} + 2 (\varphi - \Pi_\tau^{(\beta)} \varphi)_{xy} (\Pi_\tau^{(\beta)} \varphi)_{xy}$$

$$+ (\varphi - \Pi_\tau^{(\beta)} \varphi)_{yy} (\Pi_\tau^{(\beta)} \varphi)_{yy} \Big) dx dy$$

$$= 2 \iint_\tau \text{div} \begin{pmatrix} \nabla (\varphi - \Pi_\tau^{(\beta)} \varphi) \cdot \nabla (\Pi_\tau^{(\beta)} \varphi)_x \\ \nabla (\varphi - \Pi_\tau^{(\beta)} \varphi) \cdot \nabla (\Pi_\tau^{(\beta)} \varphi)_y \end{pmatrix} dx dy$$

$$= 2 \oint_{\partial \tau} \begin{pmatrix} \nabla (\varphi - \Pi_\tau^{(\beta)} \varphi) \cdot \nabla (\Pi_\tau^{(\beta)} \varphi)_x \\ \nabla (\varphi - \Pi_\tau^{(\beta)} \varphi) \cdot \nabla (\Pi_\tau^{(\beta)} \varphi)_y \end{pmatrix} \cdot n \, ds$$

$$= 2 \oint_{\partial \tau} \nabla (\varphi - \Pi_\tau^{(\beta)} \varphi) \cdot \nabla (\nabla \Pi_\tau^{(\beta)} \varphi \cdot n) \, ds$$

$$= 2 \oint_{\partial \tau} \nabla (\varphi - \Pi_\tau^{(\beta)} \varphi) \cdot \begin{pmatrix} 2a_1 & a_2 \\ a_2 & 2a_3 \end{pmatrix} n \, ds.$$

Here, Lemma 1 yields

$$\int_{\gamma_k} (\varphi - \Pi_\tau^{(\beta)} \varphi)_x \, ds = \int_{\gamma_k} (\varphi - \Pi_\tau^{(\beta)} \varphi)_y \, ds = 0, \qquad k = 1, 2, 3,$$
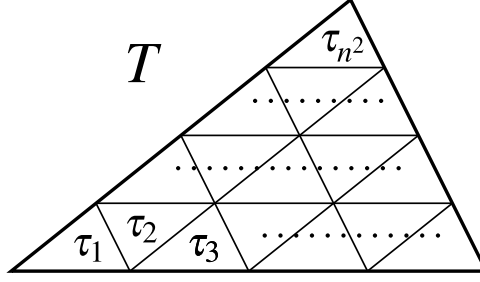
which leads us to the conclusion. $\square$

Figure 1: Divide T into n^2 congruent small triangles

4　Our method to bound the constants

We divide triangle T into n^2 congruent small triangles $\tau_1, \cdots, \tau_{n^2}$ (Fig. 1). We assume that each τ_k is open set, namely, does not contains its boundary and define

$$T' = \bigcup_{k=1}^{n^2} \tau_k.$$

Then we define $\Pi^{(\alpha)} u$ for $u \in H^1(T)$ and $\Pi^{(\beta)} u$ for $u \in H^2(T)$ as follows:

$$\Pi^{(\alpha)} u|_{\tau_k} = \Pi_{\tau_k}^{(\alpha)} u, \qquad \Pi^{(\beta)} u|_{\tau_k} = \Pi_{\tau_k}^{(\beta)} u.$$

Note that $\Pi^{(\alpha)} u$ and $\Pi^{(\beta)} u$ are not always continuous on T.

By solving finite dimensional generalized eigenvalue problems, we can obtain following constants:

$$C_1^{(n)}(T) = \sup_{u \in V^{1,1}(T) \backslash 0} \frac{\|\Pi^{(\alpha)} u\|_{L^2(T')}}{\|\nabla \Pi^{(\alpha)} u\|_{L^2(T')}}, \quad C_2^{(n)}(T) = \sup_{u \in V^{1,2}(T) \backslash 0} \frac{\|\Pi^{(\alpha)} u\|_{L^2(T')}}{\|\nabla \Pi^{(\alpha)} u\|_{L^2(T')}},$$

$$C_3^{(n)}(T) = \sup_{u \in V^2(T) \backslash 0} \frac{\|\Pi^{(\beta)} u\|_{L^2(T')}}{|\Pi^{(\beta)} u|_{H^2(T')}}, \qquad C_4^{(n)}(T) = \sup_{u \in V^2(T) \backslash 0} \frac{\|\nabla \Pi^{(\beta)} u\|_{L^2(T')}}{|\Pi^{(\beta)} u|_{H^2(T')}}.$$

With respect to these constants, we have the following theorem:

Theorem 2.

$$C_1(T) \le \sqrt{\frac{n^2}{n^2 - 1}} \, C_1^{(n)}(T), \qquad C_2(T) \le \sqrt{\frac{n^2}{n^2 - 1}} \, C_2^{(n)}(T),$$

$$C_3(T) \le \sqrt{\frac{n^4}{n^4 - 1}} \, C_3^{(n)}(T), \qquad C_4(T) \le \sqrt{\frac{n^2}{n^2 - 1}} \, C_4^{(n)}(T),$$

$$C_4(T) \le \sqrt{C_4^{(n)}(T)^2 + \frac{C_2(T)^2}{n^2}},$$

Proof. From Lemma 2, for $u \in V^{1,j}(T), \ \ j = 1, 2$, we have

$$
\begin{aligned}
\|u\|_{L^2(T)} &\leq \|\Pi^{(\alpha)}u\|_{L^2(T')} + \|u - \Pi^{(\alpha)}u\|_{L^2(T')} \\
&= \|\Pi^{(\alpha)}u\|_{L^2(T')} + \sqrt{\sum_{k=1}^{n^2} \|u - \Pi_{\tau_k}^{(\alpha)}u\|_{L^2(\tau_k)}^2} \\
&\leq C_j^{(n)}(T)\, \|\nabla\Pi^{(\alpha)}u\|_{L^2(T')} + \frac{C_j(T)}{n}\sqrt{\sum_{k=1}^{n^2} \|\nabla(u - \Pi_{\tau_k}^{(\alpha)}u)\|_{L^2(\tau_k)}^2} \\
&\leq \sqrt{C_j^{(n)}(T)^2 + \frac{C_j(T)^2}{n^2}} \sqrt{\sum_{k=1}^{n^2} \left(\|\nabla\Pi_{\tau_k}^{(\alpha)}u\|_{L^2(\tau_k)}^2 + |\nabla(u - \Pi_{\tau_k}^{(\alpha)}u)|_{L^2(\tau_k)}^2 \right)} \\
&= \sqrt{C_j^{(n)}(T)^2 + \frac{C_j(T)^2}{n^2}} \sqrt{\sum_{k=1}^{n^2} \|\nabla u\|_{L^2(\tau_k)}^2} \\
&= \sqrt{C_j^{(n)}(T)^2 + \frac{C_j(T)^2}{n^2}}\, \|\nabla u\|_{L^2(T)}.
\end{aligned}
$$

Furthermore, from Lemma 3, for $u \in V^2(T)$,

$$
\begin{aligned}
\|u\|_{L^2(T)} &\leq \|\Pi^{(\beta)}u\|_{L^2(T')} + \|u - \Pi^{(\beta)}u\|_{L^2(T')} \\
&= \|\Pi^{(\beta)}u\|_{L^2(T')} + \sqrt{\sum_{k=1}^{n^2} \|u - \Pi_{\tau_k}^{(\beta)}u\|_{L^2(\tau_k)}^2} \\
&\leq C_3^{(n)}(T)\, |\Pi^{(\beta)}u|_{H^2(T')} + \frac{C_3(T)}{n^2}\sqrt{\sum_{k=1}^{n^2} |u - \Pi_{\tau_k}^{(\beta)}u|_{H^2(\tau_k)}^2} \\
&\leq \sqrt{C_3^{(n)}(T)^2 + \frac{C_3(T)^2}{n^4}} \sqrt{\sum_{k=1}^{n^2} \left(|\Pi_{\tau_k}^{(\beta)}u|_{H^2(\tau_k)}^2 + |u - \Pi_{\tau_k}^{(\beta)}u|_{H^2(\tau_k)}^2 \right)} \\
&= \sqrt{C_3^{(n)}(T)^2 + \frac{C_3(T)^2}{n^4}} \sqrt{\sum_{k=1}^{n^2} |u|_{H^2(\tau_k)}^2} \\
&= \sqrt{C_3^{(n)}(T)^2 + \frac{C_3(T)^2}{n^4}}\, |u|_{H^2(T)}
\end{aligned}
$$

and

$$
\begin{aligned}
\|\nabla u\|_{L^2(T)} &\leq \|\nabla\Pi^{(\beta)}u\|_{L^2(T')} + \|\nabla(u - \Pi^{(\beta)}u)\|_{L^2(T')} \\
&= \|\nabla\Pi^{(\beta)}u\|_{L^2(T')} + \sqrt{\sum_{k=1}^{n^2} \|\nabla(u - \Pi_{\tau_k}^{(\beta)}u)\|_{L^2(\tau_k)}^2}
\end{aligned}
$$

$$\leq C_4^{(n)}(T)\,|\Pi^{(\beta)}u|_{H^2(T')} + \frac{C_4(T)}{n}\sqrt{\sum_{k=1}^{n^2}|u - \Pi_{\tau_k}^{(\beta)}u|_{H^2(\tau_k)}^2}$$

$$\leq \sqrt{C_4^{(n)}(T)^2 + \frac{C_4(T)^2}{n^2}}\sqrt{\sum_{k=1}^{n^2}\left(|\Pi_{\tau_k}^{(\beta)}u|_{H^2(\tau_k)}^2 + |u - \Pi_{\tau_k}^{(\beta)}u|_{H^2(\tau_k)}^2\right)}$$

$$= \sqrt{C_4^{(n)}(T)^2 + \frac{C_4(T)^2}{n^2}}\sqrt{\sum_{k=1}^{n^2}|u|_{H^2(\tau_k)}^2}$$

$$= \sqrt{C_4^{(n)}(T)^2 + \frac{C_4(T)^2}{n^2}}\,|u|_{H^2(T)}$$

holds. Using Lemma 1, we can also evaluate $\|\nabla(u - \Pi^{(\beta)}u)\|_{L^2(T')}$ in the first line of the previous expression by

$$\|\nabla(u - \Pi^{(\beta)}u)\|_{L^2(T')} = \sqrt{\sum_{k=1}^{n^2}\left(\|(u - \Pi_{\tau_k}^{(\beta)}u)_x\|_{L^2(\tau_k)}^2 + \|(u - \Pi_{\tau_k}^{(\beta)}u)_y\|_{L^2(\tau_k)}^2\right)}$$

$$\leq \frac{C_2(T)}{n}\sqrt{\sum_{k=1}^{n^2}\left(\|\nabla(u - \Pi_{\tau_k}^{(\beta)}u)_x\|_{L^2(\tau_k)}^2 + \|\nabla(u - \Pi_{\tau_k}^{(\beta)}u)_y\|_{L^2(\tau_k)}^2\right)}$$

$$= \frac{C_2(T)}{n}\sqrt{\sum_{k=1}^{n^2}|u - \Pi_{\tau_k}^{(\beta)}u|_{H^2(\tau_k)}^2}.$$

From above evaluations, we have the following:

$$C_1(T) \leq \sqrt{C_1^{(n)}(T)^2 + \frac{C_1(T)^2}{n^2}}, \qquad C_2(T) \leq \sqrt{C_2^{(n)}(T)^2 + \frac{C_2(T)^2}{n^2}},$$

$$C_3(T) \leq \sqrt{C_3^{(n)}(T)^2 + \frac{C_3(T)^2}{n^4}}, \qquad C_4(T) \leq \sqrt{C_4^{(n)}(T)^2 + \frac{C_4(T)^2}{n^2}},$$

$$C_4(T) \leq \sqrt{C_4^{(n)}(T)^2 + \frac{C_2(T)^2}{n^2}},$$

which leads us to the conclusion. $\qquad\qquad\square$

This result shows that we can bound the constants $C_1(T) \sim C_4(T)$ by means of $C_1^{(n)}(T) \sim C_4^{(n)}(T)$. We can compute $C_1^{(n)}(T) \sim C_4^{(n)}(T)$ numerically and also, obtain guaranteed results via the numerical verification method. The numerical results show that the sharp and explicit upper bounds are obtained by our method and the formulas introduced in Section 1.

5   Circumradius and $C_4(T)$

In Section 1, we claimed that the following estimate holds for the interpolation constant $C_4(T)$:

$$C_4(T) < K_4(T) = \sqrt{\frac{A^2 B^2 C^2}{16 S^2} - \frac{A^2 + B^2 + C^2}{30} - \frac{S^2}{5}\left(\frac{1}{A^2} + \frac{1}{B^2} + \frac{1}{C^2}\right)},$$

where A, B, C are the edge length of triangle T and S is the area of T. Since the circumradius of T is given by

$$R(T) = \frac{ABC}{4S},$$

we have the estimation

$$C_4(T) < R(T).$$

This fact is full of interesting suggestions for the error analysis in the finite element method. See [9, 10] for the details.

6   Conclusion

We present a simple method to obtain sharp upper bounds for the interpolation error constants over the given triangular elements. These constants are important for analysis of interpolation error and especially for the error analysis in the finite element method. Guaranteed upper bounds for these constants are obtained via the numerical verification method. Furthermore, we introduce remarkable formulas for the upper bounds of these constants. By the method explained in this paper, we can only prove these formulas for the given triangles. However, using some continuation techniques and asymptotic analysis, we can prove the formulas for any triangle. We will explain the details in the presentaion.

Acknowledgements

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ESTIMATES FOR RADIAL SOLUTIONS OF THE HOMOGENEOUS LANDAU EQUATION WITH COULOMB POTENTIAL

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1. INTRODUCTION

We consider the Cauchy problem for the homogeneous Landau equation: such equation takes the general form

$$\partial_t f(v,t) = Q(f,f), \quad f(v,0) = f_{in}(v), \quad v \in \mathbb{R}^3, \quad t > 0, \tag{1.1}$$

where $Q(f,f)$ is a quadratic operator known as the Landau collisional operator

$$Q(f,f) = \operatorname{div}\left(\int_{\mathbb{R}^3} A(v-y)\left(f(y)\nabla_v f(v) - f(v)\nabla_y f(y)\right) dy\right). \tag{1.2}$$

The term $A(v)$ denotes a positive and symmetric matrix

$$A(v) := C_\gamma \left(\operatorname{Id} - \frac{v \otimes v}{|v|^2}\right)\varphi(|v|), \quad v \neq 0, \quad C_\gamma > 0,$$

which acts as the projection operator onto the space orthogonal to the vector v. The function $\varphi(|v|)$ is a scalar valued function determined from the original Boltzmann kernel describing how particles interact. If the interaction strength between particles at a distance r is proportional to r^{-s}, then

$$\varphi(|v|) := |v|^{\gamma+2}, \quad \gamma = \frac{(s-5)}{(s-1)}. \tag{1.3}$$

Any solution to (1.1)-(1.2) is an integrable and nonnegative scalar field $f(v,t) : \mathbb{R}^3 \times [0,T] \to \mathbb{R}^+$. Equation (1.1) describes the evolution of a plasma in spatially homogeneous regimes, which means that the density function f depends only on the velocity component v. Landau's original intent in deriving this approximation was to make sense of the Boltzmann collisional operator, which always diverges when considering purely grazing collisions.

The Cauchy problem for (1.1)-(1.3) is very well understood for the case of hard potentials, which correspond to $\gamma \geq 0$ above. Desvillettes and Villani showed the existence of global classical solutions for hard potentials and studied its long time behavior, see [3, 4, 11] and references therein. In this case there is a unique global smooth solution, which converges exponentially to an equilibrium distribution, known as the Maxwellian function

$$\mathcal{M}(v) = \frac{1}{(2\pi)^{3/2}} e^{-\frac{|v|^2}{2}}.$$

Analyzing the soft potentials case, $\gamma < 0$, has proved to be more difficult: using a probabilistic approach, the authors in [12, 5, 1] show uniqueness and existence of weak solutions for $\gamma \in [-2,0]$. For $\gamma \in [-3,-2]$ it is known (i) existence for small time or (ii) global in time existence with smallness assumption on initial data [1, 2].

Villani [10] introduced the so called H-solutions, which enjoy (weak) a priori bounds in a weighted Sobolev space. However, the issue of their uniqueness and regularity (i.e. no finite time break down occurs) has remained open, even for smooth initial data: see [11, Chapter 1, Chapter 5] for further discussion.

Guo in [8] employs a completely different approach based on perturbation theory for the existence of periodic solutions to the spatially inhomogeneous Landau equation in $\mathbb{R}^3$. He shows that if the initial data is sufficiently close to the unique equilibrium in a certain high Sobolev norm then a unique global solution exists. Moreover, as remarked in [8], this approach also extends to the case of potentials (1.3) where γ might even take values below -3.

Due to the lack of a global well-posedness theory, several conjectures about possible finite-time blow up for general initial data have been made throughout the years. In [11] Villani discussed the possibility that (1.1)-(1.3) could blow up for $\gamma = -3$. Note that for smooth solutions (1.1)-(1.3) with $\gamma = -3$ can be rewritten as

$$\partial_t f = \operatorname{div}(A[f]\nabla f - f\nabla a[f]) = \operatorname{Tr}(A[f]D^2 f) + f^2, \tag{1.4}$$

where

$$A[f] := A(v) * f = \frac{1}{8\pi|v|}\left(\operatorname{Id} - \frac{v \otimes v}{|v|^2}\right) * f, \quad \Delta a = -f.$$

Equation (1.4) can be thought of as a quasi-linear nonlocal heat equation. Supports for blow-up conjectures were given by the fact that (1.4) is reminiscent of the well studied semilinear heat equation

$$\partial_t f = \Delta f + f^2. \tag{1.5}$$

Blow up for (1.5) is known to happen for every L^p norm, $p > 3/2$, see [6].

However, despite the apparent similarities, equation (1.4) behaves differently from (1.5). The Landau equation admits a richer class of equilibrium solution: every Maxwellian $\mathcal{M}$ solves $Q(\mathcal{M}, \mathcal{M}) = 0$ which holds, in particular, for those with arbitrarily large mass.

From a different perspective, Krieger-Strain [9] considered an isotropic version of (1.4)

$$\partial_t f = a[f]\Delta f + \alpha f^2, \tag{1.6}$$

and showed global existence of smooth radial solutions starting from radial initial data when $\alpha < 2/3$. This range for α later was expanded to any $\alpha < 74/75$ by means of a non-local inequality obtained by Gressman, Krieger and Strain [7]. Note that when $\alpha = 1$, the above equation can be written in divergence form,

$$\partial_t f = \operatorname{div}(a[f]\nabla f - f\nabla a[f]). \tag{1.7}$$

These results put in evidence how a non-linear equation with a non-local diffusivity such as (1.7) behaves drastically different from (and better than) (1.5).

Our main results in this manuscript are twofold. The first one gives necessary conditions for the finite time blow up of solutions to (1.4). The second (unconditional) result says that solutions to (1.7) do not blow up at all, and in fact become instantaneously smooth (even for initial data that might be initially unbounded).

Both theorems deal only with radially symmetric, decreasing initial conditions, in general we assume that

$$\begin{cases} f_{\text{in}} \geq 0, \quad \int_{\mathbb{R}^3} f_{\text{in}} \, dv = 1, \quad \int_{\mathbb{R}^3} f_{\text{in}} |v|^2 \, dv = 3, \\ H = \int_{\mathbb{R}^3} f \log(f) \, dv < \infty, \quad \text{and} \quad |v| \leq |w| \Rightarrow f_{\text{in}}(v) \geq f_{\text{in}}(w). \end{cases} \tag{1.8}$$

The normalization of the initial data is standard and follows a standard change of variables. The main results are the following.

Theorem 1.1. *Let f_{in} be as in* (1.8) *and such that $f_{\text{in}} \in L^\infty$. Then there exists $T_0 > 0$ and $f : \mathbb{R}^3 \times (0, T_0) \to \mathbb{R}_+$ such that f is smooth and solves* (1.4) *for $t \in (0, T_0)$, with $f(\cdot, 0) = f_{\text{in}}$. Moreover, T_0 is maximal in the sense that either $T_0 = \infty$ or else the $L^{3/2}$ norm of f accumulates near $v = 0$ as $t \to T_0^-$, in particular*

$$\lim_{t \to T_0^-} \|f(\cdot, t)\|_{L^p(B_1)} = \infty, \quad \forall\, p > 3/2.$$

In fact, the above theorem is a consequence of the following sharper result.

Theorem 1.2. *There is a constant ε_0, with $\varepsilon_0 \geq 1/96$, such that if above $T_0 < \infty$ then*

$$\limsup_{r \to 0^+} \sup_{t \in (0, T_0)} \left\{ r^2 \frac{\int_{B_r} f(v, t) \, dv}{\int_{B_r} a[f](v, t) \, dv} \right\} \geq \varepsilon_0.$$

Neither of the above theorems are enough to guarantee long time existence of classical solutions to (1.4). However, Theorem 1.2 suggests that (1.4) is in some sense "critical" for regularity. It can be shown (see Proposition **??**) that for any nonnegative $f \in L^1(\mathbb{R}^3)$

$$r^2 \frac{\int_{B_r} f(v) \, dv}{\int_{B_r} a[f](v) \, dv} \leq 3, \quad \forall\, r > 0.$$

In particular, if the ε_0 in Theorem 1.2 could be shown to be at least 3 (or in general if the upper bound in the last inequality could be improved to something less than ε_0) it would immediately follow that solutions to the Landau equation (1.4) cannot blow up in finite time. It is not clear if this can be guaranteed for general f without at least using some partial time regularization.

On the other hand, methods used in the proof of Theorem 1.1 and Theorem 1.2 yield long time existence for the isotropic Landau equation (1.7) (again, in the radial case).

Theorem 1.3. *Let f_{in} be as in* (1.8) *and such that for some $p > 6$,*

$$f_{\text{in}} \in L^p_{weak}(\mathbb{R}^3)$$

Then, there exists a function $f : \mathbb{R}^3 \times \mathbb{R}_+ \to \mathbb{R}$, smooth for positive times, with $f(\cdot, 0) = f_{\text{in}}$, and which solves (for $t > 0$),

$$\partial_t f = a[f] \Delta f + f^2.$$

We approach the analysis from the point of view of nonlinear parabolic equations. The nonlocal dependence of the coefficients on the solution prevents the equation from satisfying a comparison principle: if v_0 is a contact point of two functions f and g, i.e. $f(v_0) = g(v_0)$ and everywhere else $f(v) \leq g(v)$, it does not follow that $Q(f, f)(v_0) \leq Q(g, g)(v_0)$. More precisely, for the case where $Q(f, f)$ corresponds to (1.2) one cannot expect an inequality such as

$$\text{Tr}(A[f] D^2 f)(v_0) \leq \text{Tr}(A[g] D^2 g)(v_0).$$

In fact due to the nonlocality of A one only has $A[f](v_0) \leq A[g](v_0)$. Equality $A[f](v_0) = A[g](v_0)$ holds only when $f \equiv g$ for every $v \in \mathbb{R}^3$. The maximum principle is not useful either,

since at a maximum point for f we only obtain $\partial_t f \leq -f \Delta a[f]$, which does not rule out growth of the maximum of f. The same observations apply to $Q(f,f)$ corresponding to (1.7).

On the other hand, if one could construct (using only properties of f that are independent of t) a function $U(v)$ such that

$$\text{Tr}(A[f])D^2 U) + fU \leq 0 \text{ in } \mathbb{R}^3$$

$$(\text{ respectively, } a[f]\Delta U + fU \leq 0 \text{ in } \mathbb{R}^3 \text{)},$$

then the comparison principle (for linear parabolic equations) would guarantee that $f \leq cU$ for all times provided $f(t=0) \leq cU$. Our main observation is that (under radial symmetry) the above can be made to work with $U(v) = |v|^{-\gamma}$, $\gamma \in (0,1)$. From here higher local integrability of f can be propagated, and from there higher regularity follows by standard elliptic regularization.

2. RADIAL SYMMETRY

Proposition 2.1. *Suppose f_{in} and $g(\cdot, t)$ are both radially symmetric, and let $Q(\cdot, \cdot)$ denote either $Q_{\mathcal{L}}$ or $Q_{\mathcal{KS}}$. Then any solution of the linear Cauchy problem*

$$\partial_t f = Q(g, f), \quad f(v, 0) = f_{in}(v),$$

is radially symmetric for all t. Furthermore, if f_{in} and g are radially decreasing, then so is f.

Let $h : \mathbb{R}^3 \to \mathbb{R}_+$, define

$$A^*[h](v) := (A[h](v)\hat{v}, \hat{v}), \quad v \neq 0, \quad \hat{v} := v|v|^{-1}. \tag{2.1}$$

There are two useful expressions for $A^*[h]$ and $a[h]$ when h is radially symmetric.

Proposition 2.2. *Let $h \in L^1(\mathbb{R}^3)$ be radially symmetric and non-negative. Then*

$$A^*[h](v) = \frac{1}{12\pi|v|^3} \int_{B_{|v|}} h(w)|w|^2 \, dw + \frac{1}{12\pi} \int_{B_{|v|}^c} \frac{h(w)}{|w|} \, dw, \tag{2.2}$$

$$a[h](v) = \frac{1}{4\pi|v|} \int_{B_{|v|}} h(w) \, dw + \frac{1}{4\pi} \int_{B_{|v|}^c} \frac{h(w)}{|w|} \, dw. \tag{2.3}$$

The second formula above is simply the classical formula for the Newtonian potential in the case of radial symmetry; the formula for $A^*[h]$ is new and the proof can be found in the Appendix.

Lemma 2.3. *Let $h \in L^1(\mathbb{R}^3)$ be a non-negative, decreasing radial function.*

(1) If

$$\int_{B_{R_1} \setminus B_{R_0}} h \, dv \geq \delta > 0,$$

for some $\delta > 0$ and $0 < R_0 < R_1$ then,

$$A[h](v) \geq \frac{\delta R_0^2}{12\pi(1 + R_1^3)} \frac{1}{1 + |v|^3} \mathbb{I}. \tag{2.4}$$

(2) If h is bounded, i.e. if $\|h\|_{L^\infty(\mathbb{R}^3)} = h(0) < +\infty$, it holds

$$A[h](v) \leq a[h]\mathbb{I} \leq 2 \left(\frac{\|h\|_{L^\infty(\mathbb{R}^3)} + \|h\|_{L^1(\mathbb{R}^3)}}{1 + |v|} \right) \mathbb{I}. \tag{2.5}$$

Proof. (1) Let $A^*[h]$ be as in (2.2). If $|v| \geq R_1$, then

$$A^*[h](v) \geq \frac{1}{12\pi|v|^3} \int_{B_{R_1}} h(w)|w|^2 \, dw \geq \frac{1}{12\pi|v|^3} \int_{B_{R_1} \setminus B_{R_0}} h(w)|w|^2 \, dw$$

$$\geq \frac{R_0^2}{12\pi|v|^3} \int_{B_{R_1} \setminus B_{R_0}} h(w,t) \, dw \geq \frac{\delta R_0^2}{12\pi|v|^3}.$$

Note that Proposition (2.2) guarantees that $A^*[h]$ is radially decreasing. Thus,

$$A^*[h](v) \geq \frac{\delta R_0^2}{12\pi R_1^3}, \quad \forall \, v \in B_{R_1}.$$

Combining both estimates, we conclude that

$$A^*[h](v) \geq \frac{\delta R_0^2}{12\pi(1 + R_1^3)} \frac{1}{1 + |v|^3}.$$

(2) If $h \in L^\infty$, we may use (2.2) to obtain the estimate

$$A[h] \leq a[h](v)\mathbb{I} \quad \leq \quad \left(\frac{h(0)}{4\pi|v|} \int_{B_{|v|}} dw + \frac{1}{4\pi} \int_{B_1^c} h(w) \, dw + \frac{1}{4\pi} \int_{B_1} \frac{h(w)}{|w|} \, dw \right) \mathbb{I}$$

$$\leq \quad \left(\|h\|_{L^\infty(\mathbb{R}^3)} + \|h\|_{L^1(\mathbb{R}^3)} \right) \mathbb{I}, \quad \text{if } |v| \leq 1,$$

and

$$A[h] \leq a[h](v)\mathbb{I} \leq \quad \left(\frac{\|h\|_{L^1(\mathbb{R})}}{2\pi|v|} \right) \mathbb{I}, \quad \text{if } |v| \geq 1.$$

$\square$

Proposition 2.4. *Let h be a positive and radially symmetric and decreasing function. For any $\gamma \in (0,1)$ define $U_\gamma(v)$ as*

$$U_\gamma(v) := |v|^{-\gamma}.$$

Then, for $Q = Q_\mathcal{L}$ or $Q = Q_{\mathcal{KS}}$

$$Q(h, U_\gamma) \leq U_\gamma \left(-\tfrac{1}{3}\gamma(1 - \gamma)a[h]|v|^{-2} + h \right).$$

3. Pointwise bounds and proof of Theorem 1.1

3.1. **Conditional pointwise bound.** The first lemma of this section (Lemma 3.2) is the key argument for the proofs of Theorem 1.1 and Theorem 1.3. It consists of a barrier argument based on the observation that the function $U(v) = |v|^{-\gamma}$ is a supersolution for the elliptic operator $Q(f, \cdot)$ under certain assumptions on f (this is where the radial symmetry and monotonicity is needed). It affords control of certain spatial L^p-norms of the solution, and from these higher regularity will follow by standard elliptic estimates (Lemma 3.3).

First, we prove an elementary proposition that will be of use in proving the key lemma.

Proposition 3.1. *If h is a non-negative, radially symmetric and decreasing function,*

$$\frac{h(v)}{a[h](v)} \leq 8 \sup_{r \leq |v|} \left\{ r^2 \frac{\int_{B_r} h(w) \, dw}{\int_{B_r} a[h](w) \, dw} \right\} |v|^{-2}, \quad \forall \, v \in \mathbb{R}^3.$$

Lemma 3.2. *Suppose $f : \mathbb{R}^3 \times [0,T] \to \mathbb{R}_+$ is a classical solution of* (**??**). *Let $\gamma \in (0,1)$, suppose there exists some $R_0 > 0$ such that*

$$r^2 \frac{\int_{B_r} g(w,t)\,dw}{\int_{B_r} a[g](w,t)\,dw} \leq \tfrac{1}{24}\gamma(1-\gamma), \ \ \forall\, r \leq R_0, \ \ t \leq T. \tag{3.1}$$

Then,

$$f(v,t) \leq \max\left\{ \tfrac{3}{4\pi} R_0^{\gamma-3}, (\tfrac{3}{4\pi})^{\gamma/3}\|f_{\mathrm{in}}\|_{L^{3/\gamma}_{\mathrm{weak}}} \right\} |v|^{-\gamma}, \ \ \text{in } B_{R_0} \times [0,T].$$

In particular, the conclusion of the lemma holds for some $R_0 > 0$ whenever there is a modulus of continuity $\omega(r)$ and some $R_1 > 0$ such that

$$\sup_{r<|v|}\sup_{t\in[0,T]} \left\{ r^2 \frac{\int_{B_r} g(w,t)\,dw}{\int_{B_r} a[g](w,t)\,dw} \right\} \leq \omega(|v|), \ \ \forall\, 0 < |v| \leq R_1. \tag{3.2}$$

The next lemma says that any solution f to (1.4) or (1.7) is a bounded function for all times provided f satisfies (3.2).

Lemma 3.3. *Let $f : \mathbb{R}^3 \times [0,T] \to \mathbb{R}$ be a radially symmetric, radially decreasing solution to* (1.4) *(or* (1.7)*) with initial data as in* (1.8) *and such that for some $R_0 > 0$ we have*

$$r^2 \frac{\int_{B_r} f(w,t)\,dw}{\int_{B_r} a[f](w,t)\,dw} \leq \tfrac{1}{24}\gamma(1-\gamma), \ \ \forall\, r \leq R_0, \ \ t \leq T.$$

Or, assume that there is some modulus of continuity $\omega(r)$ such that,

$$\sup_{r<|v|}\sup_{t\in[0,T]} \left\{ r^2 \frac{\int_{B_r} f(w,t)\,dw}{\int_{B_r} a[f](w,t)\,dw} \right\} \leq \omega(|v|), \ \ \forall\, 0 < r \leq R_0. \tag{3.3}$$

Then,

$$\sup_{t\in[\frac{1}{2}T,T]} \|f(\cdot,t)\|_{L^\infty(\mathbb{R}^3)} \leq C_0. \tag{3.4}$$

For some constant C_0 depending only on f_{in}, T and R_0.

4. Mass comparison and proof of Theorem 1.3

In this section we apply the ideas from previous sections to construct global solutions (in the radial, monotone case) for equation (1.7), namely

$$\partial_t f = a[f]\Delta f + f^2.$$

In view of Lemma 3.3, the fact that $T_0 = \infty$ in Theorem 1.1 results from a bound of any $L^p(\mathbb{R}^3)$-norm of f, with $p > 3/2$. For (1.7) the bound of any $L^p(\mathbb{R}^3)$-norm of f, with $p > 3/2$ will be proven by a barrier argument done at the level of the *mass function* of $f(v,t)$, which is defined by

$$M_f(r,t) = \int_{B_r} f(v,t)\,dv, \quad (r,t) \in \mathbb{R}_+ \times (0,T_0).$$

Depending on which problem f solves, the associated function $M_f(r,t)$ solves a one-dimensional parabolic equation with diffusivity given by $A^*[f]$ or $a[f]$.

Proposition 4.1. *Let f be a solution of* (1.4) *(resp.* (1.7)*) in $\mathbb{R}^3 \times [0, T_0]$, then $M(r,t)$ solves*

$$\partial_t M_f = A^* \partial_{rr} M_f + \frac{2}{r}\left(\frac{M_f}{8\pi r} - A^*\right)\partial_r M_f \quad \text{in } \mathbb{R}_+ \times (0, T_0) \tag{4.1}$$

$$\left(\text{resp.} \quad \partial_t M_f = a \partial_{rr} M_f + \frac{2}{r}\left(\frac{M_f}{8\pi r} - a\right)\partial_r M_f \quad \text{in } \mathbb{R}_+ \times (0, T_0)\right). \tag{4.2}$$

Define the linear parabolic operator L in $\mathbb{R}_+ \times (0, T)$ as

$$Lh := \partial_t h - a \partial_{rr} h - \frac{2}{r}\left(\frac{M_f}{8\pi r} - a[f]\right)\partial_r h.$$

The above proposition simply says that $LM_f = 0$ in $\mathbb{R}_+ \times (0, T)$. The next proposition identifies suitable supersolutions for L.

Proposition 4.2. *If $m \in [0, 2]$ and $h(r,t) = r^m$ then $Lh \geq 0$ in $\mathbb{R}_+ \times (0, T)$.*

Proof of Theorem 1.3. Assume $f_{\text{in}} \in L^\infty$. As the bound for $f(v, t)$ will not rely on the L^∞ norm of f_{in} but a L^p_{weak} norm of f_{in} the existence of a solution for unbounded initia data in L^p ($p > 6$) will follow by a standard density argument.

Since $p > 6$, there is some $\alpha > 0$ and some $C_0 > 0$ (depending only on $\|f\|_{L^p_{\text{weak}}}$) such that

$$M_{f_{in}}(r, 0) = \int_{B_r} f_{in}\, dv \leq C_0 r^{1+\alpha}.$$

Moreover, since $f(\cdot, t)$ has total mass 1 for every $t > 0$, we also have

$$M_f(r, t) \leq 1, \quad \forall\, r > 0, t \in (0, T).$$

Proposition 4.2 says that $h = Cr^{1+\alpha}$ is a supersolution of the parabolic equation solved by M_f in $\mathbb{R}_+ \times (0, T)$. Then, choosing $C := \max\{C_0, 1\}$ the comparison principle yields

$$M_f(r, t) \leq h(r) = Cr^{1+\alpha} \quad \text{for } r \in (0, 1),\ t \in (0, T). \tag{4.3}$$

Since $f(v, t)$ is radially symmetric and decreasing, bound (4.3) implies that $f(|v|, t) \leq \frac{3C}{4\pi}\frac{1}{|v|^{2-\alpha}}$ for $v \in B_1$ and $t \in (0, T)$; hence there is some $p' > 3/2$ and some $C_{p'} > 0$ such that

$$\|f(\cdot, t)\|_{L^{p'}(B_1)} \leq C_{p'}, \quad \forall\, t \in (0, T).$$

Then Lemma 3.2 says that $f(v, t)$ is bounded in $\mathbb{R}^3 \times (0, T_0)$. By Lemma 3.3 it follows that $T_0 = +\infty$ so the solution is global in time. $\qquad\square$

The method of the proof for Theorem 1.3 fails short in preventing finite time blow up for (1.4). In any case, it at least gives another criterium for blow-up, the proof of which is essentially the same as that of Theorem 1.3.

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Two dimensional viscous free surface flow down an inclined plane

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1   Introduction

In this talk we consider the two dimensional motion of a viscous incompressible fluid flowing down an inclined plane with an angle of inclination α $(0 < \alpha < \frac{\pi}{2})$ under the effect of gravity. The fluid motion is governed by the Navier-Stokes equations

$$\partial_t \boldsymbol{u} + (\boldsymbol{u}, \nabla)\boldsymbol{u} - \frac{1}{\mathcal{R}}\Delta\boldsymbol{u} + \frac{1}{\mathcal{R}}\nabla p = \frac{1}{\mathcal{F}^2}\boldsymbol{f}_e,$$
$$\operatorname{div}\boldsymbol{u} = 0 \quad \text{in } 0 < x_2 < 1 + \eta(x_1, t), \quad t > 0,$$

with the boundary conditions

$$\boldsymbol{u} = 0 \quad \text{on } x_2 = 0,$$
$$\partial_t \eta + u_1 \partial_1 \eta - u_2 = 0 \quad \text{on } x_2 = 1 + \eta(x_1, t),$$
$$p n_j - (\partial_k u_j + \partial_j u_k) n_k + 2\sigma \csc\alpha \frac{\partial_1^2 \eta}{\left(1 + (\partial_1 \eta)^2\right)^{\frac{3}{2}}} n_j = p_a n_j, \quad j = 1, 2,$$
$$\text{on } x_2 = 1 + \eta(x_1, t),$$

where $\boldsymbol{f}_e = (\sin\alpha, -\cos\alpha)$ is the external force and (n_1, n_2) is the outward unit normal of the free surface at $(x_1, 1 + \eta(x_1, t))$. For physical background see [3]. This problem is written in dimensionless variables (see [4] for nondimensionalization). Here $\mathcal{R}$ is the Reynolds number and $\mathcal{F}$ is the Froude number. The nondimensionalization constants σ and p_a denote the surface tension and the atmospheric pressure respectively. From now on we consider disturbances which are periodic in the streamwise coordinate x_1 from the stationary flow

$$\boldsymbol{U} = (2x_2 - x_2^2, 0), \quad P = p_a - 2\cot\alpha(x_2 - 1) \quad \text{in } 0 < x_2 < 1.$$

Set $\boldsymbol{u} = \boldsymbol{v} + \boldsymbol{U}$, $p = q + P$, then we derive the equations governing disturbances.

$$\partial_t \eta + (v_1 + 1 - \eta^2)\partial_1 \eta - v_2 = 0 \quad \text{on } x_2 = 1 + \eta(x_1, t), \quad x_1 \in \mathbb{T},$$
$$\partial_t \boldsymbol{v} - \frac{1}{\mathcal{R}}\Delta\boldsymbol{v} + (\boldsymbol{U}, \nabla)\boldsymbol{v} + (\boldsymbol{v}, \nabla)\boldsymbol{U} + \frac{1}{\mathcal{R}}\nabla q = -(\boldsymbol{v}, \nabla)\boldsymbol{v},$$
$$\operatorname{div}\boldsymbol{v} = 0 \quad \text{in } 0 < x_2 < 1 + \eta(x_1, t), \quad x_1 \in \mathbb{T}.$$

Here $\mathbb{T} = \mathbb{R}/(\frac{2\pi}{\ell}\mathbb{Z})$ denote the torus for periodicity. The boundary condition are

$$\boldsymbol{v} = 0 \quad \text{on } x_2 = 0$$

and

$$(q - 2\cot\alpha\eta)\begin{pmatrix} n_1 \\ n_2 \end{pmatrix} - \begin{pmatrix} 2\partial_1 v_1 & \partial_1 v_2 + \partial_2 \boldsymbol{v}_1 - 2\eta \\ \partial_1 v_2 + \partial_2 \boldsymbol{v}_1 - 2\eta & 2\partial_2 v_2 \end{pmatrix}\begin{pmatrix} n_1 \\ n_2 \end{pmatrix}$$
$$+ 2\sigma \csc\alpha \frac{\partial_1^2 \eta}{(1 + (\partial_1 \eta)^2)^{\frac{3}{2}}}\begin{pmatrix} n_1 \\ n_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{on } x_2 = 1 + \eta(x_1, t).$$

We transform the problem in the unknown domain

$$\Omega(t) = \{(x_1, x_2)\,;\, 0 < x_2 < 1 + \eta(x_1, t),\ \ x_1 \in \mathbb{T}\},\ \ t \geq 0$$

to the one in the fixed domain

$$\Omega = \{(x_1', x_2')\,;\, x_1' \in \mathbb{T},\ \ 0 < x_2' < 1\}$$

by use of the unknown function $\eta(x_1', t)$. For each $t \geq 0$ we define $\Theta : \Omega \to \Omega(t)$ by

$$\Theta(x_1', x_2' : t) = (x_1',\ x_2' + x_2'\tilde{\eta}(x_1', x_2', t)),\ \ 0 < x_2' < 1.$$

Here $\tilde{\eta}$ is the extension of η to $\mathbb{T} \times (-\infty, 0)$ defined by

$$\tilde{\eta}(x_1', x_2', t) = \sum_{\xi \in \mathbb{Z}0} \frac{\eta^{(\xi)}(t)}{1 + |\xi|^2 (x_2' - 1)^2} \exp(i\xi \cdot x_1'),$$

where $\tilde{\eta}(t)$ are the coefficients of the Fourier series expansion of $\eta(\cdot, t)$. We assume that the velocity $\boldsymbol{v}$ on $\Omega(t)$ is given by

$$v_j = \frac{1}{J}\theta_{j,k} v_k'$$

in terms of $\boldsymbol{v}'$ defined on Ω. $(\theta_{j,k})$ is the Jacobian matrix of Θ and J is the Jacobian $\det(\theta_{j,k})$. We set $q'(x_1', x_2', t) = q(\Theta(x_1', x_2', t))$. This transformation is defined as in a same way as in [4]. With these definitions we derive the equations for $\eta, \boldsymbol{v}', q'$ in Ω :

$$\partial_t \eta + \partial_1 \eta - v_2 = \eta^2 \partial_1 \eta \quad \text{on} \quad S_F = \{(x_1, 1) \in \bar{\Omega}\,;\, x_1 \in \mathbb{T}\},\ \ t > 0, \tag{1.1}$$

$$\partial_t \boldsymbol{v} - \frac{1}{\mathcal{R}}\Delta \boldsymbol{v} + (\boldsymbol{U}, \nabla)\boldsymbol{v} + (\boldsymbol{v}, \nabla)\boldsymbol{U} + \frac{1}{\mathcal{R}}\nabla q = F_0(\eta, \boldsymbol{v}) + \frac{1}{\mathcal{R}}Q\nabla q, \tag{1.2}$$

$$\text{div } \boldsymbol{v} = 0 \quad \text{in} \quad \Omega,\ \ t > 0, \tag{1.3}$$

where

$$Q = \begin{pmatrix} -\partial_2(x_2'\eta) & -\partial_1(x_2'\eta) \\ 0 & \dfrac{\partial_2(x_2'\eta)}{J} \end{pmatrix}$$

and

$$
\begin{aligned}
F_{0.j} = {} & \frac{\partial_2(x_2\partial_t\tilde{\eta})}{J}v_j - \frac{\delta_{j2}}{J}v_\ell\partial_\ell(x_2\partial_t\tilde{\eta}) + (x_2\partial_t\tilde{\eta})\left(-\frac{\partial_2^2(x_2\tilde{\eta})}{J^2}v_j + \frac{\delta_{j2}}{J^2}v_\ell\partial_{\ell2}^2(x_2\tilde{\eta}) + \frac{1}{J}\partial_2 v_j\right) \\
& -\big(2x_2(1-x_2)\tilde{\eta} + (x_2\tilde{\eta})^2\big)\partial_1 v_1 \\
& -(x_2 + x_2\tilde{\eta})(2 - (x_2 + x_2\tilde{\eta}))\left(-\frac{\partial_1(x_2\tilde{\eta})}{J}\partial_2 v_j - \frac{\partial_{12}^2(x_2\tilde{\eta})}{J}v_j + \frac{\partial_1(x_2\tilde{\eta})\partial_2^2(x_2\tilde{\eta})}{J^2}v_j\right) \\
& +2\delta_{j1}x_2\tilde{\eta}v_2 \\
& -2(1 - x_2(1+\tilde{\eta}))\left(\delta_{j1}v_m\partial_m(x_2\tilde{\eta}) - \delta_{2j}\frac{\partial_1(x_2\tilde{\eta})}{J}(v_2 + v_m\partial_m(x_2\tilde{\eta}))\right) \\
& +\frac{1}{\mathcal{R}}\left((\zeta_{\ell c}\zeta_{mc} - \delta_{\ell c}\delta_{mc})\partial_{\ell m}^2 v_j + \zeta_{\ell c}\partial_\ell\zeta_{mc}\partial_m v_j + 2J\zeta_{jk}\zeta_{\ell c}\zeta_{mc}\partial_\ell\left(\frac{\theta_{kn}}{J}\right)\partial_m v_n\right. \\
& \left.+J\zeta_{jk}\zeta_{\ell c}\partial_\ell\zeta_{mc}\partial_m\left(\frac{\theta_{kn}}{J}\right)v_n + J\zeta_{jk}\zeta_{\ell c}\zeta_{mc}v_n\partial_{\ell m}^2\left(\frac{\theta_{kn}}{J}\right)\right) \\
& -\frac{1}{J}v_m\partial_m v_j - \zeta_{jk}v_m v_\ell\partial_m\left(\frac{\theta_{k\ell}}{J}\right),\quad j = 1, 2.
\end{aligned}
\tag{1.4}
$$

Here we omit primes and collect linear term in the left hand side. By (1.1), we replace $\partial_t\eta$ in (1.4) by $v_2 + \eta^2\partial_1\eta$. The 2×2 matrix $(\zeta_{j,k})$ denotes the inverse of the Jacobian matrix $(\theta_{j,k})$. The explicit forms of their components are

$$\theta_{jk} = \delta_{jk} + \delta_{2j}\partial_k(x_2\tilde{\eta}), \qquad \zeta_{jk} = \delta_{jk} - \frac{\delta_{2j}}{J}\partial_k(x_2\tilde{\eta}).$$

The boundary condition on the bottom is

$$\boldsymbol{v} = \boldsymbol{0} \quad \text{on} \quad S_B = \{(x_1, 0) \in \bar{\Omega}\,; \ x_1 \in \mathbb{T}\}, \quad t > 0. \tag{1.5}$$

The conditions on the upper boundary $S_F = \{(x_1, 1) \in \bar{\Omega}\,; \ x_1 \in \mathbb{T}\}$ are written as follows

$$\partial_2 v_1 + \partial_1 v_2 - 2\eta = F_1(\eta, \boldsymbol{v}) \quad \text{on} \quad S_F, \quad t > 0, \tag{1.6}$$

$$q - 2\partial_2 v_2 - (2\cot\alpha - \sigma\csc\alpha\partial_1^2)\eta = F_2(\eta, \boldsymbol{v}) \quad \text{on} \quad S_F, \quad t > 0, \tag{1.7}$$

where

$$
\begin{aligned}
F_1 = {}& -\partial_1\left(\frac{1}{J}v_m(\theta_{2m} - J\delta_{2m})\right) - (\zeta_{k1} - \delta_{k1})\partial_k\left(v_m\frac{\theta_{2m}}{J}\right) \\
& -\partial_2\left(\frac{1}{J}v_m(\theta_{1m} - J\delta_{1m})\right) - (\zeta_{k2} - \delta_{k2})\partial_k\left(v_m\frac{\theta_{1m}}{J}\right) \\
& +2\partial_1\eta\left(\zeta_{k1}\partial_k\left(v_m\frac{\theta_{1m}}{J}\right) - \zeta_{k2}\partial_k\left(v_m\frac{\theta_{2m}}{J}\right)\right) \\
& +(\partial_1\eta)^2\left(\zeta_{k1}\partial_k\left(v_m\frac{\theta_{2m}}{J}\right) + \zeta_{k2}\partial_k\left(v_m\frac{\theta_{1m}}{J}\right) - 2\eta\right), \\
F_2 = {}& 2\sigma\csc\alpha\left(1 - \frac{1}{(1 + (\partial_1\eta)^2)^{\frac{3}{2}}}\right)\partial_1^2\eta + \frac{1}{1 + (\partial_1\eta)^2}\bigg\{-2(\partial_1\eta)^2\partial_2 v_2 \\
& +\partial_2\left(\frac{1}{J}v_m(\theta_{2m} - J\delta_{2m})\right) + (\zeta_{k2} - \delta_{k2})\partial_k\left(v_m\frac{\theta_{2m}}{J}\right) \\
& -2\partial_1\eta\left(\zeta_{k1}\partial_k\left(v_m\frac{\theta_{2m}}{J}\right) + \zeta_{k2}\partial_k\left(v_m\frac{\theta_{1m}}{J}\right) - 2\eta\right) \\
& +(\partial_1\eta)^2\zeta_{k1}\partial_k\left(v_m\frac{\theta_{1m}}{J}\right)\bigg\}.
\end{aligned}
$$

The full nonlinear equations (1.1) – (1.7) are solved numerically. The detail of the result of this simulation will be submitted for publication elsewhere, we summarize the method of numerical calculations as followed. In order to satisfy the solenoidal condition (1.3), the stream function $\psi(x_1, x_2, t)$ are introduced as follows:

$$v_1 = -\partial_2\psi, \quad v_2 = \partial_1\psi.$$

The actual time integrations are performed using the equations given in terms of ψ with spectral transform method. The v_1, v_2, η, q, ψ are expanded x_1 direction by Fourier expansion and x_2 direction by the Chebychev polynomials. The time integration is performed using Adams predictor-collector scheme.

2   Linearized problem

In this section we deal with the linearized problem of (1.1)– (1.7) as follows.

$$\partial_t\eta + \partial_1\eta - u_2 = g_0 \quad \text{on } S_F, \tag{2.1}$$

$$\partial_t\boldsymbol{u} - \frac{1}{\mathcal{R}}\Delta\boldsymbol{u} + \frac{1}{\mathcal{R}}\nabla p + (\boldsymbol{U}, \nabla)\boldsymbol{u} + (\boldsymbol{u}, \nabla)\boldsymbol{U} = \boldsymbol{f}_0 \quad \text{in } \Omega, \tag{2.2}$$

$$\text{div } \boldsymbol{u} = 0 \quad \text{in } \Omega, \tag{2.3}$$

$$\boldsymbol{u} = 0 \quad \text{on } S_B \tag{2.4}$$

$$\partial_2 u_1 + \partial_1 u_2 - 2\eta = 0 \quad \text{on } S_F, \tag{2.5}$$

$$p - 2\partial_2 u_2 - (2\cot\alpha - 2\sigma\csc\alpha\partial_1^2)\eta = 0 \quad \text{on } S_F. \tag{2.6}$$

To eliminate the pressure from (2.2) we use the orthogonal projection onto the L^2 orthogonal complement of the space of gradients. We introduce the projection orthogonal to the following space

$$\mathcal{G}^0 = \{\nabla\phi\,; \ \phi \in H^1(\Omega), \ \ \phi = 0 \ \ \text{on } S_F\}.$$

Define the orthogonal projection by $\mathbb{P}^0 : L^2(\Omega) \to (\mathcal{G}^0)^\perp$.

Lemma 2.1. *Let $r \geq 0$, i) $\mathbb{P}^0$ is a bounded operator on $H^r(\Omega)$. ii) Suppose $\phi \in H^1(\Omega)$. Then $\mathbb{P}^0(\nabla\phi) = \nabla\psi$, where ψ satisfies*

$$\psi = \phi \text{ on } S_F, \quad \partial_2\psi = 0 \text{ on } S_B, \quad \Delta\psi = 0 \text{ in } \Omega.$$

See [1], page 369 for the proof of Lemma 2.1.

We proceed to formulate the linearized problem (2.1)–(2.6). Applying $\mathbb{P}^0$ to (2.2) we have

$$\partial_t \boldsymbol{u} - \frac{1}{\mathcal{R}}\mathbb{P}^0\Delta\boldsymbol{u} + \frac{1}{\mathcal{R}}\mathbb{P}^0\nabla p + \mathbb{P}^0((\boldsymbol{U},\nabla)\boldsymbol{u} + (\boldsymbol{u},\nabla)\boldsymbol{U}) = \mathbb{P}^0\boldsymbol{f}_0.$$

Using Lemma 2.1 and finding p on S_F from (2.6), we can write this as follows

$$\partial_t \boldsymbol{u} - \frac{1}{\mathcal{R}}\mathbb{P}^0\Delta\boldsymbol{u} + \frac{1}{\mathcal{R}}\nabla p_1 + \frac{1}{\mathcal{R}}\nabla p_2 + \mathbb{P}^0((\boldsymbol{U},\nabla)\boldsymbol{u} + (\boldsymbol{u},\nabla)\boldsymbol{U}) = \mathbb{P}^0\boldsymbol{f}_0 \tag{2.7}$$

with

$$\Delta p_j = 0 \quad \text{in } \Omega, \quad \partial_2 p_j = 0 \quad \text{on } S_B, \quad j = 1,2,$$
$$p_1 = 2\partial_2 u_2 \qquad\qquad \text{on } S_F,$$
$$p_2 = (2\cot\alpha - 2\sigma\csc\alpha\partial_1^2)\eta \quad \text{on } S_F.$$

Collecting the terms depending on $\boldsymbol{u}$ in (2.7), we define the operator A by

$$A\boldsymbol{u} = -\frac{1}{\mathcal{R}}\mathbb{P}^0\Delta\boldsymbol{u} + \mathbb{P}^0((\boldsymbol{U},\nabla)\boldsymbol{u} + (\boldsymbol{u},\nabla)\boldsymbol{U}) + \frac{1}{\mathcal{R}}\nabla p_1.$$

For $\boldsymbol{u} \in \mathbb{P}^0 H^0(\Omega)$, set $R : \boldsymbol{u} \to u_2|_{S_F}$, the restriction of the second component of $\boldsymbol{u}$ to S_F. The formal adjoint R^* with respect to L^2 inner is given the gradient $\nabla\psi$ of the solution

$$\nabla\psi = 0 \quad \text{in } \Omega, \qquad \psi = \phi \quad \text{on } S_F, \qquad \partial_1\psi = 0 \quad \text{on } S_B$$

for a given $\phi \in H^{1/2}(\mathbb{T})$. By this notation we can rewrite (2.1), (2.7) and (2.6) as follows.

$$\partial_t\eta + \partial_1\eta - u_2 = g_0$$
$$\partial_t\boldsymbol{u} - A\boldsymbol{u} + \frac{1}{\mathcal{R}}R^*((2\cot\alpha - 2\sigma\csc\alpha\partial_1^2)\eta) = \mathbb{P}^0\boldsymbol{f}_0.$$

We now introduce the 2×2 matrix of operators

$$G\begin{pmatrix} \eta \\ \boldsymbol{u} \end{pmatrix} = \begin{pmatrix} -\partial_1 & R \\ \frac{1}{\mathcal{R}}R^*(-2\cot\alpha + 2\sigma\csc\alpha\partial_1^2) & -A \end{pmatrix}\begin{pmatrix} \eta \\ \boldsymbol{u} \end{pmatrix}.$$

Then the above system is rewritten as

$$\frac{d}{dt}\begin{pmatrix} \eta \\ \boldsymbol{u} \end{pmatrix} - G\begin{pmatrix} \eta \\ \boldsymbol{u} \end{pmatrix} = \begin{pmatrix} g_0 \\ \mathbb{P}^0\boldsymbol{f}_0 \end{pmatrix}.$$

For $r \geq 0$ we set

$$H_0^r(\mathbb{T}) = \left\{\phi \in H^r(\mathbb{T}) \,;\, \int_{\mathbb{T}}\phi(x)dx = 0\right\}.$$

For the domain of the operator G, we set

$$\mathcal{D}(G) = \Big\{(\eta,\boldsymbol{u}) \in H_0^{\frac{3}{2}}(\mathbb{T}) \times \mathbb{P}^0 H^0(\Omega)\,;\, \eta \in H_0^{\frac{5}{2}}(\mathbb{T}), \quad \boldsymbol{u} \in H^2(\Omega),$$
$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on } S_B, \quad \partial_1 u_2 + \partial_2 u_1 - 2\eta = 0 \quad \text{on } S_F\Big\}.$$

The operator G has the sectorial property stated as follows.

Theorem 2.2 ([5]). *There is a $\gamma > 0$ such that, if $\lambda \in \mathbb{C}$ satisfies $\mathrm{Re}\,\lambda \geq \gamma$ there exists the inverse $(\lambda - G)^{-1}$ in X with $(\lambda - G)^{-1}X = \mathcal{D}(G)$ and its operator norm satisfying*

$$\big|(\lambda - G)^{-1}\big|_X \leq \frac{C_1}{|\lambda|},$$

where $X = H_0^{\frac{3}{2}}(\mathbb{T}) \times \mathbb{P}^0 H^0(\Omega)$.

Remark 2.3. *We can give the same result by considering the linear problem in the slab $\mathbb{T} \times (0,1)$, in order to get accurate estimates.*

From this we see that the resolvent operator G is compact in X. As a consequence we can show

Corollary 2.4 ([5])**.** *We can take $\theta \in (\frac{\pi}{2}, \pi)$ so that, for $\lambda \in \mathbb{C}$ with $|arg(\lambda - \gamma)| \leq \theta$, $(\lambda - G)^{-1}$ exists and satisfies*

$$\left| (\lambda - G)^{-1} \right|_X \leq \frac{C_2}{|\lambda|}.$$

We will give the sketch of the proof of Theorem 2.2 in this talk. For the detailed proof of Theorem 2.2 and Corollary 2.4, see [5]. If the data from X is more regular, the solution gets higher regularity.

Proposition 2.5 ([5])**.** *Let $s \geq 2$. Assume that λ satisfies the same condition as in Corollary 2.4. Suppose that $\boldsymbol{f} \in \mathbb{P}^0 H^{s-2}(\Omega)$, $g_0 \in H_0^{s-\frac{1}{2}}(\mathbb{T})$. Then the solution*

$$\begin{pmatrix} \eta \\ \boldsymbol{u} \end{pmatrix} = (\lambda - G)^{-1} \begin{pmatrix} g_0 \\ \boldsymbol{f} \end{pmatrix}$$

satisfies

$$\|\boldsymbol{u}\|_{H^s(\Omega)} + |\lambda|^{\frac{s}{2}} \|\boldsymbol{u}\|_{H^0(\Omega)} + \|\eta\|_{H^{s+\frac{1}{2}}(\mathbb{T})} + |\lambda|^{\frac{s}{2}+\frac{1}{4}} \|\eta\|_{H^0(\mathbb{T})}$$

$$\leq C_3 \left(\|\boldsymbol{f}\|_{H^{s-2}(\Omega)} + |\lambda|^{\frac{s}{2}-1} \|\boldsymbol{f}\|_{H^0(\Omega)} + \|g_0\|_{H^{s-\frac{1}{2}}(\mathbb{T})} + |\lambda|^{\frac{s}{2}-\frac{1}{4}} \|g_0\|_{H^0(\mathbb{T})} \right).$$

This is proved in the same way as in section 4 of [2].

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On an approximate scheme for a distance function of evolving interfaces

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1 Introduction

Distance function and eikonal equation. A distance function to a set is a well-known and fundamental tool to measure how close a point is to the given set. When $\Omega \subset \mathbf{R}^n$ is open, it is known that the distance function to its boundary, i.e., $\mathrm{dist}(x, \partial\Omega) = \inf_{y \in \partial\Omega} |x - y|$ is a unique non-negative viscosity solution of the eikonal equation

$$|\nabla u(x)| = 1 \quad \text{in } \Omega \tag{1.1}$$

under the homogeneous Dirichlet boundary condition $u = 0$ on $\partial\Omega$. See, e.g., [1, Corollary 3.4.5 (i)(ii), Remark 5.6.1] for properties of a sub- and superdifferential of the distance function and [4] for the comparison principle.

We next consider evolving interfaces $\Gamma_t \subset \mathbf{R}^n$ and the (signed) distance function $d = d(x, t)$ to the interfaces. In this case, how can we characterize d as a solution of a time-dependent partial differential equation? A big difference from the stationary case is that d can be discontinuous with respect to the time variable. As is known, if we fix a time, $d(\cdot, t)$ is a Lipschitz continuous function, whereas d is not continuous in general as a function of (x, t). Indeed, when the interface has an extinction point, the distance to the nearest interface becomes instantaneously large near the extinction point (Example 2.3). For this reason, it is impossible to get d via any problem whose unique solution is continuous.

The goal of this work is to present an approximate scheme for a possibly discontinuous distance function by continuous viscosity solutions of a certain Hamilton-Jacobi equation with a parameter. An evolving interface Γ_t we study is given as the zero level set of the solution w of the initial value problem of

$$w_t(x, t) = H(x, \nabla w(x, t)) \quad \text{in } \mathbf{R}^n \times (0, T). \tag{1.2}$$

To approximate the distance function d to the interface, we introduce a new equation of the form

$$u_t^\theta(x, t) = H(x, \nabla u^\theta(x, t)) + \theta\beta(u^\theta(x, t))(1 - |\nabla u^\theta(x, t)|) \quad \text{in } \mathbf{R}^n \times (0, T). \tag{1.3}$$

Here $\theta > 0$ is a parameter and $\beta(r)$ is a smooth approximation of the sign function such as $\beta(r) = r/\sqrt{r^2 + \delta^2}$ with $\delta > 0$. Roughly speaking, the limit $\theta \to \infty$ forces $(1 - |\nabla u^\theta(x, t)|)$ to be close to 0 except on the zero level set of u^θ, i.e., $|\nabla u^\theta(x, t)| \approx 1$ for $\theta > 0$ large. If we further know that the zero level set of u^θ is the same as that of the solution w of (1.2) and hence is equal to Γ_t, then we would get a convergence of u^θ to the distance function d, which is known to be a solution of (1.1) with the homogeneous Dirichlet boundary condition on the interface. We aim to justify this formal observation by using the theory of viscosity solutions especially when the distance function has a discontinuity. This talk is based on the paper [3].

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Reinitialization. This work was originally motivated by the reinitialization algorithm for level set equations ([6, 5]). In the literature "reinitialization" usually refers to the idea of stopping the process of solving a level set equation such as (1.2) regularly in time and changing its solution at the stopping time so that we obtain a function which approximates the signed distance function to the zero level set of the solution. In the numerical study of (1.2), when the gradient of the solution is close to zero, it becomes difficult to compute precisely the zero level set of it. The reinitialization is used to overcome such an issue since the gradient of the distance function is 1 and thus away from 0.

We briefly explain the main idea of the reinitialization algorithm. Consider the corrector equation

$$u_t(x,t) = \text{sign}(u(x,t))(1 - |\nabla u(x,t)|), \tag{1.4}$$

where $\text{sign}(\cdot)$ is the sign function defined as $\text{sign}(r) = 1$ if $r > 0$, $\text{sign}(r) = -1$ if $r < 0$ and $\text{sign}(r) = 0$ if $r = 0$. The solution of this equation asymptotically converges to its steady state (1.1), a characteristic property of the distance function. Also, the relation $\text{sign}(0) = 0$ guarantees that the initial zero level set is not distorted by solving (1.4) since $u_t(x,t) = 0$ on the zero level.

A corrector equation we use is a slight modification (continuous version) of (1.4), that is,

$$u_t(x,t) = \beta(u(x,t))(1 - |\nabla u(x,t)|). \tag{1.5}$$

As in [6], the idea is to solve (1.2) and (1.5) periodically in time. We first solve (1.2) for a period of $k_1 \Delta t$ and then (1.5) for $k_2 \Delta t$, where k_1, k_2 and Δt are positive constants. One period will be completed at a time step of length $\varepsilon = (k_1 + k_2)\Delta t$. We are thus led to the following combined Hamiltonian

$$H_{12}(x,\tau,r,p) := \begin{cases} H(x,p) & \text{if } i < \tau \leq i + \dfrac{k_1}{k_1 + k_2}, \\ \beta(r)(1 - |p|) & \text{if } i + \dfrac{k_1}{k_1 + k_2} < \tau \leq i + 1 \end{cases} \quad (i = 0, 1, \dots),$$

and the associated equation

$$u_t^\varepsilon(x,t) = H_{12}\left(x, \frac{t}{\varepsilon}, u^\varepsilon(x,t), \nabla u^\varepsilon(x,t)\right) \quad \text{in } \mathbf{R}^n \times (0,T). \tag{1.6}$$

One would expect that solving the two equations infinitely often would force the solution of the reinitialization algorithm to converge to the signed distance function. Therefore we study the limit of the solutions u^ε as $\varepsilon \to 0$. This is a homogenization problem with the Hamiltonian H_{12} being 1-periodic and discontinuous in the fast variable $\tau = t/\varepsilon$. Since the limit above is taken for $\Delta t \to 0$ (and consequently $\varepsilon \to 0$), two free parameters still remain; namely k_1 and k_2. In fact, it turns out that ([3, Theorem 2.3]) the solutions u^ε of (1.6) converge, as $\varepsilon \to 0$, to the solution $\bar{u}^\theta$ of

$$\bar{u}_t^\theta(x,t) = \bar{H}(x, \bar{u}^\theta(x,t), \nabla \bar{u}^\theta(x,t)) \quad \text{in } \mathbf{R}^n \times (0,T)$$

with

$$\bar{H}(x,r,p) := \frac{1}{1+\theta}\{H(x,p) + \theta\beta(r)(1 - |p|)\}.$$

Here $\theta = k_2/k_1$ is the ratio of length of the time intervals in which the equations (1.2) and (1.5) are solved. After rescaling $u^\theta(x,t) = \bar{u}^\theta(x, (1+\theta)t)$, we obtain (1.3). If we solve the corrector equation (1.5) in a larger interval than the one we solve the original (1.2), we can expect a convergence to the steady state. For this reason we study the limit as $\theta \to \infty$ of the solutions of (1.3).

2 Hamilton-Jacobi equations

Let $u_0 : \mathbf{R}^n \to \mathbf{R}$ be a Lipschitz continuous initial datum, and we study (1.2) and (1.3) under this fixed initial condition. The function $H : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$ in (1.2) called a Hamiltonian is assumed to satisfy

- **Continuity:**
$$|H(x,p) - H(y,p)| \leqq L_1(1+|p|)|x-y|, \quad |H(x,p) - H(x,q)| \leqq L_2|p-q|$$
  for some L_1, $L_2 > 0$;

- **Geometricity:** $H(x, \lambda p) = \lambda H(x,p)$ for all $\lambda > 0$.

The continuity assumption guarantees well-posedness of the initial value problems of both (1.2) and (1.3) in the usual viscosity sense, and the unique viscosity solution is continuous. For the viscosity solution w of (1.2) we define

$$\Gamma_t := \{x \in \mathbf{R}^n \mid w(x,t) = 0\}, \quad D_t^{\pm} := \{x \in \mathbf{R}^n \mid \pm w(x,t) > 0\} \quad \text{for } t \in [0,T),$$

which represent an interface and the inside/outside of the interface respectively. The signed distance function associated with this interface is

$$d(x,t) := \begin{cases} \mathrm{dist}(x, \Gamma_t) & \text{if } x \in D_t^+, \\ -\mathrm{dist}(x, \Gamma_t) & \text{if } x \notin D_t^+. \end{cases}$$

Remark 2.1. The Lipschitz continuity of H with respect to p guarantees the property of finite speed of propagation of the interface, and then the constant L_2 gives the maximal speed. This property implies that there is no emerging point of the interface, whereas an extinction point can appear and lead to discontinuity of d; see Example 2.3.

For numerical purposes, it will be useful that the distance function possesses different signs across the interface. However, since d is decomposed as $d = d_+ - d_-$, where d_+ and d_- stand for the plus and the minus part of d respectively, we may assume that d is non-negative. For this reason, in what follows we assume that the initial datum u_0 is non-negative, so that w is also non-negative by the comparison principle. This implies $D_t^- = \emptyset$ and $d(x,t) = \mathrm{dist}(x, \Gamma_t) \geqq 0$. In the general case where d can take negative values, we apply the theorems below to d_+ and d_-.

We now prepare barrier functions, which control the behavior of the solutions u^{θ}.

Proposition 2.2 (Barriers). *There exist some $\varepsilon, L > 0$, independent of $\theta > 0$, such that*

$$\varepsilon w \leqq u^{\theta} \leqq Ld \quad \text{in } \mathbf{R}^n \times (0,T). \tag{2.1}$$

In the proof we show that εw and Ld are, respectively, a viscosity subsolution and a viscosity supersolution of (1.3) to get the above inequalities by the comparison principle. Since $\Gamma_t = \{w(\cdot, t) = 0\} = \{d(\cdot, t) = 0\}$, the inequalities (2.1) imply that $\Gamma_t = \{u^{\theta}(\cdot, t) = 0\}$ for every $\theta > 0$. In other words, the solution u^{θ} of (1.3) preserves the zero level set of the solution w of (1.2).

The following example shows that the distance function can be discontinuous when the interface has an extinction point.

Example 2.3 (Discontinuity of the distance function). We study (1.2) with $H(x, p) = |p|$. This describes a phenomenon where the interface moves at a uniform speed 1. In this case the viscosity solution of (1.2) is represented as

$$w(x, t) = \max_{|x-y| \leqq t} u_0(y). \tag{2.2}$$

This formula is derived from the corresponding optimal control problem. We now let the dimension n be 1 and take the initial datum as $u_0(x) = \max\{(1-|x-2|)_+, (1-|x+2|)_+\}$. Then (2.2) implies

$$w(x, t) = \min\{\max\{(t+1-|x-2|)_+, (t+1-|x+2|)_+\}, 1\}.$$

We therefore have

$$\Gamma_t = \{w(\cdot, t) = 0\} = \begin{cases} \{|x| \geqq t+3\} \cup \{|x| \leqq 1-t\} & \text{if } t \leqq 1, \\ \{|x| \geqq t+3\} & \text{if } t > 1 \end{cases}$$

and

$$d(x, t) = \begin{cases} \max\{(t+1-|x-2|)_+, (t+1-|x+2|)_+\} & \text{if } t \leqq 1, \\ (t+3-|x|)_+ & \text{if } t > 1. \end{cases}$$

See Figure 1 for the graph of d. Thus d is discontinuous on $\ell := \{(x, 1) \mid -2 < x < 2\}$; more precisely, d is not upper semicontinuous but lower semicontinuous on ℓ. In this example, $0 \in \Gamma_1$ is an extinction point of the interface.
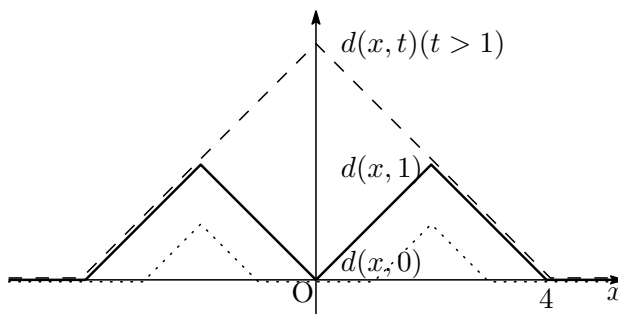

Figure 1: The graph of d.

3   Convergence results

Continuous distance function.   To illustrate the idea of the proof, we first present a convergence result to a continuous distance function d, which is much easier to handle than a general (possibly discontinuous) distance function. When d is continuous, it is uniformly approximated by the solutions u^θ of (1.3).

Theorem 3.1. *If d is continuous, then u^θ converges to d locally uniformly in $\mathbf{R}^n \times (0, T)$ as $\theta \to \infty$.*

In the proof we compare the distance function d and the half-relaxed limits of u^θ, which are weak notions of the limit for a sequence of functions defined as

$$\overline{u}(x, t) := \limsup_{(y, s, \theta) \to (x, t, \infty)} u^\theta(y, s), \quad \underline{u}(x, t) := \liminf_{(y, s, \theta) \to (x, t, \infty)} u^\theta(y, s).$$

Then $\overline{u}(\cdot, t)$ and $\underline{u}(\cdot, t)$ are, respectively, a viscosity subsolution and a viscosity supersolution of the eikonal equation (1.1) in D_t^+. To apply the comparison principle to these limits and d, we need to further know the boundary data of the limits. Despite the fact that the zero level set of u^θ is Γ_t for every $\theta > 0$, it is not clear if the same relation holds for the limits of u^θ. Now, thanks to the existence of barrier functions (2.1) together with the continuity of d, it follows that $\Gamma_t = \{\overline{u}(\cdot, t) = 0\} = \{\underline{u}(\cdot, t) = 0\}$.

The comparison principle is thus applicable, and we obtain $\overline{u} \leq d$ and $d \leq \underline{u}$. Since we always have $\underline{u} \leq \overline{u}$ by the definitions of them, combining these three inequalities, we conclude the locally uniform convergence of u^θ to d.

General distance function. If the distance function d is discontinuous, we cannot expect that the continuous solutions u^θ of (1.3) will converge locally uniformly to d. Among three inequalities in the previous paragraph, $d \leq \underline{u}$ and $\underline{u} \leq \overline{u}$ still hold even if d is discontinuous, but $\overline{u} \leq d$ is not true. This is because the zero level set of $\overline{u}$ is not necessarily Γ_t, which prevents us to apply the comparison principle for (1.1). We can however show a weaker notion of convergence to d; namely a convergence to d from below in time as follows:

Theorem 3.2. $\displaystyle \lim_{\substack{(y,s,\theta) \to (x,t,\infty) \\ s \leq t}} u^\theta(y, s) = d(x, t)$ *for all* $(x, t) \in \mathbf{R}^n \times (0, T)$.

This result is shown by introducing a notion of a half-relaxed limit from below in time:

$$\overline{u}'(x, t) := \limsup_{\substack{(y,s,\theta) \to (x,t,\infty) \\ s \leq t}} u^\theta(y, s).$$

The reason why this notion of the limit is successful is that d is always continuous from below in time; see, e.g., [2, Proposition 2.1 (ii)] for the case of a motion by mean curvature. Due to this we have $\Gamma_t = \{\overline{u}'(\cdot, t) = 0\}$.

The following lemma guarantees that the new limit $\overline{u}'$ still satisfies the viscosity inequality, which enables us to apply the comparison principle between $\overline{u}'$ and d.

Lemma 3.3. $\overline{u}'(\cdot, t)$ *is a viscosity subsolution of* (1.1) *in* D_t^+.

The main tool of the proof is the fact that the viscosity inequality can be extended up to the terminal time. This fact is used for the upper half-relaxed limit of u^θ on $(0, \hat{t}]$, which is equal to $\overline{u}(x, t)$ if $t < \hat{t}$ and $\overline{u}'(x, t)$ if $t = \hat{t}$. Since Lemma 3.3 yields $\overline{u}' \leq d$, we conclude Theorem 3.2. As byproducts of this proof, we also get

- for every $t \in (0, T)$, $u^\theta(\cdot, t)$ converges to $d(\cdot, t)$ locally uniformly in $\mathbf{R}^n$ as $\theta \to \infty$;

- $\underline{u} = d$ in $\mathbf{R}^n \times (0, T)$.

The former means a locally uniform convergence at a fixed time while the latter is a convergence in the sense of the lower half-relaxed limit.

Another equation. Finally, we propose another equation whose solution has a gradient bound away from 0 on the zero level set. Consider

$$u_t^\theta(x, t) = H(x, \nabla u^\theta(x, t)) + \theta \beta(u^\theta(x, t))(1 - |\nabla u^\theta(x, t)|)_+ \quad \text{in } \mathbf{R}^n \times (0, T). \tag{3.1}$$

Here we have replaced $(1 - |\nabla u^\theta(x, t)|)$ in (1.3) by its plus part $(1 - |\nabla u^\theta(x, t)|)_+$. Although the solutions u^θ of (3.1) do not converge to d, we have the following estimate:

Theorem 3.4. $d \leqq \sup_{\theta > 0} u^\theta \leqq Ld$ *in* $\mathbf{R}^n \times (0, T)$ *for some* $L > 0$.

This is derived from the inequality $d \leqq \underline{u}$ and the monotonicity of the solutions with respect to θ, i.e., $u^{\theta_1} \leqq u^{\theta_2}$ if $\theta_1 \leqq \theta_2$.

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A Degenerate Isoperimetric Problem and Traveling Waves to a Bi-stable Hamiltonian System

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Abstract:
We examine here a non-standard type of isoperimetric problem. Given a continuous function $F : \mathbb{R}^2 \to [0, \infty)$ vanishing at two points, $\mathbf{p}_+$ and $\mathbf{p}_-$ in the plane, we seek a curve $\gamma : [0, 1] \to \mathbb{R}^2$ that minimizes the distance between these two points in the metric having F as its conformal factor, subject to a constraint associated with Euclidean area. That is, we seek a solution to the variational problem

$$\inf_{\gamma} E(\gamma) \quad \text{with} \quad E(\gamma) := \int_0^1 F(\gamma) \left| \gamma' \right| dt, \tag{0.1}$$

where competitors $\gamma : [0, 1] \to \mathbb{R}^2$ must satisfy $\gamma(0) = \mathbf{p}_-$, $\gamma(1) = \mathbf{p}_+$ as well as the constraint

$$\int_{\gamma} \omega_0 = const. \quad \text{with } \omega_0 \text{ given by the 1-form } \omega_0 = -p_2 dp_1. \tag{0.2}$$

Since $d\omega_0$ is just the standard Euclidean area form $dp_1\, dp_2$, the isoperimetric nature of the minimizing curve becomes evident. What makes this particular isoperimetric problem non-standard is both the degeneracy of the conformal factor at the two "wells" $\mathbf{p}_-$ and $\mathbf{p}_+$ and the fact that length is measured with respect to a metric given by F while area is measured with respect to the Euclidean metric. There is a vast literature on isoperimetric problems with assorted assumptions on the conformal factor or "density," though to our knowledge none address this combination of degeneracy and mixture of metrics. We mention, for example, [5, 6, 7, 9, 12] but of course there are many, many others.

One motivation for our investigation is the connection between such isoperimetric curves–should they exist–and the existence of traveling wave solutions to a Hamiltonian system associated with the energy functional

$$H(u) := \int \frac{1}{2} |\nabla u|^2 + W(u) \quad \text{where } W(u) = F^2(u).$$

The theory of heteroclinic connections to bi-stable gradient-type reaction-diffusion systems in the form of standing or traveling waves is by now very well-developed in both the scalar and vector-valued settings, including for example, [1, 2, 3, 4, 8, 10, 11, 14] to name but a few studies. Here, however, rather than seeking traveling wave solutions to a gradient flow $u_t = -\delta H(u)$, we pursue traveling wave solutions to a Hamiltonian flow associated with H, namely,

$$\mathbb{J}u_t = \Delta u - \nabla_u W(u) \quad \text{where} \quad \mathbb{J} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and } u = \left(u^{(1)}(x, t), u^{(2)}(x, t) \right), \tag{0.3}$$

which conserves the value of H over time. Such a solution would take the form $u = u(x, t) = U(x_1 - \nu t)$ for some wave speed ν with $U : \mathbb{R} \to \mathbb{R}^2$ then required to solve

$$-\nu \mathbb{J}U' = U'' - \nabla_u W(U) \quad \text{on } (-\infty, \infty) \quad \text{with } U(\pm\infty) = \mathbf{p}_\pm. \tag{0.4}$$

This problem is itself variational in nature. At least formally, solutions are critical points of the constrained minimization problem

$$\inf_u H(u) \quad \text{among competitors } u : \mathbb{R} \to \mathbb{R}^2 \text{ satisfying:}$$

$$U(\pm\infty) = \mathbf{p}_\pm \text{ and the constraint } - \int_{\mathbb{R}} u^{(2)}(u^{(1)})' = A \qquad (0.5)$$

for some $A \in \mathbb{R}$, with the wave speed arising as a Lagrange multiplier associated with the constraint. It turns out that minimizers of this problem can be found if one identifies minimizers of the isoperimetric problem (0.1), very much in the spirit of [13, 14].

This talk then is primarily devoted to the study of (0.1). Our main result here is the existence of a minimizing curve under certain assumptions on the behavior of W near its minima $\mathbf{p}_+$ and $\mathbf{p}_-$. We deal with the degeneracy of the conformal factor by first focusing on the problem of finding a constrained minimizer of E joining a non-degenerate point in the plane to either of the wells of W. This one-well isoperimetric problem is solved uniquely for W taking the form of a non-negative quadratic vanishing at the well. For convenience, in this one-well setting we take the zero of W to be at the origin. Somewhat surprisingly, the optimal curves are spirals in some cases.

To illustrate how the spiral shape arises, we give a simple derivation of their form for the one-well problem with a radial quadratic potential, $W(p) = |p|^2$, with $p = (p_1, p_2)$. Suppose we wish to join $(1, 0)$ to the origin and we make the assumption that the optimal curve can be parametrized by polar radius $r \in (0, 1)$ in the form $z(r) = re^{i\theta(r)}$, and we normalize $\theta(1) = 0$. Then the problem reduces to minimizing the length functional

$$E(z) = \int_0^1 r \sqrt{1 + r^2[\theta'(r)]^2} \, dr \qquad \text{subject to the constraint} \qquad \int_0^1 \frac{r^2}{2} \theta'(r) \, dr = A,$$

for given A. By defining $w = z^2(r) = r^2 e^{i2\theta(r)} = \rho e^{i\phi(\rho)}$, $\rho \in (0, 1)$, the problem is further reduced to minimizing

$$L(w) := \int_0^1 |w'(\rho)| \, d\rho = \int_0^1 \sqrt{1 + \rho^2[\phi'(\rho)]^2} \, d\rho, \quad \text{with} \quad \frac{1}{4} \int_0^1 \rho\phi'(\rho) \, d\rho = A.$$

By Jensen's inequality (applied with the convex function $h(x) = \sqrt{1 + x^2}$), we obtain an isoperimetric inequality of the form,

$$L(w) \geq \left(1 + \left[\int_0^1 \rho \, \phi'(\rho) \, d\rho \right]^2 \right)^{\frac{1}{2}} = (1 + 16A^2)^{\frac{1}{2}},$$

for any such $w(\rho) = z^2(r)$, with equality holding exactly when $\rho\phi'(\rho) = 4A$. Integrating and returning to the original parametrization, the optimal curve is the spiral

$$z(r) = re^{i4A \ln r}, \quad r \in (0, 1).$$

The rigorous derivation of the minimizing curve without the assumption that $\theta = \theta(r)$ and for more general quadratics, follows substantially different methods, using calibrations.

We now describe the result for general quadratic W a bit more precisely. We suppose that $F(p) = \sqrt{W(p)}$ where $W : \mathbb{R}^2 \to \mathbb{R}$ is given by the quadratic of the form

$$W(p) = p^T H_W p. \tag{0.6}$$

Here H_W is a constant, real, symmetric, positive definite 2×2 matrix. We denote by λ_1^2 and λ_2^2 the two positive eigenvalues of the matrix H_W and express all points in $\mathbb{R}^2$ and all curves in the plane with respect to the orthonormal basis $\{v_1, v_2\}$ of eigenvectors of H_W. In particular, then F takes the form

$$F(p) = \sqrt{\lambda_1^2 p_1^2 + \lambda_2^2 p_2^2}$$

and we let

$$E(\gamma) := \int_0^1 F(\gamma) \left| \gamma' \right| dt \tag{0.7}$$

For any $A \in \mathbb{R}$ and any $\mathbf{p}_0 \in \mathbb{R}^2$, we introduce the admissible class

$$\mathcal{S}_{A,\mathbf{p}_0} := \left\{ \gamma : [0,1] \to \mathbb{R}^2 : \gamma \text{ locally Lipschitz}, \; \gamma(0) = \mathbf{p}_0, \; \gamma(1) = (0,0), \; \mathcal{P}(\gamma) = A \right\}, \tag{0.8}$$

where

$$\mathcal{P}(\gamma) := - \int_0^1 \gamma^{(2)}(\gamma^{(1)})' \, dt, \quad \text{and where we write} \quad \gamma = (\gamma^{(1)}, \gamma^{(2)}).$$

We then cast the one-well isoperimetric problem as

$$m_0 := \inf_{\mathcal{S}_{A,\mathbf{p}_0}} E(\gamma). \tag{0.9}$$

To state the result we recall the definition of the 1-form $\omega_0 := -p_2 dp_1$ and introduce another 1-form

$$\omega_1 := \frac{1}{\lambda_1 + \lambda_2} \left(-\lambda_2 p_2 dp_1 + \lambda_1 p_1 dp_2 \right). \tag{0.10}$$

We observe that

$$\mathcal{P}(\gamma) = \int_\gamma \omega_0 \quad \text{and that} \quad d\omega_0 = d\omega_1 = dp_1 dp_2. \tag{0.11}$$

We fix any reference curve γ_0 starting at $(0,0)$ and ending at $\mathbf{p}_0$ (for instance, a line segment or the geodesic joining these points in the degenerate metric). Then for any curve $\gamma \in \mathcal{S}_{A,\mathbf{p}_0}$ we conclude from (0.11) that

$$\int_{\gamma \cup \gamma_0} (\omega_0 - \omega_1) = 0$$

and consequently,

$$\mathcal{P}(\gamma) = A \quad \text{if and only if} \quad \int_\gamma \omega_1 = A - C(\gamma_0, \mathbf{p}_0) =: \tilde{A} \tag{0.12}$$

where $C(\gamma_0, \mathbf{p}_0) := \int_{\gamma_0} (\omega_0 - \omega_1)$.

We can then establish:

Theorem 1. *The unique solution to the minimization* (0.9) *within the class* $\mathcal{S}_{A,\mathbf{p}_0}$ *is the curve* γ_β *defined as the integral curve of the vector field*

$$V_\beta(p) := (\cos\beta)\,\Theta(p) - (\sin\beta)\,R(p) \tag{0.13}$$

that joins $\mathbf{p}_0$ *to the origin. Here*

$$R(p) := \frac{(\lambda_1 p_1, \lambda_2 p_2)}{F(p)^2}, \qquad \Theta(p) := \frac{(-\lambda_2 p_2, \lambda_1 p_1)}{F(p)^2},$$

and β *is selected so that*

$$\frac{\tilde{r}(\mathbf{p}_0)}{\lambda_1 + \lambda_2}\,\cot\beta = \tilde{A}. \tag{0.14}$$

We remark that while in general, a (constrained) geodesic associated with a critical point of (0.9) should satisfy a second order system of ODE's, it is noteworthy that here the optimal curve satisfies a first order system.

We can broaden the one-well existence result to cover certain W that are analytic near the well and whose Taylor development begins with the kind of quadratic considered above. We won't state this generalization precisely here. The existence proof for an isoperimetric curve in this more general setting also comes through a type of calibration argument.

We next comment on the need for an assumption of non-degeneracy of the potential W at the wells. It turns out that if the Taylor development of W at a well (i.e. at the origin in the one-well setting) vanishes to quadratic order (i.e. if $\lambda_1 = \lambda_2 = 0$), then there will not exist a solution to (0.9). In short, the cost of length in the metric having density $F = \sqrt{W}$ is so cheap near the origin that minimizing sequences tend to simply wrap more and more tightly around the origin in accumulating the requisite area. In light of this phenomenon, some kind of non-degeneracy assumption for $D^2 W$ is seen as a necessary ingredient for existence.

With existence of a solution to the one-well isoperimetric problem in hand for appropriate W, we can then establish the existence of curves solving the two-well problem (0.1)-(0.2). Essentially, the approach for existence of such a minimizer comes via the Direct Method. When an element of a minimizing sequence approaches a well, it can be replaced by the solution to the one-well problem described above, satisfying an appropriate area constraint. This one-well solution satisfies a bound on its Euclidean arclength and since a Euclidean arclength bound on a minimizing sequence is easy away from the wells, using that F is strictly bounded away from zero, one obtains the needed compactness to pass to a subsequence.

We now return to one of the motivation for our study of the two-well isoperimetric problem, namely the pursuit of traveling wave solutions to (0.3) solving (0.4). For this purpose, it turns out we need to know our isoperimetric curve does not "visit" either of the wells too soon. That is, we must be sure there is no bubbling event, whereby a minimal curve passes through $\mathbf{p}_+$ or $\mathbf{p}_-$ more than once. One way to guarantee this is to pick an area constraint sufficiently close to that which corresponds to the geodesic γ_0 joining the two wells. In other words, one proves:

Theorem 2. *Assume W is given by a non-degenerate quadratic near the two wells $\mathbf{p}_+$ and $\mathbf{p}_-$. Let γ_0 be an unconstrained minimizer of $E(\gamma)$ among locally Lipschitz curves $\gamma : [0,1] \to \mathbb{R}^2$ satisfying $\gamma(0) = \mathbf{p}_-$ and $\gamma(1) = \mathbf{p}_+$. Then there exists a number $\varepsilon_0 > 0$ such that for any ε with $0 < |\varepsilon| < \varepsilon_0$, any solution γ_* to*

$$\inf\left\{E(\gamma):\ \gamma:[0,1]\to\mathbb{R}^2\ \text{locally Lipschitz},\ \gamma(0)=\mathbf{p}_-,\ \gamma(1)=\mathbf{p}_+,\ \mathcal{P}(\gamma)=\mathcal{P}(\gamma_0)+\varepsilon\right\} \tag{0.15}$$

satisfies $\gamma_*(t) \notin \{\mathbf{p}_-, \mathbf{p}_+\}$ *for all* $t \in (0, 1)$. *That is,* γ_* *has no bubbles.*

With this "well-behaved" isoperimetric curve in hand, we let $y : (0, 1) \to \mathbb{R}$ be defined through

$$y(t) := \frac{1}{\sqrt{2}} \int_{\frac{1}{2}}^{t} \frac{|\gamma'(\tau)|}{\sqrt{W(\gamma(\tau))}} \, d\tau = \frac{L}{\sqrt{2}} \int_{\frac{1}{2}}^{t} \frac{1}{\sqrt{W(\gamma(\tau))}} \, d\tau. \tag{0.16}$$

Then we define say $U = U(y)$ mapping $\mathbb{R}$ to $\mathbb{R}^2$ via $U(y) = \gamma(t(y))$ so that U will satisfy the condition

$$\left|U'(y)\right| = \sqrt{2} \sqrt{W(U(y))}. \tag{0.17}$$

Now we can make a connection between isoperimetric curves and traveling waves as stated (loosely) below:

Theorem 3. *For* ε_0 *given by Theorem 2 and for any non-zero* $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, *let* $\gamma_* = \gamma_*(t)$ *be any minimizer of* (0.15). *We take this curve to be parametrized by constant velocity* $\left|\gamma'_*(t)\right| = L$, *where* L *is its Euclidean arc length so that* $\gamma_* : [0, 1] \to \mathbb{R}^2$. *Let* $U_* : \mathbb{R} \to \mathbb{R}^2$ *given by* $U_*(y) := \gamma_*(t(y))$ *be the reparametrization of* γ_* *given by* (0.16). *Then we have:*

(i) The function U_* *solves* (0.5) *with* $A = \mathcal{P}(\gamma_0) + \varepsilon$.
(ii) Let C *denote the set of all heteroclinic connections between* $\mathbf{p}_-$ *and* $\mathbf{p}_+$. *Then for all non-zero* $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ *such that*

$$\mathcal{P}(\gamma) \neq \mathcal{P}(\gamma_0) + \varepsilon \quad \text{for all } \gamma \in C, \tag{0.18}$$

the function U_* *solves* (0.4) *with non-zero speed* v.

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Mathematical entropy for hyperbolic balance laws and applications

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Abstract

We introduce a notion of the mathematical entropy for hyperbolic systems of balance laws with (not necessarily symmetric) relaxation. As applications, we deal with the dissipative Timoshenko system, the Euler-Maxwell system and the Euler-Cattaneo-Maxwell system. Also we observe that the dissipative structure of each system is of the regularity-loss type and investigate the corresponding decay property. Furthermore, we prove the global existence and asymptotic stability of solutions for small initial data.

1 Introduction

We consider hyperbolic systems of balance laws

$$w_t + \sum_{j=1}^{n} f^j(w)_{x_j} = g(w). \tag{1.1}$$

Here w is the unknown m-vector valued function of time t and space variable $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$, f^j and g are given m-vector valued smooth functions of $w \in \mathcal{O}_w$, where $\mathcal{O}_w$ is a convex open set in $\mathbb{R}^m$.

The notion of the mathematical entropy was first introduced by Godunov [6] (cf. [5]) for hyperbolic systems of conservation laws. This notion was then extended in [11] (cf. [9, 10]) to hyperbolic-parabolic systems of conservation laws. Also, a similar extension was done in [13] (cf. [2, 26]) for hyperbolic systems of balance laws (1.1) with symmetric relaxation.

On the other hand, recently, several interesting model systems which are written in the form of (1.1) but with non-symmetric relaxation attract our attention. Among them, we mention the dissipative Timoshenko system, the isentropic Euler-Maxwell system, and the Euler-Cattaneo-Maxwell system. Our first aim of this note is to refine the definition of the mathematical entropy in [13] such that it covers those systems with non-symmetric relaxation.

Our second aim is to review the recent mathematical theory developed in the above interesting model systems with non-symmetric relaxation. Especially, we observe that the dissipative structure of these systems is of the regularity-loss type which is characterized by

$$\operatorname{Re} \lambda(i\xi) \leq -c|\xi|^2/(1+|\xi|^2)^2 \tag{1.2}$$

and then investigate the corresponding decay property; here $\lambda(i\xi)$ denotes the eigenvalues of the linearized system in the Fourier space. Furthermore, we report the results on the global existence and optimal decay of solutions for small initial data in Sobolev (or Besov) spaces with less regularity.

Finally, we emphasize the importance of our Euler-Cattaneo-Maxwell system. There are several models of the Euler-Maxwell system with thermal effects, which usually include thermal damping (cf. [4]). However, such models may not have the mathematical entropy and hence they may not be consistent with the classical thermodynamics. We believe that the thermal effect described by our Cattaneo law will give a positive answer to such a difficulty.

2   Mathematical entropy and symmetrization

We give a definition of the mathematical entropy for hyperbolic balance laws (1.1) with not necessarily symmetric relaxation. To this end, we set

$$\mathcal{M} = \{\psi \in \mathbb{R}^m; \quad \langle \psi, g(w) \rangle = 0 \ \text{ for any } w \in \mathcal{O}_w\},$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathbb{R}^m$. Then $\mathcal{M}$ is a subspace of $\mathbb{R}^m$ such that $g(w) \in \mathcal{M}^\perp$ for any $w \in \mathcal{O}_w$, where $\mathcal{M}^\perp$ denotes the orthogonal complement of $\mathcal{M}$. Also we introduce the set $\mathcal{E}$ of equilibrium states:

$$\mathcal{E} = \{w \in \mathcal{O}_w; \quad g(w) = 0\}.$$

The following definition of the mathematical entropy was given in [12] as a refinement of the one introduced in [13].

Definition 2.1 (Mathematical entropy [12] (cf. [13]))**.** *Let $\eta = \eta(w)$ be a smooth function defined in a convex open set $\mathcal{O}_w$. Then $\eta(w)$ is called a mathematical entropy for hyperbolic balance laws (1.1) if the following four conditions hold true:*

(a) $\eta(w)$ is strictly convex in $\mathcal{O}_w$ in the sense that the Hessian $D_w^2\eta(w)$ is positive definite for $w \in \mathcal{O}_w$.

(b) $D_w f^j(w)(D_w^2\eta(w))^{-1}$ is symmetric for $w \in \mathcal{O}_w$ and $j = 1, \cdots, n$.

(c) For $w \in \mathcal{O}_w$, $w \in \mathcal{E}$ holds if and only if $u := ((D_w\eta(w))^T \in \mathcal{M}$.

(d) For $w \in \mathcal{E}$, the matrix $-D_w g(w)(D_w^2\eta(w))^{-1}$ is (not necessarily symmetric but) nonnegative definite such that its kernel space coincides with $\mathcal{M}$.

We remark that our Definition 2.1 just removes the "symmetric" property of the matrix $-D_w g(w)(D_w^2\eta(w))^{-1}$ for $w \in \mathcal{E}$, which was required in the previous definition in

[13]. This small modification enables us to use the notion of the mathematical entropy for hyperbolic balance laws (1.1) even with non-symmetric relaxation.

Next we consider a diffeomorphism $w = w(u)$ from an open set $\mathcal{O}_u$ onto $\mathcal{O}_w$. By using $w = w(u)$, we rewrite (1.1) in the form

$$A^0(u)u_t + \sum_{j=1}^{n} A^j(u)u_{x_j} = h(u), \tag{2.1}$$

where $A^0(u) = D_u w(u)$, $A^j(u) = D_u f^j(w(u)) = D_w f^j(w(u))D_u w(u)$, and $h(u) = g(w(u))$. Let us define

$$L(u) := -D_u h(u) = -D_w g(w(u))D_u w(u) \tag{2.2}$$

and call this matrix $L(u)$ the relaxation matrix.

Definition 2.2 ([12] (cf. [13])). *The system* (2.1) *is called symmetric dissipative if the following four conditions hold true:*

(a) $A^0(u)$ *is symmetric and positive definite for* $u \in \mathcal{O}_u$.

(b) $A^j(u)$ *is symmetric for* $u \in \mathcal{O}_u$ *and* $j = 1, \cdots, n$.

(c) *For* $u \in \mathcal{O}_u$, $h(u) = 0$ *holds if and only if* $u \in \mathcal{M}$.

(d) *For* $u \in \mathcal{M}$, *the relaxation matrix* $L(u)$ *is (not necessarily symmetric but) nonnegative definite such that its kernel space coincides with* $\mathcal{M}$.

We know that the symmetrization of hyperbolic balance laws is characterized by the existence of a mathematical entropy.

Theorem 2.3 ([12] (cf. [13])). *The following two statements are equivalent.*

(i) *The system* (1.1) *has a mathematical entropy.*

(ii) *There is a diffeomorphism by which* (1.1) *is transformed to a symmetric dissipative system* (2.1).

Here we briefly show that (i) implies (ii). Let $\eta(w)$ be the mathematical entropy in the sense of Definition 2.1 and set

$$u := (D_w \eta(w))^T. \tag{2.3}$$

As was shown in [11, 13], this gives a diffeomorphism $u = u(w)$ from $\mathcal{O}_w$ onto its range $\mathcal{O}_u$. We denote by $w = w(u)$ the inverse mapping. Then it becomes a diffeomorphism from $\mathcal{O}_u$ onto $\mathcal{O}_w$. Since $D_u w(u) = (D_w^2 \eta(w(u)))^{-1}$ by (2.3), we see that the associated system (2.1) becomes a symmetric dissipative system in the sence of Definition 2.2 such that

$$A^0(u) = (D_w^2 \eta(w))^{-1}, \qquad A^j(u) = D_w f^j(w)(D_w^2 \eta(w))^{-1},$$
$$L(u) = -D_w g(w)(D_w^2 \eta(w))^{-1},$$

where the right hand side is evaluated at $w = w(u)$.

Finally in this section we derive the equation of the mathematical entropy $\eta(w)$:

$$\eta(w)_t + \sum_{j=1}^{n} q^j(w)_{x_j} = \langle u, g(w) \rangle, \tag{2.4}$$

where $q^j(w)$ is the corresponding entropy flux given by $q^j(w) = \langle u, f^j(w) \rangle - \tilde{q}^j(u)$ with u in (2.3); here $\tilde{q}^j(u)$ is the potential function satisfying $(D_u \tilde{q}^j(u))^T = f^j(w(u))$ with $w(u)$ being the inverse of the mapping defined by (2.3).

3 Dissipative Timoshenko system

We consider the dissipative Timoshenko system

$$\begin{cases} \varphi_{tt} - (\varphi_x - \psi)_x = 0, \\ \psi_{tt} - \sigma(\psi_x)_x - (\varphi_x - \psi) + \gamma \psi_t = 0, \end{cases} \tag{3.1}$$

which describes the vibration of a beam (Timoshenko beam). Here $x \in \mathbb{R}$ denotes a point on the center line of the beam, and φ and ψ are the transversal displacement and the rotation angle of the beam, respectively; φ and ψ are unknown functions of time $t > 0$ and space variable $x \in \mathbb{R}$. We assume that γ is a positive constant and $\sigma(z)$ is a given smooth function of z satisfying $\sigma'(z) > 0$ for $z \in \mathbb{R}$.

As in [7, 8], we introduce new unknown functions $v = \varphi_x - \psi$, $u = \varphi_t$, $z = \psi_x$ and $y = \psi_t$, and transform (3.1) into a hyperbolic system of balance laws:

$$\begin{cases} v_t - u_x = -y, \\ y_t - \sigma(z)_x = v - \gamma y, \\ u_t - v_x = 0, \\ z_t - y_x = 0. \end{cases} \tag{3.2}$$

This is exactly in the form of (1.1) with $n = 1$, namely, we have

$$W_t + F(W)_x = G(W),$$

where $W = (v, y, u, z)^T$, $F(W) = -(u, \sigma(z), v, y)^T$ and $G(W) = (-y, v - \gamma y, 0, 0)^T$. The set of basic states is $\mathcal{O}_W = \mathbb{R}^4$. The subspace $\mathcal{M}$ and the set $\mathcal{E}$ of equilibrium states are given respectively by

$$\mathcal{M} = \mathrm{span}\{e_3, e_4\}, \qquad \mathcal{E} = \{W = (0, 0, u, z)^T; \ u, z \in \mathbb{R}\},$$

where $\{e_1, e_2, e_3, e_4\}$ denotes the standard orthonormal basis of $\mathbb{R}^4$.

By a simple computation we see that the total energy of the dissipative Timoshenko system (3.2) satisfies

$$\left\{ \frac{1}{2}(v^2 + y^2 + u^2) + S(z) \right\}_t - \{vu + y\sigma(z)\}_x + \gamma y^2 = 0, \tag{3.3}$$

where $S(z) = \int_0^z \sigma(\zeta)\,d\zeta$. This total energy is regarded as a mathematical entropy of our dissipative Timoshenko system. In fact, putting

$$\eta(W) = \frac{1}{2}(v^2 + y^2 + u^2) + S(z), \tag{3.4}$$

we can show the following result.

Proposition 3.1. *The function $\eta(W)$ in (3.4) is a mathematical entropy of the dissipative Timoshenko system (3.2) in the sense of Definition 2.1. Therefore the system (3.2) is put in a symmetric dissipative system for $U := (D_W\eta(W))^T = (v, y, u, \sigma(z))^T$.*

In the actual computation we use $W = (v, y, u, z)^T$ and rewrite (3.2) in the form

$$\tilde{A}^0(W)W_t + \tilde{A}(W)W_x + LW = 0,$$

where $\tilde{A}^0(W) = \mathrm{diag}\,(1, 1, 1, \sigma'(z))$,

$$\tilde{A}(W) = -\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \sigma'(z) \\ 1 & 0 & 0 & 0 \\ 0 & \sigma'(z) & 0 & 0 \end{pmatrix}, \qquad L = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & \gamma & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The linearized system around $W = 0$ is given by

$$A^0 W_t + A W_x + LW = 0,$$

where we simply write as $A^0 = \tilde{A}^0(0)$ and $A = \tilde{A}(0)$. It was shown in [7] that the solution W to the linearized system satisfies

$$|\hat{W}(\xi, t)| \le Ce^{-c\eta(\xi)t}|\hat{W}_0(\xi)|,$$

where $\eta(\xi) = \xi^2/(1+\xi^2)^2$, and W_0 denotes the corresponding initial data. This implies the regularity-loss property described by (1.2). Consequently, we have the following linear decay estimate:

$$\|\partial_x^k W(t)\|_{L^2} \le C(1+t)^{-1/4-k/2}\|W_0\|_{L^1} + C(1+t)^{-l/2}\|\partial_x^{k+l}W_0\|_{L^2},$$

where $k, l \ge 0$.

Concerning the nonlinear problem we have the following result on the global existence and optimal decay of solutions.

Theorem 3.2 ([15] (cf. [8])). *Suppose that $E_0 = \|W_0\|_{H^2}$ is small. Then the disipative Timoshenko system (3.2) admits a unique global solution W satisfying*

$$\|W(t)\|_{H^2}^2 + \int_0^t \|v(\tau)\|_{H^1}^2 + \|y(\tau)\|_{H^2}^2 + \|\partial_x u(\tau)\|_{L^2}^2 + \|\partial_x z(\tau)\|_{H^1}^2\,d\tau \le CE_0^2. \tag{3.5}$$

Suppose in addition that $E_1 = \|W_0\|_{H^2 \cap L^1}$ is small. Then the global solution W verifies the optimal decay estimate

$$\|W(t)\|_{L^2} \le CE_1(1+t)^{-1/4}. \tag{3.6}$$

The regularity assumption H^2 in the above theorem can be relaxed to $B_{2,1}^{3/2}$. We refer to [16, 25].

4 Isentropic Euler-Maxwell system

We consider the isentropic Euler-Maxwell system in plasma physics:

$$\begin{cases} n_t + \operatorname{div}(nu) = 0, \\ (nu)_t + \operatorname{div}(nu \otimes u) + \nabla p(n) = nf_{EM} - nu, \\ E_t - \operatorname{rot} B = nu, \\ B_t + \operatorname{rot} E = 0, \end{cases} \tag{4.1}$$

$$\operatorname{div} E = n_\infty - n, \qquad \operatorname{div} B = 0. \tag{4.2}$$

Here $n > 0$ and $u \in \mathbb{R}^3$ are the density and the velocity of the electron, respectively, while $E \in \mathbb{R}^3$ is the electric field and $B \in \mathbb{R}^3$ is the magnetic induction; these are unknown functions of time $t > 0$ and space variable $x \in \mathbb{R}^3$. The pressure $p(n)$ is a given smooth function of n satisfying $p'(n) > 0$ for $n > 0$, f_{EM} denotes the Lorentz force given by $f_{EM} = -(E + u \times B)$, and n_∞ is the ion density in the background which is assumed to be a positive constant. The Euler-Maxwell system (4.1), (4.2) describes the dynamics of compressible electrons in plasma physics under the interaction of the magnetic and electric fields via the Lorentz force.

We note that any solution of (4.1) satisfies (4.2) for all $t > 0$ if (4.2) holds initially. Our system (4.1) is written in the form of (1.1) with $n = 3$, namely, we have

$$W_t + \sum_{j=1}^3 F^j(W)_{x_j} = G(W), \tag{4.3}$$

where $W = (n, nu, E, B)^T$,

$$\sum_{j=1}^3 F^j(W)\xi_j = \big(n(u \cdot \xi),\ nu(u \cdot \xi) + p(n)\xi,\ B\Omega_\xi,\ -E\Omega_\xi\big)^T,$$

and $G(W) = (0, -n(E + u \times B) - nu, nu, 0)^T$. Here and below, u, E, B and ξ denotes row vectors in $\mathbb{R}^3$, and Ω_ξ is the skew-symmetric matrix defined by

$$\Omega_\xi = \begin{pmatrix} 0 & -\xi_3 & \xi_2 \\ \xi_3 & 0 & -\xi_1 \\ -\xi_2 & \xi_1 & 0 \end{pmatrix}$$

for $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$, so that we have $\xi \times B = -B\Omega_\xi$ (as a row vector). The set of basic states for our system is $\mathcal{O}_W = \{W = (n, nu, E, B)^T;\ n > 0,\ u, E, B \in \mathbb{R}^3\} \subset \mathbb{R}^{10}$. Let $\{e_1, \cdots, e_4, \phi_1, \cdots, \phi_6\}$ be the standard orthonormal basis of $\mathbb{R}^{10}$; e_1, $\{e_2, e_3, e_4\}$, $\{\phi_1, \phi_2, \phi_3\}$ and $\{\phi_4, \phi_5, \phi_6\}$ are corresponding to n, nu, E and B, respectively. Then the subspace $\mathcal{M}$ and the set $\mathcal{E}$ of equilibrium states are given by

$$\mathcal{M} = \operatorname{span}\{e_1, \phi_4, \phi_5, \phi_6\}, \qquad \mathcal{E} = \{W = (n, 0, 0, B)^T;\ n > 0,\ B \in \mathbb{R}^3\},$$

respectively.

We see that the total energy of the isentropic Euler-Maxwell system (4.1) satisfies the equation

$$\left\{ n\big(\Phi(n) + \frac{1}{2}|u|^2\big) + \frac{1}{2}(|E|^2 + |B|^2) \right\}_t$$
$$+ \operatorname{div}\left\{ nu\big(\Phi(n) + \frac{1}{2}|u|^2\big) + p(n)u + E \times B \right\} + n|u|^2 = 0, \tag{4.4}$$

where $\Phi(n) = \int^n \frac{p(\zeta)}{\zeta^2}\, d\zeta$. We note that $n(\Phi(n) + \frac{1}{2}|u|^2)$ is the total energy of the electron and $\frac{1}{2}(|E|^2 + |B|^2)$ is the electro-magnetic energy. We set

$$\eta(W) = n\big(\Phi(n) + \frac{1}{2}|u|^2\big) + \frac{1}{2}(|E|^2 + |B|^2). \tag{4.5}$$

This total energy becomes a mathematical entropy of our isentropic Euler-Maxwell system (4.1). In fact, we have:

Proposition 4.1. *The function $\eta(W)$ in (4.5) is a mathematical entropy of the isentropic Euler-Maxwell system (4.1) in the sense of Definition 2.1. Therefore the system (4.1) is put in a symmetric dissipative system for $U := (D_W \eta(W))^T$.*

We note the above U is given explicitly as $U = \big(a(n) - \frac{1}{2}|u|^2, u, E, B\big)^T$, where $a(n) := \Phi(n) + \frac{p(n)}{n}$. Here we use $V = (n, u, E, B)^T$ which is a physically natural state variable. Then the system (4.1) can be written in the form

$$\bar{A}^0(V)V_t + \sum_{j=1}^{3} \bar{A}^j(V)V_{x_j} + \bar{L}(V)V = 0, \tag{4.6}$$

where

$$\bar{A}^0(V) = \begin{pmatrix} \frac{p'(n)}{n} & & & \\ & nI & & \\ & & I & \\ & & & I \end{pmatrix}, \qquad \bar{L}(V) = \begin{pmatrix} 0 & & & \\ & n(I - \Omega_B) & nI & \\ & -nI & O & \\ & & & O \end{pmatrix},$$

$$\sum_j \bar{A}^j(V)\xi_j = \begin{pmatrix} \frac{p'(n)}{n}(u \cdot \xi) & p'(n)\xi & & \\ p'(n)\xi^T & n(u \cdot \xi)I & & \\ & & O & -\Omega_\xi \\ & & \Omega_\xi & O \end{pmatrix}.$$

The system (4.1) or (4.6) admits a constant equilibrium state $V_\infty = (n_\infty, 0, 0, B_\infty)^T \in \mathcal{E}$, where $B_\infty \in \mathbb{R}^3$ is an arbitrarily fixed constant vector. We linearize (4.6) around the constant state V_∞. The linearized system for the perturbation $Z = V - V_\infty$ is given by

$$A^0 Z_t + \sum_{j=1}^{3} A^j Z_{x_j} + LZ = 0,$$

where $A^0 = \bar{A}^0(V_\infty)$, $A^j = \bar{A}^j(V_\infty)$ and $L = \bar{L}(V_\infty)$. It was shown in [3, 18] that the solution Z to the linearized system satisfies

$$|\hat{Z}(\xi,t)| \le Ce^{-c\eta(\xi)t}|\hat{Z}_0(\xi)|, \tag{4.7}$$

where $\eta(\xi) = |\xi|^2/(1+|\xi|^2)^2$, and Z_0 denotes the initial data satisfying (4.2) (in the modified version). This implies the regularity-loss property described by (1.2). As the consequence, we have the following linear decay estimate:

$$\|\partial_x^k Z(t)\|_{L^2} \le C(1+t)^{-3/4-k/2}\|Z_0\|_{L^1} + C(1+t)^{-l/2}\|\partial_x^{k+l} Z_0\|_{L^2},$$

where k, $l \ge 0$.

Concerning the nonlinear problem we have the following result on the global existence and optimal decay of solutions.

Theorem 4.2 ([24] (cf. [3, 18, 19])). *Suppose that the initial data V_0 satisfy (4.2) and that $E_0 = \|V_0 - V_\infty\|_{H^3}$ is small. Then the isentropic Euler-Maxwell system (4.1) admits a unique global solution V satisfying*

$$\|(V - V_\infty)(t)\|_{H^3}^2 + \int_0^t \|(n - n_\infty, u)(\tau)\|_{H^3}^2 + \|E(\tau)\|_{H^2}^2 + \|\partial_x B(\tau)\|_{H^1}^2 \, d\tau \le CE_0^2. \tag{4.8}$$

Suppose in addition that $E_1 = \|V_0 - V_\infty\|_{H^3 \cap L^1}$ is small. Then the global solution V verifies the optimal decay estimate

$$\|(V - V_\infty)(t)\|_{L^2} \le CE_1(1+t)^{-3/4}. \tag{4.9}$$

The regularity assumption H^3 in the above theorem can be relaxed to $B_{2,1}^{5/2}$. For the details, we refer to [23].

5 Euler-Cattaneo-Maxwell system

In this section we consider the non-isentropic Euler-Maxwell system. As in the usual isentropic model treated in the previous section, we introduce the frictional damping for the momentum equation of electron. Also we take into account the thermal effect which is described by the Cattaneo law, but we do not introduce the "energy damping" of electron, which was usually assumed in the previous works.

Our non-isentropic Euler-Maxwell system (which we call the Euler-Cattaneo-Maxwell

system in this note) is written in the form

$$
\begin{cases}
n_t + \operatorname{div}(nu) = 0, \\
(nu)_t + \operatorname{div}(nu \otimes u) + \nabla p = nf_{EM} - \dfrac{nu}{\tau_1}, \\
\left\{ n\left(e + \dfrac{1}{2}|u|^2\right) + \dfrac{1}{2}(|E|^2 + |B|^2) \right\}_t \\
\qquad + \operatorname{div}\left\{ nu\left(e + \dfrac{1}{2}|u|^2\right) + pu + q + E \times B \right\} = 0, \\
\tau_0 q_t + \dfrac{q}{\kappa \theta^2} - \nabla\left(\dfrac{1}{\theta}\right) = 0, \\
E_t - \operatorname{rot} B = nu, \\
B_t + \operatorname{rot} E = 0,
\end{cases}
\tag{5.1}
$$

$$
\operatorname{div} E = n_\infty - n, \qquad \operatorname{div} B = 0.
\tag{5.2}
$$

Here $n > 0$, $u \in \mathbb{R}^3$, $\theta > 0$ and $q \in \mathbb{R}^3$ are the density, the velocity, the absolute temperature and the heat flow of the electron, respectively, while $E \in \mathbb{R}^3$ is the electric field and $B \in \mathbb{R}^3$ is the magnetic induction; these are unknown functions of $t > 0$ and $x \in \mathbb{R}^3$. The pressure p and the internal energy e of the electron field are given smooth functions of (n, θ) satisfying the second law of thermodynamics

$$
de = \theta ds - pd\left(\dfrac{1}{n}\right),
$$

where s denotes the entropy of the electron field which is also a smooth function of (n, θ). It is assumed that $p_n = \frac{\partial p}{\partial n} > 0$, $p_\theta = \frac{\partial p}{\partial \theta} > 0$ and $e_\theta = \frac{\partial e}{\partial \theta} > 0$ for $n > 0$ and $\theta > 0$. The Lorentz force $f_{EM} = -(E + u \times B)$ is the same as in the previous section and acts as an outer force in the electron field, and $\kappa > 0$ denotes the coefficient of thermal conductivity which is assume to be smooth in (n, θ). The positive constant n_∞ denotes the ion density in the background, and $\tau_0 > 0$ and $\tau_1 > 0$ are relaxation parameters.

We note that any solution of (5.1) satisfies (5.2) for all $t > 0$ if (5.2) holds initially. Our system (5.1) is written in the form of (1.1) with $n = 3$, namely, we have (4.3), where $W = (n, nu, \Sigma, \tau_0 q, E, B)^T$,

$$
\sum_{j=1}^{3} F^j(W)\xi_j = \left(n(u \cdot \xi),\ nu(u \cdot \xi) + p\xi,\ \Gamma \cdot \xi,\ -\dfrac{1}{\theta}\xi,\ B\Omega_\xi,\ -E\Omega_\xi \right)^T,
$$

and $G(W) = (0,\ -n(E + u \times B) - \frac{1}{\tau_1}nu,\ 0,\ -\frac{1}{\kappa\theta^2}q,\ nu,\ 0)^T$. Here Σ and Γ denote the total energy and the corresponding flux: $\Sigma = n\left(e + \frac{1}{2}|u|^2\right) + \frac{1}{2}(|E|^2 + |B|^2)$ and $\Gamma \cdot \xi = n(u \cdot \xi)(e + \frac{1}{2}|u|^2) + p(u \cdot \xi) + q \cdot \xi + (E \times B) \cdot \xi$. For this system, the set of basic states is $\mathcal{O}_W = \{ W = (n, nu, \Sigma, \tau_0 q, E, B)^T;\ n, \theta > 0,\ u, q, E, B \in \mathbb{R}^3 \} \subset \mathbb{R}^{14}$. Let $\{e_1, \cdots, e_8, \phi_1, \cdots, \phi_6\}$ be the standard orthonormal basis of $\mathbb{R}^{14}$; e_1, $\{e_2, e_3, e_4\}$, e_5 and $\{e_6, e_7, e_8\}$ are corresponding to n, nu, Σ and $\tau_0 q$, respectively, while $\{\phi_1, \phi_2, \phi_3\}$

and $\{\phi_4, \phi_5, \phi_6\}$ correspond to E and B, respectively. Then the subspace $\mathcal{M}$ and the set $\mathcal{E}$ of equilibrium states are given by

$$\mathcal{M} = \text{span}\,\{e_1, e_5, \phi_4, \phi_5, \phi_6\},$$
$$\mathcal{E} = \{W = (n, 0, \Sigma_0, 0, 0, B)^T;\ n, \theta > 0,\ B \in \mathbb{R}^3\},$$

respectively, where $\Sigma_0 = ne + \frac{1}{2}|B|^2$.

We compute the equation of the entropy s. Then combining it with the equation of the heat flow q, we find that

$$\left(-ns + \frac{1}{2}\tau_0|q|^2\right)_t - \text{div}\left(nus + \frac{q}{\theta}\right) + \frac{n|u|^2}{\tau_1\theta} + \frac{|q|^2}{\kappa\theta^2} = 0, \tag{5.3}$$

which is the equation of the modified negative entropy. This modified negative entropy can be regarded as a mathematical entropy of the system (5.1). In fact we have:

Proposition 5.1. *The modified negative entropy $\eta(W) := -ns + \frac{1}{2}\tau_0|q|^2$ is a mathematical entropy of the Euler-Catteneo-Maxwell system* (5.1) *in the sense of Definition 2.1. Therefore the system* (5.1) *is put in a symmetric dissipative system for* $U := (D_W\eta(W))^T$.

We note the above U is given explicitly as $U = \frac{1}{\theta}\left(b - \frac{1}{2}|u|^2,\ u,\ -1,\ \theta q,\ E,\ B\right)^T$, where $b := e + \frac{p}{n} - \theta s$. In the actual computations we use $V = (n, u, \theta, q, E, B)$ as the unknown vector of the system (5.1).

It was observed in [12] that the dissipative structure of the Euler-Cattaneo-system (5.1) is of the regularity-loss type and is characterized by (1.2). Therefore, as in the case of the isentropic Euler-Maxwell system, we have the pointwise estimate (4.7) and the corresponding linear decay estimate for linear solutions. Furthermore, we can show the global existence and stability of nonlinear solutions for small initial data in H^3. For the details, we refer to [12]. On the other hand, the work on the optimal decay of nonlinear solutions with less regularity is now in progress.

Finally in this note, we remark that the general theory was already establised when the relaxation matrix is symmetric. We refer the reader to [17, 20, 1] for dissipative structure and linear decay property, and [14] and [21, 22] for global existence and optimal decay of nonlinear solutions in Sobolev spaces and Besov spaces, respectively.

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Remarks on the effect of spatial expansion
for nonlinear partial differential equations

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Abstract

The Cauchy problem of a semilinear partial differential equation which generalizes the Schrödinger equation and the parabolic equation is considered, and the global and blow-up solutions are shown in Sobolev spaces. The variance of the scale-function of the space is considered. The role of spatial variance on the problem is studied, and some dissipative properties of the equation are remarked.

1   Introduction

We consider the Cauchy problem of a semilinear partial differential equation, and we show the global and blow-up solutions in Sobolev spaces. We introduce the equation. We denote the spatial dimension by $n \geq 1$, the Planck constant by $\hbar := h/2\pi$, the mass by $m > 0$. Let $\sigma \in \mathbb{R}$, $a_0 > 0$, $a_1 \in \mathbb{R}$. We put $T_0 := \infty$ when $(1 + \sigma)a_1 \geq 0$, $T_0 := 2a_0/n(1 + \sigma)a_1$ when $(1 + \sigma)a_1 < 0$. We define a scale-function $a(t)$ for $t \in [0, T_0)$ by

$$a(t) := \begin{cases} a_0 \left(1 + \frac{n(1+\sigma)a_1 t}{2a_0}\right)^{2/n(1+\sigma)} & \text{if } \sigma \neq -1, \\ a_0 \exp\left(\frac{a_1 t}{a_0}\right) & \text{if } \sigma = -1, \end{cases} \tag{1.1}$$

where we note that $a_0 = a(0)$ and $a_1 = \partial_t a(0)$. We define the weight function $w(t) := (a_0/a(t))^{n/2}$. We use the change of variable $s = s(t) := \int_0^t a^{-2}(\tau)d\tau$, and

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we put $S_0 := s(T_0)$. We use the convention $a(s) := a(t(s))$ and $w(s) := w(t(s))$ for $s \in [0, S_0)$ as far as there is no fear of confusion. A direct computation shows

$$S_0 = \begin{cases} \frac{1}{2a_0 a_1} & \text{if } a_1 > 0 \text{ and } \sigma = -1, \\ \frac{2}{a_0 a_1 (4 - n(1+\sigma))} & \text{if } a_1(1+\sigma)\left(1 - \frac{4}{n(1+\sigma)}\right) < 0 \text{ and } \sigma \neq -1, \\ \infty & \text{otherwise.} \end{cases}$$

For $\lambda \in \mathbb{C}$, $1 \leq p < \infty$, $-\pi/2 < \omega \leq \pi/2$, $0 \leq \mu_0 < n/2$, and $0 < S \leq S_0$, we consider the Cauchy problem given by

$$\begin{cases} \pm i \frac{2m}{\hbar} \partial_s u(s,x) + e^{-2i\omega} \Delta u(s,x) - \lambda e^{-2i\omega} a(s)^2 \left(|uw|^{p-1} u\right)(s,x) = 0, \\ u(0, \cdot) = u_0(\cdot) \in H^{\mu_0}(\mathbb{R}^n) \end{cases} \tag{1.2}$$

for $(s,x) \in [0, S) \times \mathbb{R}^n$, where $i := \sqrt{-1}$, $\Delta := \sum_{j=1}^{n} \partial^2/\partial x_j^2$, and $H^{\mu_0}(\mathbb{R}^n)$ denotes the Sobolev space of order $\mu_0 \geq 0$. The double sign $\pm$ is in same order. We say that u is a global solution of (1.2) if it exists on $[0, S_0)$.

When the scale-function is a constant $a(\cdot) = 1$, we have $s = t$ and $S_0 = T_0 = \infty$. The first equation in (1.2) corresponds to the Schrödinger equation

$$\pm i \frac{2m}{\hbar} \partial_s u(s,x) + \Delta u(s,x) + \lambda \left(|u|^{p-1} u\right)(s,x) = 0 \tag{1.3}$$

for $\omega = 0$, and to the parabolic equation

$$\frac{2m}{\hbar} \partial_s u(s,x) - \Delta u(s,x) \pm \lambda \left(|u|^{p-1} u\right)(s,x) = 0 \tag{1.4}$$

for $\omega = \pm \pi/4$.

Let us consider the well-posedness of (1.2). For any real numbers $2 \leq q, r \leq \infty$, we say that the pair (q, r) is admissible if it satisfies $1/r + 2/nq = 1/2$ and $(q, r, n) \neq (2, \infty, 2)$. For $\mu_0 \geq 0$ and two admissible pairs $\{(q_j, r_j)\}_{j=1,2}$, we define a function space

$$X^{\mu_0}([0, S)) := \{u \in C([0, S), H^{\mu_0}(\mathbb{R}^n)); \ \max_{\mu = 0, \mu_0} \|u\|_{X^{\mu}([0,S))} < \infty\},$$

where

$$\|u\|_{X^{\mu}([0,S))} := \begin{cases} \|u\|_{L^{\infty}((0,S), L^2(\mathbb{R}^n)) \cap \bigcap_{j=1,2} L^{q_j}((0,S), L^{r_j}(\mathbb{R}^n))} & \text{if } \mu = 0, \\ \|u\|_{L^{\infty}((0,S), \dot{H}^{\mu}(\mathbb{R}^n)) \cap \bigcap_{j=1,2} L^{q_j}((0,S), \dot{B}^{\mu}_{r_j 2}(\mathbb{R}^n))} & \text{if } \mu > 0. \end{cases}$$

Here, $\dot{H}^{\mu}(\mathbb{R}^n)$ and $\dot{B}^{\mu}_{r_j 2}(\mathbb{R}^n)$ are the homogeneous Sobolev and Besov spaces, respectively. Since the propagator of the linear part of the first equation in (1.2) is written as $\exp(\pm i\hbar \exp(-2i\omega)s\Delta/2m)$, we assume $\pm \sin 2\omega \geq 0$ to define it for tempered distributions. We note that the scaling critical number of p for (1.2) is $p(\mu_0) := 1 + 4/(n - 2\mu_0)$ when $a(\cdot) = 1$.

Theorem 1.1. *Let $n \geq 1$, $\lambda \in \mathbb{C}$, $0 \leq \mu_0 < n/2$, and $1 \leq p \leq p(\mu_0)$. Let ω satisfies $-\pi/2 < \omega \leq \pi/2$, and $\pm \sin 2\omega \geq 0$. Assume $\mu_0 < p$ if p is not an odd number. There exist two admissible pairs $\{(q_j, r_j)\}_{j=1,2}$ with the following properties.*

(1) (Local solutions.) For any $u_0 \in H^{\mu_0}(\mathbb{R}^n)$, there exist $S > 0$ with $S \leq S_0$ and a unique local solution u of (1.2) in $X^{\mu_0}([0,S))$. Here, S depends only on the norm $\|u_0\|_{\dot{H}^{\mu_0}(\mathbb{R}^n)}$ when $p < p(\mu_0)$, while S depends on the profile of u_0 when $p = p(\mu_0)$. The solutions depend on the initial data continuously.

(2) (Small global solutions.) Let $p = p(\mu_0)$, $\sigma \in \mathbb{R}$, $\mu_0 \geq 0$, $a_1 \geq 0$, or let $1 < p < p(\mu_0)$, $\sigma = -1$, $\mu_0 > 0$, $a_1 > 0$. If $\|u_0\|_{\dot{H}^{\mu_0}(\mathbb{R}^n)}$ is sufficiently small, then the solution u obtained in (1) is a global solution, namely, $S = S_0$.

When $\lambda \in \mathbb{R}$, we are able to use the conservation law to show global solutions for large data in $H^1(\mathbb{R}^n)$. The global solutions for the complex Ginzburg-Landau equation are shown in [3, Theorem 3.1]. The blow-up solutions for initial data with negative energy are obtained by the concavity of an auxiliary function, the virial identity, and the Heisenberg uncertainty principle. We refer to [2, Section 6.5] for (1.3), [4, Theorem 5.3] for (1.4), [1, Theorem 1.8] for (1.3) in the hyperbolic space. We show some fundamental results in this talk.

To prove the results for global and blow-up solutions, we use two dissipative properties. One is from the parabolic structure of the equation when $\pm \sin 2\omega > 0$. The other is from the scale function $a(\cdot)$ when $\partial_t a(0) = a_1 \neq 0$. Even if the equation does not have the parabolic structure when $\pm \sin 2\omega = 0$, the latter is very effective to obtain the global solutions. The energy estimate shows the dissipative property for the first equation in (1.2) if $\lambda a_1 (p - 1 - 4/n) > 0$.

In this talk, the notation $A \lesssim B$ denotes the inequality $A \leq CB$ for some constant $C > 0$ which is not essential in our argument. For any real number $1 \leq r \leq \infty$, its conjugate number is denoted by r' with $1/r + 1/r' = 1$. For $\mu \in \mathbb{R}$ and $1 \leq r, m \leq \infty$, we use the Lebesgue space $L^r(\mathbb{R}^n)$, the Sobolev space $H^{\mu,r}(\mathbb{R}^n)$, the Besov space $B_{rm}^\mu(\mathbb{R}^n)$, and their homogeneous spaces $\dot{H}^{\mu,r}(\mathbb{R}^n)$ and $\dot{B}_{rm}^\mu(\mathbb{R}^n)$.

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Unconditional uniqueness of solutions for nonlinear dispersive equations

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1. Introduction

We investigate *unconditional uniqueness* (UU) of solutions to the Cauchy problem for general nonlinear dispersive equations on the torus $\mathbb{T}^d := \mathbb{R}^d/(2\pi\mathbb{Z})^d$ or on $\mathbb{R}^d$. We consider Sobolev spaces $H^s := (1-\Delta)^{-s/2}L^2$ as the spaces of initial data. Here, UU in H^s means uniqueness of the solutions (in the sense of distribution) in $C([0,T];H^s)$ for initial data in H^s. Hereafter, for a Banach space X we write $C_T X$ to denote $C([0,T];X)$.

Two critical regularity exponents may arise in this problem. First, if the equation is invariant under the scaling transformation, then the scale-invariant Sobolev regularity $s = s_s$ is initially expected to be the lowest regularity that admits the well-posedness of the Cauchy problem. (However, there are many cases where the Cauchy problem becomes ill-posed at some regularity higher than the scaling.) Secondly, there exists the regularity threshold $s = s_e$ below which the nonlinear part does not make sense in the distributional framework. Therefore, we naturally focus on UU in H^s for $s \geq \max\{s_s, s_e\}$. For instance, if we consider the nonlinear Schrödinger equation (NLS) with the cubic nonlinearity

$$i\partial_t u + \Delta u = |u|^2 u, \qquad (t,x) \in [0,T] \times \mathbb{R}^d,$$

the equation is invariant under the scaling $u(t,x) \mapsto u_\lambda(t,x) := \lambda u(\lambda^2 t, \lambda x)$ $(\lambda > 0)$, which preserves the $\dot{H}^s(\mathbb{R}^d)$ norm if $s = s_s := \frac{d}{2} - 1$, whereas the embedding $H^s(\mathbb{R}^d) \hookrightarrow L^3(\mathbb{R}^d)$ holds if and only if $s \geq \frac{d}{6}$, which suggests that $s_e = \frac{d}{6}$. Hence, for this equation we consider UU in H^s only for $s \geq \max\{\frac{d}{2} - 1, \frac{d}{6}\}$.

We also note that UU is sometimes trivial, especially if the solution is obtained by an iteration argument in $C_T H^s$ itself. For the above cubic NLS, this corresponds to the case $s > \frac{d}{2}$ for which $H^s(\mathbb{R}^d)$ is an algebra. The concept of *unconditional well-posedness* (i.e., well-posedness with unconditional uniqueness) was introduced by T. Kato [9], who pointed out that UU becomes meaningful in the case that the solution is obtained by iteration but using an auxiliary function space in addition to $C_T H^s$. In the NLS example, one can still construct solutions for $s < \frac{d}{2}$ in a certain range by using the so-called Strichartz estimates, but then uniqueness is obtained initially in the intersection of $C_T H^s$ with some mixed Lebesgue space $L_T^p L^q$ used as an auxiliary space, and to establish UU often requires an additional argument.

There are many results on UU in the non-periodic case (i.e., the Cauchy problem on $\mathbb{R}^d$). For NLS (with general power-type nonlinearities), the first result of T. Kato [9] has been improved by Furioli-Terraneo [3], Rogers [13], Su Win-Y. Tsutsumi [15], and recently by Han-Fang [5]. These results settled the UU problem for most of $s \geq \max\{s_s, s_e\}$. For other equations see e.g. Zhou [17] (KdV equation), Su Win [14] (cubic derivative NLS), and Masmoudi-Nakanishi [12] (Zakharov system).

Compared to the non-periodic case, UU seems less studied in the periodic settings. For the KdV equation UU was obtained by Babin et al. [2], which was followed by

Kwon-Oh [11] (modified KdV equation) and T.K. Kato-Tsugawa [10] (fifth order KdV-type equations). Recently, Guo et al. [4] proved UU for one-dimensional periodic cubic NLS. In these results, proof is based on successive applications of integration by parts in the time variable. This can be regarded as a variant of Poincaré-Dulac Normal Form Reduction (NFR); we refer to [1] for details of the Poincaré-Dulac NFR.

It is worth noticing that Guo et al. [4] had to invoke NFR *infinitely* many times to make all the nonlinear estimates closed in $C_T H^s$, in contrast to the previous works for the KdV-type equations in which, despite of the derivative losses in the nonlinearities, the results were obtained by applying such integration-by-parts procedure *finitely* many times. Such a difference comes from the difference of *resonance structure* between the NLS and the KdV type equations.

We also note that in the NFR argument, just the $C_T H^s$-norm is basically used in the estimates, whereas many of UU results in the non-periodic case also use some auxiliary function spaces (e.g. Strichartz-type spaces or the Fourier restriction spaces), which are designed to be large enough to contain $C_T H^s$ so that desired nonlinear estimates hold.

2. Main Results

Let us concentrate on the periodic case, i.e., $x \in \mathbb{T}^d$. Our aim is to generalize the infinite NFR machinery of [4] so that it can be applied to a wide range of nonlinear dispersive equations.

There are many potential difficulties in this machinery. Some of them are as follows:

(a) Each application of NFR will produce higher and higher order nonlinear terms. For instance, in the case of cubic NLS, nonlinear terms of order $2k + 3$ will appear after the k-th application of NFR. Then, one needs to establish multilinear estimates with higher and higher degrees of nonlinearities.

(b) The resonance structures in nonlinear terms are different from each other and become more and more complicated as the degrees of them increase. Note that NFR can be applied only to the *non-resonant* part of nonlinear terms.

(c) The number of terms after the k-th NFR grows in a factorial order $(k!)^C$, which is faster than an exponential order C^k.

(d) One has to justify the limiting procedure of 'applying NFR indefinitely', namely, find the *limit equation* and show that any distributional solution of the original equation in $C_T H^s$ is also a solution of it.

Guo et al. [4] could deal with the above (a)–(d) (actually they just gave a 'hint' for (d)) for the simplest NLS, i.e., in the one-dimensional cubic case, by explicitly writing down all the nonlinear terms and making delicate resonance/non-resonance decompositions of them. Since their proof was highly dependent on simplicity of the equation, their argument has not been applied to other equations, even to the *two-dimensional* cubic NLS.

Our main result, as stated below, treats more general nonlinear dispersive equations and gives two sufficient conditions which allow the infinite NFR machinery to work. Each sufficient condition consists of several simple multilinear estimates, and we can show that these estimates are actually enough to yield all the required multilinear estimates by an induction on the degree, and also enough to justify the limit equation.

We use Fourier series expansion to move to the frequency space and consider the following abstract equation:

$$\partial_t \omega_n(t) = \sum_{n=n_1+\cdots+n_p} e^{it\phi} m \omega_{n_1}(t) \omega_{n_2}(t) \cdots \omega_{n_p}(t) + \mathcal{R}[\omega]_n(t), \quad n \in \mathbb{Z}^d, \tag{1}$$

where $\phi = \phi(n, n_1, \ldots, n_p) \in \mathbb{R}$ denotes the phase part, $m = m(n, n_1, \ldots, n_p) \in \mathbb{C}$ is the multiplier, and $\mathcal{R}[\omega]$ is the remainder terms. For example, the KdV equation

$$\partial_t u + \partial_x^3 u = \partial_x(u^2), \qquad (t,x) \in [0,T] \times \mathbb{T}$$

is, by setting $\omega_n(t) := \int_0^{2\pi} [U(-t)u(t)](x)e^{-inx}\,dx$ with $U(t)$ being the propagator for the Airy equation, equivalent to

$$\partial_t \omega_n(t) = \frac{in}{2\pi} \sum_{n=n_1+n_2} e^{it(n^3-n_1^3-n_2^3)} \omega_{n_1}(t) \omega_{n_2}(t), \qquad (t,n) \in [0,T] \times \mathbb{Z},$$

which is of the form (1) with $p = 2$, $\phi = n^3 - n_1^3 - n_2^3$, $m = in/2\pi$, and $\mathcal{R} = 0$. In such a way, nonlinear dispersive equations can be represented as (1) if we assume that *the nonlinearity is a polynomial in u, $\bar{u}$ and derivatives of them with constant coefficients.* The initial data $\omega_n(0)$ is now given in weighted ℓ^2 spaces $\ell_s^2 := \langle n \rangle^{-s} \ell^2(\mathbb{Z}^d)$ instead of H^s, and unconditional uniqueness for the original equation in H^s is replaced with that for (1) in ℓ_s^2.

Our main theorem is as follows:

Theorem 1. *Let $s \in \mathbb{R}$ and assume that the following estimates for $\mathcal{R}[\omega]$ hold;*

$$\left\| \mathcal{R}[\omega] \right\|_{C_T \ell_s^2} \leq C\left(\|\omega\|_{C_T \ell_s^2} \right),$$
$$\left\| \mathcal{R}[\omega] - \mathcal{R}[\widetilde{\omega}] \right\|_{C_T \ell_s^2} \leq C\left(\|\omega\|_{C_T \ell_s^2}, \|\widetilde{\omega}\|_{C_T \ell_s^2} \right) \|\omega - \widetilde{\omega}\|_{C_T \ell_s^2}.$$

Assume further that for some Banach space X of functions on $\mathbb{Z}^d$ satisfying $\|\omega\|_X \leq C\|\omega\|_{\ell_s^2}$, we have one of the following [A], [B]. Then, UU for (1) holds in ℓ_s^2.

[A] There exists $\delta > 0$ such that

$$(A1) \quad \Big\| \sum_{n=n_1+\cdots+n_p} \frac{|m|}{\langle \phi \rangle^{1/2}} \omega_{n_1}^{(1)} \cdots \omega_{n_p}^{(p)} \Big\|_{\ell_s^2} \leq C \prod_{j=1}^{p} \|\omega^{(j)}\|_{\ell_s^2},$$

$$(A2) \quad \Big\| \sum_{n=n_1+\cdots+n_p} \frac{|m|}{\langle \phi \rangle^{1-\delta}} \omega_{n_1}^{(1)} \cdots \omega_{n_p}^{(p)} \Big\|_{X} \leq C \min_{1 \leq j \leq p} \|\omega^{(j)}\|_{X} \prod_{\substack{l=1 \\ l \neq j}}^{p} \|\omega^{(l)}\|_{\ell_s^2},$$

$$(A3) \quad \Big\| \sum_{n=n_1+\cdots+n_p} |m| \omega_{n_1}^{(1)} \cdots \omega_{n_p}^{(p)} \Big\|_{X} \leq C \prod_{j=1}^{p} \|\omega^{(j)}\|_{\ell_s^2}.$$

[B] *There exist $s_1, s_2 \in \mathbb{R}$ satisfying $s_1 < s < s_2$ such that*

$$(B1) \quad \sup_{\mu \in \mathbb{Z}} \Big\| \sum_{\substack{n=n_1+\cdots+n_p \\ \mu \leq \phi < \mu+1}} |m| \omega_{n_1}^{(1)} \cdots \omega_{n_p}^{(p)} \Big\|_{\ell_{s_1}^2} \leq C \prod_{j=1}^p \|\omega^{(j)}\|_{\ell_{s_1}^2},$$

$$(B1)' \quad \Big\| \sum_{n=n_1+\cdots+n_p} |m| \omega_{n_1}^{(1)} \cdots \omega_{n_p}^{(p)} \Big\|_{\ell_{s_2}^2} \leq C \prod_{j=1}^p \|\omega^{(j)}\|_{\ell_{s_2}^2},$$

$$(B2) \quad \sup_{\mu \in \mathbb{Z}} \Big\| \sum_{\substack{n=n_1+\cdots+n_p \\ \mu \leq \phi < \mu+1}} |m| \omega_{n_1}^{(1)} \cdots \omega_{n_p}^{(p)} \Big\|_X \leq C \min_{1 \leq j \leq p} \|\omega^{(j)}\|_X \prod_{\substack{l=1 \\ l \neq j}}^p \|\omega^{(l)}\|_{\ell_{s_1}^2},$$

$$(B2)' \quad \Big\| \sum_{n=n_1+\cdots+n_p} |m| \omega_{n_1}^{(1)} \cdots \omega_{n_p}^{(p)} \Big\|_X \leq C \min_{1 \leq j \leq p} \|\omega^{(j)}\|_X \prod_{\substack{l=1 \\ l \neq j}}^p \|\omega^{(l)}\|_{\ell_{s_2}^2},$$

$$(B3) \quad \Big\| \sum_{n=n_1+\cdots+n_p} |m| \omega_{n_1}^{(1)} \cdots \omega_{n_p}^{(p)} \Big\|_X \leq C \prod_{j=1}^p \|\omega^{(j)}\|_{\ell_s^2}. \qquad \text{(same as (A3))}$$

With the above multilinear estimates, the infinite NFR machinery proceeds as follows. The estimate (A1) or (B1)+(B1)' is the main estimate to obtain ℓ_s^2-control for all the nonlinear terms in each NFR step, except for one 'less regular' term. Then (A2) or (B2)+(B2)' allow us to show that this term vanishes in the X norm in the limit equation. The estimate (A3) or (B3) ensures that the nonlinear terms make sense in the framework of distribution.

It is essential in the proof of unconditional uniqueness to notice that one cannot rely on approximation by good solutions (decaying faster) and needs to justify every formal calculation for a solution in $C_T \ell_s^2$ itself, because a general solution in $C_T \ell_s^2$ is not necessarily approximated by good solutions. However, this is done by using (A2)+(A3) or (B2)+(B2)'+(B3).

Remarks. (i) A similar statement holds in the case of multiple nonlinear terms (in which case we need to use the same Banach space X for each term), or for systems of equations.

(ii) The remainder term $\mathcal{R}[\omega]$ basically includes easily controlled terms or the specific part of the main term which is in itself easily estimated but causes trouble in establishing the multilinear estimates (A1)–(A5) or (B1)–(B3) if it remains in the main term. It is sometimes important to find such a problematic part in the main term and move it to $\mathcal{R}[\omega]$ before carrying on NFR.

(iii) At this moment, it is not clear whether a similar argument is applicable to non-periodic problems. In fact, the only issue is justification of formal calculations. In the periodic case, the equation combined with the estimate (A3) or (B3) shows that $\omega_n(t)$ is a C^1 function for each fixed n. However, this is no longer true for each $n \in \mathbb{R}^d$ in the non-periodic case, and one has to verify the product rule for the time differentiation or the exchange of the integral and the time differentiation in the framework of distribution.

3. Applications

(i) NLS with odd power nonlinearity, general dimension:

$$i\partial_t u + \Delta u = \pm |u|^{2k}u, \qquad (t,x) \in [0,T] \times \mathbb{T}^d, \quad k,d \in \mathbb{N}. \tag{2}$$

We can prove (B1)–(B3) with $X = \ell^\infty(\mathbb{Z}^d)$ and obtain the following:

Theorem 2. *UU holds for* (2) *in* $H^s(\mathbb{T}^d)$ *if (i)* $s > \frac{d^2}{2(d+3)}$ *($k = 1$ and $2 \leq d \leq 5$), (ii)* $s > s_c = \frac{d}{2} - \frac{1}{k}$ *and* $s \geq s_e = \frac{d(2k-1)}{2(2k+1)}$ *($k \geq 2$ or $d \geq 6$).*

(ii) Zakharov system:

$$i\partial_t u + \Delta u = nu, \quad \partial_t^2 n - \Delta n = \Delta(|u|^2), \qquad (u,n): [0,T] \times \mathbb{T}^d \to \mathbb{C} \times \mathbb{R}. \tag{3}$$

Reducing to a first order system and applying Theorem 1 [A] with $X = \ell_0^2 \times \ell_{-1}^2$, we have

Theorem 3. *If $d = 1, 2$, then UU holds for* (3) *in the energy space, i.e. for* $\big(u(0), n(0), \partial_t n(0)\big) \in H^1(\mathbb{T}^d) \times L^2(\mathbb{T}^d) \times H^{-1}(\mathbb{T}^d)$.

(iii) cubic derivative NLS:

$$i\partial_t u + \partial_x^2 u = \pm i\partial_x(|u|^2 u), \qquad (t,x) \in [0,T] \times \mathbb{T}. \tag{4}$$

We apply Theorem 1 [A] with $X = \ell_{s-1}^2$ to an equivalent equation obtained by the gauge transformation (cf. [6, 7]).

Theorem 4. *UU holds for* (4) *in* H^s, $s > \frac{1}{2}$.

Note that the best regularity in which well-posedness is valid is $s = \frac{1}{2}$ ([16, 8]), while $s_s = 0$ and $s_e = \frac{1}{3}$ for (4). (Modified) Benjamin-Ono equation can be also treated with a gauge transformation, but our proof with Theorem 1 needs to be slightly modified.

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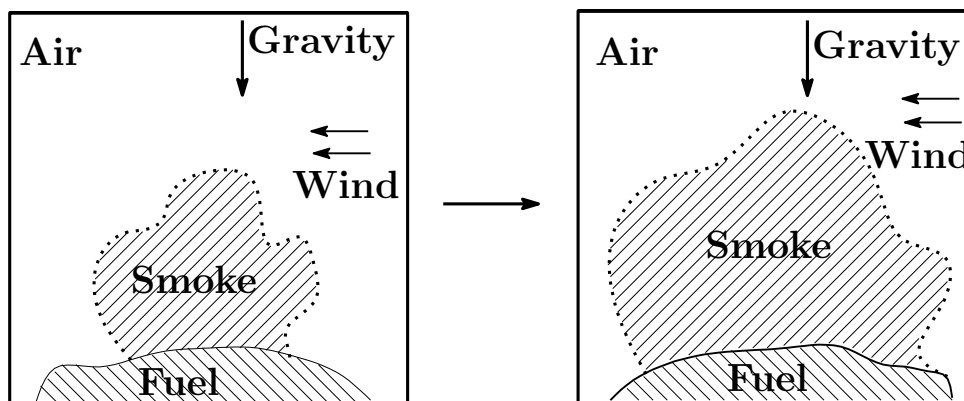# Introduction to mathematical modeling of soap bubble and smoke in air

Hajime Koba[*]

1. Introduction

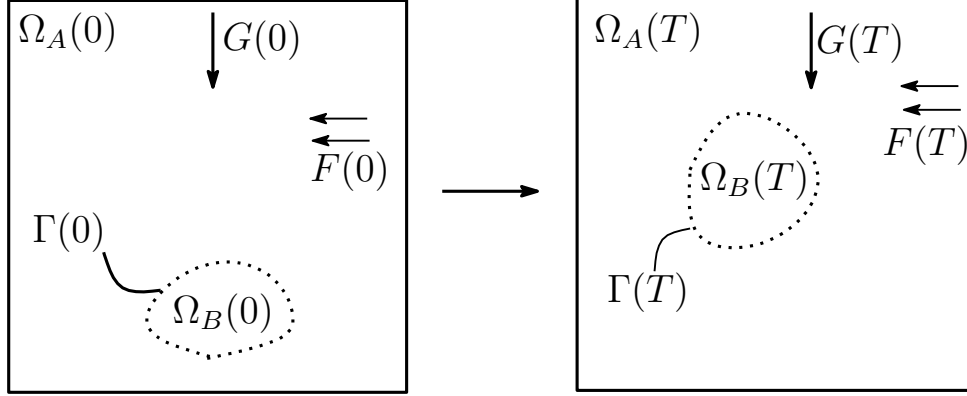We are interested in mathematical models of both a soap bubble and smoke in air:
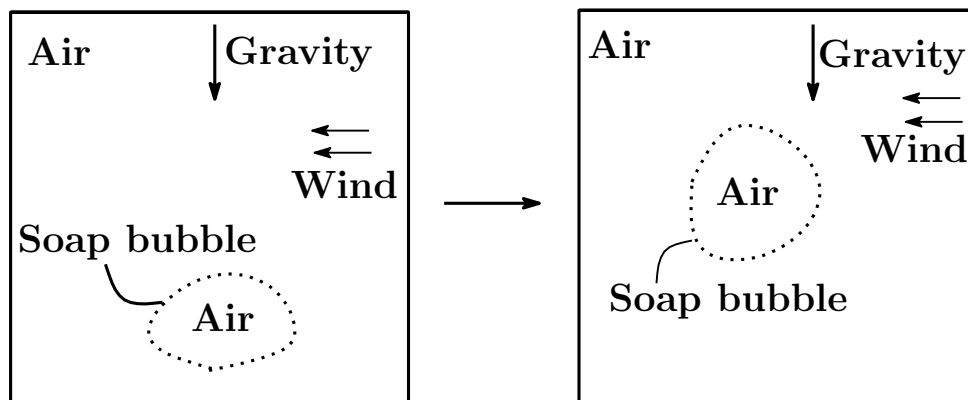
Soap bubble in air

Smoke in air

Surface flow (interfacial flow) and surface tension (interfacial tension) play an important role in a soap bubble in air and smoke in air. Now we consider a soap bubble in air. In order to make a mathematical model of a soap bubble in air, we formalize as follows:

Here $\Omega_A(t)$, $\Omega_B(t)$ are domains with a moving boundary, $\Gamma(t)$ is an evolving hypersurface, $G(t)$ is the gravity, $F(t)$ is an external force.

Now we introduce some notation.

$$
\begin{cases}
\rho_A : \text{density of fluid in } \Omega_A(t), \\
v_A : \text{velocity of fluid in } \Omega_A(t), \\
p_A : \text{pressure of fluid in } \Omega_A(t), \\
e_A : \text{internal energy in } \Omega_A(t), \\
q_A : \text{heat flux in } \Omega_A(t), \\
\theta_A : \text{temperature of fluid in } \Omega_A(t),
\end{cases}
\qquad
\begin{cases}
\rho_B : \text{density of fluid in } \Omega_B(t), \\
v_B : \text{velocity of fluid in } \Omega_B(t), \\
p_B : \text{pressure of fluid in } \Omega_B(t), \\
e_B : \text{internal energy in } \Omega_B(t), \\
q_B : \text{heat flux in } \Omega_B(t), \\
\theta_B : \text{temperature of fluid in } \Omega_B(t),
\end{cases}
$$

$$
\begin{cases}
\rho : \text{density of fluid on } \Gamma(t), \\
v : \text{velocity of fluid on } \Gamma(t), \\
p : \text{pressure of fluid on } \Gamma(t), \\
e : \text{internal energy on } \Gamma(t), \\
q : \text{heat flux on } \Gamma(t), \\
\theta : \text{temperature of fluid on } \Gamma(t),
\end{cases}
\qquad
\begin{cases}
\sigma : \text{surface tension on } \Gamma(t), \\
u : \text{relative velocity of fluid on } \Gamma(t), \\
w : \text{motion velocity of } \Gamma(t), \\
[[\ \cdot \ ]] : \text{ jump condition },
\end{cases}
$$

$$
\begin{cases}
\mu_A, \lambda_A : \text{viscosity coefficients of fluid in } \Omega_A(t), \\
\kappa_A : \text{ thermal diffusivity of fluid in } \Omega_A(t), \\
\mu_B, \lambda_B : \text{viscosity coefficients of fluid in } \Omega_B(t), \\
\kappa_B : \text{ thermal diffusivity of fluid in } \Omega_B(t), \\
\mu, \lambda : \text{viscosity coefficients of fluid on } \Gamma(t), \\
\kappa : \text{ thermal diffusivity of fluid on } \Gamma(t).
\end{cases}
$$

We would like to model a soap bubble in air from a mathematical point of view. In other words, we would like to derive relationships among the above symbols from mathematical assumptions. To this end, we shall go through the following steps:

Step1: Deriving the dominant equations for the motion of fluid in a domain with a moving boundary

Step2: Deriving the dominant equations for the motion of fluid on an evolving hypersurface:[Inviscid/Viscous]

$$\begin{cases} \text{Incompressible fluid} \\ \text{Compressible fluid (Barotorpic)} \\ \text{Compressible fluid (Internal energy/Temperature/Heat transfer)} \end{cases}$$

Step3: Soap bubble in air / Smoke in air

Here an evolving hypersurface means that the surface is moving or the shape of the surface is changing along with the time.

In this talk, we focus on the second step. We introduce various fluid flow systems on an evolving hypersurface derived by applying our energetic variational approach.

2.   Known results (Fixed hypersurface·Manifold)

Let us introduce the Euler system on a manifold derived by Arnold [2, 3] and the Navier-Stokes system on a manifold introduced by Taylor [8]. Let $\mathcal{M}$ be a 2-dimensional closed Riemannian manifold, and let $\mu > 0$ be a viscosity coefficients of fluid on $\mathcal{M}$. Let u be a velocity on $\mathcal{M}$, and let p be a pressure associated with u. Assume that u is a 1-form on $\mathcal{M}$ and that p is a function on $\mathcal{M}$.
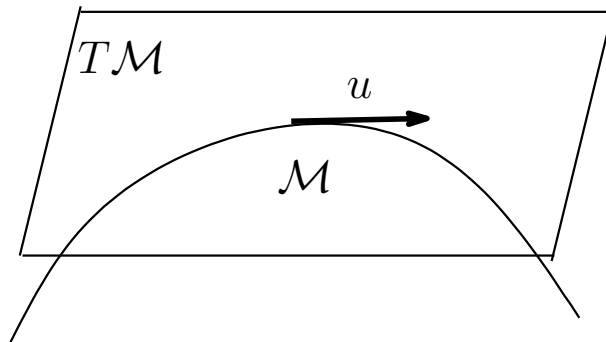
Arnold [2, 3] applied the Lie group of diffeomorphisms to derive the following Euler system on a manifold $\mathcal{M}$:

$$\begin{cases} u_t + \nabla_u u + \mathrm{grad}_{\mathcal{M}} p = 0, \\ \mathrm{div}_{\mathcal{M}} u = 0. \end{cases}$$

See also Ebin-Marsden [4].

Taylor [8] introduced the following Navier-Stokes system, derived from their physical sense, on a manifold $\mathcal{M}$:

$$(2\text{-}1) \quad \begin{cases} u_t + \nabla_u u + \mathrm{grad}_{\mathcal{M}} p = \mu(\Delta_{\mathcal{M}} u + Ku), \\ \mathrm{div}_{\mathcal{M}} u = 0. \end{cases}$$

Mitsumatsu and Yano [7] also derived the system (2-1) from an energetic point of view. Arnaudon and Cruzeiro [1] applied stochastic variational approach to derive the system (2-1).

Here

$$\begin{cases} \mathcal{M} & : \text{2-dimensional closed Riemannian manifold,} \\ u & : \text{1-form on } \mathcal{M}\text{[fluid flow in surface],} \\ p & : \text{function on } \mathcal{M}\text{[pressure associated with } u\text{],} \\ \mu > 0 & : \text{viscosity coefficient,} \\ \Delta_{\mathcal{M}} & : \text{the Borhner-Laplacian,} \\ K & : \text{the Gaussian curvature (the Ricci curvature),} \\ \mathrm{grad}_{\mathcal{M}} & : \text{gradient operator on } \mathcal{M}, \\ \mathrm{div}_{\mathcal{M}} & : \text{divergence operator on } \mathcal{M}, \\ \nabla_u u & : \text{covariant derivative.} \end{cases}$$

3.　Main results (Evolving hypersurface)

Let us state our main results. We first explain difficult points to derive the dominant equations for the motion of fluid on an evolving hypersurface.

Point1: To characterize incompressible and compressible fluid on an evolving hypersurface or a moving hypersurface

Point2: To deal with normal direction of fluid flow on an evolving hypersurface

Point3: To derive viscous terms of fluid flow systems on an evolving hypersurface

In order to overcome these difficulties, we use the Riemannian metric and our energetic variational approach. In this talk we explain them in detail. Note that we cannot directly apply the method from [8, 7, 1] to derive a fluid flow system on an evolving hypersurface.

Before stating our main results, we introduce notation. Let $x = (x_1, x_2, x_3)$ be the spatial variable and t be the time variable. Let $\Gamma(t)$ be a hypersurface in $\mathbb{R}^3$ depending on time $t \in [0, T]$ for some $T \in (0, \infty]$. Let $w = (w_1(x, t), w_2(x, t), w_3(x, t))$ be a motion velocity of $\Gamma(t)$. Let $u = (u_1(x, t), u_2(x, t), u_3(x, t))$ be a relative velocity on $\Gamma(t)$.
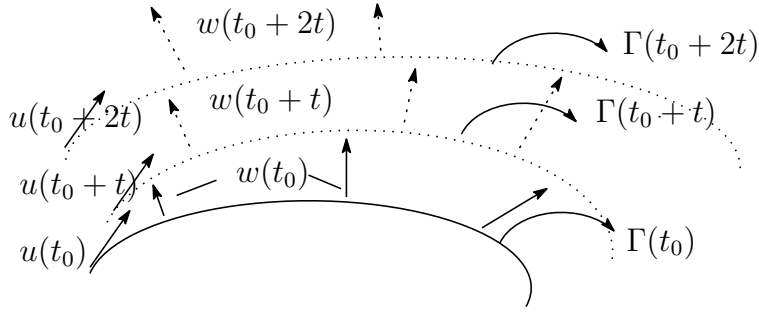
The velocity

$$v = (v_1(x, t), v_2(x, t), v_3(x, t)) := u + w$$

is called a *total velocity* of the fluid particle at x. The symbol q denotes a *total pressure* or a *pressure associated with v*. Let ρ, θ be the density and temperature of fluid on $\Gamma(t)$, respectively. Write

$$\mathcal{S}_T = \{(x, t) \in \mathbb{R}^4; \ (x, t) \in \bigcup_{0 < t < T} \left\{ \Gamma(t) \times \{t\} \right\} \}.$$

We assume that $\Gamma(t)$ is a 2-dimensional closed manifold for each fixed $t \in [0, T]$. Set

$$C^\infty(\mathcal{S}_T) = \{f : \mathcal{S}_T \to \mathbb{R}; \ f = g|_{\mathcal{S}_T} \text{ for some } g \in C^\infty(\mathbb{R}^4)\},$$
$$C_0^\infty(\mathcal{S}_T) = \{f \in C^\infty(\mathcal{S}_T); \ \text{supp } f(\cdot, t) \subset \Gamma(t)\}.$$

We suppose that

$$\rho, u, w, v, q, \theta \in C^\infty(\mathcal{S}_T).$$

Applying our energy methods (Least Action Principle and Minimum Dissipation Energy Principle), we derive

Incompressible fluid [Motion is unknown](Giga[*]-Liu[†]-K)

$$\begin{cases} \rho_0 D_t v - \mathrm{div}_\Gamma D_\Gamma^{\mu,0}(v) + \mathrm{grad}_\Gamma q + qHn = 0, \\ \mathrm{div}_\Gamma v = 0. \end{cases}$$

Incompressible fluid [Motion is given](Giga[*]-Liu[†]-K)

$$(3\text{-}1) \begin{cases} \rho_0 D_t^\Gamma v - P_\Gamma \mathrm{div}_\Gamma D_\Gamma^{\mu,0}(v) + \mathrm{grad}_\Gamma q = 0, \\ \mathrm{div}_\Gamma v = 0. \end{cases}$$

Compressible fluid [Barotropic](K)

$$\begin{cases} D_t^N \rho + \mathrm{div}_\Gamma(\rho v) = 0, \\ D_t^N(\rho v) + \mathrm{div}_\Gamma\{\rho v \otimes v - D_\Gamma^{\mu,\lambda}(v) + P_\Gamma q\} = 0, \\ q = q(\rho). \end{cases}$$

Compressible fluid [Heat transfer](K)

$$\begin{cases} D_t^N \rho + \mathrm{div}_\Gamma(\rho v) = 0, \\ D_t^N(\rho v) + \mathrm{div}_\Gamma\{\rho v \otimes v - D_\Gamma^{\mu,\lambda}(v) + P_\Gamma q\} = 0, \\ D_t^N(\rho \theta) + \mathrm{div}_\Gamma(\rho \theta v - \kappa \mathrm{grad}_\Gamma \theta) = 0, \\ q = q(\rho, \theta). \end{cases}$$

Here

$$\begin{cases} \rho, \rho_0: \text{ density,} \\ v: \text{ velocity,} \\ q: \text{ pressure ,} \\ \theta: \text{ temperature,} \\ \mu, \lambda: \text{ viscosity,} \\ \kappa: \text{ thermal diffusivity,} \\ H: \text{ mean curvature,} \\ n: \text{ unit outer normal vector,} \end{cases} \quad \begin{cases} \mathrm{div}_\Gamma: \text{ surface divergence,} \\ \mathrm{grad}_\Gamma: \text{ surface gradient,} \\ P_\Gamma: \text{ orthogonal projection to tangent,} \\ D_t: \text{ material derivative,} \\ D_t^\Gamma: \text{ surface material derivative,} \\ D_t^N: \text{ time derivative with Neumann,} \\ D_\Gamma^{\mu,\lambda}(\cdot): \text{ surface deformation tensor} \end{cases}$$

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4. Remark (Comparison with previous models)

Let us compare the system (3-1) with the Euler system derived by Arnold [2, 3] and the Navier-Stokes system introduced by Taylor [8]. If $\rho_0 = 1$, $D_\Gamma^{\mu,0}(v) \equiv 0$, and v is a tangent vector on the surface, the incompressible fluid system (3-1) is noting but the Euler system on a manifold. The system (3-2) with $\rho_0 = 1$, $D_\Gamma^{\mu,0}(v) \equiv 0$, and $P_\Gamma v = v$, is same as the Euler system on a manifold derived by Arnold. However, when $\rho_0 = 1$, $\mu > 0$, and $P_\Gamma v = v$, our model (3-1) is different from the Navier-Stokes system on a manifold introduced by Taylor. In this talk, we will talk about it if time permits.

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On an approximation scheme
for oscillatory interfacial motions

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Abstract

We present a method for approximating interfacial motions under curvature dependent acceleration. The method is a thresholding algorithm which utilizes evolution by a wave equation to obtain the desired interfacial dynamics. Oscillating interfaces can be described by curvature dependent accelerations and, from the point of view of applications, since interfaces in nature are often observed to oscillate (e.g., elastic membranes, soap bubbles, and liquid droplets), we remark this class of motions includes interesting physical phenomena. The order of convergence and numerical results of the algorithm, including an investigation of the volume preserving motions, will also be shown.

Keywords: curvature dependent acceleration, threshold dynamics, interfacial motion

1   Introduction

From cavitation, to the motion of DNA strands and soap bubbles, oscillating interfaces play an important role in applications. Mean curvature flow is a well-known interfacial motion and refers to the evolution of a hypersurface, whose normal velocity is given by the mean curvature of the interface (see for example [8]). There are several methods for computing such motions, including front-tracking and phase-field approaches; yet another approach is the so-called Bence-Merriman-Osher algorithm (BMO) [7]. Here we will focus on developing a BMO-type method for interfacial accelerations. That is, we focus on the case where the acceleration acting on the interface, in the normal direction, is given by the mean curvature (see [6]).

In particular, we consider interfacial dynamics governed by the following equation of motion:

$$A = -\kappa \boldsymbol{n}, \tag{1}$$

where A is the interfacial normal acceleration, κ is the mean curvature, and $\boldsymbol{n}$ is the unit normal vector to the interface, pointing inward. At time zero, we assume that the interface is a smooth closed hypersurface Γ_0, surrounding a region E_0, and which is accompanied by a smooth initial velocity field $\boldsymbol{v}_0$ (directed normal to the interface so that there is no tangential velocity).

Remark. For a parameterized Jordan curve, we remark that the differential equation governing the evolution can be expressed:

$$\begin{cases} \alpha''(t,s) = -\kappa(s) & (t,s) \in (0,T) \times [0,1], \\ \alpha'(t=0,s) = v_0(s) & s \in [0,1], \\ \alpha(t=0,s) = \gamma(s) & s \in [0,1]. \end{cases}$$

2 Threshold dynamics of a vector valued wave equation

Our approximation method is threshold dynamical, also known as a BMO algorithm [7], and is written in terms of a single vector-valued wave equation. Choosing a small time step Δt, we find a function $\boldsymbol{u} : \Omega \to \mathbf{R}^{N-1}$ solving:

$$\begin{cases} \boldsymbol{u}_{tt} = c^2 \Delta \boldsymbol{u} & \text{in } (0, \Delta t) \times \Omega, \\ \frac{\partial \boldsymbol{u}}{\partial \nu} = 0 & \text{on } (0, \Delta t) \times \partial \Omega, \\ \boldsymbol{u}_t(0, x) = \boldsymbol{v}_0 & \text{in } \Omega, \\ \boldsymbol{u}(t = 0, x) = 2\boldsymbol{z}_\epsilon^0 - \boldsymbol{z}_\epsilon^{-\Delta t} & \text{in } \Omega, \end{cases} \tag{2}$$

where N denotes the number of phases, Ω is a smooth bounded domain in $\mathbf{R}^d$, $\boldsymbol{v}_0$ is an appropriate initial velocity, c^2 is a wave speed depending on the dimension d (see the remark at the end of this section), and the initial condition is defined by the following signed-distance interpolated vector field:

$$\boldsymbol{z}_\epsilon^t(x) = \sum_{i=1}^N \boldsymbol{p}_i \chi_{\{d_i^t(x) > \epsilon/2\}} + \frac{1}{\epsilon}\left(\frac{\epsilon}{2} + d_i^t(x)\right)\boldsymbol{p}_i \chi_{\{-\epsilon/2 \leq d_i^t(x) \leq \epsilon/2\}}. \tag{3}$$

Here, $d_i^t(x)$ denotes the signed distance function to the boundary of phase i at location x and time t:

$$d_i^t(x) = \begin{cases} \inf_{y \in \partial P_k^t} ||x - y|| & \text{if } x \in P_k^t, \\ -\inf_{y \in \partial P_k^t} ||x - y|| & \text{otherwise,} \end{cases} \tag{4}$$

where χ_E is the characteristic function of the set E, and $\boldsymbol{p}_i$ is the ith coordinate vector of a *regular simplex* in $\mathbf{R}^{N-1}$, $i = 1, ..., N$. We remark that, when $N = 2$, equation (2) is scalar.

At time Δt, in a process called *thresholding*, each phase region $P_i^{\Delta t}$ is evolved as follows:

$$P_i^{\Delta t} = \{x \in \Omega : \boldsymbol{u}(\Delta t, x) \cdot \boldsymbol{p}_i \geq \boldsymbol{u}(\Delta t, x) \cdot \boldsymbol{p}_k, \text{for all } k \in \{1, ..., N\}\}. \tag{5}$$

The vector field $\boldsymbol{z}_\epsilon^0$ is then constructed using the boundaries of these sets and the initial condition for the wave equation is updated. The procedure is then repeated and one can show that the geometric evolution of the interface is approximately (1) in the case $d = 2$. In particular, we have the following (the details can be found in [5]):

Theorem. *Let $\epsilon > 0$ be sufficiently small, k denote a time step, and $2\boldsymbol{z}_\epsilon^k - \boldsymbol{z}_\epsilon^{k-1}$ evolve by the wave equation with $\boldsymbol{v}_0 = 0$ and $c^2 = 2$, for a time $t > 0$. Then the boundaries of the phase regions evolve with normal acceleration $v' = -\kappa + O(t)$ and velocity $v(t) = v(0) - t\kappa + O(t^2))$.*

Remark 1. The hyperbolic mean curvature flow considered here can be interpreted as describing the motion of a thin membrane whose evolution is determined by a force, given in the direction normal to the membrane, proportional to the mean curvature at each point. Considering the motion of a scalar valued membrane, one has the nonlinear wave equation:

$$\begin{cases} u_{tt} = \text{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) & \text{in } (0, T) \times \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } (0, \Delta t) \times \partial \Omega, \\ u_t(0, x) = v_0 & \text{in } \Omega, \\ u(t = 0, x) = u_0 & \text{in } \Omega, \end{cases}$$

where $T > 0$ denotes the final time and Ω is a given domain. When the curvature (given by the minimal surface operator) is approximated by the Laplacian, one has the wave equation. With this interpretation, the oscillating membrane motion considered here is no longer scalar, but in the normal direction.

Remark 2. The $c^2 = 2$ wave speed may seem unnatural at first glance, but this is easily clarified by considering a simple example (also see [5]). In polar coordinates, the solution of the wave equation with a radially symmetric initial condition can be written as follows:

$$u(\tau, r) = 2r_{n-1} - r_{n-2} - r + c^2 \int_0^\tau \int_0^s \left(-\frac{d-1}{r}\right) dt\, ds + O(\tau^3),$$

where τ denotes time, and r_n denotes the distance from the origin to the interface at time step n. Since the interface is embedded in the zero level set of u we observe that, in the case of planar curves, one should take $c^2 = 2$. Moreover, when $d = 3$, one should take $c^2 = 1$.

3 The signed-distance interpolated vector field

Here we will record a few properties satisfied by the initial vector field. To this end, let the interface be given by the boundaries of regions labelled P_k, $k = 1, ..., N$, and consider locations where the distance from the interface is less than or equal to $\epsilon/2 \geq 0$, as shown in figure 1. Let
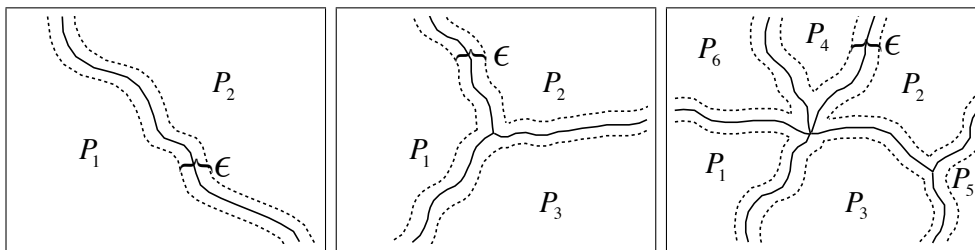

Figure 1: (Left) Two-phase interface. (Center) A junction. (Left) A network.

$\boldsymbol{u}$ denote a corresponding vector field, constructed by (3), at a time step n. Figure 2 shows two illustrations of this vector field, for $\epsilon = 0$ and $\epsilon > 0$.

Away from the interfaces, at any location x, we remark that there is precisely one signed distance with a positive distance. Moreover, on interfaces, at junctions, and networks (see figure 1) these distances are zero.

Loosely speaking, this vector field is built so that the thresholding procedure (5) generates signed distance functions near the interface. This establishes a vector-type BMO approach to the algorithms in [1], where it is carified that the 1-d profile along the direction normal to the interface can be any odd, monotone function.

Let $\boldsymbol{u}$ evolve by the vector valued wave equation. Well-known representation formulae for the solutions are available and we remark that the evolution at a location x depends only on information local in both time and space. This fact allows one to formally construct multiphase algorithms. In particular, one can evolve N independent wave equations, one for each phase. Moreover, as follows, this approach is equivalent to the current threshold dynamics. That is, one considers the following scalar valued function:

$$w^j = \epsilon \frac{N-1}{N} \left[\boldsymbol{u} \cdot \boldsymbol{p}^j + \frac{2-N}{2(N-1)} \right].$$

Then, when x is close to a single interface, say ∂P_k, we have

$$\boldsymbol{u}(x) = \frac{1}{\epsilon}\left(\frac{\epsilon}{2} + d_k^n(x)\right)\boldsymbol{p}^k + \frac{1}{\epsilon}\left(\frac{\epsilon}{2} + d_i^n(x)\right)\boldsymbol{p}^i.$$

When $\boldsymbol{p}^j$ is equal to one of $\boldsymbol{p}^k$ or $\boldsymbol{p}^i$ (without loss of generality we can assume $\boldsymbol{p}^j = \boldsymbol{p}^k$), one has

$$w^j = \epsilon\frac{N-1}{N}\left[\frac{d_k^n(x)}{\epsilon} - \frac{d_i^n(x)}{\epsilon(N-1)}\right] = d_k^n(x),$$

for sufficiently small ϵ. As we have shown, near interfaces (in ϵ neighborhoods), the local profile
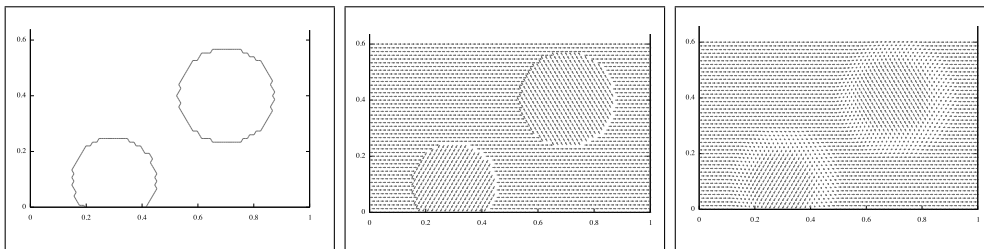

Figure 2: (Left) The initial configurations. (Center) The vector field, $\epsilon = 0$. (Right) The signed-distance vector field, $\epsilon > 0$.

in the direction of the outer normal is given by a signed distance function. Moreover, the interpolated vector field can be used in computing multiphase motions and allows one to investigate the volume preserving multiphase motions, as in [4]. By solving the vector-type wave equation with this initial condition, we can also show that the evolution via our algorithm agrees with that obtained by independently solving wave equations. Computing the acceleration and the Laplacian, we have

$$w_{tt}^j - \Delta w^j = \epsilon\frac{N-1}{N}\left[\boldsymbol{u}_{tt} - \Delta\boldsymbol{u}\right]\cdot\boldsymbol{p}_j = 0,$$

since $\boldsymbol{u}$ solves the vector type wave equation.

4   A numerical analysis

In this section, we will examine the numerical error of our method, as applied to a test problem. In particular, we check the behavior of the method as applied to the hyperbolic mean curvature flow of a circle with zero initial velocity. Then the radius of the circle satisfies the following ordinary differential equation:

$$\begin{cases} r'' = -\frac{1}{r} \\ r'(0) = 0, r(0) = r_0, \end{cases} \tag{6}$$

where the initial condition is obtained from a target radius $r_0 = 0.2381$. The output is a list of interface locations$\{(P_i^x, P_i^y)\}_i$ and we take the relative error as a numerical approximation to the follow value:

$$\frac{\int_0^{t^*} |r_u(t) - r(s)|^2\, ds}{\int_0^{t^*} |r(s)|^2\, ds}, \tag{7}$$

Resolution	Error (%)	Order
8×8	0.13	-
16×16	0.12	0.85
32×32	0.11	0.18
64×64	0.08	0.38
128×128	0.04	0.93
256×256	0.02	1.06

Table 1: Error table for the test problem (6).

where r_u is the average distance along the zero level set of the solution to the wave equation to the origin, and t^* is the extinction time of the numerical solution. The results are displayed in table 1 and figure 3 shows the corresponding evolutions.

Our computations make use of a Delaunay triangulation of the domain, and we remark that the finite element method's approximation of the initial condition is also a source of error. This is particularly evident on coarse grids. Moreover, when keeping the time step to grid size ratio fixed, it is not possible to set the number of curves until the radius obtained by the finite element method vanishes. That is, the method produces a time-discrete evolution (a set of curves) whose length becomes zero at an indeterminable time. Also, since the test problem corresponds to a two phase evolution, one can take the interpolation parameter larger (here, $\epsilon = 0.52$), without altering the evolution (one just needs to take the interpolation distance larger than the diameter of a few elements).
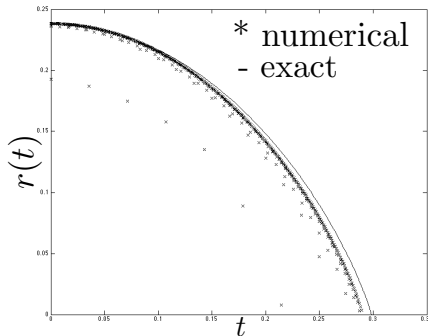

Figure 3: Evolutions corresponding to the error table. Dots correspond to numerical solutions, and the exact solution is shown as a curve.

5   Numerical experiments

Our thresholding approach has the advantage that there is no need to calculate curvatures, and singularities are implicitly handled by the partial differential equation. Moreover, it is extremely simple to implement (one need only solve the wave equation). We have constructed the corresponding numerical method and figure 4 below shows evolution under (1), with zero initial velocity, as obtained by our algorithm. Although it is rather difficult to observe in the figure, we note the oscillation of the interface. Figure 5 shows the numerical evolution of four phase regions.
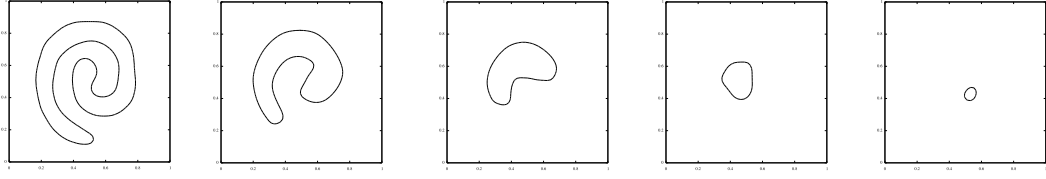
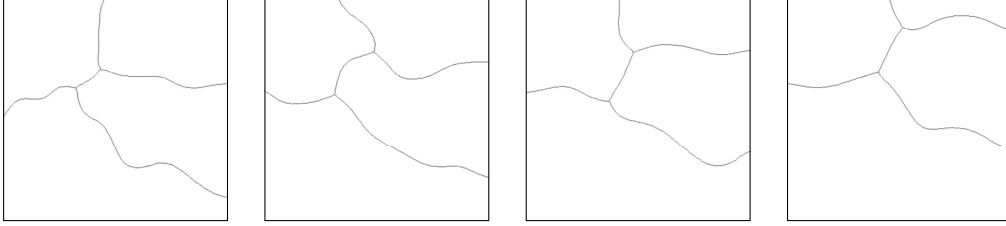**Figure 4:** Evolution by hyperbolic mean curvature flow (time is from left to right).

Figure 5: Multiphase evolution by HMCF (time is from left to right).

5.1 Investigation of volume preserving motions

By use of a hyperbolic *minimizing movements*, we are also able to formally compute volume preserving motions. In the two phase case, we formulate the problem as finding a function $u \in H^1(\Omega)$ to minimize functionals of the following type:

$$\mathcal{F}_n(u) = \int_\Omega \left(\frac{|u - 2u_{n-1} + u_{n-2}|^2}{2\tau^2} + \frac{|\nabla u|^2}{2} \right) dx + \frac{1}{\tilde{\epsilon}} \left| V - \int_\Omega \chi_{\{u>0\}} dx \right|^2, \tag{8}$$

where $\tau > 0$ is the time discretization, u_{n-1}, u_{n-2} are given functions (constructed using the initial velocity), and $\tilde{\epsilon} > 0$ is the penalty parameter for the volume constraint.

Functional values are approximated by means of the finite element method, where we use $\mathbf{P}^1$ finite elements. Accordingly, over each element e, u has the following form:

$$u(x, y)|_e = \alpha x + \beta y + \gamma,$$

where α, β and γ are real numbers. The finite element method approach to minimization of functionals involving penalties is to approximate the infinite dimensional problem (8) by a finite dimensional (although the number of variables is large) minimization:

$$\mathcal{F}_n(u) \approx \sum_{j=1}^M \int_{e_j} \left(\frac{\left|u^j - 2u_{n-1}^j + u_{n-2}^j\right|^2}{2\tau^2} + \frac{|\nabla u^j|^2}{2} \right) dx + \frac{1}{\tilde{\epsilon}} \left| V - \sum_{j=1}^M \int_{e_j} \chi_{\{u^i>0\}} dx \right|^2,$$

where M is the number of elements in the triangulation, and u^j denotes the $\mathbf{P}^1$ function over element e_j. Utilizing this approach, we are able to recover the evolution by a constrained wave equation. The numerical results are shown in figure 6, where we again note the oscillatory motion of the interface.

For the unconstrained HMCF, one can also show the convergence of the minimizing movements (see, for example [3]). In particular, one can obtain a uniform energy estimate for wave-type problems from the minimality condition. That is, with $\mathcal{F}_k(u)$ denoting the k-th functional
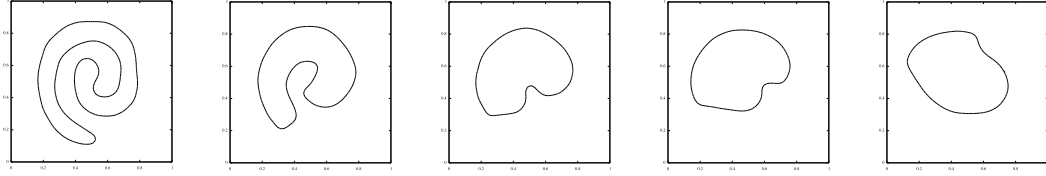
Figure 6: Evolution by volume preserving hyperbolic mean curvature flow (time is from left to right).

of the wave-type minimizing movement (without a penalty for the volume constraint), the minimality condition states that $\mathcal{F}_k(\boldsymbol{u}_k) \leq \mathcal{F}_k((1-\theta)\boldsymbol{u}_k + \theta\boldsymbol{u}_{k-1})$. Then one has

$$0 \leq \lim_{\theta \downarrow 0} \frac{\mathcal{F}_k((1-\theta)\boldsymbol{u}_k + \theta\boldsymbol{u}_{k-1}) - \mathcal{F}_k(\boldsymbol{u}_k)}{\theta} \tag{9}$$

which readily yields (see [2])

$$\left\|\frac{\boldsymbol{u}_k - \boldsymbol{u}_{k-1}}{h}\right\|_{L^2(\Omega)}^2 + C_p\left(\|\nabla\boldsymbol{u}_k\|_{L^2(\Omega)}^2 + \|\boldsymbol{u}_k\|_{L^2(\Omega)}^2\right) \leq C_0, \tag{10}$$

where, since the initial condition is a Lipschitz vector field, C_0 is finite and depends only on the initial conditions. This allows one to show the convergence of the minimizing movement.

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Shadow system approach to a 3-component Lotka-Volterra system with diffusion

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1 Introduction

The term harmful algal bloom (in short, HAB) indicates an algal bloom that has negative impacts on other organisms via the production of toxins, mechanical damage, or by other means. HAB includes different types of taxa such as dinoagellates, diatoms, and cyanobacteria (commonly known as blue-green algae). The latter are of special importance because of their potential impact on drinking or recreational waters. In fact, they can produce a variety of potent toxins called cyanotoxins (e.g. Falconer and Humpage [1]). These compounds have been found to be hepatotoxic or neurotoxic for a wide range of organisms, including humans, and several intoxication cases have been reported worldwide (Jochimsen et al. [5]). Therefore, in the recent years, the formation of toxic blooms of cyanobacteria in lakes and rivers has been causing more and more concern. Ecological evidence suggest that toxic and nontoxic species of freshwater phytoplankton hardly coexist in absence of other species. In particular, competition experiments have shown that the toxic strain of Microcystis is a very poor competitor for light (Kardinaal et al. [7]). In these experiments the toxic strain always lost the competition against the nontoxic one, even when given a strong initial advantage. Then a natural question is: how can these species survive and actually bloom? Regarding this question, it is ecologically discussed that toxin-producing Microcystis has overall an inhibitory effect on the growth of most herbivore taxa. Nevertheless, zooplankton usually grazes on both toxic and nontoxic species (Fulton and Pearl [3]). This is interesting, since the toxic or noxious chemicals produced by blue green algae may inhibit feeding and, over long term, cause mortality of zooplankton (Porter and Orcutt [10], Lampert [8], [9], Fulton and Paerl [2]). In particular, while a few species like the rotifer Brachionus calyciorus and the cladoceran Bosmina longirostris apparently make no great distinction between toxic and non toxic prey, the feeding rates of other small-bodied cladocerans, rotifers, and copepods seem to be strongly related to the toxicity of Microcystis (Fulton and Paerl [2]). These observations suggest that predator and toxic prey have an inhibitory effect on each other. Then another naive question is: can the existence of such interaction promote the spatial pattern formation and local algal blooms? This ecological question motivates us to theoretically understand the mechanism behind the formation of spatial blooms of toxic plankton. For this purpose, we proposed a two-prey (toxic and nontoxic phytoplankton)-one-predator (zooplankton) Lotka-Volterra system with diffusion in a previous paper Scotti et al. [11], under the assumptions that (i) in absence of the predator, the toxic prey is a weaker competitor for common resources than the non-toxic one, and (ii) depending on the toxicity and another parameter, the predator is more or less inclined to avoid the toxic prey. The dynamics discussed above are described by the following reaction-diffusion system:

$$
\begin{cases}
\dfrac{\partial P}{\partial t} = r_1 P \left(1 - \dfrac{P + aX}{K_1} - Z \right) + D_1 \Delta P \\[2ex]
\dfrac{\partial X}{\partial t} = r_2 X \left(1 - \dfrac{X + bP}{K_2} - dZ \right) + D_2 \Delta X \\[2ex]
\dfrac{\partial Z}{\partial t} = r_3 Z (P - \mu X - 1) + D_3 \Delta Z,
\end{cases}
\tag{1.1}
$$

where $P(t,x), X(t,x)$ and $Z(t,x)$ are respectively the population densities of the nontoxic and toxic phytoplankton and of the zooplankton, and the parameters $r_i, D_i (i = 1,2,3), K_1, K_2, a, b, d$ and μ are all positive constants. Ecologically, an important parameter in (1.1) is μ, which is the rate of *toxicity*. We assume that the predation rate of the toxic prey d decreases in μ. Here we simply specify $d = d(\mu) = 1/(1 + \mu)$. The ecological explanation of (1.1) is discussed in [11].

By using suitable transformations of time and space, (1.1) can be rewritten as

$$\begin{cases} \dfrac{\partial P}{\partial t} = P\left(1 - \dfrac{P + aX}{K_1} - Z\right) + \Delta P \\[2mm] \dfrac{\partial X}{\partial t} = rX\left(1 - \dfrac{X + bP}{K_2} - d(\mu)Z\right) + \sigma\Delta X \\[2mm] \dfrac{\partial Z}{\partial t} = RZ(P - \mu X - 1) + D\Delta Z, \end{cases} \tag{1.2}$$

where $r = r_2/r_1$, $R = r_3/r_1$, $\sigma = D_2/D_1$ and $D = D_3/D_1$ are respectively the ratios of the growth rates and the diffusion rates of the nontoxic phytoplankton and the toxic phytoplankton and/or the zooplankton.

2 A mathematical model

In this paper, simply assuming that $r = 1$, $K_1 = K_2 = K$ and $\sigma = 1$ in (1.2), we consider the one dimensional system of (1.2) in a finite interval $0 < x < L$, that is,

$$\begin{cases} P_t = P\left(1 - \dfrac{P + aX}{K} - Z\right) + P_{xx} \\[2mm] X_t = X\left(1 - \dfrac{X + bP}{K} - d(\mu)Z\right) + X_{xx} \qquad t > 0, \ 0 < x < L \\[2mm] Z_t = RZ(P - \mu X - 1) + DZ_{xx}, \end{cases} \tag{2.1}$$

where a, b, K, μ, R and D are positive constants. We treat (2.1) under the Neumann boundary conditions

$$P_x = X_x = Z_x = 0, \qquad t > 0, \ x = 0, \ L \tag{2.2}$$

and the initial conditions

$$P(0,x) = P_0(x) \geqq 0, \ X(0,x) = X_0(x) \geqq 0, \ Z(0,x) = Z_0(x) \geqq 0, \quad 0 \leqq x \leqq L. \tag{2.3}$$

For the system (2.1), we impose the following assumptions:

(**A1**) $\qquad K > 1$,

which implies that predator and nontoxic prey coexist in the absence of toxic prey.

(**A2**) $\qquad a < 1 < b$,

which implies that in the absence of predator, the nontoxic prey is a superior competitor for resources with respect to the toxic one. Assumption (**A1**) corresponds to the coexistence of two species of (nontoxic) phytoplankton and zooplankton (e.g. Hutchinson [4]). Assumption (**A2**) is based on the idea that toxic strains are eventually outcompeted by nontoxic ones (e.g. Kardinaal et al. [7], Lampert [8]).

For (2.1) with (2.2), we easily find that the spatially constant equilibrium solution $E_3 = (1, 0, (K-1)/K)$ exists for any $\mu > 0$. Instead of (**A1**), we assume K to satisfy
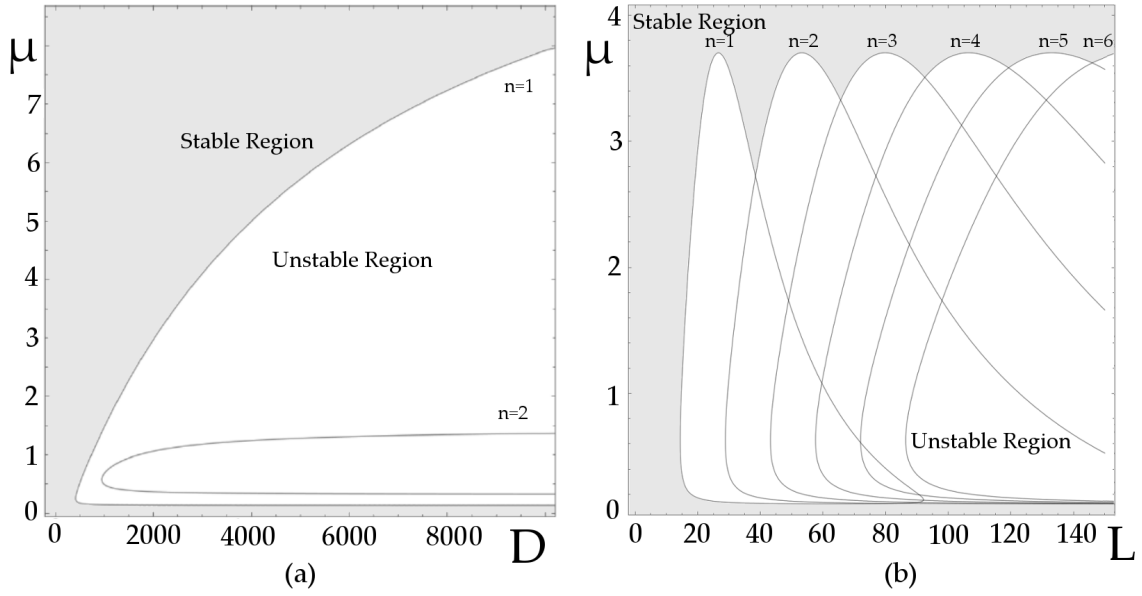
Figure 2.1 Bifurcation curves of E_4 in the (D, μ)-plane ($L = 30$) (a) and the (L, μ)-plane ($D = 2500$) (b), where $a = 0.95$, $b = 1.2$, $K = 2.9$ and $R = 0.43$. The curve n corresponds to the n-mode bifurcations, where the zero solution of the linearized problem of (2.1) with (2.2) around E_4 destabilizes under the nth eigenmode $\cos(n\pi x/L)$ perturbation.
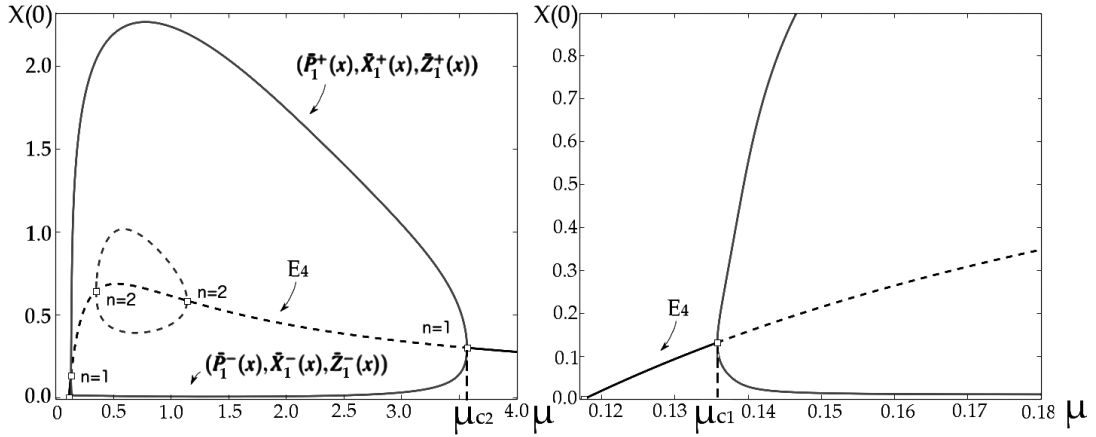

Figure 2.2 Global structure of equilibrium solutions of (2.1) with (2.2) when μ is varied, where $L = 30$, $D = 2500$. The other parameters are the same as the ones in Figure 2.1. Solid (resp. dashed) lines represent stable (resp. unstable) equilibrium solutions of (2.1) with (2.2). The right figure is a magnification of the left one where μ is close to μ_{c1}.

(A3) $K > b$.

Putting $\mu_c = (b-1)/(K-b)$, we know that E_3 is stable for $0 < \mu < \mu_c$, while it is unstable for $\mu_c < \mu$. Furthermore, when μ increases, a spatially positive constant equilibrium solution of (2.1) with (2.2), say, $E_4 = (\bar{P}_\mu, \bar{X}_\mu, \bar{Z}_\mu)$ super-critically bifurcates from E_3 at $\mu = \mu_c$, that is, E_4 exists for $\mu > \mu_c$. Here we assume R suitably large to require that E_4 is stable for $\mu > \mu_c$ in the ODEs corresponding to (2.1) in the absence of diffusion. In addition to (A2) - (A3), we assume μ to satisfy
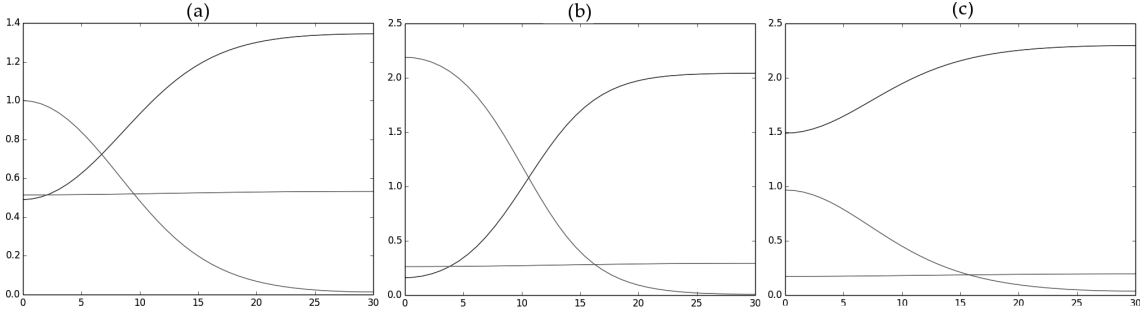
(A4) $\mu > \mu_c$.

Figure 2.3 1-mode equilibrium solutions $(\bar{P}_1^+(x), \bar{X}_1^+(x), \bar{Z}_1^+(x))$ of (2.1) with (2.2) for (a) $\mu = 0.15$, (b) $\mu = 0.5$ and (c) $\mu = 3.1$. The other parameters are the same as the ones in Figure 2.1 and $D = 2500$. Here $\bar{P}_1^+, \bar{X}_1^+$ and $\bar{Z}_1^+$ are drawn in blue, green and red colors, respectively.
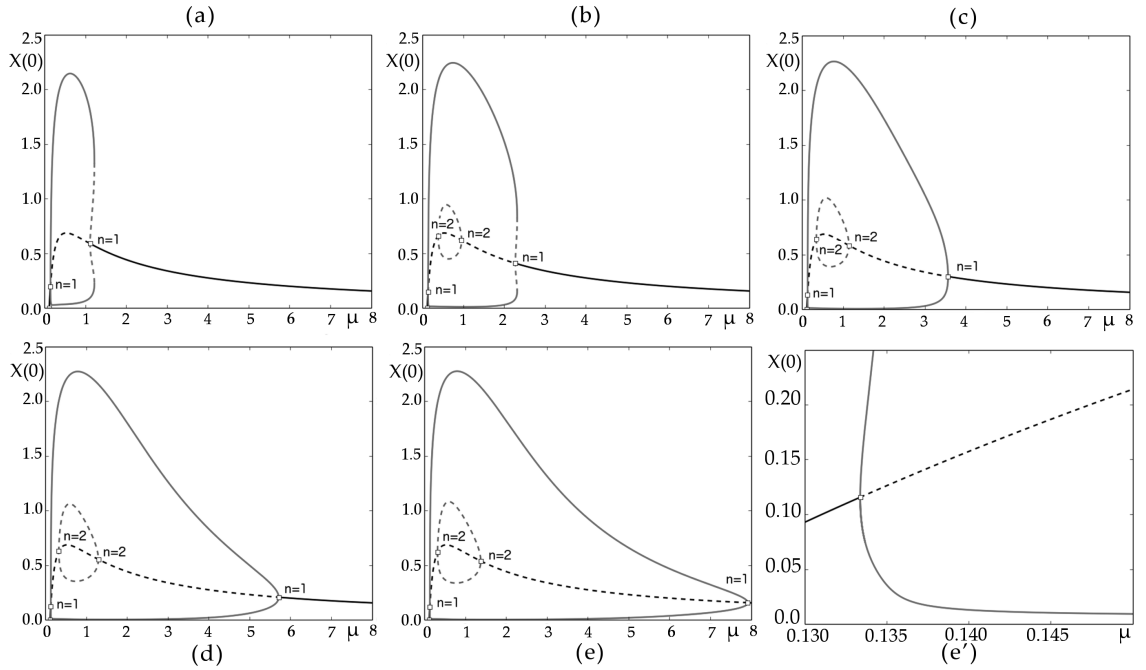

Figure 2.4 Dependency of D on the global structures of equilibrium solutions of the system (2.1) with (2.2) when $L = 30$. (a) $D = 800$, (b) $D = 1500$, (c) $D = 2500$, (d) $D = 5000$, (e) $D = 10000$ and (e′) is a magnification of (e) around $\mu = \mu_{c1}$. The other parameters are the same as the ones in Figure 2.1. Solid (resp. dashed) lines represent stable (resp. unstable) equilibrium solutions of (2.1) with (2.2).

Here we note that E_4 is not necessarily stable in (2.1) with (2.2), that is, the stability of E_4 depends on μ, R, D and L. Figures 2.1(a) and (b) show respectively the linearized stability of E_4 in the (D, μ) plane ($L = 30$) and (L, μ) plane ($D = 2500$) for suitably fixed a, b, K and R ([11]). When $\mu (> \mu_c)$ is small, E_4 is stable for any fixed D (or L), while when μ is suitably large, E_4 loses its stability as D (or L) increases. This destabilization is called diffusion-induced instability as stated by Turing ([12]). In fact, suppose that D is suitably large (for instance, $D = 2500$). By suing AUTO, we can show the global structure of equilibrium solutions of (2.1) with (2.2) when μ is varied (see Figure 2.2). This figure shows the occurrence of two bifurcation values of μ, say μ_{c1} and μ_{c2} ($\mu_{c1} < \mu_{c2}$), at which spatially non-constant 1-mode equilibrium solutions, say $(\bar{P}_1^\pm(x), \bar{X}_1^\pm(x), \bar{Z}_1^\pm(x))$ super-critically bifurcate from E_4, and exist and are stable for $\mu_{c1} < \mu < \mu_{c2}$, where $(\bar{P}_1^+(x), \bar{X}_1^+(x), \bar{Z}_1^+(x))$ is shown in Figure 2.3 and $(\bar{P}_1^-(x), \bar{X}_1^-(x), \bar{Z}_1^-(x))$

is given by $(\bar{P}_1^+(L-x), \bar{X}_1^+(L-x), \bar{Z}_1^+(L-x))$. Here a 1-mode equilibrium solution stands for a new non-constant equilibrium solution which appears via the 1-mode bifurcation. These numerical results indicate that a HAB (stable non-constant equilibrium solutions with large amplitude) does not appear for either small or large μ, while it appears for intermediate μ.

We now address the following question: *can we show the existence and stability of such non-constant equilibrium solutions of (2.1) and (2.2) analytically?* One of the ways is to assume that D is rather large in (2.1) so that the problem (2.1) with (2.2) and (2.3) becomes simpler, because one can expect $Z(t,x)$ to be spatially homogeneous. In Figure 2.4, we show the dependency of D on the structures of equilibrium solutions of (2.1) with (2.2) when μ is globally varied. We notice that they do not qualitatively change if D becomes larger and larger, as shown in Figures 2.4(c) - (e). This result motivates us to study the limiting system of (2.1) with (2.2) as $D \to \infty$ in order to discuss the existence and stability of non-constant equilibrium solutions.

3 The shadow system

We formally derive the limiting system from (2.1) with (2.2) as $D \to +\infty$. We first start by dividing the third equation of (2.1) by D so that we obtain

$$\frac{1}{D}Z_t = \frac{R}{D}Z(P - \mu X - 1) + Z_{xx}. \tag{3.1}$$

If we assume that Z_t, P, X and Z remain bounded as $D \to +\infty$, then (3.1) implies that the limit of Z, say, ξ satisfies

$$\xi_{xx} = 0. \tag{3.2}$$

It follows from (3.2) and the boundary conditions (2.2) that ξ must be spatially constant. On the other hand, integrating the equation of Z in (2.1) on the interval $[0,L]$ with respect to x, we get

$$\frac{\partial}{\partial t} \int_0^L Z dx = \int_0^L RZ(P - \mu X - 1)dx \tag{3.3}$$

by the boundary conditions (2.2) and then

$$L\xi_t = R\xi \left(\int_0^L P dx - \mu \int_0^L X dx - L \right),$$

that is,

$$\xi_t = R\xi \left(\frac{1}{L} \int_0^L P dx - \frac{\mu}{L} \int_0^L X dx - 1 \right)$$

by (3.2) and (3.3). Consequently when $D \to +\infty$, we formally obtain the following limiting system, which is called a *shadow system*, of (2.1) with (2.2) for the unknowns $(P(t,x), X(t,x), \xi(t))$:

$$\begin{cases} P_t = P\left(1 - \dfrac{P+aX}{K} - \xi\right) + P_{xx} \\[2mm] X_t = X\left(1 - \dfrac{X+bP}{K} - d(\mu)\xi\right) + X_{xx}, \qquad t > 0,\ 0 < x < L \\[2mm] \xi_t = R\xi\left(\dfrac{1}{L}\int_0^L P dx - \dfrac{\mu}{L}\int_0^L X dx - 1\right) \end{cases} \tag{3.4}$$

with

$$(P_x, X_x)(t,0) = (0,0) = (P_x, X_x)(t,L), \qquad t > 0. \tag{3.5}$$

In order to obtain equilibrium solutions of (3.4) with (3.5), we first assume ξ to be suitably fixed and consider the first two equations of (3.4) with (3.5), which are the well known

competition-diffusion system for two species. If we could find an equilibrium solution $(\bar{P}(x;\xi), \bar{X}(x;\xi))$ of the system and substitute it into the third equation of (3.4), we obtain the equation of ξ only

$$H(\xi) \equiv \frac{1}{L} \int_0^L \bar{P}dx - \frac{\mu}{L} \int_0^L \bar{X}dx - 1 = 0. \tag{3.6}$$

If we can find $\bar{\bar{\xi}}$ to satisfy (3.6), we get an equilibrium solution $(\bar{P}(x;\bar{\bar{\xi}}), \bar{X}(x;\bar{\bar{\xi}}), \bar{\bar{\xi}})$ of the shadow system (3.4) with (3.5).

In order to obtain equilibrium solutions of the competition-diffusion system for two species, we can apply the results by Kan-on [6]. For this purpose, we introduce a new parameter $\varepsilon = 1/L^2$ into (3.4) with (3.5) to normalize the interval $[0, L]$ to the unit one $[0, 1]$. Then, the shadow system (3.4) with (3.5) is rewritten to the rescaled normal system with a parameter ε as follows:

$$\begin{cases} P_t = P\left(1 - \dfrac{P + aX}{K} - \xi\right) + \varepsilon P_{xx} \\[2mm] X_t = X\left(1 - \dfrac{X + bP}{K} - d(\mu)\xi\right) + \varepsilon X_{xx}, & t > 0,\ 0 < x < 1 \\[2mm] \xi_t = R\xi\left(\displaystyle\int_0^1 Pdx - \mu \int_0^1 Xdx - 1\right) \end{cases} \tag{3.7}$$

with

$$(P_x, X_x)(t, 0) = (0, 0) = (P_x, X_x)(t, 1), \qquad t > 0. \tag{3.8}$$

In this section, we fix μ arbitrarily to satisfy $\mu > \mu_c$ and ignore the μ-dependency of equilibrium solutions of the shadow system (3.4) with (3.5).

3.1 2-component competition system of (3.7) with (3.8) for fixed ξ

In this subsection, we look for equilibrium solutions of the following 2-component competition system of (3.7) with (3.8) for a fixed ξ when ε is a free parameter:

$$\begin{cases} P_t = P\left(1 - \dfrac{P + aX}{K} - \xi\right) + \varepsilon P_{xx} \\[2mm] X_t = X\left(1 - \dfrac{X + bP}{K} - d(\mu)\xi\right) + \varepsilon X_{xx}, & t > 0,\ 0 < x < 1 \\[2mm] (P_x, X_x)(t, 0) = (0, 0) = (P_x, X_x)(t, 1), & t > 0. \end{cases} \tag{3.9}$$

Here we note that, depending on the parameter values a, b, K, μ and ξ, the dynamics of solutions of (3.9) possesses three different cases, (a) mono stability, (b) bistability and (c) coexistence, as shown in Figure 3.1. Since our concern is to find non-constant equilibrium solutions of (3.9), we restrict values of the parameters a, b, K, μ and ξ to satisfy the case (b) in Figure 3.1, because this is the only case for which such solutions exist (Zhou and Pao [13], Kan-On [6]). Figure 3.1(b) is equivalent to assume

(A5) $ab > 1$,

which is identical to $0 < \dfrac{1-a}{1-ad(\mu)} < \dfrac{b-1}{b-d(\mu)} < 1$. We define the interval I by

$$I = (\underline{\xi}, \bar{\xi}) \quad \text{with} \quad \underline{\xi} = \frac{1-a}{1-ad(\mu)} \quad \text{and} \quad \bar{\xi} = \frac{b-1}{b-d(\mu)} \tag{3.10}$$
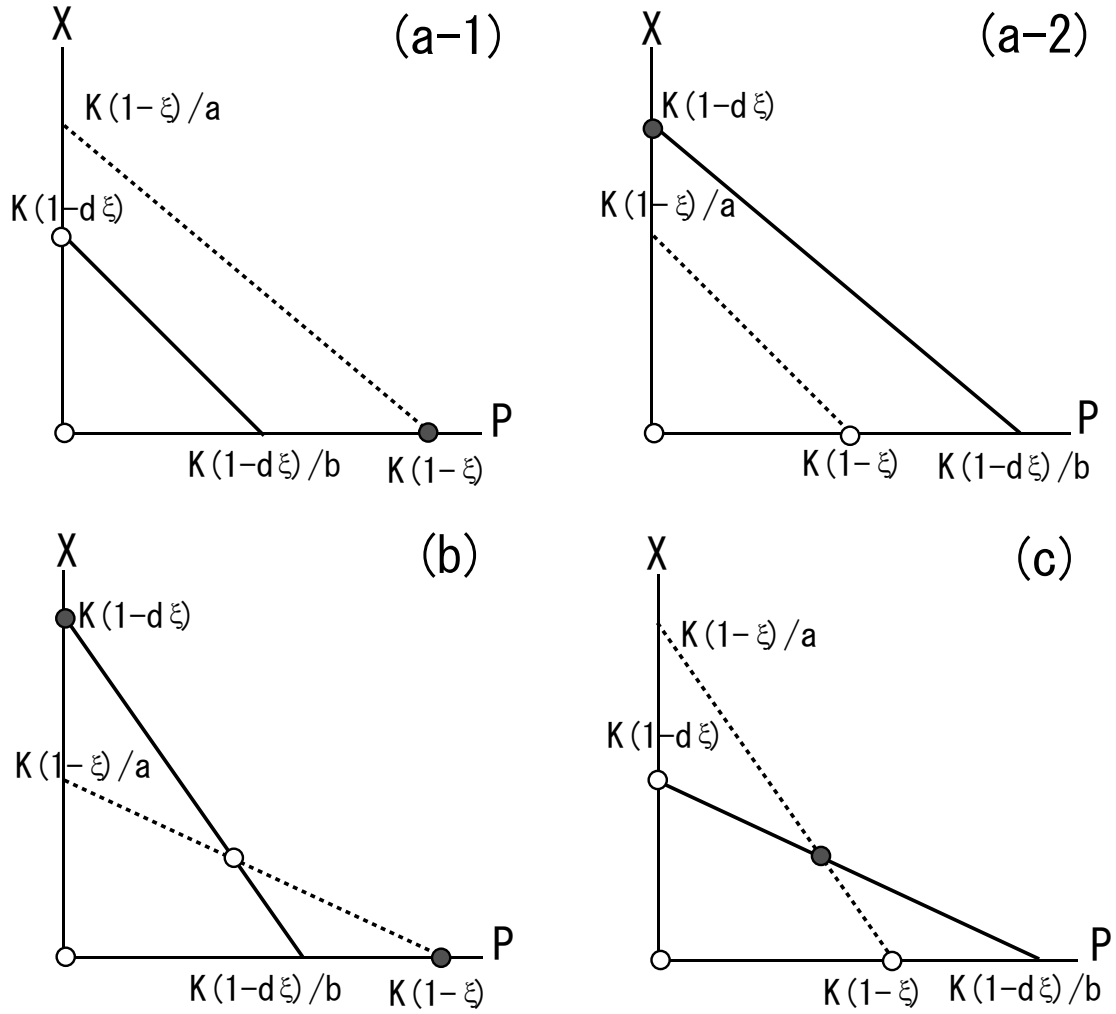
Figure 3.1 Three different structures of the nullclines of (3.9) with $d = d(\mu) = 1/(1+\mu)$. (a-1) *P*-monostability, (a-2) *X*-monostability, (b) bistability and (c) coexistence. The red and white circles stand for stable and unstable equilibrium solutions of (3.9), respectively.

and assume ξ to satisfy $\xi \in I$. Then the system (3.9) has a positive constant equilibrium solution $(\bar{P}(\xi), \bar{X}(\xi))$ which is explicitly represented as

$$\bar{P}(\xi) = \frac{1-a-(1-ad(\mu))\xi}{1-ab}, \quad \bar{X}(\xi) = \frac{1-b+(b-d(\mu))\xi}{1-ab}. \tag{3.11}$$

Remark 1 We easily find that

$$\lim_{\xi \to \underline{\xi}}(\bar{P}(\xi), \bar{X}(\xi)) = \left(\frac{1-d(\mu)}{b-d(\mu)}, 0\right) \quad \text{and} \quad \lim_{\xi \to \bar{\xi}}(\bar{P}(\xi), \bar{X}(\xi)) = \left(0, \frac{1-d(\mu)}{1-ad(\mu)}\right).$$

Taking ε as a bifurcation parameter, we look for non-constant equilibrium solutions of (3.9), which are bifurcated from the constant solution $(\bar{P}(\xi), \bar{X}(\xi))$. By simple calculation, it can be easily checked that for arbitrarily fixed $\xi \in I$, the linearized operator of (3.9) around $(\bar{P}(\xi), \bar{X}(\xi))$ has the zero eigenvalue at $\varepsilon = \varepsilon_0^n(\xi) = (1/n\pi)^2 Q_0(\xi)$ with

$$Q_0(\xi) = \frac{-(\bar{P}(\xi)+\bar{X}(\xi))+\sqrt{(\bar{X}(\xi)-\bar{P}(\xi))^2+4ab\bar{P}(\xi)\bar{X}(\xi)}}{2K} \tag{3.12}$$
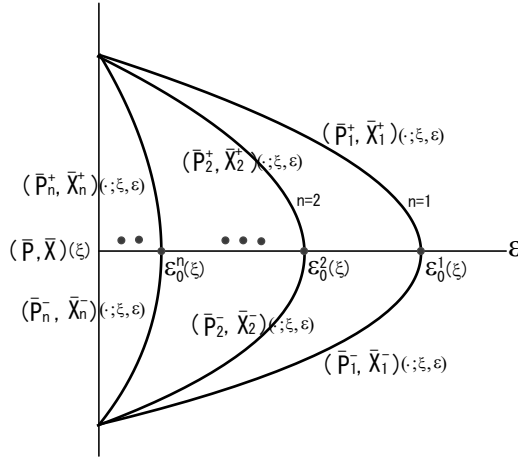
Figure 3.2 Schematic global structure of the constant and non-constant equilibrium solutions $(\bar{P}(\xi),\bar{X}(\xi))$ and $(\bar{P}_n^{\pm}(x;\xi,\varepsilon),\bar{X}_n^{\pm}(x;\xi,\varepsilon))$ $(n=1,2,\cdots)$ of (3.9) when ε is varied.

and the corresponding eigenfunction $\mathbf{u}\cos(n\pi x)$ for each $n\in\mathbf{N}$, where $\mathbf{u}=(u_1,u_2)^T$ is an eigenvector of the matrix

$$A(\xi)\equiv\begin{bmatrix}\pi^2\varepsilon_0^1(\xi)+\bar{P}(\xi)/K & a\bar{P}(\xi)/K\\b\bar{X}(\xi)/K & \pi^2\varepsilon_0^1(\xi)+\bar{X}(\xi)/K\end{bmatrix}\tag{3.13}$$

corresponding to the zero eigenvalue. Here we used the relation $\varepsilon_0^n(\xi)=\varepsilon_0^1(\xi)/n^2$ for each $n\in\mathbf{N}$ and normalized u_1 to 1 so that $u_2=-(K\pi^2\varepsilon_0^1(\xi)+\bar{P}(\xi))/(a\bar{P}(\xi))$.

Then the following results on equilibrium solutions of (3.9) are given, as shown in Figure 3.2:

Proposition 1 (Kan-on [6]) *. Assume (A2) - (A5) and consider the problem (3.9) with arbitrarily fixed $\xi\in I$. For each $n\in\mathbf{N}$, we have the followings:*
(i) *For any $\varepsilon\in(0,\varepsilon_0^n(\xi))$, there exists a pair of spatially non-constant positive equilibrium solutions $(\bar{P}_n^{\pm}(x;\xi,\varepsilon),\bar{X}_n^{\pm}(x;\xi,\varepsilon))$ of (3.9).*
(ii) *$(\bar{P}_n^{\pm}(x;\xi,\varepsilon),\bar{X}_n^{\pm}(x;\xi,\varepsilon))$ are C^1-class functions from $(0,\varepsilon_0^n(\xi))$ to $C^2[0,1]\times C^2[0,1]$.*
(iii) *(3.9) has no positive equilibrium solutions other than $(\bar{P}(\xi),\bar{X}(\xi))$ and $(\bar{P}_j^{\pm}(x;\xi,\varepsilon),\bar{X}_j^{\pm}(x;\xi,\varepsilon))$ $(j=1,2,\cdots,n)$ for any $\varepsilon\in[\varepsilon_0^{n+1}(\xi),\varepsilon_0^n(\xi))$.*
(iv) $\lim\limits_{\varepsilon\to\varepsilon_0^n(\xi)}(\bar{P}_n^{\pm}(x;\xi,\varepsilon),\bar{X}_n^{\pm}(x;\xi,\varepsilon))=(\bar{P}(\xi),\bar{X}(\xi))$.

3.2 Existence of non-constant equilibrium solutions of the shadow system

Under the assumptions (**A2**) - (**A5**), we have the following main theorem by virtue of Proposition 1:

Theorem 1 *Assume (A2) - (A5). For each $n\in\mathbf{N}$, the following results hold :*
(i) *For any $\varepsilon\in(0,\varepsilon_0^n(\bar{\xi}^*))$, there exists at least a pair of spatially non-constant n-mode equilibrium solutions $(\bar{P}_n^{\pm}(x;\varepsilon),\bar{X}_n^{\pm}(x;\varepsilon),\bar{\xi}_n^{\pm}(\varepsilon))$ of (3.7) with (3.8).*
(ii) *$(\bar{P}_n^{\pm}(x;\varepsilon),\bar{X}_n^{\pm}(x;\varepsilon))$ are C^1-class functions from $(0,\varepsilon_0^n(\bar{\xi}^*))$ to $C^2[0,1]\times C^2[0,1]$.*
(iii) $\lim\limits_{\varepsilon\to\varepsilon_0^n(\bar{\xi}^*)}(\bar{P}_n^{\pm}(x;\varepsilon),\bar{X}_n^{\pm}(x;\varepsilon),\bar{\xi}_n^{\pm}(\varepsilon))=(\bar{P}^*,\bar{X}^*,\bar{\xi}^*)$.

Unfortunately, this theorem does not give any information on the number of non-constant equilibrium solutions of (3.7) with (3.8). It really depends on the functional form of $H(\xi;\varepsilon)$ when ε is suitably fixed.

In this talk, we will give a simple proof of this theorem and discuss the dependency of spatially non-constant 1-mode equilibrium solutions on the toxicity μ. Furthermore, we will consider the bifurcation property and the stability of these non-constant equilibrium solutions.

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Title:
Weak solutions to the Navier-Stokes initial boundary value problem
in exterior domains and initial data in the $L(3,\infty)$-space

Abstract:
The talk concerns the existence of weak solutions to the Navier-Stokes initial
boundary value problem in exterior domains:

$$u_t + u \cdot \nabla u - \Delta u = -\nabla \pi_u, \ \nabla \cdot u = 0, \ \text{in } (0,t) \times \Omega$$
$$u(t,x) = 0 \text{ on } (0,T) \times \partial\Omega, \ u(0,x) = u_\circ(x) \text{ on } \{0\} \times \Omega, \tag{1}$$

where $v_t := \frac{\partial v}{\partial t}$ e $v \cdot \nabla w := v_k \frac{\partial w}{\partial x_k}$. In literature we find the existence of
a weak solution for data $u_\circ \in L^2(\Omega)$. Recently, an existence theorem for
data $u_\circ$ which only have finite Dirichlet norm is proved in paper [10]. This
result appears interesting because it realizes a bridge between unsteady so-
lutions and the class of existence of time periodic solutions and of stationary
solutions of the Navier-Stokes equations. In this note, we study the exis-
tence of weak solutions corresponding to data $L(3,\infty)(\Omega)$ (Lorentz's space).
Beyond the connection, again, with the steady (see [8]) and time periodic
solutions (see [12]), we look for further developments concerning the space-
time asymptotic behaviors of the solutions to problem (1), already proposed
for the first time in [2, 3] for the weak solutions of the L^2-theory. Employing
some ideas contained in paper [10], we are able to prove the following result:

Theorem 0.1. *Let $u_\circ(x) \in \mathbb{L}(3,\infty)(\Omega)$. There exists a weak solution u
to problem (1) such that, for all $T > 0$, $u \in L^4(0,T;\mathbb{L}(3,\infty)(\Omega))$, $u \in
L^2(0,T;J^{1,2}_{\ell oc}(\overline{\Omega}))$ and, for some $T_0 > 0$, $u \in C(0,T_0;\mathbb{L}(3,\infty)(\Omega))$. Finally,
for such a weak solution the Leray structure theorem holds.*

$\mathbb{L}(3,\infty)(\Omega)$:=completion of $\mathscr{C}_0(\Omega)$ in $L(3,\infty)(\Omega)$ and $J^{1,2}_{\ell oc}(\overline{\Omega}) := \{u :
u \in W^{1,2}_{\ell oc}(\overline{\Omega}), \gamma_{\partial\Omega}(u) = 0 \text{ and } \nabla \cdot u = 0\}$ ($\mathscr{C}_0(\Omega)$ is the vector space of
hydrodynamics test functions).

To better explain our results, we make a short review of some known
results on the topic.

1 Weak solutions L^2-theory.

We begin introducing the definition of Hopf weak solution whose set is de-
noted by $\mathcal{H}$

Definition 1.1. *A field u, such that $u : (0,T) \times \Omega \to \mathbb{R}^n$, is said a weak solution to problem (1) if*

   i) for all $T > 0$, $u \in L^2(0,T; J^{1,2}(\Omega))$,

$$\|u(t)\|_2^2 + 2 \int\limits_0^t \|\nabla u(\tau)\|_2^2 d\tau \leq \|u_\circ\|_2^2, \; \forall t > 0,$$

   ii) $\lim\limits_{t \to 0} \|u(t) - u_\circ\|_2 = 0$,

   iii) u satisfies the equation:

$$\int\limits_0^T \Big[(u, \varphi_\tau) - (\nabla u, \nabla \varphi) + (u \cdot \nabla \varphi, u) \Big] d\tau + (u_\circ, \varphi(0)) = 0,$$

$$\textit{for all } \varphi \in C_0^1([0,T); \mathscr{C}_0(\Omega)).$$

In the following we denote (*a priori*) the subsets $\mathcal{L}$ and $\mathcal{CKN}$ of the set $\mathcal{H}$ of weak solutions, one due to Leray in [9] and the other to Caffarelli-Kohn-Nirenberg in [1], that we formalize by means of the following definitions.

Definition 1.2. *A field $u : (0,T) \times \Omega \to \mathbb{R}^3$ is said a Leray weak solution to problem (1) if u is a weak solution and the following inequality holds: (energy relation in strong form)*

$$\|u(t)\|_2^2 + 2 \int\limits_0^t \|\nabla u(\tau)\|_2^2 d\tau \leq \|u(s)\|_2^2, \; t \geq s, \; \text{for } s = 0 \text{ and a.e. in } s \geq 0, \tag{2}$$

$$\Omega \subseteq \mathbb{R}^d, \; d = 3.$$

Definition 1.3. *A field $u : (0,T) \times \Omega \to \mathbb{R}^3$ is said a suitable weak solution to problem (1) if u is a weak solution and the following inequality holds:*
for some field $\pi_u : (0,T) \times \Omega \to \mathbb{R}$ (pressure field) there holds:

$$\int\limits_\Omega |u(t)|^2 \phi(t) dx + 2 \int\limits_s^t \int\limits_\Omega |\nabla u|^2 \phi \, dx d\tau \leq \int\limits_\Omega |u(s)|^2 \phi(s) dx$$

$$+ \int\limits_s^t \int\limits_\Omega |u|^2 (\phi_\tau + \Delta \phi) dx d\tau + \int\limits_s^t \int\limits_\Omega (|u|^2 + 2\pi_u) u \cdot \nabla \phi \, dx d\tau, \tag{3}$$

for all $t \geq s$, for $s = 0$ and a.e. in $s \geq 0$, and for all nonnegative $\phi \in C_0^\infty(\mathbb{R} \times \mathbb{R}^3)$.

The difference in the weak formulations is connected to the three different energy relations. Actually they imply a different regularity and also different qualitative properties, as, for example, stability properties.

The existence of weak solutions $(\mathcal{H})$ has been proved by Hopf for an initial boundary value problem in an arbitrary domain $\Omega \subseteq \mathbb{R}^n$. Instead, in the cases of $\mathcal{L}$ and of $\mathcal{CKN}$ some restrictions are needed on the domain Ω. We refer to papers [11] and [5], respectively, for such questions.

The sets $\mathcal{H} \supseteq \mathcal{L} \supseteq \mathcal{CKN}$ of weak solutions are not empty as proved by means of the following theorems that we state in the case of the Cauchy problem of the Navier-Stokes system, but whose validity is known for the IBVP (1):

Theorem 1.1. [Hopf's existence theorem] - *For all $u_\circ \in J^2(\Omega)$ there exists a weak solution $u(t,x)$ such that, for all $\psi \in J^2(\Omega)$, $(u(t),\psi)$ is a continuous function of t.*

Theorem 1.2. *For all $u_\circ \in J^2(\mathbb{R}^3)$ there exists a Leray weak solution such that*

$$
\int_s^t \Big[(u,\zeta_\tau) - (\nabla u,\nabla\zeta) + (u\cdot\nabla\zeta,u) + (\pi_u,\nabla\zeta)\Big]d\tau + (u(s),\zeta(s)) \tag{4}
$$
$$
= (u(t),\zeta(t)),
$$

for all $\zeta \in C_0^1([0,T];C_0^\infty(\mathbb{R}^3))$, with $\pi_u \in L^r(0,T;L^s(\mathbb{R}^3))$, provided that $s \in (1,3]$ and $r := \frac{1}{3}\frac{2s}{s-1}$.

Theorem 1.3. *For all $u_\circ \in J^2(\mathbb{R}^3)$, there exists a suitable weak solution such that*

$$
\int_s^t \Big[(u,\zeta_\tau) - (\nabla u,\nabla\zeta) + (u\cdot\nabla\zeta,u) + (\pi_u,\nabla\zeta)\Big]d\tau + (u(s),\zeta(s)) \tag{5}
$$
$$
= (u(t),\zeta(t)),
$$

for all $\zeta \in C_0^1([0,T];C_0^\infty(\mathbb{R}^3))$, with $\pi_u \in L^r(0,T;L^s(\mathbb{R}^3))$, provided that $s \in (1,3]$ and $r := \frac{1}{3}\frac{2s}{s-1}$.

Now we would like to recall the regularity properties and the space time pointwise stability of a weak solution belonging to $\mathcal{CKN}$.

2 Regularity and space time pointwise stability of a $\mathcal{CKN}$ solution.

We limit ourselves to the Cauchy problem. A first contribute concerning the regularity of a weak solution $u \in \mathcal{L}$ is due to Leray in [9], that is the so called *structure theorem*:

Theorem 2.1. *If $u \in \mathcal{L}$, there exists a set of indices $\mathcal{A} \subseteq \mathbb{N}$ such that, for all $l \in \mathcal{A}$, u is a regular inside the disjoint cylinders $(\Theta_l, T_l) \times \mathbb{R}^3$ with $T_l \leq \infty$. Moreover, setting $O = \cup_{l \in \mathcal{A}} (\Theta_l, T_l)$, then, the complement of O in $\mathbb{R}^+$ has finite $\frac{1}{2}$-Hausdorff measure, and $\vartheta := \sup\limits_{l} \Theta_l \leq c\|u_\circ\|_2^4$.*

We give some comments on the above statement.

REMARK 2.1.

The regularity claimed in the theorem is the classical one that we deduce (*a priori*) locally in time for initial data belonging [1] to $J^{1,2}(\Omega)$.

For all $l \in \mathcal{A}$, $u \in \mathcal{L}$ is regular in $(\Theta_l, T_l) \times \mathbb{R}^3$, the cardinality of $\mathcal{A}$ is at most the one of $\mathbb{N}$ and $(\vartheta, \infty) \subset O$.

In particular, the regularity allows to consider $\|u(t)\|_\infty$ and to give an asymptotic behavior of the kind:

$$\|u(t)\|_\infty \leq c\|u_\circ\|_2 t^{-\frac{3}{4}}, t \geq \vartheta.$$

By several authors Theorem 2.1 has been extended to domains $\Omega \subset \mathbb{R}^3$. □

If in the structure theorem the partial regularity is given on $(0, \infty)$, in paper [1] the partial regularity is given also in the space. In this connection we start giving:

Definition 2.1. *We say that (t, x) is a singular point for a weak solution (u, π_u) if $u \notin L^\infty$ in any neighborhood of (t, x); the remaining points, where $u \in L^\infty(I(t, x))$ for some neighborhood $I(t, x)$, are called regular points. We denote by $\tilde{S}$ the set of all the possible singular points (t, x).*

We introduce the parabolic cylinders

$$Q_r(t, x) := \{(\tau, y) : t - r^2 < \tau < t \text{ and } |y - x| < r\}.$$

---

[1] $J^{1,2}(\Omega) :=$ completion of $\mathscr{C}_0(\Omega)$ in $W^{1,2}(\Omega)$.

Definition 2.2 (Parabolic Hausdorff measure). *For all $X \subset \mathbb{R} \times \mathbb{R}^3$, and $\eta \geq 0$ we set*

$$\mathscr{P}^\eta(X) := \lim_{\varepsilon \to 0} \mathscr{P}_\varepsilon^\eta(X),$$

where

$$\mathscr{P}_\varepsilon^\eta(X) := \inf \Big\{ \sum_{i=1}^\infty r_i^\eta : X \subset \bigcup_{i=1}^\infty Q_{r_i}, r_i < \varepsilon \Big\}$$

Denoted by $\mathscr{H}$ the usual Hausdorff measure, we get $\mathscr{H}^\eta(X) \leq c(\eta)\mathscr{P}^\eta(X)$. The following result holds:

Theorem 2.2. *For any suitable weak solution u ($\mathcal{CKN}$) to the Navier-Stokes Cauchy problem the associated singular set satisfies $\mathscr{P}^1(\widetilde{S}) = 0$.*

In the recent paper [3] the following result concerning the poitwise stability is proved:

Theorem 2.3. *Let $u_\circ \in J^2(\mathbb{R}^3)$ and, for some $\alpha \in [1,3)$ and $R_0 > 0$, let be*

$$|u_\circ(x)| \leq U_\circ |x|^{-\alpha}, \ for \ |x| > R_0. \tag{6}$$

Let (u, π_u) be a suitable weak solution to the Navier-Stokes Cauchy problem. Then, there exists a constant $M \geq 1$ such that

$$|u(t,x)| \leq c(u_\circ)|x|^{-\alpha}, \ for \ all \ (t,x) \in (0,\infty) \times \mathbb{R}^3 \setminus B_{MR_0}, \tag{7}$$

where M is independent of $u_\circ$.

Corollary 2.1. *For a solution of Theorem 2.3 we get:*

$$k \in [0, \alpha], \quad |u(t,x)| \leq c(u_\circ)c|x|^{-\alpha+k}t^{-\frac{k}{2}}, (t,x) \in (0,\infty) \times (\mathbb{R}^3 \setminus B_{MR_0}). \tag{8}$$

Some comments on the last statements.

REMARK 2.2.

Under our assumptions, estimates (7) are an improvement of regularity results furnished in [1] in the case of the Cauchy problem (see Theorem D and its Corollary p.820 of [1]).

Theorem 2.3 is an improvement of the ones given in [4] in very stronger hypotheses.

As far as we know such results are the first for weak solutions.

3 The assumption $u_\circ \in \mathbb{L}(3, \infty)$ and its connection with the pointwise stability: Theorem 0.1.

In Theorem 2.3 we have a double assumption on the initial data $u_\circ$. Actually, we require that initial data belongs to J^2 and, for some $\alpha \in [1, 3)$ and $R_0 > 0$, $|u_\circ(x)| \leq U_\circ |x|^{-\alpha}$, for $|x| > R_0$. Of course, the assumptions play a different role. The former hypothesis is made in order to obtain the existence (of a suitable weak solution) with no restriction on the data and the latter to achieve the spatial decay from $t > 0$. However, roughly speaking, since Ω is exterior, for $\alpha \in [1, \frac{3}{2}]$, the former assumption seems to clash with the latter. Therefore, we would like to make our assumptions more natural and coherent. The following one

$$u_\circ \in \mathbb{L}(3, \infty) \tag{9}$$

seems to be the suitable assumption for all $\alpha \in [1, 3)$.

We would like to make precise a concept: assumption (9) is not connected with questions of scaling invariant data and regularity of solutions, for which our theorem does not give contributions. The regularity of a solution of Theorem 0.1 is just the one of a suitable weak solution given in section 2.

Why is it made assumption (9) and not $u_\circ \in \mathbb{L}(q, \infty)$ for some $q \in (1, 3)$? Because assumption (9) gives the opportunity to discuss the nonlinear term in a way useful for our aims.

Our Theorem 0.1 is just the first step, that is the existence of solutions assuming data (9) in exterior domains. Under the assumption (6), the problem of pointwise stability will be object of a forthcoming paper jointly with F. Crispo. As matter of course, under assumptions (6) and (9) the asymptotic result (7) will become sharp as in the case of small data. Assumption (9) includes the case of $\alpha = 1$ with no further assumption. Moreover, this case achieves a special interest for its analogy with steady and time periodic solutions [6, 12] and, very recent, for questions of stability [7]. However, our result is just a starting point, since our solution tends to zero at infinity while the steady problem can be considered for $v_\infty \neq 0$ too.

Finally, we insert a further comment on Theorem 0.1. We perform a solution u as

$$u := V + v,$$

where $V \in \mathbb{L}(3, \infty)$ uniformly in $t > 0$ and $v \in J^2$ uniformly in $t > 0$. Taking into account that the unbounded nature of the domain makes the two different bounds not comparable, a solution of Theorem 0.1 has not a metric with respect to which it is bounded and depends on the initial data

uniformly in time, as in the case of the L^2-theory or in the case of small data in $\mathbb{L}(3, \infty)$ or $\mathbb{L}^3$. Hence, although the result is restricted to the exterior of a ball, the pointwse stability achieves further interest since it realizes a dependence on the initial data for this kind of weak solution.

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