Dualities and evolutes of fronts in hyperbolic 2-space and de Sitter 2-space

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Abstract

We consider the differential geometry of evolutes of singular curves in hyperbolic 2-space and de Sitter 2-space. Firstly, as an application of the basic Legendrian duality theorems, we give the definitions of fronts in hyperbolic 2-space or de Sitter 2-space, respectively. We also give the notions of moving frames along the fronts. By using the moving frames, we define the evolutes of spacelike fronts and timelike fronts, and investigate the geometric properties of these evolutes. As results, these evolutes can be viewed as wavefronts from the viewpoint of Legendrian singularity theory. At last, we study the relationships among these evolutes.

Keywords: evolute; spacelike front; timelike front; hyperbolic 2-space; de Sitter 2-space.

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1 Introduction

This paper is a part of our research projects about the differential geometry of evolutes of singular curves in different ambient space forms. Notions of evolutes (or, focal sets) of regular curves in Euclidean plane or 3-space are classical topics in differential geometry. As well known, the evolute of a regular plane curve is defined as the locus of the center of osculating circle of the original curve. The radius of the osculating circle of a regular plane curve is $1/\kappa$, where κ is the curvature of the curve. Unfortunately, if the curve is not regular at some point, then we can not define the evolute at this point as the classical way. The second author of this paper, however, had presented an alternative method for the studying of evolutes of singular curves in Euclidean plane [4, 5]. They firstly define frontals (or fronts) in Euclidean plane and Legendrian curves (or Legendrian immersions) in the unit tangent bundle of $\mathbb{R}^2$. The differential geometric properties of the frontal is studied in [3]. The most difference between a regular curve and a frontal is that the frontal might exist singular points. A key tool for studying of the frontal is so called moving frame defined in the unit tangent bundle. By using the moving frame, they defined a pair of smooth functions like as the curvature of a regular curve and called the pair the

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curvature of the Legendrian curve. As results, the existence and uniqueness for the Legendrian curve which take this curvature as the associated curvature are established. Furthermore, they used the moving frame and the curvature of the Legendrian immersion to give a new definition of an evolute of the front. We remark that this new definition on the evolute is consistent with the classical one when the curve is a regular curve. They also studied the evolutes of smooth curves in sphere 2-space as applications of this method [13]. In this paper, we proceed with this way to investigate the evolutes of smooth curves in hyperbolic 2-space and de Sitter 2-space. As it to be expected, the situation presents certain peculiarities when compared with the Euclidean case and the sphere case. For instance, in our case the evolutes of smooth spacelike curves in hyperbolic 2-space (or, de Sitter 2-space) are split into hyperbolic 2-space and de Sitter 2-space.

The organization of this paper is as follows. In §2, we prepare some basic notions on regular curves in hyperbolic 2-space and de Sitter 2-space, respectively. We first review the properties of the evolutes of regular curves in hyperbolic 2-space which developed by S. Izumiya and his collaborators in [8] (for regular hypersurfaces case please see [9, 10]). Moreover, by using a similar way to that of [8], we study the evolutes of spacelike regular curves and timelike regular curves in de Sitter 2-space, respectively. In §3, we give a brief review on the basic Legendrian duality theorems appeared in [2, 6, 7]. Especially, Δ_1-duality and Δ_5-duality are very helpful in this paper. We define the spacelike frontals (or, spacelike fronts) in hyperbolic 2-space and de Sitter 2-space, and spacelike Legendrian curves (or, spacelike Legendrian immersions) by using the Δ_1-duality. We also use the Δ_5-duality to define the timelike frontals (or, timelike fronts) in de Sitter 2-space and timelike Legendrian curves (or, timelike Legendrian immersions). The basic properties of the frontals are discussed. We give the definitions of evolutes of spacelike fronts in hyperbolic 2-space, spacelike and timelike fronts in de Sitter 2-space in §4, respectively. We also study the geometric properties of these evolutes in this section. In the last section, §5, we investigate the relationships among the evolutes of these fronts in hyperbolic 2-space and de Sitter 2-space.

We shall assume throughout the whole paper that all maps and manifolds are C^∞ unless the contrary is explicitly stated.

2 The evolutes of regular curves

In this section, we investigate the basic properties of evolutes of regular curves in hyperbolic 2-space or de Sitter 2-space, respectively. Firstly, we will prepare some notions in Minkowski space. For details of Lorentzian geometry, see [12].

Let $\mathbb{R}^3 = \{(x_1, x_2, x_3) | x_i \in \mathbb{R},\ i = 1, 2, 3\}$ be a 3-dimensional vector space. For any vectors $\boldsymbol{x} = (x_1, x_2, x_3)$ and $\boldsymbol{y} = (y_1, y_2, y_3)$ in $\mathbb{R}^3$, the *pseudo scalar product* of $\boldsymbol{x}$ and $\boldsymbol{y}$ is defined to be $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = -x_1y_1 + x_2y_2 + x_3y_3$. We call $(\mathbb{R}^3, \langle , \rangle)$ the *Minkowski* 3*-space* and write $\mathbb{R}^3_1$ instead of $(\mathbb{R}^3, \langle , \rangle)$.

We say that a non-zero vector $\boldsymbol{x}$ in $\mathbb{R}^3_1$ is *spacelike*, *lightlike* or *timelike* if $\langle \boldsymbol{x}, \boldsymbol{x} \rangle > 0$, $\langle \boldsymbol{x}, \boldsymbol{x} \rangle = 0$ or $\langle \boldsymbol{x}, \boldsymbol{x} \rangle < 0$ respectively. The *norm* of the vector $\boldsymbol{x} \in \mathbb{R}^3_1$ is defined by $\|\boldsymbol{x}\| = \sqrt{|\langle \boldsymbol{x}, \boldsymbol{x} \rangle|}$.

For any $\boldsymbol{x} = (x_1, x_2, x_3), \boldsymbol{y} = (y_1, y_2, y_3) \in \mathbb{R}^3_1$, we define a vector $\boldsymbol{x} \wedge \boldsymbol{y}$ by

$$\boldsymbol{x} \wedge \boldsymbol{y} = \begin{vmatrix} -\boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix},$$

where $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ is the canonical basis of $\mathbb{R}^3_1$. For any $\boldsymbol{w} \in \mathbb{R}^3_1$, we can easily check that

$$\langle \boldsymbol{w}, \boldsymbol{x} \wedge \boldsymbol{y} \rangle = \det(\boldsymbol{w}, \boldsymbol{x}, \boldsymbol{y}),$$

so that $\boldsymbol{x} \wedge \boldsymbol{y}$ is pseudo-orthogonal to both $\boldsymbol{x}$ and $\boldsymbol{y}$. Moreover, if $\boldsymbol{x}$ is a timelike vector, $\boldsymbol{y}$ is a spacelike vector and $\boldsymbol{x} \wedge \boldsymbol{y} = \boldsymbol{z}$, then by a straightforward calculation we have

$$\boldsymbol{z} \wedge \boldsymbol{x} = \boldsymbol{y},\ \boldsymbol{y} \wedge \boldsymbol{z} = -\boldsymbol{x}.$$

If $\boldsymbol{x}$ is a spacelike vector, $\boldsymbol{y}$ is a timelike vector and $\boldsymbol{x} \wedge \boldsymbol{y} = \boldsymbol{z}$, then by a straightforward calculation we have

$$\boldsymbol{z} \wedge \boldsymbol{x} = -\boldsymbol{y},\ \boldsymbol{y} \wedge \boldsymbol{z} = \boldsymbol{x}.$$

If both $\boldsymbol{x}$, $\boldsymbol{y}$ are a spacelike vectors and $\boldsymbol{x} \wedge \boldsymbol{y} = \boldsymbol{z}$, then by a straightforward calculation we have

$$\boldsymbol{z} \wedge \boldsymbol{x} = -\boldsymbol{y},\ \boldsymbol{y} \wedge \boldsymbol{z} = -\boldsymbol{x}.$$

For a vector $\boldsymbol{v} \in \mathbb{R}^3_1$ and a real number c, we define the *plane* with the pseudo-normal $\boldsymbol{v}$ by

$$P(\boldsymbol{v}, c) = \{\boldsymbol{x} \in \mathbb{R}^3_1 | \langle \boldsymbol{x}, \boldsymbol{v} \rangle = c\}.$$

We call $P(\boldsymbol{v}, c)$ a *timelike plane*, *spacelike plane* or *lightlike plane* if $\boldsymbol{v}$ is *spacelike, timelike or lightlike*, respectively.

We define *hyperbolic* 2*-space* by

$$H^2(-1) = \{\boldsymbol{x} \in \mathbb{R}^3_1 \mid \langle \boldsymbol{x}, \boldsymbol{x} \rangle = -1\},$$

de Sitter 2*-space* by

$$S^2_1 = \{\boldsymbol{x} \in \mathbb{R}^3_1 \mid \langle \boldsymbol{x}, \boldsymbol{x} \rangle = 1\},$$

(*open*) *lightcone* at the origin by

$$LC^* = \{\boldsymbol{x} \in \mathbb{R}^3_1 \setminus \{\boldsymbol{0}\} | \langle \boldsymbol{x}, \boldsymbol{x} \rangle = 0\}.$$

We consider a curve given by the intersection of $H^2(-1)$ (or, S^2_1) with the plane $P(\boldsymbol{v}, c)$ as follows:

$$HP(\boldsymbol{v}, c) = H^2(-1) \cap P(\boldsymbol{v}, c)\ (or, DP(\boldsymbol{v}, c) = S^2_1 \cap P(\boldsymbol{v}, c))$$

and call it the *hyperbolic* (*or, de Sitter*) *ellipse*, *hyperbolic* (*or, de Sitter*) *parabola* or *hyperbolic* (*or, de Sitter*) *hyperbola* if $\boldsymbol{v}$ is timelike, lightlike or spacelike, respectively.

We study the evolutes of regular curves in hyperbolic 2-space or de Sitter 2-space, respectively, in the following.

2.1 The evolutes of regular curves in hyperbolic 2-space

We firstly give a brief review on differential geometry of regular curves in $H^2(-1)$. For details please see [8]. Let $\boldsymbol{\gamma}_h : I \to H^2(-1)$ be a regular curve, we have $||\dot{\boldsymbol{\gamma}}_h(t)|| \neq 0$, where $\dot{\boldsymbol{\gamma}}_h(t) = (d\boldsymbol{\gamma}_h/dt)(t)$. Denoted by $\boldsymbol{t}_h(t) = \dot{\boldsymbol{\gamma}}_h(t)/||\dot{\boldsymbol{\gamma}}_h(t)|| \in S^2_1$ the unit spacelike tangent vector. We can define a unit spacelike vector $\boldsymbol{e}_h(t)$ by $\boldsymbol{e}_h(t) = \boldsymbol{\gamma}_h(t) \wedge \boldsymbol{t}_h(t)$ and call it the *normal vector* of $\boldsymbol{\gamma}_h$, then we have a pseudo orthonormal frame $\{\boldsymbol{\gamma}_h, \boldsymbol{t}_h, \boldsymbol{e}_h\}$ of $\mathbb{R}^3_1$ along $\boldsymbol{\gamma}_h$. By the standard arguments, we can give the following *hyperbolic Frenet-Serret type formula*:

$$\begin{pmatrix} \dot{\boldsymbol{\gamma}}_h(t) \\ \dot{\boldsymbol{t}}_h(t) \\ \dot{\boldsymbol{e}}_h(t) \end{pmatrix} = ||\dot{\boldsymbol{\gamma}}_h(t)|| \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & \kappa_h(t) \\ 0 & -\kappa_h(t) & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{\gamma}_h(t) \\ \boldsymbol{t}_h(t) \\ \boldsymbol{e}_h(t) \end{pmatrix},$$

where $\kappa_h(t) = \frac{\det(\boldsymbol{\gamma}_h(t), \dot{\boldsymbol{\gamma}}_h(t), \ddot{\boldsymbol{\gamma}}_h(t))}{||\dot{\boldsymbol{\gamma}}_h(t)||^3}$, we call it the *hyperbolic geodesic curvature.* We remark that since $\boldsymbol{\gamma}_h$ is a regular curve in $H^2(-1)$, it may admit the arc length parametrization $s = s(t)$. Therefore, we can assume that $\boldsymbol{\gamma}_h(s)$ is a unit speed curve. To the convenience of calculation, however, we stick to the general parametrization in this paper.

Under the assumption that $\kappa_h(t) \neq \pm 1$, we define the evolute of $\boldsymbol{\gamma}_h$ as follows:

$$E_v(\boldsymbol{\gamma}_h) : I \to \mathbb{R}^3_1,\ E_v(\boldsymbol{\gamma}_h)(t) = \pm \frac{1}{\sqrt{|\kappa_h^2(t) - 1|}} \left(\kappa_h(t)\boldsymbol{\gamma}_h(t) + \boldsymbol{e}_h(t)\right).$$

In the case $\kappa_h^2(t) > 1$, $E_v(\boldsymbol{\gamma}_h)(t)$ is located in $H^2(-1)$, we call it the *hyperbolic evolute* of $\boldsymbol{\gamma}_h$ and denote it by $E_v^h(\boldsymbol{\gamma}_h)(t)$. If $0 \leq \kappa_h^2(t) < 1$, it is in S^2_1, we call it the *de Sitter evolute* of $\boldsymbol{\gamma}_h$ and denote it by $E_v^d(\boldsymbol{\gamma}_h)(t)$. Then we have the following proposition ([8], Proposition 4.1).

Proposition 2.1 *Suppose that $\boldsymbol{\gamma}_h : I \to H^2(-1)$ is a regular curve with $\kappa_h^2(t) \neq 1$. Then $\dot{\kappa}_h(t) \equiv 0$ if and only if $E_v^h(\boldsymbol{\gamma}_h)(t)$ or $E_v^d(\boldsymbol{\gamma}_h)(t)$ are constant vectors. Under this condition, $\boldsymbol{\gamma}_h$ is a part of a hyperbolic ellipse or a part of a hyperbolic hyperbola, respectively.*

If $\boldsymbol{v}_0 = E_v^h(\boldsymbol{\gamma}_h)(t_0)$ and $c_0 = \mp\kappa_h(t_0)/\sqrt{|\kappa_h^2(t_0) - 1|}$, then we have $\boldsymbol{\gamma}_h$ and $HP(\boldsymbol{v}_0, c_0)$ are at least 3-point contact at $\boldsymbol{\gamma}_h(t_0)$, see [8]. In this case, we call $HP(\boldsymbol{v}_0, c_0)$ the *osculating hyperbolic ellipse* (*or, osculating hyperbolic hyperbola*). Its center $\boldsymbol{v}_0$ is called the *center of hyperbolic geodesic curvature.* Therefore, the evolutes of $\boldsymbol{\gamma}_h$ is the locus of the center of hyperbolic geodesic curvature.

2.2 The evolutes of regular spacelike curves in de Sitter 2-space

We now consider the differential geometry of regular spacelike curves in S^2_1. Let $\boldsymbol{\gamma}_d : I \to S^2_1$ be a regular curve. The regular curve $\boldsymbol{\gamma}_d$ is said to be spacelike if $\dot{\boldsymbol{\gamma}}_d(t)$ is a spacelike vector at any $t \in I$, where $\dot{\boldsymbol{\gamma}}_d(t) = (d\boldsymbol{\gamma}_d/dt)(t)$. We call such curve a *spacelike curve.* Denoted by $\boldsymbol{t}_d(t) = \dot{\boldsymbol{\gamma}}_d(t)/||\dot{\boldsymbol{\gamma}}_d(t)|| \in S^2_1$ the unit spacelike tangent vector. We can define a unit timelike vector $\boldsymbol{e}_d(t)$ by $\boldsymbol{e}_d(t) = \boldsymbol{\gamma}_d(t) \wedge \boldsymbol{t}_d(t)$ and call it the *normal vector* of $\boldsymbol{\gamma}_d$, then we have a pseudo orthonormal frame $\{\boldsymbol{\gamma}_d, \boldsymbol{t}_d, \boldsymbol{e}_d\}$ of $\mathbb{R}^3_1$ along $\boldsymbol{\gamma}_d$. By the standard arguments, we can give the following *spacelike de Sitter Frenet-Serret type formula*:

$$\begin{pmatrix} \dot{\boldsymbol{\gamma}}_d(t) \\ \dot{\boldsymbol{t}}_d(t) \\ \dot{\boldsymbol{e}}_d(t) \end{pmatrix} = ||\dot{\boldsymbol{\gamma}}_d(t)|| \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & \kappa_d(t) \\ 0 & \kappa_d(t) & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{\gamma}_d(t) \\ \boldsymbol{t}_d(t) \\ \boldsymbol{e}_d(t) \end{pmatrix},$$

where $\kappa_d(t) = \frac{\det(\boldsymbol{\gamma}_d(t), \dot{\boldsymbol{\gamma}}_d(t), \ddot{\boldsymbol{\gamma}}_d(t))}{||\dot{\boldsymbol{\gamma}}_d(t)||^3}$, we call it the *spacelike de Sitter geodesic curvature.*

Under the assumption that $\kappa_d(t) \neq \pm 1$, we define the evolute of $\boldsymbol{\gamma}_d$ as follows:

$$E_v(\boldsymbol{\gamma}_d) : I \to \mathbb{R}^3_1,\ E_v(\boldsymbol{\gamma}_d)(t) = \pm \frac{1}{\sqrt{|\kappa_d^2(t) - 1|}} \left(\kappa_d(t)\boldsymbol{\gamma}_d(t) - \boldsymbol{e}_d(t)\right).$$

In the case $\kappa_d^2(t) > 1$, $E_v(\boldsymbol{\gamma}_h)(t)$ is located in S^2_1, we call it the *de Sitter evolute* of $\boldsymbol{\gamma}_d$ and denote it by $E_v^d(\boldsymbol{\gamma}_d)(t)$. If $0 \leq \kappa_d^2(t) < 1$, it is in $H^2(-1)$, we call it the *hyperbolic evolute* of $\boldsymbol{\gamma}_d$ and denote it by $E_v^h(\boldsymbol{\gamma}_d)(t)$. Then we have the following proposition.

Proposition 2.2 *Suppose that $\boldsymbol{\gamma}_d : I \to S^2_1$ be a regular spacelike curve with $\kappa_d^2(t) \neq 1$. Then $\dot{\kappa}_d(t) \equiv 0$ if and only if $E_v^h(\boldsymbol{\gamma}_d)(t)$ or $E_v^d(\boldsymbol{\gamma}_d)(t)$ are constant vectors. Under this condition, $\boldsymbol{\gamma}_d$ is a part of a de Sitter ellipse or a part of a de Sitter hyperbola, respectively.*

The proof of this proposition is similar to that of Proposition 4.1 in [8], so we omit it.

We assume that $\boldsymbol{v}_0 = E_v^h(\boldsymbol{\gamma}_d)(t_0)$ and $c_0 = \pm\kappa_d(t_0)/\sqrt{|\kappa_d^2(t_0)-1|}$, then we have $\boldsymbol{\gamma}_d$ and $DP(\boldsymbol{v}_0, c_0)$ are at least 3-point contact at $\boldsymbol{\gamma}_d(t_0)$. In this case, we call $DP(\boldsymbol{v}_0, c_0)$ the *osculating de Sitter ellipse* (*or, osculating de Sitter hyperbola*). Its center $\boldsymbol{v}_0$ is called the *center of spacelike de Sitter geodesic curvature*. Therefore, the evolutes of $\boldsymbol{\gamma}_d$ is the locus of the center of spacelike de Sitter geodesic curvature.

2.3 The evolutes of regular timelike curves in de Sitter 2-space

Finally, we consider the differential geometry of regular timelike curves in S_1^2. Let $\boldsymbol{\gamma}_T : I \to S_1^2$ be a regular curve. The regular curve $\boldsymbol{\gamma}_T$ is said to be timelike if $\dot{\boldsymbol{\gamma}}_T(t)$ is a timelike vector at any $t \in I$, where $\dot{\boldsymbol{\gamma}}_T(t) = (d\boldsymbol{\gamma}_T/dt)(t)$. We call such curve the *timelike curve.* Denoted by $\boldsymbol{t}_T(t) = \dot{\boldsymbol{\gamma}}_T(t)/||\dot{\boldsymbol{\gamma}}_T(t)|| \in H^2(-1)$ the unit timelike tangent vector. We can define a unit spacelike normal vector $\boldsymbol{e}_T(t)$ by $\boldsymbol{e}_T(t) = \boldsymbol{\gamma}_T(t) \wedge \boldsymbol{t}_T(t)$ and call it the *normal vector* of $\boldsymbol{\gamma}_T$. Then we have a pseudo orthonormal frame $\{\boldsymbol{\gamma}_T, \boldsymbol{t}_T, \boldsymbol{e}_T\}$ of $\mathbb{R}_1^3$ along $\boldsymbol{\gamma}_T$. By the standard arguments, we can give the following *timelike de Sitter Frenet-Serret type formula*:

$$\begin{pmatrix} \dot{\boldsymbol{\gamma}}_T(t) \\ \dot{\boldsymbol{t}}_T(t) \\ \dot{\boldsymbol{e}}_T(t) \end{pmatrix} = ||\dot{\boldsymbol{\gamma}}_T(t)|| \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & \kappa_T(t) \\ 0 & \kappa_T(t) & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{\gamma}_T(t) \\ \boldsymbol{t}_T(t) \\ \boldsymbol{e}_T(t) \end{pmatrix},$$

where $\kappa_T(t) = \frac{\det\left(\boldsymbol{\gamma}_T(t), \dot{\boldsymbol{\gamma}}_T(t), \ddot{\boldsymbol{\gamma}}_T(t)\right)}{||\dot{\boldsymbol{\gamma}}_T(t)||^3}$, we call it the *timelike de Sitter geodesic curvature.*

We define the evolute of $\boldsymbol{\gamma}_T$ in de Sitter space as follows:

$$E_v^d(\boldsymbol{\gamma}_T) : I \to S_1^2,\ E_v^d(\boldsymbol{\gamma}_T)(t) = \pm\frac{1}{\sqrt{\kappa_T^2(t)+1}}\left(\kappa_T(t)\boldsymbol{\gamma}_T(t) - \boldsymbol{e}_T(t)\right).$$

We call it the *spacelike de Sitter evolute* of $\boldsymbol{\gamma}_T$. Then we have the following proposition.

Proposition 2.3 *Suppose that $\boldsymbol{\gamma}_T : I \to S_1^2$ be a regular timelike curve. Then $\dot{\kappa}_T(t) \equiv 0$ if and only if $E_v^d(\boldsymbol{\gamma}_T)(t)$ is a constant vector. Under this condition, $\boldsymbol{\gamma}_T$ is a part of a de Sitter hyperbola.*

The proof of this proposition is also similar to that of Proposition 4.1 in [8], so we omit it.

We assume that $\boldsymbol{v}_0 = E_v^d(\boldsymbol{\gamma}_T)(t_0)$ and $c_0 = \pm\kappa_T(t_0)/\sqrt{\kappa_T^2(t_0)+1}$ then we have $\boldsymbol{\gamma}_T$ and $DP(\boldsymbol{v}_0, c_0)$ are at least 3-point contact at $\boldsymbol{\gamma}_T(t_0)$. In this case, we call $DP(\boldsymbol{v}_0, c_0)$ the *osculating de Sitter hyperbola.* Its center $\boldsymbol{v}_0$ is called the *center of timelike de Sitter geodesic curvature.* Therefore, the spacelike de Sitter evolutes of $\boldsymbol{\gamma}_T$ is the locus of the center of timelike de Sitter geodesic curvature.

3 The frontals in hyperbolic 2-space and de Sitter 2-space

In this section, we consider the differential geometry of smooth curves in hyperbolic 2-space or de Sitter 2-space, respectively. If the curve have singular points, we can not define the pseudo orthonormal Frenet frame at these singular points. We also can not use the Frenet-Serret type formula to study the properties of the original curve. In order to overcome this difficulty, we take advantage of the way developed by the second author of this paper in [3] instead of the classical way. We give the detailed descriptions about this way as follows.

3.1 The spacelike frontals in hyperbolic 2-space

We firstly consider the differential geometry of curves in $H^2(-1)$. Let $\boldsymbol{\gamma}_h : I \to H^2(-1)$ be a smooth curve. We call $\boldsymbol{\gamma}_h$ the *spacelike frontal* in $H^2(-1)$, if there exists a smooth mapping $\boldsymbol{\gamma}_h^d : I \to S_1^2$, such that the pair $(\boldsymbol{\gamma}_h, \boldsymbol{\gamma}_h^d) : I \to \Delta_1$ satisfies $(\boldsymbol{\gamma}_h(t), \boldsymbol{\gamma}_h^d(t))^*\theta = 0$ for all $t \in I$. Here

$$\Delta_1 = \{(\boldsymbol{v}, \boldsymbol{w}) \mid \langle \boldsymbol{v}, \boldsymbol{w} \rangle = 0 \ \} \subset H^2(-1) \times S_1^2$$

is the Δ_1-duality defined by S. Izumiya in [6, 7] and θ is a canonical contact 1-form on Δ_1. The condition $(\boldsymbol{\gamma}_h(t), \boldsymbol{\gamma}_h^d(t))^*\theta = 0$ is equivalent to $\langle \dot{\boldsymbol{\gamma}}_h(t), \boldsymbol{\gamma}_h^d(t) \rangle = 0$, for all $t \in I$. We call $(\boldsymbol{\gamma}_h, \boldsymbol{\gamma}_h^d)$ the *spacelike Legendrian curve* in Δ_1. Moreover, if $(\boldsymbol{\gamma}_h, \boldsymbol{\gamma}_h^d)$ is an immersion, we call $\boldsymbol{\gamma}_h$ the *spacelike front* in $H^2(-1)$ and $(\boldsymbol{\gamma}_h, \boldsymbol{\gamma}_h^d)$ the *spacelike Legendrian immersion* in Δ_1.

Let $(\boldsymbol{\gamma}_h, \boldsymbol{\gamma}_h^d)$ be a spacelike Legendrian curve in Δ_1. If $\boldsymbol{\gamma}_h$ is singular at a point t_0 in $H^2(-1)$, then we can not define the Frenet-Serret formula at this point. By the definition of the spacelike Legendrian curve, however, the $\boldsymbol{\gamma}_h^d$ is always well defined even if at a singular point of $\boldsymbol{\gamma}_h$. Let $\boldsymbol{\gamma}_h^s(t) = \boldsymbol{\gamma}_h(t) \wedge \boldsymbol{\gamma}_h^d(t) \in S_1^2$. We have a moving frame $\{\boldsymbol{\gamma}_h, \boldsymbol{\gamma}_h^d, \boldsymbol{\gamma}_h^s\}$ which called the *hyperbolic Legendrian Frenet frame* of $\mathbb{R}_1^3$ along $\boldsymbol{\gamma}_h$. By the standard arguments, we have the following *hyperbolic Legendrian Frenet-Serret type formula*:

$$\begin{pmatrix} \dot{\boldsymbol{\gamma}}_h(t) \\ \dot{\boldsymbol{\gamma}}_h^d(t) \\ \dot{\boldsymbol{\gamma}}_h^s(t) \end{pmatrix} = \begin{pmatrix} 0 & 0 & m_h(t) \\ 0 & 0 & n_h(t) \\ m_h(t) & -n_h(t) & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{\gamma}_h(t) \\ \boldsymbol{\gamma}_h^d(t) \\ \boldsymbol{\gamma}_h^s(t) \end{pmatrix},$$

where $m_h(t) = \langle \dot{\boldsymbol{\gamma}}_h(t), \boldsymbol{\gamma}_h^s(t) \rangle$ and $n_h(t) = \langle \dot{\boldsymbol{\gamma}}_h^d(t), \boldsymbol{\gamma}_h^s(t) \rangle$. We call the pair (m_h, n_h) the *spacelike hyperbolic Legendrian curvature* of spacelike Legendrian curve $(\boldsymbol{\gamma}_h, \boldsymbol{\gamma}_h^d)$. We remark that if $(\boldsymbol{\gamma}_h, \boldsymbol{\gamma}_h^d)$ is a spacelike Legendrian curve (respectively, spacelike Legendrian immersion) with the spacelike hyperbolic Legendrian curvature (m_h, n_h), then both $(\boldsymbol{\gamma}_h, -\boldsymbol{\gamma}_h^d)$ and $(-\boldsymbol{\gamma}_h, \boldsymbol{\gamma}_h^d)$ are spacelike Legendrian curves (respectively, spacelike Legendrian immersions) with the spacelike hyperbolic Legendrian curvatures $(-m_h, n_h)$ and $(m_h, -n_h)$, respectively.

We will characterize the properties of (m_h, n_h) as follows.

Proposition 3.1 *If $(\boldsymbol{\gamma}_h, \boldsymbol{\gamma}_h^d) : I \to \Delta_1$ is a spacelike Legendrian curve with the spacelike hyperbolic Legendrian curvature (m_h, n_h), then (m_h, n_h) depends on the parametrization of $(\boldsymbol{\gamma}_h, \boldsymbol{\gamma}_h^d)$.*

Proof. Let $(\bar{\boldsymbol{\gamma}}_h, \bar{\boldsymbol{\gamma}}_h^d) : \bar{I} \to \Delta_1$ be a spacelike Legendrian curve and $(\bar{m}_h, \bar{n}_h)$ be its spacelike hyperbolic Legendrian curvature. Suppose that $t : \bar{I} \to I$ is a (positive) change of parameter, that is, t is surjective and has a positive derivative at every point. We assume that $(\bar{\boldsymbol{\gamma}}_h(u), \bar{\boldsymbol{\gamma}}_h^d(u)) = (\boldsymbol{\gamma}_h(t(u)), \boldsymbol{\gamma}_h^d(t(u)))$ for all $u \in \bar{I}$, then we have

$$\begin{cases} \bar{m}_h(u)\bar{\boldsymbol{\gamma}}_h^s(u) = \dot{\bar{\boldsymbol{\gamma}}}_h(u) = m_h(t(u))\dot{t}(u)\boldsymbol{\gamma}_h^s(t(u)), \\ \bar{n}_h(u)\bar{\boldsymbol{\gamma}}_h^s(u) = \dot{\bar{\boldsymbol{\gamma}}}_h^d(u) = n_h(t(u))\dot{t}(u)\boldsymbol{\gamma}_h^s(t(u)). \end{cases}$$

It follows from $\bar{\boldsymbol{\gamma}}_h^s(u) = \boldsymbol{\gamma}_h^s(t(u))$, we have

$$\begin{cases} \bar{m}_h(u) = m_h(t(u))\dot{t}(u), \\ \bar{n}_h(u) = n_h(t(u))\dot{t}(u). \end{cases}$$

□

Proposition 3.2 *Suppose that $(\gamma_h, \gamma_h^d) : I \to \Delta_1$ is a spacelike Legendrian immersion with the spacelike hyperbolic Legendrian curvature (m_h, n_h). Then $(m_h(t), n_h(t)) \neq (0, 0)$ if and only if $(\dot{\gamma}_h(t), \dot{\gamma}_h^d(t)) \neq (0, 0)$, for all $t \in I$.*

Proof. By the hyperbolic Legendrian Frenet-Serret type formula, this assertion holds. $\square$

Example 3.3 Let γ_h be a regular curve in $H^2(-1)$ with the hyperbolic geodesic curvature κ_h. If we take $\gamma_h^d = \boldsymbol{e}_h$, then (γ_h, γ_h^d) is a spacelike Legendrian curve with the spacelike hyperbolic Legendrian curvature $(-||\dot{\gamma}_h||, ||\dot{\gamma}_h||\kappa_h)$. In fact, it is a spacelike Legendrian immersion. Moreover, by a straightforward calculation, we have $n_h(t) = |m_h(t)|\kappa_h(t)$ for all $t \in I$. In this case, we have $n_h(t) = 0$ if and only if $\kappa_h(t) = 0$.

Example 3.4 Let $\gamma_h : I \to H^2(-1)$ be $\gamma_h(t) = (\sqrt{1 + t^{2n} + t^{2m}}, t^n, t^m)$, where $m = n + k$, $m, n, k \in \mathbb{N}$. It is obviously that the origin is the singular point of γ_h. We assume that

$$\gamma_h^d(t) = \frac{1}{\sqrt{k^2t^{2m} + m^2t^{2k} + n^2}}(kt^m\sqrt{1 + t^{2n} + t^{2m}}, kt^{m+n} + mt^k, kt^{2m} - n).$$

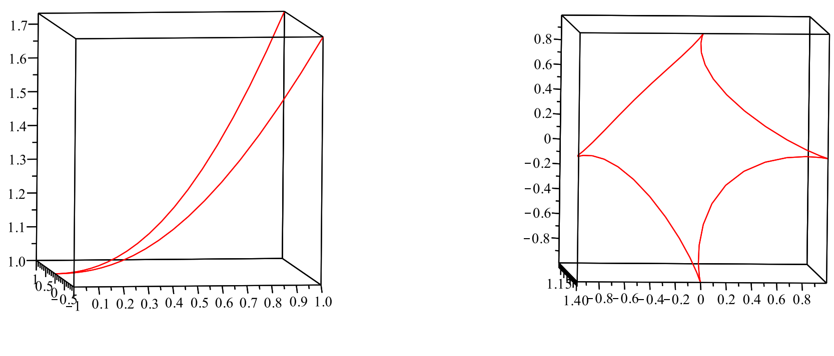
By a straightforward calculation, we have

$$\gamma_h^s(t) = \frac{\sqrt{1 + t^{2n} + t^{2m}}}{\sqrt{k^2t^{2m} + m^2t^{2k} + n^2}}\left(\frac{mt^{2k+n} + nt^n}{\sqrt{1 + t^{2n} + t^{2m}}}, n, mt^k\right)$$

and (γ_h, γ_h^d) is a spacelike Legendrian curve with the spacelike hyperbolic Legendrian curvature (m_h, n_h), where

$$\begin{aligned} m_h(t) &= \frac{t^{n-1}\sqrt{k^2t^{2m} + m^2t^{2k} + n^2}}{\sqrt{1 + t^{2n} + t^{2m}}}, \\ n_h(t) &= \frac{kt^{k-1}\left(k^2t^{2m+2n} + m(m+n)t^{2m} + n(m+n)t^{2n} + mn\right)}{(k^2t^{2m} + m^2t^{2k} + n^2)\sqrt{1 + t^{2n} + t^{2m}}}. \end{aligned}$$

Moreover, if $k = 1$, then (γ_h, γ_h^d) is a spacelike Legendrian immersion. In the case when $n = 2$ and $m = 3$, we call γ_h the *hyperbolic* $3/2$*-cusp*, see Figure 1 (i).



(i) hyperbolic 3/2-cusp (ii) hyperbolic astroid

Figure 1

Example 3.5 Let $I=[0,2\pi)$. We define $\boldsymbol{\gamma}_h : I \to H^2(-1)$ by

$$\boldsymbol{\gamma}_h(t) = \left(\sqrt{1+\cos^6 t+\sin^6 t}, \cos^3 t, \sin^3 t\right)$$

and call it the *hyperbolic astroid*, see Figure 1 (ii). It is obviously that $\boldsymbol{\gamma}_h$ is singular at point $t = 0, \pi/2, \pi$ and $3\pi/2$. We assume that

$$\boldsymbol{\gamma}_h^d(t) = \frac{1}{\sqrt{1+\sin^2 t\cos^2 t}}\left(\sin t\cos t\sqrt{1+\sin^6 t+\cos^6 t}, \sin t(1+\cos^4 t), \cos t(1+\sin^4 t)\right).$$

By a straightforward calculation, we have

$$\boldsymbol{\gamma}_h^s(t) = \frac{1}{\sqrt{1+\sin^2 t\cos^2 t}}\left(\sin^2 t-\cos^2 t, -\cos t\sqrt{1+\sin^6 t+\cos^6 t}, \sin t\sqrt{1+\sin^6 t+\cos^6 t}\right)$$

and $(\boldsymbol{\gamma}_h, \boldsymbol{\gamma}_h^d)$ is a spacelike Legendrian curve with the spacelike hyperbolic Legendrian curvature (m_h, n_h), where

$$\begin{aligned} m_h(t) &= \frac{3\sin t\cos t\sqrt{1+\sin^2 t\cos^2 t}}{\sqrt{1+\sin^6 t+\cos^6 t}}, \\ n_h(t) &= \frac{3\sin^4 t\cos^4 t+3\sin^2 t\cos^2 t-1-\sin^6 t-\cos^6 t}{(1+\sin^2 t\cos^2 t)\sqrt{1+\sin^6 t+\cos^6 t}}. \end{aligned}$$

Moreover, $(\boldsymbol{\gamma}_h, \boldsymbol{\gamma}_h^d)$ is also a spacelike Legendrian immersion.

Let $(\boldsymbol{\gamma}_h, \boldsymbol{\gamma}_h^d) : I \to \Delta_1$ be a spacelike Legendrian curve in Δ_1. We define a mapping $\boldsymbol{\gamma}_h^\phi : I \to H^2(-1)$ by

$$\boldsymbol{\gamma}_h^\phi(t) = \cosh\phi\,\boldsymbol{\gamma}_h(t) + \sinh\phi\,\boldsymbol{\gamma}_h^d(t),$$

where ϕ is any fixed real number. We call it the *hyperbolic parallel* of the frontal $\boldsymbol{\gamma}_h$ (cf. [10]). Then we have the following assertion.

Proposition 3.6 *Let $(\boldsymbol{\gamma}_h, \boldsymbol{\gamma}_h^d) : I \to \Delta_1$ be a spacelike Legendrian curve with the spacelike hyperbolic Legendrian curvature (m_h, n_h). For any fixed real number ϕ, $(\boldsymbol{\gamma}_h^\phi, (\boldsymbol{\gamma}_h^\phi)^d) : I \to \Delta_1$ is a spacelike Legendrian curve with the spacelike hyperbolic Legendrian curvature (m_h^ϕ, n_h^ϕ), where*

$$(\boldsymbol{\gamma}_h^\phi)^d : I \to S_1^2,\ (\boldsymbol{\gamma}_h^\phi)^d(t) = \sinh\phi\,\boldsymbol{\gamma}_h(t) + \cosh\phi\,\boldsymbol{\gamma}_h^d(t),$$

$$m_h^\phi(t) = \cosh\phi\, m_h(t) + \sinh\phi\, n_h(t),\ n_h^\phi(t) = \sinh\phi\, m_h(t) + \cosh\phi\, n_h(t).$$

Moreover, $(m_h^\phi)^2(t) - (n_h^\phi)^2(t) = m_h^2(t) - n_h^2(t)$ for all $t \in I$.

Proof. It is obviously that $(\boldsymbol{\gamma}_h^\phi(t), (\boldsymbol{\gamma}_h^\phi)^d(t)) \in \Delta_1$ and $\langle \dot{\boldsymbol{\gamma}}_h^\phi(t), (\boldsymbol{\gamma}_h^\phi)^d(t)\rangle = 0$ for all $t \in I$. By the definition of spacelike Legendrian curve, $(\boldsymbol{\gamma}_h^\phi, (\boldsymbol{\gamma}_h^\phi)^d)$ is a spacelike Legendrian curve in Δ_1. On the other hand, we have $\boldsymbol{\gamma}_h^\phi(t) \wedge (\boldsymbol{\gamma}_h^\phi)^d(t) = \boldsymbol{\gamma}_h^s(t)$. Hence $\{\boldsymbol{\gamma}_h^\phi, (\boldsymbol{\gamma}_h^\phi)^d, \boldsymbol{\gamma}_h^s\}$ is a hyperbolic Legendrian Frenet frame of $\mathbb{R}_1^3$ along $\boldsymbol{\gamma}_h^\phi$. Moreover,

$$\dot{\boldsymbol{\gamma}}_h^\phi(t) = (\cosh\phi\, m_h(t) + \sinh\phi\, n_h(t))\boldsymbol{\gamma}_h^s(t),\ \dot{(\boldsymbol{\gamma}_h^\phi)^d}(t) = (\sinh\phi\, m_h(t) + \cosh\phi\, n_h(t))\boldsymbol{\gamma}_h^s(t).$$

According to the hyperbolic Legendrian Frenet-Serret type formula, we have

$$m_h^\phi(t) = \cosh\phi\, m_h(t) + \sinh\phi\, n_h(t),\ n_h^\phi(t) = \sinh\phi\, m_h(t) + \cosh\phi\, n_h(t)$$

are the spacelike hyperbolic Legendrian curvature. By a direct calculation, we have $(m_h^\phi)^2(t) - (n_h^\phi)^2(t) = m_h^2(t) - n_h^2(t)$ for all $t \in I$. □

We call $(\boldsymbol{\gamma}_h^\phi, (\boldsymbol{\gamma}_h^\phi)^d)$ the *spacelike parallel Legendrian curve.*

3.2 The spacelike frontals in de Sitter 2-space

We now consider the differential geometry of spacelike curves in S^2_1. Suppose $\boldsymbol{\gamma}_d : I \to S^2_1$ is a spacelike curve at regular points $t \in I$, namely, $\dot{\boldsymbol{\gamma}}_d(t)$ is a spacelike vector at the regular points. We call $\boldsymbol{\gamma}_d$ the *spacelike frontal* in S^2_1 if there exists a smooth mapping $\boldsymbol{\gamma}^h_d : I \to H^2(-1)$ such that the pair $(\boldsymbol{\gamma}^h_d, \boldsymbol{\gamma}_d) : I \to \Delta_1$ satisfies $(\boldsymbol{\gamma}^h_d(t), \boldsymbol{\gamma}_d(t))^*\theta = 0$ for all $t \in I$. The condition $(\boldsymbol{\gamma}^h_d(t), \boldsymbol{\gamma}_d(t))^*\theta = 0$ is equivalent to $\langle \dot{\boldsymbol{\gamma}}_d(t), \boldsymbol{\gamma}^h_d(t)\rangle = 0$, for all $t \in I$. We call $(\boldsymbol{\gamma}_d, \boldsymbol{\gamma}^h_d)$ the *spacelike Legendrian curve* in Δ_1. Moreover, if $(\boldsymbol{\gamma}_d, \boldsymbol{\gamma}^h_d)$ is an immersion, we call $\boldsymbol{\gamma}_d$ the *spacelike front* in S^2_1 and $(\boldsymbol{\gamma}_d, \boldsymbol{\gamma}^h_d)$ the *spacelike Legendrian immersion* in Δ_1.

Let $(\boldsymbol{\gamma}_d, \boldsymbol{\gamma}^h_d)$ be a spacelike Legendrian curve in Δ_1 and $\boldsymbol{\gamma}^s_d(t) = \boldsymbol{\gamma}_d(t) \wedge \boldsymbol{\gamma}^h_d(t) \in S^2_1$. We have a moving frame $\{\boldsymbol{\gamma}_d, \boldsymbol{\gamma}^h_d, \boldsymbol{\gamma}^s_d\}$ which called the *spacelike de Sitter Legendrian Frenet frame* of $\mathbb{R}^3_1$ along $\boldsymbol{\gamma}_d$. By the standard arguments, we have the following *spacelike de Sitter Legendrian Frenet-Serret type formula*:

$$\begin{pmatrix} \dot{\boldsymbol{\gamma}}_d(t) \\ \dot{\boldsymbol{\gamma}}^h_d(t) \\ \dot{\boldsymbol{\gamma}}^s_d(t) \end{pmatrix} = \begin{pmatrix} 0 & 0 & m_d(t) \\ 0 & 0 & n_d(t) \\ -m_d(t) & n_d(t) & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{\gamma}_d(t) \\ \boldsymbol{\gamma}^h_d(t) \\ \boldsymbol{\gamma}^s_d(t) \end{pmatrix},$$

where $m_d(t) = \langle \dot{\boldsymbol{\gamma}}_d(t), \boldsymbol{\gamma}^s_d(t)\rangle$ and $n_d(t) = \langle \dot{\boldsymbol{\gamma}}^h_d(t), \boldsymbol{\gamma}^s_d(t)\rangle$. We call the pair (m_d, n_d) the *spacelike de Sitter Legendrian curvature* of the spacelike Legendrian curve $(\boldsymbol{\gamma}_d, \boldsymbol{\gamma}^h_d)$. We remark that if $(\boldsymbol{\gamma}_d, \boldsymbol{\gamma}^h_d)$ is a spacelike Legendrian curve (respectively, spacelike Legendrian immersion) with the spacelike de Sitter Legendrian curvature (m_d, n_d), then both $(\boldsymbol{\gamma}_d, -\boldsymbol{\gamma}^h_d)$ and $(-\boldsymbol{\gamma}_d, \boldsymbol{\gamma}^h_d)$ are spacelike Legendrian curves (respectively, spacelike Legendrian immersions) with the spacelike de Sitter Legendrian curvatures $(-m_d, n_d)$ and $(m_d, -n_d)$, respectively.

We can also characterize the properties of the spacelike de Sitter Legendrian curvature (m_d, n_d) by the similar arguments with the spacelike hyperbolic Legendrian curvature (m_h, n_h). We summarize here as follows.

Proposition 3.7 *Suppose that $(\boldsymbol{\gamma}_d, \boldsymbol{\gamma}^h_d)$ is a spacelike Legendrian curve with the spacelike de Sitter Legendrian curvature (m_d, n_d), then (m_d, n_d) depends on the parametrization of $(\boldsymbol{\gamma}_d, \boldsymbol{\gamma}^h_d)$.*

The proof is almost the same with Proposition 3.1, so that we omit it.

Proposition 3.8 *If $(\boldsymbol{\gamma}_d, \boldsymbol{\gamma}^h_d)$ be a spacelike Legendrian immersion with the spacelike de Sitter Legendrian curvature (m_d, n_d), then $(m_d(t), n_d(t)) \neq (0, 0)$ if and only if $(\dot{\boldsymbol{\gamma}}_d(t), \dot{\boldsymbol{\gamma}}^h_d(t)) \neq (0, 0)$, for all $t \in I$.*

Example 3.9 Let $\boldsymbol{\gamma}_d$ be a regular spacelike curve in S^2_1 with the spacelike de Sitter geodesic curvature κ_d. If we take $\boldsymbol{\gamma}^h_d = \boldsymbol{e}_d$, then $(\boldsymbol{\gamma}_d, \boldsymbol{\gamma}^h_d)$ is a spacelike Legendrian curve in Δ_1 with the spacelike de Sitter Legendrian curvature $(\|\dot{\boldsymbol{\gamma}}_d\|, \|\dot{\boldsymbol{\gamma}}_d\|\kappa_d)$. In fact, it is a spacelike Legendrian immersion in Δ_1. Moreover, it follows from the spacelike de Sitter Legendrian Frenet-Serret type formula, we have $n_d(t) = |m_d(t)|\kappa_d(t)$ for all $t \in I$. Then we have $n_d(t) = 0$ if and only if $\kappa_d(t) = 0$.

Let $(\boldsymbol{\gamma}_d, \boldsymbol{\gamma}^h_d)$ be a spacelike Legendrian curve in Δ_1. We define a mapping $\boldsymbol{\gamma}^\phi_d : I \to S^2_1$ by

$$\boldsymbol{\gamma}^\phi_d(t) = \cosh\phi\,\boldsymbol{\gamma}_d(t) + \sinh\phi\,\boldsymbol{\gamma}^h_d(t),$$

where ϕ is any fixed real number. We call it *de Sitter parallel* of the spacelike frontal $\boldsymbol{\gamma}_d$ (cf. [10]). Then we have the following assertion.

Proposition 3.10 *Let* $(\boldsymbol{\gamma}_d, \boldsymbol{\gamma}_d^h)$ *be a spacelike Legendrian curve in* Δ_1 *with the spacelike de Sitter Legendrian curvature* (m_d, n_d)*. For any fixed real number* ϕ*,* $(\boldsymbol{\gamma}_d^\phi, (\boldsymbol{\gamma}_d^\phi)^h)$ *is a spacelike Legendrian curve in* Δ_1 *with the spacelike de Sitter Legendrian curvature* (m_d^ϕ, n_d^ϕ)*, where*

$$(\boldsymbol{\gamma}_d^\phi)^h : I \to H^2(-1),\ (\boldsymbol{\gamma}_d^\phi)^h(t) = \sinh\phi\boldsymbol{\gamma}_d(t) + \cosh\phi\boldsymbol{\gamma}_d^h(t),$$

$$m_d^\phi(t) = \cosh\phi m_d(t) + \sinh\phi n_d(t),\ n_d^\phi(t) = \sinh\phi m_d(t) + \cosh\phi n_d(t).$$

Moreover, $(m_d^\phi)^2(t) - (n_d^\phi)^2(t) = m_d^2(t) - n_d^2(t)$ *for all* $t \in I$*.*

Proof. It is obviously that $((\boldsymbol{\gamma}_d^\phi)^h(t), \boldsymbol{\gamma}_d^\phi(t)) \in \Delta_1$ and $\langle \dot{\boldsymbol{\gamma}}_d^\phi(t), (\boldsymbol{\gamma}_d^\phi)^h(t)\rangle = 0$. By the definition of spacelike Legendrian curve, $(\boldsymbol{\gamma}_d^\phi, (\boldsymbol{\gamma}_d^\phi)^h)$ is a spacelike Legendrian curve in Δ_1. On the other hand, we have $\boldsymbol{\gamma}_d^\phi(t)\wedge(\boldsymbol{\gamma}_d^\phi)^h(t) = \boldsymbol{\gamma}_d^s(t)$. Hence $\{\boldsymbol{\gamma}_d^\phi, (\boldsymbol{\gamma}_d^\phi)^h, \boldsymbol{\gamma}_d^s\}$ is a spacelike de Sitter Legendrian Frenet frame of $\mathbb{R}_1^3$ along $\boldsymbol{\gamma}_d^\phi$. Moreover,

$$\dot{\boldsymbol{\gamma}}_d^\phi(t) = (\cosh\phi m_d(t) + \sinh\phi n_d(t))\boldsymbol{\gamma}_d^s(t),\ \dot{\overline{(\boldsymbol{\gamma}_d^\phi)^h}}(t) = (\sinh\phi m_d(t) + \cosh\phi n_d(t))\boldsymbol{\gamma}_d^s(t).$$

According to the spacelike de Sitter Legendrian Frenet-Serret type formula, we have

$$m_d^\phi(t) = \cosh\phi m_d(t) + \sinh\phi n_d(t),\ n_d^\phi(t) = \sinh\phi m_d(t) + \cosh\phi n_d(t)$$

are the spacelike de Sitter Legendrian curvature. By a direct calculation, we have $(m_d^\phi)^2(t) - (n_d^\phi)^2(t) = m_d^2(t) - n_d^2(t)$ for all $t \in I$. □

3.3 The timelike frontals in de Sitter 2-space

We now consider the differential geometry of timelike curves in S_1^2. Let $\boldsymbol{\gamma}_T : I \to S_1^2$ be a timelike curve at regular points $t \in I$, namely, $\dot{\boldsymbol{\gamma}}_T(t)$ is a timelike vector at the regular point. We call $\boldsymbol{\gamma}_T$ the *timelike frontal* in S_1^2 if there exists a smooth mapping $\boldsymbol{\gamma}_T^d : I \to S_1^2$, such that the pair $(\boldsymbol{\gamma}_T, \boldsymbol{\gamma}_T^d) : I \to \Delta_5$ satisfies $(\boldsymbol{\gamma}_T(t), \boldsymbol{\gamma}_T^d(t))^*\alpha = 0$ for all $t \in I$. Here

$$\Delta_5 = \{(\boldsymbol{v}, \boldsymbol{w}) \mid \langle \boldsymbol{v}, \boldsymbol{w}\rangle = 0\ \} \subset S_1^2 \times S_1^2$$

is the Δ_5-duality defined by the first author and S. Izumiya in [2, 6] and α is a canonical contact 1-form on Δ_5. The condition $(\boldsymbol{\gamma}_T(t), \boldsymbol{\gamma}_T^d(t))^*\alpha = 0$ is equivalent to $\langle \dot{\boldsymbol{\gamma}}_T(t), \boldsymbol{\gamma}_T^d(t)\rangle = 0$, for all $t \in I$. We call $(\boldsymbol{\gamma}_T, \boldsymbol{\gamma}_T^d)$ the *timelike Legendrian curve* in Δ_5. Moreover, if $(\boldsymbol{\gamma}_T, \boldsymbol{\gamma}_T^d)$ is an immersion, we call $\boldsymbol{\gamma}_T$ the *timelike front* in S_1^2 and $(\boldsymbol{\gamma}_T, \boldsymbol{\gamma}_T^d)$ the *timelike Legendrian immersion* in Δ_5.

Let $(\boldsymbol{\gamma}_T, \boldsymbol{\gamma}_T^d)$ be a timelike Legendrian curve in Δ_5 and $\boldsymbol{\gamma}_T^h(t) = \boldsymbol{\gamma}_T(t) \wedge \boldsymbol{\gamma}_T^d(t) \in H^2(-1)$. We have a moving frame $\{\boldsymbol{\gamma}_T, \boldsymbol{\gamma}_T^d, \boldsymbol{\gamma}_T^h\}$ which called the *timelike de Sitter Legendrian Frenet frame* of $\mathbb{R}_1^3$ along $\boldsymbol{\gamma}_T$. By the standard arguments, we have the following *timelike de Sitter Legendrian Frenet-Serret type formula*:

$$\begin{pmatrix} \dot{\boldsymbol{\gamma}}_T(t) \\ \dot{\boldsymbol{\gamma}}_T^d(t) \\ \dot{\boldsymbol{\gamma}}_T^h(t) \end{pmatrix} = \begin{pmatrix} 0 & 0 & m_T(t) \\ 0 & 0 & n_T(t) \\ m_T(t) & n_T(t) & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{\gamma}_T(t) \\ \boldsymbol{\gamma}_T^d(t) \\ \boldsymbol{\gamma}_T^h(t) \end{pmatrix},$$

where $m_T(t) = -\langle \dot{\boldsymbol{\gamma}}_T(t), \boldsymbol{\gamma}_T^h(t)\rangle$ and $n_T(t) = -\langle \dot{\boldsymbol{\gamma}}_T^d(t), \boldsymbol{\gamma}_T^h(t)\rangle$. We call the pair (m_T, n_T) the *timelike de Sitter Legendrian curvature* of timelike Legendrian curve $(\boldsymbol{\gamma}_T, \boldsymbol{\gamma}_T^d)$. We remark that if $(\boldsymbol{\gamma}_T, \boldsymbol{\gamma}_T^d)$ is a timelike Legendrian curve (respectively, timelike Legendrian immersion) in Δ_5

with the timelike de Sitter Legendrian curvature (m_T, n_T), then both $(\boldsymbol{\gamma}_T, -\boldsymbol{\gamma}_T^d)$ and $(-\boldsymbol{\gamma}_T, \boldsymbol{\gamma}_T^d)$ are timelike Legendrian curves (respectively, timelike Legendrian immersions) in Δ_5 with the timelike de Sitter Legendrian curvatures $(-m_T, n_T)$ and $(m_T, -n_T)$, respectively.

We can also characterize the properties of the timelike de Sitter Legendrian curvature (m_T, n_T) by the similar arguments with the spacelike de Sitter Legendrian curvature (m_d, n_d). We summarize them and omit the proofs as follows.

Proposition 3.11 *Suppose that* $(\boldsymbol{\gamma}_T, \boldsymbol{\gamma}_T^d) : I \to \Delta_5$ *is a timelike Legendrian curve with the timelike de Sitter Legendrian curvature* (m_T, n_T). *Then* (m_T, n_T) *depends on the parametrization of* $(\boldsymbol{\gamma}_T, \boldsymbol{\gamma}_T^d)$.

Proposition 3.12 *If* $(\boldsymbol{\gamma}_T, \boldsymbol{\gamma}_T^d) : I \to \Delta_5$ *is a timelike Legendrian immersion with the timelike de Sitter Legendrian curvature* (m_T, n_T), *then we have* $(m_T(t), n_T(t)) \neq (0, 0)$ *if and only if* $(\dot{\boldsymbol{\gamma}}_T(t), \dot{\boldsymbol{\gamma}}_T^d(t)) \neq (0, 0)$, *for all* $t \in I$.

Example 3.13 Let $\boldsymbol{\gamma}_T$ be a regular timelike curve in S_1^2 with the timelike de Sitter geodesic curvature κ_T. If we take $\boldsymbol{\gamma}_T^d = \boldsymbol{e}_T$, then $(\boldsymbol{\gamma}_T, \boldsymbol{\gamma}_T^d)$ is a timelike Legendrian curve in Δ_5 with the timelike de Sitter Legendrian curvature $(\|\dot{\boldsymbol{\gamma}}_T\|, \|\dot{\boldsymbol{\gamma}}_T\|\kappa_T)$. In fact, it is a timelike Legendrian immersion in Δ_5. Moreover, we have $n_T(t) = |m_T(t)|\kappa_T(t)$ for all $t \in I$. Therefore, we have $n_T(t) = 0$ if and only if $\kappa_T(t) = 0$.

Let $(\boldsymbol{\gamma}_T, \boldsymbol{\gamma}_T^d) : I \to \Delta_5$ be a timelike Legendrian curve in Δ_5. We define $\boldsymbol{\gamma}_T^\theta : I \to S_1^2$ by

$$\boldsymbol{\gamma}_T^\theta(t) = \cos\theta \boldsymbol{\gamma}_T(t) + \sin\theta \boldsymbol{\gamma}_T^d(t),$$

where $\theta \in [0, 2\pi)$ is a fixed number. We call it the *timelike parallel* of the timelike frontal $\boldsymbol{\gamma}_T$. Then we have the following assertion.

Proposition 3.14 *Let* $(\boldsymbol{\gamma}_T, \boldsymbol{\gamma}_T^d) : I \to \Delta_5$ *be a timelike Legendrian curve with the timelike de Sitter Legendrian curvature* (m_T, n_T). *For any fixed* $\theta \in [0, 2\pi)$, $(\boldsymbol{\gamma}_T^\theta, (\boldsymbol{\gamma}_T^\theta)^d) : I \to \Delta_5$ *is a timelike Legendrian curve with the timelike de Sitter Legendrian curvature* (m_T^θ, n_T^θ), *where*

$$(\boldsymbol{\gamma}_T^\theta)^d : I \to S_1^2, \ (\boldsymbol{\gamma}_T^\theta)^d(t) = -\sin\theta \boldsymbol{\gamma}_T(t) + \cos\theta \boldsymbol{\gamma}_T^d(t),$$

$$m_T^\theta(t) = \cos\theta m_T(t) + \sin\theta n_T(t), \ n_T^\theta(t) = -\sin\theta m_T(t) + \cos\theta n_T(t).$$

Moreover, $(m_T^\theta)^2(t) + (n_T^\theta)^2(t) = m_T^2(t) + n_T^2(t)$ *for all* $t \in I$.

Proof. It is obviously that $(\boldsymbol{\gamma}_T^\theta(t), (\boldsymbol{\gamma}_T^\theta)^d(t)) \in \Delta_5$ and $\langle \dot{\boldsymbol{\gamma}}_T^\theta(t), (\boldsymbol{\gamma}_T^\theta)^d(t) \rangle = 0$. By the definition of timelike Legendrian curve, $(\boldsymbol{\gamma}_T^\theta, (\boldsymbol{\gamma}_T^\theta)^d)$ is a timelike Legendrian curve in Δ_5. On the other hand, we have $\boldsymbol{\gamma}_T^\theta(t) \wedge (\boldsymbol{\gamma}_T^\theta)^d(t) = \boldsymbol{\gamma}_T^h(t)$. Hence $\{\boldsymbol{\gamma}_T^\theta, (\boldsymbol{\gamma}_T^\theta)^d, \boldsymbol{\gamma}_T^h\}$ is a timelike de Sitter Legendrian Frenet frame of $\mathbb{R}_1^3$ along $\boldsymbol{\gamma}_T^\theta$. Moreover,

$$\dot{\boldsymbol{\gamma}}_T^\theta(t) = (\cos\theta m_T(t) + \sin\theta n_T(t))\boldsymbol{\gamma}_T^h(t), \ (\dot{\boldsymbol{\gamma}}_T^\theta)^d(t) = (-\sin\theta m_T(t) + \cos\theta n_T(t))\boldsymbol{\gamma}_T^h(t).$$

Therefore

$$m_T^\theta(t) = \cos\theta m_T(t) + \sin\theta n_T(t), \ n_T^\theta(t) = -\sin\theta m_T(t) + \cos\theta n_T(t)$$

are the timelike de Sitter Legendrian curvature and $(m_T^\theta)^2(t) + (n_T^\theta)^2(t) = m_T^2(t) + n_T^2(t)$ for all $t \in I$. □

We call $(\boldsymbol{\gamma}_T^\theta, (\boldsymbol{\gamma}_T^\theta)^d)$ the *timelike parallel Legendrian curve.*

4 The evolutes of fronts in hyperbolic 2-space and de Sitter 2-space

4.1 The evolutes of spacelike fronts in hyperbolic 2-space

We firstly consider the geometric meanings of evolutes of spacelike fronts in $H^2(-1)$. Let $(\boldsymbol{\gamma}_h, \boldsymbol{\gamma}_h^d) : I \to \Delta_1$ be a spacelike Legendrian immersion with the spacelike hyperbolic Legendrian curvature (m_h, n_h) which satisfies $n_h^2(t) \neq m_h^2(t)$ for all $t \in I$. We define a mapping $\mathcal{E}_v(\boldsymbol{\gamma}_h) : I \to \mathbb{R}_1^3$ by

$$\mathcal{E}_v(\boldsymbol{\gamma}_h)(t) = \pm \frac{1}{\sqrt{|n_h^2(t) - m_h^2(t)|}} \left(n_h(t)\boldsymbol{\gamma}_h(t) - m_h(t)\boldsymbol{\gamma}_h^d(t)\right)$$

and call it the *totally evolute* of $\boldsymbol{\gamma}_h$ in $\mathbb{R}_1^3$. We remark that if $n_h^2(t) > m_h^2(t)$, then $\mathcal{E}_v(\boldsymbol{\gamma}_h)(t) \in H^2(-1)$. In this case, we denote it by $\mathcal{E}_v^h(\boldsymbol{\gamma}_h)$ and call it the *hyperbolic evolute* of $\boldsymbol{\gamma}_h$. Moreover, if $n_h^2(t) < m_h^2(t)$, then $\mathcal{E}_v(\boldsymbol{\gamma}_h)(t) \in S_1^2$. We rewrite it as $\mathcal{E}_v^d(\boldsymbol{\gamma}_h)$ and call it the *de Sitter evolute* of $\boldsymbol{\gamma}_h$. By a direct calculation, we have the following properties about the totally evolutes of $\boldsymbol{\gamma}_h$.

Proposition 4.1 *Suppose that $(\boldsymbol{\gamma}_h, \boldsymbol{\gamma}_h^d) : I \to \Delta_1$ is a spacelike Legendrian immersion with the spacelike hyperbolic Legendrian curvature (m_h, n_h). Then the totally evolute $\mathcal{E}_v(\boldsymbol{\gamma}_h)$ of $\boldsymbol{\gamma}_h$ is independent on the parametrization of $(\boldsymbol{\gamma}_h, \boldsymbol{\gamma}_h^d)$.*

Proof. According to the proof of Proposition 3.1, if we take $t : \bar{I} \to I$ as a (positive) change of parameter, that is, t is surjective and has a positive derivative at every point. Then we have

$$\begin{cases} \bar{m}_h(u) = m_h(t(u))\dot{t}(u), \\ \bar{n}_h(u) = n_h(t(u))\dot{t}(u). \end{cases}$$

Therefore,

$$\begin{aligned} \mathcal{E}_v(\boldsymbol{\gamma}_h)(u) &= \pm \frac{1}{\sqrt{|\bar{n}_h^2(u) - \bar{m}_h^2(u)|}} \left(\bar{n}_h(u)\boldsymbol{\gamma}_h(u) - \bar{m}_h(u)\boldsymbol{\gamma}_h^d(u)\right) \\ &= \pm \frac{1}{\sqrt{|n_h^2(t(u)) - m_h^2(t(u))|\dot{t}^2(u)}} \left(n_h(t(u))\boldsymbol{\gamma}_h(t(u)) - m_h(t(u))\boldsymbol{\gamma}_h^d(t(u))\right) \dot{t}(u) \\ &= \mathcal{E}_v(\boldsymbol{\gamma}_h)(t). \end{aligned}$$

□

Remark 4.2 *Let $(\boldsymbol{\gamma}_h, \boldsymbol{\gamma}_h^d) : I \to \Delta_1$ be a spacelike Legendrian immersion.*

(i) *If we take $-\boldsymbol{\gamma}_h$ instead of $\boldsymbol{\gamma}_h$, then the totally evolute of $\boldsymbol{\gamma}_h$ does not change.*

(ii) *If we take $-\boldsymbol{\gamma}_h^d$ instead of $\boldsymbol{\gamma}_h^d$, then the totally evolute of $\boldsymbol{\gamma}_h$ does not change.*

Proposition 4.3 *Let $\boldsymbol{\gamma}_h : I \to H^2(-1)$ be a regular curve in $H^2(-1)$ with the hyperbolic geodesic curvature κ_h which satisfies $\kappa_h \neq \pm 1$. Then we have the following assertions*:

(i) *If $\kappa_h^2(t) > 1$, then $E_v^h(\boldsymbol{\gamma}_h)(t) = \mathcal{E}_v^h(\boldsymbol{\gamma}_h)(t)$.*

(ii) *If $\kappa_h^2(t) < 1$, then $E_v^d(\boldsymbol{\gamma}_h)(t) = \mathcal{E}_v^d(\boldsymbol{\gamma}_h)(t)$.*

Proof. Without loss of generality, by Example 3.3, we take $\boldsymbol{\gamma}_h^d = \boldsymbol{e}_h$, then $(\boldsymbol{\gamma}_h, \boldsymbol{\gamma}_h^d)$ is a spacelike Legendrian immersion with the spacelike hyperbolic Legendrian curvature $(-||\dot{\boldsymbol{\gamma}}_h||, ||\dot{\boldsymbol{\gamma}}_h||\kappa_h)$. It follows from the definiton of the totally evolute $\boldsymbol{\gamma}_h$, we have

$$\mathcal{E}_v(\boldsymbol{\gamma}_h)(t) = \pm\frac{1}{\sqrt{|n_h^2(t) - m_h^2(t)|}}\left(n_h(t)\boldsymbol{\gamma}_h(t) - m_h(t)\boldsymbol{\gamma}_h^d(t)\right)$$

$$= \pm\frac{1}{\sqrt{|\kappa_h^2(t) - 1|}}\left(\kappa_h(t)\boldsymbol{\gamma}_h(t) + \boldsymbol{e}_h(t)\right).$$

Since $\kappa_h^2(t) > 1$ if and only if $n_h^2(t) > m_h^2(t)$, we have $E_v^h(\boldsymbol{\gamma}_h)(t) = \mathcal{E}_v^h(\boldsymbol{\gamma}_h)(t)$. Moreover, $\kappa_h^2(t) < 1$ if and only if $n_h^2(t) < m_h^2(t)$, we have $E_v^d(\boldsymbol{\gamma}_h)(t) = \mathcal{E}_v^d(\boldsymbol{\gamma}_h)(t)$. □

According to the above proposition, we have shown that the definition of the totally evolute of $\boldsymbol{\gamma}_h$ is consistent with the definition of the evolute of $\boldsymbol{\gamma}_h$ when $\boldsymbol{\gamma}_h$ is a regular curve in $H^2(-1)$.

Proposition 4.4 *Suppose that $(\boldsymbol{\gamma}_h, \boldsymbol{\gamma}_h^d) : I \to \Delta_1$ be a spacelike Legendrian immersion with the spacelike hyperbolic Legendrian curvature (m_h, n_h). Then we have the following assertions*:

(i) *If t_0 is a singular point of $\boldsymbol{\gamma}_h$, then $\mathcal{E}_v^h(\boldsymbol{\gamma}_h)(t_0) = \pm\boldsymbol{\gamma}_h(t_0)$.*

(ii) *If t_0 is a singular point of $\boldsymbol{\gamma}_h^d$, then $\mathcal{E}_v^d(\boldsymbol{\gamma}_h)(t_0) = \pm\boldsymbol{\gamma}_h^d(t_0)$.*

(iii) (a) *If $n_h^2(t) > m_h^2(t)$, then $(\mathcal{E}_v^h(\boldsymbol{\gamma}_h), \boldsymbol{\gamma}_h^s) : I \to \Delta_1$ is a spacelike Legendrian immersion with the spacelike hyperbolic Legendrian curvature*

$$\left(\frac{\dot{m}_h n_h - m_h \dot{n}_h}{n_h^2 - m_h^2}, \pm\sqrt{n_h^2 - m_h^2}\right).$$

(b) *If $n_h^2(t) < m_h^2(t)$, then $(\mathcal{E}_v^d(\boldsymbol{\gamma}_h), \boldsymbol{\gamma}_h^s) : I \to \Delta_5$ is a timelike Legendrian immersion with the timelike de Sitter Legendrian curvature*

$$\left(\frac{m_h \dot{n}_h - \dot{m}_h n_h}{m_h^2 - n_h^2}, \pm\sqrt{m_h^2 - n_h^2}\right).$$

Proof. (i) Since t_0 is a singular point of $\boldsymbol{\gamma}_h$, we have $m_h(t_0) = 0$. It follows that

$$\mathcal{E}_v^h(\boldsymbol{\gamma}_h)(t_0) = \pm\frac{1}{\sqrt{n_h^2(t_0)}} n_h(t_0)\boldsymbol{\gamma}_h(t_0) = \pm\boldsymbol{\gamma}_h(t_0).$$

(ii) Since t_0 is a singular point of $\boldsymbol{\gamma}_h^d$, we have $n_h(t_0) = 0$. It follows that

$$\mathcal{E}_v^d(\boldsymbol{\gamma}_h)(t_0) = \pm\frac{1}{\sqrt{m_h^2(t_0)}} m_h(t_0)\boldsymbol{\gamma}_h^d(t_0) = \pm\boldsymbol{\gamma}_h^d(t_0).$$

(iii) We firstly suppose that $n_h^2(t) > m_h^2(t)$ and denote that

$$\widetilde{\boldsymbol{\gamma}}_h = \mathcal{E}_v^h(\boldsymbol{\gamma}_h),\ \widetilde{\boldsymbol{\gamma}}_h^d = \boldsymbol{\gamma}_h \wedge \boldsymbol{\gamma}_h^d = \boldsymbol{\gamma}_h^s,\ \widetilde{\boldsymbol{\gamma}}_h^s = \widetilde{\boldsymbol{\gamma}}_h \wedge \widetilde{\boldsymbol{\gamma}}_h^d.$$

By a straightforward calculation, we have

$$\dot{\widetilde{\boldsymbol{\gamma}}}_h(t) = \frac{\dot{m}_h(t)n_h(t) - m_h(t)\dot{n}_h(t)}{n_h^2(t) - m_h^2(t)}\left(\frac{\pm 1}{\sqrt{n_h^2(t) - m_h^2(t)}}\left(m_h(t)\boldsymbol{\gamma}_h(t) - n_h(t)\boldsymbol{\gamma}_h^d(t)\right)\right),$$

$$\dot{\widetilde{\gamma}}_h^d(t) = \dot{\gamma}_h^s(t) = \pm\sqrt{n_h^2(t) - m_h^2(t)}\left(\frac{\pm 1}{\sqrt{n_h^2(t) - m_h^2(t)}}\left(m_h(t)\gamma_h(t) - n_h(t)\gamma_h^d(t)\right)\right)$$

and

$$\widetilde{\gamma}_h^s(t) = \frac{\pm 1}{\sqrt{n_h^2(t) - m_h^2(t)}}\left(m_h(t)\gamma_h(t) - n_h(t)\gamma_h^d(t)\right).$$

It follows that

$$\langle \dot{\widetilde{\gamma}}_h(t), \dot{\widetilde{\gamma}}_h(t)\rangle = \left(\frac{\dot{m}_h(t)n_h(t) - m_h(t)\dot{n}_h(t)}{n_h^2(t) - m_h^2(t)}\right)^2 \geq 0,$$

$$\langle \dot{\widetilde{\gamma}}_h(t), \widetilde{\gamma}_h^d(t)\rangle = \langle \dot{\mathcal{E}}_v^h(\gamma_h)(t), \gamma_h^s(t)\rangle = 0, \ (\dot{\widetilde{\gamma}}_h(t), \dot{\widetilde{\gamma}}_h^d(t)) \neq (0,0)$$

and

$$\widetilde{m}_h(t) = \langle \dot{\widetilde{\gamma}}_h(t), \widetilde{\gamma}_h^s(t)\rangle = \frac{\dot{m}_h(t)n_h(t) - m_h(t)\dot{n}_h(t)}{n_h^2(t) - m_h^2(t)},$$

$$\widetilde{n}_h(t) = \langle \dot{\widetilde{\gamma}}_d(t), \widetilde{\gamma}_h^s(t)\rangle = \pm\sqrt{n_h^2(t) - m_h^2(t)}.$$

This means that $(\widetilde{\gamma}_h, \widetilde{\gamma}_h^d) = (\mathcal{E}_v^h(\gamma_h), \gamma_h^s) : I \to \Delta_1$ is a spacelike Legendrian immersion with the spacelike hyperbolic Legendrian curvature $\left(\dfrac{\dot{m}_h n_h - m_h \dot{n}_h}{n_h^2 - m_h^2}, \pm\sqrt{n_h^2 - m_h^2}\right)$. Therefore the assertion (a) of (iii) holds.

Moreover, if $n_h^2(t) < m_h^2(t)$, then

$$\dot{\mathcal{E}}_v^d(\gamma_h)(t) = \frac{m_h(t)\dot{n}_h(t) - \dot{m}_h(t)n_h(t)}{m_h^2(t) - n_h^2(t)}\left(\frac{\pm 1}{\sqrt{m_h^2(t) - n_h^2(t)}}\left(m_h(t)\gamma_h(t) - n_h(t)\gamma_h^d(t)\right)\right).$$

We have

$$\langle \dot{\mathcal{E}}_v^d(\gamma_h)(t), \dot{\mathcal{E}}_v^d(\gamma_h)(t)\rangle = -\left(\frac{m_h(t)\dot{n}_h(t) - \dot{m}_h(t)n_h(t)}{m_h^2(t) - n_h^2(t)}\right)^2 \leq 0.$$

Thus $\dot{\mathcal{E}}_v^d(\gamma_h)(t)$ is a timelike vector at a regular point of $\mathcal{E}_v^d(\gamma_h)$ in S_1^2. We denote that

$$\widetilde{\gamma}_T = \mathcal{E}_v^d(\gamma_h), \ \widetilde{\gamma}_T^d = \gamma_h^s, \ \widetilde{\gamma}_T^h = \widetilde{\gamma}_T \wedge \widetilde{\gamma}_T^d.$$

By using almost the same arguments as above, we have

$$\dot{\widetilde{\gamma}}_T^d(t) = \dot{\gamma}_h^s(t) = \pm\sqrt{m_h^2(t) - n_h^2(t)}\left(\frac{\pm 1}{\sqrt{m_h^2(t) - n_h^2(t)}}\left(m_h(t)\gamma_h(t) - n_h(t)\gamma_h^d(t)\right)\right),$$

$$\widetilde{\gamma}_T^h(t) = \frac{\pm 1}{\sqrt{m_h^2(t) - n_h^2(t)}}\left(m_h(t)\gamma_h(t) - n_h(t)\gamma_h^d(t)\right),$$

$$\langle \dot{\widetilde{\gamma}}_T(t), \widetilde{\gamma}_T^d(t)\rangle = \langle \dot{\mathcal{E}}_v^d(\gamma_h)(t), \gamma_h^s(t)\rangle = 0, \ (\dot{\widetilde{\gamma}}_T(t), \dot{\widetilde{\gamma}}_T^d(t)) \neq (0,0)$$

and

$$\widetilde{m}_T(t) = \langle \dot{\widetilde{\gamma}}_T(t), \widetilde{\gamma}_T^h(t)\rangle = \frac{m_h(t)\dot{n}_h(t) - \dot{m}_h(t)n_h(t)}{m_h^2(t) - n_h^2(t)},$$

$$\widetilde{n}_T(t) = \langle \dot{\widetilde{\gamma}}_T(t), \widetilde{\gamma}_T^h(t)\rangle = \pm\sqrt{m_h^2(t) - n_h^2(t)}.$$

This means that $(\widetilde{\gamma}_T, \widetilde{\gamma}_T^d) = (\mathcal{E}_v^d(\gamma_h), \gamma_h^s) : I \to \Delta_5$ is a timelike Legendrian immersion with the timelike de Sitter Legendrian curvature $\left(\dfrac{m_h \dot{n}_h - \dot{m}_h n_h}{m_h^2 - n_h^2}, \pm\sqrt{m_h^2 - n_h^2}\right)$. Therefore the assertion (b) of (iii) holds. □

Let $(\boldsymbol{\gamma}_h, \boldsymbol{\gamma}_h^d) : I \to \Delta_1$ be a spacelike Legendrian immersion with the spacelike hyperbolic Legendrian curvature (m_h, n_h) which satisfies $n_h^2 \neq m_h^2$. We now explain the hyperbolic or de Sitter evolute of the spacelike front $\boldsymbol{\gamma}_h : I \to H^2(-1)$ as a wavefront from the viewpoint of Legendrian singularity theory [1, 11, 14] as follows. We define a function $H^T : I \times H^2(-1) \to \mathbb{R}$ by $H^T(t, \boldsymbol{v}) = \langle \boldsymbol{\gamma}_h^s(t), \boldsymbol{v} \rangle$ and call it the *hyperbolic timelike height function* on the spacelike Legendrian immersion $(\boldsymbol{\gamma}_h, \boldsymbol{\gamma}_h^d)$. We also define another function $H^S : I \times S_1^2 \to \mathbb{R}$ by $H^S(t, \boldsymbol{v}) = \langle \boldsymbol{\gamma}_h^s(t), \boldsymbol{v} \rangle$ and call it the *hyperbolic spacelike height function* on the spacelike Legendrian immersion $(\boldsymbol{\gamma}_h, \boldsymbol{\gamma}_h^d)$. By a straightforward calculation, we have the following proposition.

Proposition 4.5 *Let $(\boldsymbol{\gamma}_h, \boldsymbol{\gamma}_h^d) : I \to \Delta_1$ be a spacelike Legendrian immersion with the spacelike hyperbolic Legendrian curvature (m_h, n_h).*

(i) *Suppose that $\boldsymbol{v} \in H^2(-1)$ and $n_h^2(t) > m_h^2(t)$ for all $t \in I$:*

(a) $H^T(t, \boldsymbol{v}) = 0$ *if and only if there exist real numbers a and b such that $\boldsymbol{v} = a\boldsymbol{\gamma}_h(t) + b\boldsymbol{\gamma}_h^d(t)$.*

(b) $H^T(t, \boldsymbol{v}) = (\partial H^T/\partial t)(t, \boldsymbol{v}) = 0$ *if and only if $\boldsymbol{v} = \mathcal{E}_v^h(\boldsymbol{\gamma}_h)(t)$.*

(ii) *Suppose that $\boldsymbol{v} \in S_1^2$ and $n_h^2(t) < m_h^2(t)$ for all $t \in I$:*

(a) $H^S(t, \boldsymbol{v}) = 0$ *if and only if there exist real numbers a and b such that $\boldsymbol{v} = a\boldsymbol{\gamma}_h(t) + b\boldsymbol{\gamma}_h^d(t)$.*

(b) $H^S(t, \boldsymbol{v}) = (\partial H^S/\partial t)(t, \boldsymbol{v}) = 0$ *if and only if $\boldsymbol{v} = \mathcal{E}_v^d(\boldsymbol{\gamma}_h)(t)$.*

Proof. (i) Taking $\{\boldsymbol{\gamma}_h, \boldsymbol{\gamma}_h^d, \boldsymbol{\gamma}_h^s\}$ as the moving frame of $\mathbb{R}_1^3$ along $\boldsymbol{\gamma}_h$. For all $(t, \boldsymbol{v}) \in I \times H^2(-1)$, $H^T(t, \boldsymbol{v}) = 0$ if and only if $\langle \boldsymbol{\gamma}_h^s(t), \boldsymbol{v} \rangle = 0$. This means that there exist real numbers a and b with $b^2 - a^2 = -1$ such that $\boldsymbol{v} = a\boldsymbol{\gamma}_h(t) + b\boldsymbol{\gamma}_h^d(t)$. Thus, the assertion (a) of (i) holds. Moreover, $H^T(t, \boldsymbol{v}) = (\partial H^T/\partial t)(t, \boldsymbol{v}) = 0$ if and only if $\langle m_h(t)\boldsymbol{\gamma}_h(t) - n_h(t)\boldsymbol{\gamma}_h^d(t), a\boldsymbol{\gamma}_h(t) + b\boldsymbol{\gamma}_h^d(t) \rangle = 0$. By a direct calculation, we can show that $\boldsymbol{v} = \mathcal{E}_v^h(\boldsymbol{\gamma}_h)(t)$. Therefore, the assertion (b) of (i) holds.

(ii) By almost the same arguments as the assertion (i), we can show the assertion (ii). □

One can show that $(H^T, \partial H^T/\partial t)$ is non-singular at $(t, \boldsymbol{v}) \in \mathcal{D}(H^T)$, where

$$\mathcal{D}(H^T) = \{(t, \boldsymbol{v}) | H^T(t, \boldsymbol{v}) = \frac{\partial H^T}{\partial t}(t, \boldsymbol{v}) = 0\}.$$

This means that H^T is a Morse family. Therefore, the hyperbolic evolute $\mathcal{E}_v^h(\boldsymbol{\gamma}_h)$ of the spacelike front $\boldsymbol{\gamma}_h$ is a wavefront of a Legendrian immersion generated by H^T. Moreover, the hyperbolic spacelike height function H^S is also a Morse family. Hence the de Sitter evolute $\mathcal{E}_v^d(\boldsymbol{\gamma}_h)$ of the spacelike front $\boldsymbol{\gamma}_h$ is also a wavefront of a Legendrian immersion generated by H^S.

Proposition 4.6 *Let $(\boldsymbol{\gamma}_h, \boldsymbol{\gamma}_h^d) : I \to \Delta_1$ be a spacelike Legendrian immersion with the spacelike hyperbolic Legendrian curvature (m_h, n_h). If t_0 is a singular point of $\boldsymbol{\gamma}_h$, then we have the following assertions*:

(i) *t_0 is a regular point of $\mathcal{E}_v^h(\boldsymbol{\gamma}_h)$ if and only if $\dot{m}_h(t_0) \neq 0$.*

(ii) *t_0 is a singular point of $\mathcal{E}_v^h(\boldsymbol{\gamma}_h)$ if and only if $\ddot{\boldsymbol{\gamma}}_h(t_0) = 0$.*

Proof. By a direct calculation,

$$\dot{\mathcal{E}}_v^h(\boldsymbol{\gamma}_h)(t) = \pm \frac{\dot{m}_h(t)n_h(t) - m_h(t)\dot{n}_h(t)}{n_h^2(t) - m_h^2(t)} \left(\frac{1}{\sqrt{n_h^2(t) - m_h^2(t)}} \left(m_h(t)\boldsymbol{\gamma}_h(t) - n_h(t)\boldsymbol{\gamma}_h^d(t) \right) \right).$$

Since $m_h(t_0) = 0$, we have $\dot{\mathcal{E}}_v^h(\boldsymbol{\gamma}_h)(t_0) = \pm \frac{\dot{m}_h(t_0)}{|n_h(t_0)|}\boldsymbol{\gamma}_h^d(t_0)$. Therefore, t_0 is a regular point (respectively, a singular point) of $\mathcal{E}_v^h(\boldsymbol{\gamma}_h)$ if and only if $\dot{m}_h(t_0) \neq 0$ (respectively, $\dot{m}_h(t_0) = 0$). Furthermore, since $\ddot{\boldsymbol{\gamma}}_h(t) = \dot{m}_h(t)\boldsymbol{\gamma}_h^s(t) + m_h(t)\dot{\boldsymbol{\gamma}}_h^s(t)$, we have $\ddot{\boldsymbol{\gamma}}_h(t_0) = \dot{m}_h(t_0)\boldsymbol{\gamma}_h^s(t)$. Therefore, $\dot{m}_h(t_0) = 0$ if and only if $\ddot{\boldsymbol{\gamma}}_h(t_0) = 0$. □

Remark 4.7 *If t_0 is a singular point of $\boldsymbol{\gamma}_h$, then $m_h(t_0) = 0$ and hence we can not define $\mathcal{E}_v^d(\boldsymbol{\gamma}_h)$ at this point t_0. This is the reason why we don't consider the properties of $\mathcal{E}_v^d(\boldsymbol{\gamma}_h)$ at singular point of $\boldsymbol{\gamma}_h$.*

Proposition 4.8 *Let $(\boldsymbol{\gamma}_h, \boldsymbol{\gamma}_h^d) : I \to \Delta_1$ be a spacelike Legendrian immersion with the spacelike hyperbolic Legendrian curvature (m_h, n_h).*
(i) *$n_h^2(t) > m_h^2(t)$ and $\dot{\mathcal{E}}_v^h(\boldsymbol{\gamma}_h)(t) = 0$ for all $t \in I$ if and only if $\boldsymbol{\gamma}_h(t)$ is a point or there exist a constant timelike vector $\boldsymbol{v}$ and a constant real number C with $C^2 < 1$, such that $\boldsymbol{\gamma}_h(t) \in HP(\boldsymbol{v}, -1)$ and $\boldsymbol{\gamma}_h^d(t) \in DP(\boldsymbol{v}, -C)$ for all $t \in I$.*

(ii) *$m_h^2(t) > n_h^2(t)$ and $\dot{\mathcal{E}}_v^d(\boldsymbol{\gamma}_h)(t) = 0$ for all $t \in I$ if and only if $\boldsymbol{\gamma}_h^d(t)$ is a point or there exist a constant spacelike vector $\boldsymbol{w}$ and a constant real number C with $C^2 < 1$, such that $\boldsymbol{\gamma}_h(t) \in HP(\boldsymbol{w}, C)$ and $\boldsymbol{\gamma}_h^d(t) \in DP(\boldsymbol{w}, 1)$ for all $t \in I$.*

Proof. (i) Since

$$\dot{\mathcal{E}}_v^h(\boldsymbol{\gamma}_h)(t) = \pm\frac{\dot{m}_h(t)n_h(t) - m_h(t)\dot{n}_h(t)}{n_h^2(t) - m_h^2(t)}\left(\frac{1}{\sqrt{n_h^2(t) - m_h^2(t)}}\left(m_h(t)\boldsymbol{\gamma}_h(t) - n_h(t)\boldsymbol{\gamma}_h^d(t)\right)\right),$$

we have $\dot{\mathcal{E}}_v^h(\boldsymbol{\gamma}_h)(t) = 0$ if and only if $\dot{m}_h(t)n_h(t) - m_h(t)\dot{n}_h(t) = 0$ for all $t \in I$. Therefore, there exists a constant real number C with $C^2 < 1$ such that $m_h(t) = Cn_h(t)$. In the case when $C \equiv 0$, we have $\dot{\boldsymbol{\gamma}}_h(t) \equiv 0$. This means that $\boldsymbol{\gamma}_h(t)$ is a point. Moreover, if $C \not\equiv 0$, then $\dot{\boldsymbol{\gamma}}_h(t) = C\dot{\boldsymbol{\gamma}}_h^d(t)$. It follows that $\boldsymbol{\gamma}_h(t) = C\boldsymbol{\gamma}_h^d(t) + \boldsymbol{v}$, where $\boldsymbol{v}$ is a constant timelike vector. It is obviously that $\langle \boldsymbol{\gamma}_h(t), \boldsymbol{v}\rangle = -1$ and $\langle \boldsymbol{\gamma}_h^d(t), \boldsymbol{v}\rangle = -C$ for all $t \in I$.

Conversely, if $\boldsymbol{\gamma}_h(t)$ is a point for all $t \in I$, then $m_h(t) \equiv \dot{m}_h(t) \equiv 0$. It follows that $\dot{\mathcal{E}}_v^h(\boldsymbol{\gamma}_h)(t) \equiv 0$. If $\boldsymbol{v}$ is a constant timelike vector and C is a constant real number, then $\boldsymbol{\gamma}_h(t) \in HP(\boldsymbol{v}, -1)$ and $\boldsymbol{\gamma}_h^d(t) \in HP(\boldsymbol{v}, -C)$. Since $\langle \boldsymbol{\gamma}_h(t), \boldsymbol{v}\rangle = -1$ and $\langle \boldsymbol{\gamma}_h^d(t), \boldsymbol{v}\rangle = -C$, we have $\langle \dot{\boldsymbol{\gamma}}_h(t), \boldsymbol{v}\rangle = 0$ and $\langle \dot{\boldsymbol{\gamma}}_h^d(t), \boldsymbol{v}\rangle = 0$. Therefore, $\boldsymbol{v} = \boldsymbol{\gamma}_h(t) - C\boldsymbol{\gamma}_h^d(t)$. Then $\dot{\boldsymbol{v}} = m_h(t)\boldsymbol{\gamma}_h^s(t) - Cn_h(t)\boldsymbol{\gamma}_h^s(t) = 0$. It follows that $m_h(t) = Cn_h(t)$ and $\dot{m}_h(t) = C\dot{n}_h(t)$. This means that $\dot{m}_h(t)n_h(t) - m_h(t)\dot{n}_h(t) = 0$ for all $t \in I$. Therefore, $\dot{\mathcal{E}}_v^h(\boldsymbol{\gamma}_h)(t) = 0$ for all $t \in I$.

(ii) By almost the same arguments as the proof of the assertion (i), we can show that the assertion (ii) holds. □

Proposition 4.9 *Suppose that $(\boldsymbol{\gamma}_h, \boldsymbol{\gamma}_h^d) : I \to \Delta_1$ is a spacelike Legendrian immersion with the spacelike hyperbolic Legendrian curvature (m_h, n_h) and $(\boldsymbol{\gamma}_h^\phi, (\boldsymbol{\gamma}_h^\phi)^d) : I \to \Delta_1$ is a spacelike parallel Legendrian immersion with the spacelike hyperbolic Legendrian curvature (m_h^ϕ, n_h^ϕ). Then $\mathcal{E}_v(\boldsymbol{\gamma}_h) = \mathcal{E}_v(\boldsymbol{\gamma}_h^\phi)$ for any $\phi \in \mathbb{R}$.*

Proof. According to the definition of the totally evolute of the spacelike front in $H^2(-1)$ and Proposition 3.6, we have

$$\begin{aligned}\mathcal{E}_v(\boldsymbol{\gamma}_h^\phi)(t) &= \pm\frac{1}{\sqrt{|(n_h^\phi)^2(t) - (m_h^\phi)^2(t)|}}\left(n_h^\phi(t)\boldsymbol{\gamma}_h^\phi(t) - m_h^\phi(t)(\boldsymbol{\gamma}_h^\phi)^d(t)\right)\\ &= \pm\frac{1}{\sqrt{|{n_h}^2(t) - {m_h}^2(t)|}}\left(n_h(t)\boldsymbol{\gamma}_h(t) - m_h(t)(\boldsymbol{\gamma}_h)^d(t)\right) = \mathcal{E}_v(\boldsymbol{\gamma}_h)(t)\end{aligned}$$

for any $\phi \in \mathbb{R}$. □

In the last of this subsection, we calculate the evolutes of spacelike fronts defined in Example 3.4 and Example 3.5, respectively.

Example 4.10 Let $\gamma_h : I \to H^2(-1)$ be $\gamma_h(t) = (\sqrt{1+t^4+t^6}, t^2, t^3)$ and $\gamma_h^d : I \to S_1^2$ be

$$\gamma_h^d(t) = \frac{1}{\sqrt{4+9t^2+t^6}}\left(t^3\sqrt{1+t^4+t^6}, t^5+3t, t^6-2\right).$$

By Example 3.4, we have

$$m_h(t) = \frac{t\sqrt{4+9t^2+t^6}}{\sqrt{1+t^4+t^6}},\ n_h(t) = \frac{t^{10}+15t^6+10t^4+6}{(4+9t^2+t^6)\sqrt{1+t^4+t^6}}.$$

Therefore, we have

$$\mathcal{E}_v(\gamma_h)(t) = \pm\frac{\left(6(1+t^4+t^6)^{3/2},\ 3t^8+6t^6-27t^4-6t^2,\ 6t^9+8t^7+24t^3+8t\right)}{\sqrt{|(t^{10}+15t^6+10t^4+6)^2-t^2(4+9t^2+t^6)^3|}},$$

see Figure 2.

Example 4.11 Let $\gamma_h : I \to H^2(-1)$ be

$$\gamma_h(t) = (\sqrt{1+\cos^6 t+\sin^6 t}, \cos^3 t, \sin^3 t)$$

and $\gamma_h^d : I \to S_1^2$ be

$$\gamma_h^d(t) = \frac{1}{\sqrt{1+\sin^2 t\cos^2 t}}\left(\sin t\cos t\sqrt{1+\sin^6 t+\cos^6 t}, \sin t(1+\cos^4 t), \cos t(1+\sin^4 t)\right),$$

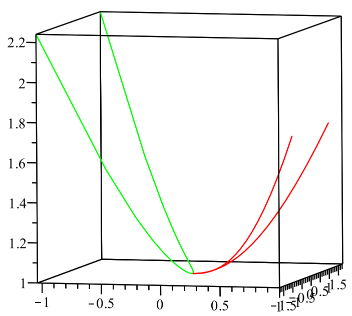
where $I = [0, 2\pi)$. By Example 3.5, we have

$$\begin{aligned} m_h(t) &= \frac{3\sin t\cos t\sqrt{1+\sin^2 t\cos^2 t}}{\sqrt{1+\sin^6 t+\cos^6 t}}, \\ n_h(t) &= \frac{3\sin^4 t\cos^4 t+3\sin^2 t\cos^2 t-1-\sin^6 t-\cos^6 t}{(1+\sin^2 t\cos^2 t)\sqrt{1+\sin^6 t+\cos^6 t}}. \end{aligned}$$

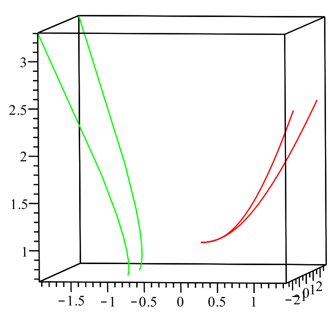
Thus, we have

$$\mathcal{E}_v(\gamma_h)(t) = \pm\frac{(a\sqrt{a},\ a\cos^3 t+b/\cos t,\ a\sin^3 t+b/\sin t)}{\sqrt{|(b-a)^2-3b(1+\sin^2 t\cos^2 t)^2|}},$$

where $a = 1+\cos^6 t+\sin^6 t,\ b = 3\sin^4 t\cos^4 t+3\sin^2 t\cos^2 t$, see Figure 3.

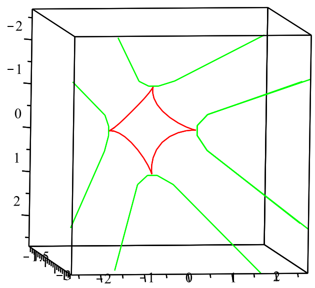


hyperbolic evolute of the hyperbolic 3/2-cusp

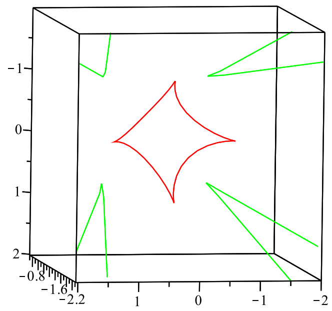


de Sitter evolute of the hyperbolic 3/2-cusp

Figure 2



hyperbolic evolute of the hyperbolic astroid



de Sitter evolute of the hyperbolic astroid

Figure 3

4.2 The evolutes of spacelike fronts in de Sitter 2-space

We now consider the geometric meanings of evolutes of spacelike fronts in S^2_1. Let $(\boldsymbol{\gamma}_d, \boldsymbol{\gamma}^h_d)$ be a spacelike Legendrian immersion in Δ_1 with the spacelike de Sitter Legendrian curvature (m_d, n_d) which satisfies $n^2_d(t) \neq m^2_d(t)$ for all $t \in I$. We define a mapping $\mathcal{E}_v(\boldsymbol{\gamma}_d) : I \to \mathbb{R}^3_1$ by

$$\mathcal{E}_v(\boldsymbol{\gamma}_d)(t) = \pm \frac{1}{\sqrt{|n^2_d(t) - m^2_d(t)|}} \big(n_d(t)\boldsymbol{\gamma}_d(t) - m_d(t)\boldsymbol{\gamma}^h_d(t)\big)$$

and call it the *totally evolute* of $\boldsymbol{\gamma}_d$ in $\mathbb{R}^3_1$. We remark that if $n^2_d(t) < m^2_d(t)$, then $\mathcal{E}_v(\boldsymbol{\gamma}_d)(t) \in H^2(-1)$. In this case, we denote it by $\mathcal{E}^h_v(\boldsymbol{\gamma}_d)$ and call it the *hyperbolic evolute* of $\boldsymbol{\gamma}_d$. Moreover, if $n^2_d(t) > m^2_d(t)$, then $\mathcal{E}_v(\boldsymbol{\gamma}_d)(t) \in S^2_1$. We rewrite it as $\mathcal{E}^d_v(\boldsymbol{\gamma}_d)$ and call it the *de Sitter evolute* of $\boldsymbol{\gamma}_d$. By a direct calculation, we have the following properties about the totally evolutes of $\boldsymbol{\gamma}_d$.

Proposition 4.12 *Suppose that $(\boldsymbol{\gamma}^h_d, \boldsymbol{\gamma}_d) : I \to \Delta_1$ is a spacelike Legendrian immersion with the spacelike de Sitter Legendrian curvature (m_d, n_d). Then the totally evolute $\mathcal{E}_v(\boldsymbol{\gamma}_d)$ of $\boldsymbol{\gamma}_d$ is independent on the parametrization of $(\boldsymbol{\gamma}_d, \boldsymbol{\gamma}^h_d)$.*

Proof. If we take $t : \bar{I} \to I$ as a (positive) change of parameter, that is, t is surjective and has a positive derivative at every point. Then we have

$$\begin{cases} \bar{m}_d(u) = m_d(t(u))\dot{t}(u), \\ \bar{n}_d(u) = n_d(t(u))\dot{t}(u). \end{cases}$$

Therefore,

$$\begin{aligned} \mathcal{E}_v(\boldsymbol{\gamma}_d)(u) &= \pm \frac{1}{\sqrt{|\bar{n}^2_d(u) - \bar{m}^2_d(u)|}} \big(\bar{n}_d(u)\boldsymbol{\gamma}_d(u) - \bar{m}_d(u)\boldsymbol{\gamma}^h_d(u)\big) \\ &= \pm \frac{1}{\sqrt{|n^2_d(t(u)) - m^2_d(t(u))|\dot{t}^2(u)}} \big(n_d(t(u))\boldsymbol{\gamma}_d(t(u)) - m_d(t(u))\boldsymbol{\gamma}^h_d(t(u))\big)\, \dot{t}(u) \\ &= \mathcal{E}_v(\boldsymbol{\gamma}_d)(t). \end{aligned}$$

□

Remark 4.13 *Let* $(\boldsymbol{\gamma}_d^h, \boldsymbol{\gamma}_d) : I \to \Delta_1$ *be a spacelike Legendrian immersion.*

(i) *If we take* $-\boldsymbol{\gamma}_d$ *instead of* $\boldsymbol{\gamma}_d$*, then the totally evolute of* $\boldsymbol{\gamma}_d$ *does not change.*

(ii) *If we take* $-\boldsymbol{\gamma}_d^h$ *instead of* $\boldsymbol{\gamma}_d^h$*, then the totally evolute of* $\boldsymbol{\gamma}_d$ *does not change.*

Proposition 4.14 *Let* $\boldsymbol{\gamma}_d : I \to S_1^2$ *be a regular spacelike curve in* S_1^2 *with the spacelike de Sitter geodesic curvature* κ_d *which satisfies* $\kappa_d \neq \pm 1$*. Then we have the following assertions*:

(i) *If* $\kappa_d^2(t) > 1$*, then* $E_v^d(\boldsymbol{\gamma}_d)(t) = \mathcal{E}_v^d(\boldsymbol{\gamma}_d)(t)$*.*

(ii) *If* $\kappa_d^2(t) < 1$*, then* $E_v^h(\boldsymbol{\gamma}_d)(t) = \mathcal{E}_v^h(\boldsymbol{\gamma}_d)(t)$*.*

Proof. Without loss of generality, by Example 3.9, we take $\boldsymbol{\gamma}_d^h = \boldsymbol{e}_d$, then $(\boldsymbol{\gamma}_d, \boldsymbol{\gamma}_d^h)$ is a spacelike Legendrian immersion with the spacelike de Sitter Legendrian curvature $(||\dot{\boldsymbol{\gamma}}_d||, ||\dot{\boldsymbol{\gamma}}_d||\kappa_d)$. It follows from the definiton of the totally evolute $\boldsymbol{\gamma}_d$, we have

$$
\begin{aligned}
\mathcal{E}_v(\boldsymbol{\gamma}_d)(t) &= \pm \frac{1}{\sqrt{|n_d^2(t) - m_d^2(t)|}} \left(n_d(t)\boldsymbol{\gamma}_d(t) - m_d(t)\boldsymbol{\gamma}_d^h(t) \right) \\
&= \pm \frac{1}{\sqrt{|\kappa_d^2(t) - 1|}} \left(\kappa_d(t)\boldsymbol{\gamma}_d(t) - \boldsymbol{e}_d(t) \right).
\end{aligned}
$$

Since $\kappa_d^2(t) > 1$ if and only if $n_d^2(t) > m_d^2(t)$, we have $E_v^d(\boldsymbol{\gamma}_d)(t) = \mathcal{E}_v^d(\boldsymbol{\gamma}_d)(t)$. Moreover, $\kappa_d^2(t) < 1$ if and only if $n_d^2(t) < m_d^2(t)$, we have $E_v^h(\boldsymbol{\gamma}_d)(t) = \mathcal{E}_v^h(\boldsymbol{\gamma}_d)(t)$. □

According to the above proposition, we have shown that the definition of the totally evolute of $\boldsymbol{\gamma}_d$ is consistent with the definition of the evolute of $\boldsymbol{\gamma}_d$ when $\boldsymbol{\gamma}_d$ is a regular spacelike curve in S_1^2.

Proposition 4.15 *If* $(\boldsymbol{\gamma}_d^h, \boldsymbol{\gamma}_d) : I \to \Delta_1$ *be a spacelike Legendrian immersion with the spacelike de Sitter Legendrian curvature* (m_d, n_d)*, then we have the following assertions*:

(i) *If* t_0 *is a singular point of* $\boldsymbol{\gamma}_d$*, then* $\mathcal{E}_v^d(\boldsymbol{\gamma}_d)(t_0) = \pm\boldsymbol{\gamma}_d(t_0)$*.*

(ii) *If* t_0 *is a singular point of* $\boldsymbol{\gamma}_d^h$*, then* $\mathcal{E}_v^h(\boldsymbol{\gamma}_d)(t_0) = \pm\boldsymbol{\gamma}_d^h(t_0)$*.*

(iii) (a) *If* $n_d^2(t) > m_d^2(t)$*, then* $(\mathcal{E}_v^d(\boldsymbol{\gamma}_d), \boldsymbol{\gamma}_d^s) : I \to \Delta_5$ *is a timelike Legendrian immersion with the timelike de Sitter Legendrian curvature*

$$
\left(\frac{m_d \dot{n}_d - \dot{m}_d n_d}{n_d^2 - m_d^2}, \pm\sqrt{n_d^2 - m_d^2} \right).
$$

(b) *If* $n_d^2(t) < m_d^2(t)$*, then* $(\mathcal{E}_v^h(\boldsymbol{\gamma}_d), \boldsymbol{\gamma}_d^s) : I \to \Delta_1$ *is a spacelike Legendrian immersion with the spacelike hyperbolic Legendrian curvature*

$$
\left(\frac{\dot{m}_d n_d - m_d \dot{n}_d}{m_d^2 - n_d^2}, \pm\sqrt{m_d^2 - n_d^2} \right).
$$

Proof. (i) Since t_0 is a singular point of $\boldsymbol{\gamma}_d$, we have $m_d(t_0) = 0$. It follows that

$$
\mathcal{E}_v^d(\boldsymbol{\gamma}_d)(t_0) = \pm \frac{1}{\sqrt{n_d^2(t_0)}} n_d(t_0)\boldsymbol{\gamma}_d(t_0) = \pm\boldsymbol{\gamma}_d(t_0).
$$

(ii) Since t_0 is a singular point of $\boldsymbol{\gamma}_d^h$, we have $n_d(t_0) = 0$. It follows that

$$
\mathcal{E}_v^h(\boldsymbol{\gamma}_d)(t_0) = \pm \frac{1}{\sqrt{m_d^2(t_0)}} m_d(t_0)\boldsymbol{\gamma}_d^h(t_0) = \pm\boldsymbol{\gamma}_d^h(t_0).
$$

(iii) We firstly assume that $n_d^2(t) > m_d^2(t)$ and denote that

$$\widetilde{\boldsymbol{\gamma}}_T = \mathcal{E}_v^d(\boldsymbol{\gamma}_d),\ \widetilde{\boldsymbol{\gamma}}_T^d = \boldsymbol{\gamma}_d^s,\ \widetilde{\boldsymbol{\gamma}}_T^h = \widetilde{\boldsymbol{\gamma}}_T \wedge \widetilde{\boldsymbol{\gamma}}_T^d.$$

By a straightforward calculation, we have

$$\dot{\widetilde{\boldsymbol{\gamma}}}_T(t) = \frac{m_d(t)\dot{n}_d(t) - \dot{m}_d(t)n_d(t)}{n_d^2(t) - m_d^2(t)} \left(\frac{\mp 1}{\sqrt{n_d^2(t) - m_d^2(t)}} \left(m_d(t)\boldsymbol{\gamma}_d(t) - n_d(t)\boldsymbol{\gamma}_d^h(t) \right) \right),$$

$$\dot{\widetilde{\boldsymbol{\gamma}}}_T^d(t) = \dot{\boldsymbol{\gamma}}_d^s(t) = \pm\sqrt{n_d^2(t) - m_d^2(t)} \left(\frac{\mp 1}{\sqrt{n_d^2(t) - m_d^2(t)}} (m_d(t)\boldsymbol{\gamma}_d(t) - n_d(t)\boldsymbol{\gamma}_d^h(t)) \right)$$

and

$$\widetilde{\boldsymbol{\gamma}}_T^h(t) = \frac{\mp 1}{\sqrt{n_d^2(t) - m_d^2(t)}} \left(m_d(t)\boldsymbol{\gamma}_d(t) - n_d(t)\boldsymbol{\gamma}_d^h(t) \right).$$

It follows that

$$\langle \dot{\widetilde{\boldsymbol{\gamma}}}_T(t), \dot{\widetilde{\boldsymbol{\gamma}}}_T(t) \rangle = -\left(\frac{m_d(t)\dot{n}_d(t) - \dot{m}_d(t)n_d(t)}{n_d^2(t) - m_d^2(t)} \right)^2 \leq 0,$$

$$\langle \dot{\widetilde{\boldsymbol{\gamma}}}_T(t), \widetilde{\boldsymbol{\gamma}}_T^d(t) \rangle = \langle \dot{\mathcal{E}}_v^d(\boldsymbol{\gamma}_d)(t), \boldsymbol{\gamma}_d^s(t) \rangle = 0,\ (\dot{\widetilde{\boldsymbol{\gamma}}}_T(t), \dot{\widetilde{\boldsymbol{\gamma}}}_T^d(t)) \neq (0,0)$$

and

$$\widetilde{m}_T(t) = -\langle \dot{\widetilde{\boldsymbol{\gamma}}}_T(t), \widetilde{\boldsymbol{\gamma}}_T^h(t) \rangle = \frac{m_d(t)\dot{n}_d(t) - \dot{m}_d(t)n_d(t)}{n_d^2(t) - m_d^2(t)},$$

$$\widetilde{n}_T(t) = -\langle \dot{\widetilde{\boldsymbol{\gamma}}}_T^d(t), \widetilde{\boldsymbol{\gamma}}_T^h(t) \rangle = \pm\sqrt{n_d^2(t) - m_d^2(t)}.$$

This means that $(\widetilde{\boldsymbol{\gamma}}_T, \widetilde{\boldsymbol{\gamma}}_T^d) = (\mathcal{E}_v^d(\boldsymbol{\gamma}_d), \boldsymbol{\gamma}_d^s) : I \to \Delta_5$ is a timelike Legendrian immersion with the timelike de Sitter Legendrian curvature $\left(\frac{m_d\dot{n}_d - \dot{m}_d n_d}{n_d^2 - m_d^2}, \pm\sqrt{n_d^2 - m_d^2} \right)$. Therefore, the assertion (a) of (iii) holds.

Moreover, we assume that $n_d^2(t) < m_d^2(t)$ and denote that

$$\widetilde{\boldsymbol{\gamma}}_h = \mathcal{E}_v^h(\boldsymbol{\gamma}_d),\ \widetilde{\boldsymbol{\gamma}}_h^d = \boldsymbol{\gamma}_d^s,\ \widetilde{\boldsymbol{\gamma}}_h^s = \widetilde{\boldsymbol{\gamma}}_h \wedge \widetilde{\boldsymbol{\gamma}}_d.$$

By a direct calculation, we can show that

$$\dot{\widetilde{\boldsymbol{\gamma}}}_h(t) = \frac{\dot{m}_d(t)n_d(t) - m_d(t)\dot{n}_d(t)}{m_d^2(t) - n_d^2(t)} \left(\frac{\mp 1}{\sqrt{m_d^2(t) - n_d^2(t)}} \left(m_d(t)\boldsymbol{\gamma}_d(t) - n_d(t)\boldsymbol{\gamma}_d^h(t) \right) \right),$$

$$\dot{\widetilde{\boldsymbol{\gamma}}}_h^d(t) = \dot{\boldsymbol{\gamma}}_d^s(t) = \pm\sqrt{m_d^2(t) - n_d^2(t)} \left(\frac{\mp 1}{\sqrt{m_d^2(t) - n_d^2(t)}} (m_d(t)\boldsymbol{\gamma}_d(t) - n_d(t)\boldsymbol{\gamma}_d^h(t)) \right)$$

and

$$\widetilde{\boldsymbol{\gamma}}_h^s(t) = \frac{\mp 1}{\sqrt{m_d^2(t) - n_d^2(t)}} \left(m_d(t)\boldsymbol{\gamma}_d(t) - n_d(t)\boldsymbol{\gamma}_d^h(t) \right).$$

Therefore, we have

$$\langle \dot{\widetilde{\boldsymbol{\gamma}}}_h(t), \dot{\widetilde{\boldsymbol{\gamma}}}_h(t) \rangle = \left(\frac{\dot{m}_d(t)n_d(t) - m_d(t)\dot{n}_d(t)}{m_d^2(t) - n_d^2(t)} \right)^2 \geq 0,$$

$$\langle \dot{\widetilde{\boldsymbol{\gamma}}}_h(t), \widetilde{\boldsymbol{\gamma}}_h^d(t) \rangle = \langle \dot{\mathcal{E}}_v^h(\boldsymbol{\gamma}_d)(t), \boldsymbol{\gamma}_d^s(t) \rangle = 0,\ (\dot{\widetilde{\boldsymbol{\gamma}}}_h(t), \dot{\widetilde{\boldsymbol{\gamma}}}_h^d(t)) \neq (0,0)$$

and

$$\widetilde{m}_h(t) = \langle \dot{\widetilde{\boldsymbol{\gamma}}}_h(t), \widetilde{\boldsymbol{\gamma}}^s_h(t) \rangle = \frac{\dot{m}_d(t)n_d(t) - m_d(t)\dot{n}_d(t)}{m_d^2(t) - n_d^2(t)},$$

$$\widetilde{n}_h(t) = \langle \dot{\widetilde{\boldsymbol{\gamma}}}^d_h(t), \widetilde{\boldsymbol{\gamma}}^s_h(t) \rangle = \pm\sqrt{m_d^2(t) - n_d^2(t)}.$$

This means that $(\widetilde{\boldsymbol{\gamma}}_h, \widetilde{\boldsymbol{\gamma}}^d_h) = (\mathcal{E}^h_v(\boldsymbol{\gamma}_d), \boldsymbol{\gamma}^s_d) : I \to \Delta_1$ is a spacelike Legendrian immersion with the spacelike hyperbolic Legendrian curvature $\left(\dfrac{\dot{m}_d n_d - m_d \dot{n}_d}{m_d^2 - n_d^2}, \pm\sqrt{m_d^2 - n_d^2}\right)$. Therefore the assertion (b) of (iii) holds. □

Let $(\boldsymbol{\gamma}_d, \boldsymbol{\gamma}^h_d)$ be a spacelike Legendrian immersion in Δ_1 with the spacelike de Sitter Legendrian curvature (m_d, n_d) which satisfies $n_d^2 \neq m_d^2$. We can also explain the hyperbolic or de Sitter evolute of the spacelike front $\boldsymbol{\gamma}_d$ as a wavefront from the viewpoint of Legendrian singularity theory as follows. We define a function $D^T : I \times H^2(-1) \to \mathbb{R}$ by $D^T(t, \boldsymbol{v}) = \langle \boldsymbol{\gamma}^s_d(t), \boldsymbol{v} \rangle$ and call it the *de Sitter timelike height function* on the spacelike Legendrian immersion $(\boldsymbol{\gamma}_d, \boldsymbol{\gamma}^h_d)$. We also define another function $D^S : I \times S^2_1 \to \mathbb{R}$ by $D^S(t, \boldsymbol{v}) = \langle \boldsymbol{\gamma}^s_d(t), \boldsymbol{v} \rangle$ and call it the *de Sitter spacelike height function* on the spacelike Legendrian immersion $(\boldsymbol{\gamma}_d, \boldsymbol{\gamma}^h_d)$. By a direct calculation, we can show the following proposition.

Proposition 4.16 *Let $(\boldsymbol{\gamma}^h_d, \boldsymbol{\gamma}_d) : I \to \Delta_1$ be a spacelike Legendrian immersion with the spacelike de Sitter Legendrian curvature (m_d, n_d).*

(i) *Suppose that $\boldsymbol{v} \in H^2(-1)$ and $n_d^2(t) < m_d^2(t)$ for all $t \in I$:*

(a) *$D^T(t, \boldsymbol{v}) = 0$ if and only if there exist real numbers a and b such that $\boldsymbol{v} = a\boldsymbol{\gamma}_d(t) + b\boldsymbol{\gamma}^h_d(t)$.*

(b) *$D^T(t, \boldsymbol{v}) = (\partial D^T/\partial t)(t, \boldsymbol{v}) = 0$ if and only if $\boldsymbol{v} = \mathcal{E}^h_v(\boldsymbol{\gamma}_d)(t)$.*

(ii) *Suppose that $\boldsymbol{v} \in S^2_1$ and $n_d^2(t) > m_d^2(t)$ for all $t \in I$:*

(a) *$D^S(t, \boldsymbol{v}) = 0$ if and only if there exist real numbers a and b such that $\boldsymbol{v} = a\boldsymbol{\gamma}_d(t) + b\boldsymbol{\gamma}^h_d(t)$.*

(b) *$D^S(t, \boldsymbol{v}) = (\partial D^S/\partial t)(t, \boldsymbol{v}) = 0$ if and only if $\boldsymbol{v} = \mathcal{E}^d_v(\boldsymbol{\gamma}_d)(t)$.*

One can show that $(D^T, \partial D^T/\partial t)$ is non-singular at $(t, \boldsymbol{v}) \in \mathcal{D}(D^T)$, where

$$\mathcal{D}(D^T) = \{(t, \boldsymbol{v}) | D^T(t, \boldsymbol{v}) = \frac{\partial D^T}{\partial t}(t, \boldsymbol{v}) = 0\}.$$

This means that D^T is a Morse family. Therefore, the hyperbolic evolute $\mathcal{E}^h_v(\boldsymbol{\gamma}_d)$ of the spacelike front $\boldsymbol{\gamma}_d$ is a wavefront of a Legendrian immersion generated by D^T. Moreover, the de Sitter spacelike height function D^S is also a Morse family. Hence the de Sitter evolute $\mathcal{E}^d_v(\boldsymbol{\gamma}_d)$ of the spacelike front $\boldsymbol{\gamma}_d$ is also a wavefront of a Legendrian immersion generated by D^S.

By almost the same arguments with Propositions 4.6, 4.8, 4.9 and Remark 4.7, we have the following propositions and remark.

Proposition 4.17 *Let $(\boldsymbol{\gamma}^h_d, \boldsymbol{\gamma}_d) : I \to \Delta_1$ be a spacelike Legendrian immersion with the spacelike de Sitter Legendrian curvature (m_d, n_d). If t_0 is a singular point of $\boldsymbol{\gamma}_d$, then we have the following assertions*:

(i) *t_0 is a regular point of $\mathcal{E}^d_v(\boldsymbol{\gamma}_h)$ if and only if $\dot{m}_d(t_0) \neq 0$.*

(ii) *t_0 is a singular point of $\mathcal{E}^d_v(\boldsymbol{\gamma}_h)$ if and only if $\ddot{\boldsymbol{\gamma}}_d(t_0) = 0$.*

Remark 4.18 *If t_0 is a singular point of $\boldsymbol{\gamma}_d$, then $m_d(t_0) = 0$ and hence we can not define $\mathcal{E}^h_v(\boldsymbol{\gamma}_h)$ at this point t_0. This is the reason why we don't consider the properties of $\mathcal{E}^h_v(\boldsymbol{\gamma}_d)$ at singular point of $\boldsymbol{\gamma}_d$.*

Proposition 4.19 *If* $(\boldsymbol{\gamma}_d^h, \boldsymbol{\gamma}_d) : I \to \Delta_1$ *be a spacelike Legendrian immersion with the spacelike de Sitter Legendrian curvature* (m_d, n_d)*, then we have the following assertions*:
(i) $n_d^2(t) > m_d^2(t)$ *and* $\dot{\mathcal{E}}_v^d(\boldsymbol{\gamma}_d)(t) = 0$ *for all* $t \in I$ *if and only if* $\boldsymbol{\gamma}_d(t)$ *is a point or there exist a constant spacelike vector* $\boldsymbol{v}$ *and a constant real number* C *with* $C^2 < 1$*, such that* $\boldsymbol{\gamma}_d(t) \in DP(\boldsymbol{v}, 1)$ *and* $\boldsymbol{\gamma}_d^h(t) \in HP(\boldsymbol{v}, C)$ *for all* $t \in I$.

(ii) $m_d^2(t) > n_d^2(t)$ *and* $\dot{\mathcal{E}}_v^h(\boldsymbol{\gamma}_d)(t) = 0$ *for all* $t \in I$ *if and only if* $\boldsymbol{\gamma}_d^h(t)$ *is a point or there exist a constant timelike vector* $\boldsymbol{w}$ *and a constant real number* C *with* $C^2 < 1$*, such that* $\boldsymbol{\gamma}_d(t) \in DP(\boldsymbol{w}, -C)$ *and* $\boldsymbol{\gamma}_d^h(t) \in HP(\boldsymbol{w}, -1)$ *for all* $t \in I$.

Proposition 4.20 *Suppose that* $(\boldsymbol{\gamma}_d, \boldsymbol{\gamma}_d^h)$ *is a spacelike Legendrian immersion with the spacelike de Sitter Legendrian curvature* (m_d, n_d) *and* $(\boldsymbol{\gamma}_d^\phi, (\boldsymbol{\gamma}_d^\phi)^h)$ *is a spacelike parallel Legendrian immersion with the spacelike de Sitter Legendrian curvature* (m_d^ϕ, n_d^ϕ)*. Then* $\mathcal{E}_v(\boldsymbol{\gamma}_d) = \mathcal{E}_v(\boldsymbol{\gamma}_d^\phi)$ *for any* $\phi \in \mathbb{R}$.

4.3 The evolutes of timelike fronts in de Sitter 2-space

Finally, we consider the geometric meanings of evolutes of timelike fronts in S_1^2. Let $(\boldsymbol{\gamma}_T, \boldsymbol{\gamma}_T^d) : I \to \Delta_5$ be a timelike Legendrian immersion with the timelike de Sitter Legendrian curvature (m_T, n_T). We define a mapping $\mathcal{E}_v^d(\boldsymbol{\gamma}_T) : I \to S_1^2$ by

$$\mathcal{E}_v^d(\boldsymbol{\gamma}_T)(t) = \pm\frac{1}{\sqrt{n_T^2(t) + m_T^2(t)}}\big(n_T(t)\boldsymbol{\gamma}_T(t) - m_T(t)\boldsymbol{\gamma}_T^d(t)\big)$$

and call it the *spacelike evolute* of $\boldsymbol{\gamma}_T$ in S_1^2. By a direct calculation, we have the following properties about the spacelike evolutes of timelike fronts in S_1^2.

Proposition 4.21 *Suppose that* $(\boldsymbol{\gamma}_T, \boldsymbol{\gamma}_T^d) : I \to \Delta_5$ *is a timelike Legendrian immersion with the timelike de Sitter Legendrian curvature* (m_T, n_T)*. Then the spacelike evolute* $\mathcal{E}_v(\boldsymbol{\gamma}_T)$ *of* $\boldsymbol{\gamma}_T$ *is independent on the parametrization of* $(\boldsymbol{\gamma}_T, \boldsymbol{\gamma}_T^d)$.

Remark 4.22 *Let* $(\boldsymbol{\gamma}_T, \boldsymbol{\gamma}_T^d) : I \to \Delta_5$ *be a timelike Legendrian immersion.*

(i) *If we take* $-\boldsymbol{\gamma}_T$ *instead of* $\boldsymbol{\gamma}_T$*, then the spacelike evolute of* $\boldsymbol{\gamma}_T$ *does not change.*

(ii) *If we take* $-\boldsymbol{\gamma}_T^d$ *instead of* $\boldsymbol{\gamma}_T^d$*, then the spacelike evolute of* $\boldsymbol{\gamma}_T$ *does not change.*

Proposition 4.23 *Let* $\boldsymbol{\gamma}_T : I \to S_1^2$ *be a regular timelike curve in* S_1^2 *with the timelike de Sitter geodesic curvature* κ_T*. Then* $E_v^d(\boldsymbol{\gamma}_T)(t) = \mathcal{E}_v^d(\boldsymbol{\gamma}_T)(t)$.

Proof. Without loss of generality, by Example 3.13, we take $\boldsymbol{\gamma}_T^d = \boldsymbol{e}_T$, then $(\boldsymbol{\gamma}_T, \boldsymbol{\gamma}_T^d)$ is a timelike Legendrian immersion with the timelike de Sitter Legendrian curvature $(||\dot{\boldsymbol{\gamma}}_T||, ||\dot{\boldsymbol{\gamma}}_T||\kappa_T)$. It follows from the definiton of the spacelike evolute of $\boldsymbol{\gamma}_T$, we have

$$\begin{aligned}
\mathcal{E}_v^d(\boldsymbol{\gamma}_T)(t) &= \pm\frac{1}{\sqrt{n_T^2(t) + m_T^2(t)}}\big(n_T(t)\boldsymbol{\gamma}_T(t) - m_T(t)\boldsymbol{\gamma}_T^d(t)\big) \\
&= \pm\frac{1}{\sqrt{\kappa_T^2(t) + 1}}\left(\kappa_T(t)\boldsymbol{\gamma}_T(t) - \boldsymbol{e}_T(t)\right) \\
&= E_v^d(\boldsymbol{\gamma}_T)(t).
\end{aligned}$$

□

According to the above proposition, we have shown that the definition of the spacelike evolute of $\boldsymbol{\gamma}_T$ is consistent with the definition of the evolute of $\boldsymbol{\gamma}_T$ when $\boldsymbol{\gamma}_T$ is a regular timelike curve in S_1^2.

Proposition 4.24 *If* $(\boldsymbol{\gamma}_T, \boldsymbol{\gamma}_T^d) : I \to \Delta_5$ *is a timelike Legendrian immersion with the timelike de Sitter Legendrian curvature* (m_T, n_T)*, then we have the following:*

(i) *If* t_0 *is a singular point of* $\boldsymbol{\gamma}_T$*, then* $\mathcal{E}_v^d(\boldsymbol{\gamma}_T)(t_0) = \pm\boldsymbol{\gamma}_T(t_0)$.

(ii) *If* t_0 *is a singular point of* $\boldsymbol{\gamma}_T^d$*, then* $\mathcal{E}_v^d(\boldsymbol{\gamma}_T)(t_0) = \pm\boldsymbol{\gamma}_T^d(t_0)$.

(iii) $(\boldsymbol{\gamma}_T^h, \mathcal{E}_v^d(\boldsymbol{\gamma}_T)) : I \to \Delta_1$ *is a spacelike Legendrian immersion with the spacelike de Sitter Legendrian curvature*

$$\left(\frac{m_T\dot{n}_T - \dot{m}_T n_T}{n_T^2 + m_T^2}, \pm\sqrt{n_T^2 + m_T^2}\right).$$

Proof. (i) Since t_0 is a singular point of $\boldsymbol{\gamma}_T$, we have $m_T(t_0) = 0$. It follows that

$$\mathcal{E}_v^d(\boldsymbol{\gamma}_T)(t_0) = \pm\frac{1}{\sqrt{n_T^2(t_0)}} n_T(t_0)\boldsymbol{\gamma}_T(t_0) = \pm\boldsymbol{\gamma}_T(t_0).$$

(ii) Since t_0 is a singular point of $\boldsymbol{\gamma}_T^d$, we have $n_T(t_0) = 0$. It follows that

$$\mathcal{E}_v^d(\boldsymbol{\gamma}_T)(t_0) = \pm\frac{1}{\sqrt{m_T^2(t_0)}} m_T(t_0)\boldsymbol{\gamma}_T^d(t_0) = \pm\boldsymbol{\gamma}_T^d(t_0).$$

(iii) We denote that $\widetilde{\boldsymbol{\gamma}}_d = \mathcal{E}_v^d(\boldsymbol{\gamma}_T)$, $\widetilde{\boldsymbol{\gamma}}_d^h = \boldsymbol{\gamma}_T^h$ and $\widetilde{\boldsymbol{\gamma}}_d^s = \widetilde{\boldsymbol{\gamma}}_d \wedge \widetilde{\boldsymbol{\gamma}}_d^h$. By a straightforward calculation, we have

$$\dot{\widetilde{\boldsymbol{\gamma}}}_d(t) = \frac{m_T(t)\dot{n}_T(t) - \dot{m}_T(t)n_T(t)}{n_T^2(t) + m_T^2(t)}\left(\frac{\pm 1}{\sqrt{n_T^2(t) + m_T^2(t)}}(m_T(t)\boldsymbol{\gamma}_T(t) + n_T(t)\boldsymbol{\gamma}_T^d(t))\right),$$

$$\dot{\widetilde{\boldsymbol{\gamma}}}_d^h(t) = \dot{\boldsymbol{\gamma}}_T^h(t) = \pm\sqrt{n_T^2(t) + m_T^2(t)}\left(\frac{\pm 1}{\sqrt{n_T^2(t) + m_T^2(t)}}(m_T(t)\boldsymbol{\gamma}_T(t) + n_T(t)\boldsymbol{\gamma}_T^d(t))\right)$$

and

$$\widetilde{\boldsymbol{\gamma}}_d^s(t) = \frac{\pm 1}{\sqrt{n_T^2(t) + m_T^2(t)}}(m_T(t)\boldsymbol{\gamma}_T(t) + n_T(t)\boldsymbol{\gamma}_T^d(t)).$$

It follows that

$$\langle\dot{\widetilde{\boldsymbol{\gamma}}}_d(t), \dot{\widetilde{\boldsymbol{\gamma}}}_d(t)\rangle = \left(\frac{m_T(t)\dot{n}_T(t) - \dot{m}_T(t)n_T(t)}{n_T^2(t) + m_T^2(t)}\right)^2 \geq 0,$$

$$\langle\dot{\widetilde{\boldsymbol{\gamma}}}_d(t), \widetilde{\boldsymbol{\gamma}}_d^h(t)\rangle = \langle\dot{\mathcal{E}}_v^d(\boldsymbol{\gamma}_T)(t), \boldsymbol{\gamma}_d^h(t)\rangle = 0,\ (\dot{\widetilde{\boldsymbol{\gamma}}}_d(t), \dot{\widetilde{\boldsymbol{\gamma}}}_d^h(t)) \neq (0, 0)$$

and

$$\widetilde{m}_d(t) = \langle\dot{\widetilde{\boldsymbol{\gamma}}}_d(t), \widetilde{\boldsymbol{\gamma}}_d^s(t)\rangle = \frac{m_T(t)\dot{n}_T(t) - \dot{m}_T(t)n_T(t)}{n_T^2(t) + m_T^2(t)},$$

$$\widetilde{n}_d(t) = \langle\dot{\widetilde{\boldsymbol{\gamma}}}_d^h(t), \widetilde{\boldsymbol{\gamma}}_d^s(t)\rangle = \pm\sqrt{n_T^2(t) + m_T^2(t)}.$$

This means that $(\widetilde{\boldsymbol{\gamma}}_d^h, \widetilde{\boldsymbol{\gamma}}_d) = (\boldsymbol{\gamma}_T^h, \mathcal{E}_v^d(\boldsymbol{\gamma}_T)) : I \to \Delta_1$ is a spacelike Legendrian immersion with the spacelike de Sitter Legendrian curvature

$$\left(\frac{m_T\dot{n}_T - \dot{m}_T n_T}{n_T^2 + m_T^2},\ \pm\sqrt{n_T^2 + m_T^2}\right).$$

□

Let $(\boldsymbol{\gamma}_T, \boldsymbol{\gamma}_T^d) : I \to \Delta_5$ be a timelike Legendrian immersion. We now define a function $F^S : I \times S_1^2 \to \mathbb{R}$ by $F^S(t, \boldsymbol{v}) = \langle\boldsymbol{\gamma}_T^h(t), \boldsymbol{v}\rangle$ and call it the *de Sitter height function* on the timelike Legendrian immersion $(\boldsymbol{\gamma}_T, \boldsymbol{\gamma}_T^d)$. By a straightforward calculation, we can show the following proposition.

Proposition 4.25 *Let $(\boldsymbol{\gamma}_T, \boldsymbol{\gamma}_T^d) : I \to \Delta_5$ be a timelike Legendrian immersion with the timelike de Sitter Legendrian curvature (m_T, n_T). For all $(t, \boldsymbol{v}) \in I \times S_1^2$, we have the following*:

(i) $F^S(t, \boldsymbol{v}) = 0$ *if and only if there exist real numbers a and b such that* $\boldsymbol{v} = a\boldsymbol{\gamma}_T(t) + b\boldsymbol{\gamma}_T^d(t)$.

(ii) $F^S(t, \boldsymbol{v}) = (\partial F^S/\partial t)(t, \boldsymbol{v}) = 0$ *if and only if* $\boldsymbol{v} = \mathcal{E}_v^d(\boldsymbol{\gamma}_T)(t)$.

One can show that $(F^S, \partial F^S/\partial t)$ is non-singular at $(t, \boldsymbol{v}) \in \mathcal{D}(F^S)$, where

$$\mathcal{D}(F^S) = \{(t, \boldsymbol{v}) | F^S(t, \boldsymbol{v}) = \frac{\partial F^S}{\partial t}(t, \boldsymbol{v}) = 0\}.$$

This means that F^S is a Morse family. Therefore, the spacelike evolute $\mathcal{E}_v^d(\boldsymbol{\gamma}_T)$ of the timelike front $\boldsymbol{\gamma}_T$ is a wavefront of a Legendrian immersion generated by F^S.

By almost the same arguments with Proposition 4.6, we have the following proposition.

Proposition 4.26 *Let $(\boldsymbol{\gamma}_T, \boldsymbol{\gamma}_T^d) : I \to \Delta_5$ be a timelike Legendrian immersion with the timelike de Sitter Legendrian curvature (m_T, n_T). If t_0 is a singular point of $\boldsymbol{\gamma}_T$, then we have the following assertions*:

(i) t_0 *is a regular point of* $\mathcal{E}_v^d(\boldsymbol{\gamma}_T)$ *if and only if* $\dot{m}_T(t_0) \neq 0$.

(ii) t_0 *is a singular point of* $\mathcal{E}_v^d(\boldsymbol{\gamma}_T)$ *if and only if* $\ddot{\boldsymbol{\gamma}}_T(t_0) = 0$.

Proposition 4.27 *Let $(\boldsymbol{\gamma}_T, \boldsymbol{\gamma}_T^d) : I \to \Delta_5$ be a timelike Legendrian immersion. $\dot{\mathcal{E}}_v^d(\boldsymbol{\gamma}_T)(t) = 0$ for all $t \in I$ if and only if $\boldsymbol{\gamma}_T(t)$ is a point or $\boldsymbol{\gamma}_T^d(t)$ is a point or there exist a constant spacelike vector $\boldsymbol{w}$ and two constant real numbers C_1 and C_2 with $(C_1, C_2) \neq (0, 0)$, such that $\boldsymbol{\gamma}_T(t) \in DP(\boldsymbol{w}, C_1)$ and $\boldsymbol{\gamma}_T^d(t) \in DP(\boldsymbol{w}, -C_2)$ for all $t \in I$.*

Proof. Since

$$\dot{\mathcal{E}}_v^d(\boldsymbol{\gamma}_T)(t) = \frac{m_T(t)\dot{n}_T(t) - \dot{m}_T(t)n_T(t)}{n_T^2(t) + m_T^2(t)} \left(\frac{\pm 1}{\sqrt{n_T^2(t) + m_T^2(t)}} (m_T(t)\boldsymbol{\gamma}_T(t) + n_T(t)\boldsymbol{\gamma}_T^d(t)) \right),$$

we have $\dot{\mathcal{E}}_v^d(\boldsymbol{\gamma}_T)(t) = 0$ if and only if $\dot{m}_T(t)n_T(t) - m_T(t)\dot{n}_T(t) = 0$ for all $t \in I$. Therefore, there exist two constant real numbers C_1 and C_2 with $(C_1, C_2) \neq (0, 0)$ such that $C_1 m_T(t) = C_2 n_T(t)$. In the case when $C_1 \equiv 0$, we have $\dot{\boldsymbol{\gamma}}_T^d(t) \equiv 0$. This means that $\boldsymbol{\gamma}_T^d(t)$ is a point. Moreover, if $C_2 \equiv 0$, we have $\dot{\boldsymbol{\gamma}}_T(t) \equiv 0$. This means that $\boldsymbol{\gamma}_T(t)$ is a point. Otherwise, we have $C_1\dot{\boldsymbol{\gamma}}_T(t) = C_2\dot{\boldsymbol{\gamma}}_T^d(t)$. So that $C_1\boldsymbol{\gamma}_T(t) = C_2\boldsymbol{\gamma}_T^d(t) + \boldsymbol{w}$, where $\boldsymbol{w}$ is a constant spacelike vector. It is obviously that $\langle \boldsymbol{\gamma}_T(t), \boldsymbol{w} \rangle = C_1$ and $\langle \boldsymbol{\gamma}_T^d(t), \boldsymbol{w} \rangle = -C_2$ for all $t \in I$.

Conversely, if $\boldsymbol{\gamma}_T(t)$ is a point for all $t \in I$, then $m_T(t) \equiv \dot{m}_T(t) \equiv 0$. It follows that $\dot{\mathcal{E}}_v^d(\boldsymbol{\gamma}_T)(t) \equiv 0$. Moreover, if $\boldsymbol{\gamma}_T^d(t)$ is a point for all $t \in I$, then $n_T(t) \equiv \dot{n}_T(t) \equiv 0$. It also follows that $\dot{\mathcal{E}}_v^d(\boldsymbol{\gamma}_T)(t) \equiv 0$. On the other hand, we assume that $\boldsymbol{w}$ is a constant spacelike vector, C_1 and C_2 are constant real numbers which satisfy $(C_1, C_2) \neq (0, 0)$, $\boldsymbol{\gamma}_T(t) \in DP(\boldsymbol{w}, C_1)$ and $\boldsymbol{\gamma}_T^d(t) \in DP(\boldsymbol{w}, -C_2)$. It follows that $\langle \boldsymbol{\gamma}_T(t), \boldsymbol{w} \rangle = C_1$ and $\langle \boldsymbol{\gamma}_T^d(t), \boldsymbol{w} \rangle = -C_2$, then $\langle \dot{\boldsymbol{\gamma}}_T(t), \boldsymbol{w} \rangle = 0$ and $\langle \dot{\boldsymbol{\gamma}}_T^d(t), \boldsymbol{w} \rangle = 0$. We denote $\boldsymbol{w} = C_1\boldsymbol{\gamma}_T(t) - C_2\boldsymbol{\gamma}_T^d(t)$, then $\dot{\boldsymbol{w}} = C_1 m_T(t)\boldsymbol{\gamma}_T^h(t) - C_2 n_T(t)\boldsymbol{\gamma}_T^h(t) = 0$. Thus we have $C_1 m_T(t) = C_2 n_T(t)$ and $C_1\dot{m}_T(t) = C_2\dot{n}_T(t)$. This means that $\dot{m}_T(t)n_T(t) - m_T(t)\dot{n}_T(t) = 0$ for all $t \in I$. Therefore, $\dot{\mathcal{E}}_v^d(\boldsymbol{\gamma}_T)(t) = 0$ for all $t \in I$. □

We also prove the following proposition by the similar arguments with Proposition 4.9.

Proposition 4.28 *Suppose that $(\boldsymbol{\gamma}_T, \boldsymbol{\gamma}_T^d) : I \to \Delta_5$ be a timelike Legendrian immersion with the timelike de Sitter Legendrian curvature (m_T, n_T) and $(\boldsymbol{\gamma}_T^\theta, (\boldsymbol{\gamma}_T^\theta)^d)$ be a timelike parallel Legendrian immersion with the timelike de Sitter Legendrian curvature (m_T^θ, n_T^θ). Then $\mathcal{E}_v^d(\boldsymbol{\gamma}_T) = \mathcal{E}_v^d(\boldsymbol{\gamma}_T^\theta)$ for any $\theta \in [0, 2\pi)$.*

5 The relationships among the evolutes of spacelike fronts and timelike fronts

We investigate the relationships among the evolutes of spacelike fronts and timelike fronts in hyperbolic 2-space and de Sitter 2-space. We firstly discuss the relationship between the totally evolute $\mathcal{E}_v(\boldsymbol{\gamma}_h)$ of a spacelike front $\boldsymbol{\gamma}_h$ in $H^2(-1)$ and the totally evolute $\mathcal{E}_v(\boldsymbol{\gamma}_d)$ of a spacelike front $\boldsymbol{\gamma}_d$ in S^2_1. As results, we can show the following theorem.

Theorem 5.1 *Suppose that* $(\boldsymbol{\gamma}_h, \boldsymbol{\gamma}_d) : I \to \Delta_1$ *is a spacelike Legendrian immersion with the spacelike hyperbolic Legendrian curvature* (m_h, n_h) *which satisfies* $m_h^2(t) \neq n_h^2(t)$ *for all* $t \in I$. *Then we have*:

(i) *If* $m_h^2(t) < n_h^2(t)$, *then* $\mathcal{E}_v^h(\boldsymbol{\gamma}_h)(t) = \mathcal{E}_v^h(\boldsymbol{\gamma}_d)(t)$.

(ii) *If* $m_h^2(t) > n_h^2(t)$, *then* $\mathcal{E}_v^d(\boldsymbol{\gamma}_h)(t) = \mathcal{E}_v^d(\boldsymbol{\gamma}_d)(t)$.

Proof. We firstly assume that $\boldsymbol{\gamma}_h^s = \boldsymbol{\gamma}_h \wedge \boldsymbol{\gamma}_d$ and $\boldsymbol{\gamma}_h^d = \boldsymbol{\gamma}_d$. Since $(\boldsymbol{\gamma}_h, \boldsymbol{\gamma}_d) : I \to \Delta_1$ is a spacelike Legendrian immersion, we have

$$\begin{cases} \dot{\boldsymbol{\gamma}}_h(t) = m_h(t)\boldsymbol{\gamma}_h^s(t), \\ \dot{\boldsymbol{\gamma}}_h^d(t) = n_h(t)\boldsymbol{\gamma}_h^s(t) = \dot{\boldsymbol{\gamma}}_d(t). \end{cases}$$

On the other hand, we denote that $\boldsymbol{\gamma}_d^s = \boldsymbol{\gamma}_d \wedge \boldsymbol{\gamma}_h$ and $\boldsymbol{\gamma}_d^h = \boldsymbol{\gamma}_h$. Then we have $\boldsymbol{\gamma}_d^s = -\boldsymbol{\gamma}_h^s$ and

$$\begin{cases} \dot{\boldsymbol{\gamma}}_d(t) = m_d(t)\boldsymbol{\gamma}_d^s(t), \\ \dot{\boldsymbol{\gamma}}_d^h(t) = n_d(t)\boldsymbol{\gamma}_d^s(t) = \dot{\boldsymbol{\gamma}}_h(t). \end{cases}$$

It is obviously that

$$m_h(t) = -n_d(t),\ n_h(t) = -m_d(t).$$

Therefore if $m_h^2(t) < n_h^2(t)$, then we have $m_d^2(t) > n_d^2(t)$ and

$$\begin{aligned} \mathcal{E}_v^h(\boldsymbol{\gamma}_h)(t) &= \pm\frac{1}{\sqrt{n_h^2(t) - m_h^2(t)}}\left(n_h(t)\boldsymbol{\gamma}_h(t) - m_h(t)\boldsymbol{\gamma}_d(t)\right) \\ &= \pm\frac{1}{\sqrt{m_d^2(t) - n_d^2(t)}}\left(n_d(t)\boldsymbol{\gamma}_d(t) - m_d(t)\boldsymbol{\gamma}_h(t)\right) = \mathcal{E}_v^h(\boldsymbol{\gamma}_d)(t). \end{aligned}$$

Moreover, if $m_h^2(t) > n_h^2(t)$, then we have $m_d^2(t) < n_d^2(t)$ and

$$\begin{aligned} \mathcal{E}_v^d(\boldsymbol{\gamma}_h)(t) &= \pm\frac{1}{\sqrt{m_h^2(t) - n_h^2(t)}}\left(n_h(t)\boldsymbol{\gamma}_h(t) - m_h(t)\boldsymbol{\gamma}_d(t)\right) \\ &= \pm\frac{1}{\sqrt{n_d^2(t) - m_d^2(t)}}\left(n_d(t)\boldsymbol{\gamma}_d(t) - m_d(t)\boldsymbol{\gamma}_h(t)\right) = \mathcal{E}_v^d(\boldsymbol{\gamma}_d)(t). \end{aligned}$$

□

On the other hand, according to the timelike de Sitter Legendrian Frenet-Serret type formula, if $\boldsymbol{\gamma}_T : I \to S^2_1$ is a timelike front in S^2_1, then we can know that $\boldsymbol{\gamma}_T^d : I \to S^2_1$ is a timelike curve in S^2_1 and $(\boldsymbol{\gamma}_T^d(t), \boldsymbol{\gamma}_T(t)) \in \Delta_5$. Therefore, $\boldsymbol{\gamma}_T^d$ is also a timelike front in S^2_1. We can define the spacelike evolute of $\boldsymbol{\gamma}_T^d$ as follows: we assume that $\widetilde{\boldsymbol{\gamma}}_T = \boldsymbol{\gamma}_T^d$, $\widetilde{\boldsymbol{\gamma}}_T^d = \boldsymbol{\gamma}_T$ and $\widetilde{\boldsymbol{\gamma}}_T^h = \widetilde{\boldsymbol{\gamma}}_T \wedge \widetilde{\boldsymbol{\gamma}}_T^d = -\boldsymbol{\gamma}_T^h$, then we have

$$\begin{cases} \widetilde{m}_T(t)\widetilde{\boldsymbol{\gamma}}_T^h(t) = \dot{\widetilde{\boldsymbol{\gamma}}}_T(t) = \dot{\boldsymbol{\gamma}}_T^d(t) = n_T(t)\boldsymbol{\gamma}_T^h(t) = -n_T(t)\widetilde{\boldsymbol{\gamma}}_T^h(t), \\ \widetilde{n}_T(t)\widetilde{\boldsymbol{\gamma}}_T^h(t) = \dot{\widetilde{\boldsymbol{\gamma}}}{}_T^d(t) = \dot{\boldsymbol{\gamma}}_T(t) = m_T(t)\boldsymbol{\gamma}_T^h(t) = -m_T(t)\widetilde{\boldsymbol{\gamma}}_T^h(t). \end{cases}$$

Therefore, $\widetilde{m}_T(t) = -n_T(t)$, $\widetilde{n}_T(t) = -m_T(t)$. Then we have

$$\mathcal{E}_v^d(\boldsymbol{\gamma}_T^d)(t) = \mathcal{E}_v^d(\widetilde{\boldsymbol{\gamma}}_T)(t) = \pm\frac{1}{\sqrt{n_T^2(t)+m_T^2(t)}}\big(-m_T(t)\boldsymbol{\gamma}_T^d(t) + n_T(t)\boldsymbol{\gamma}_T(t)\big) = \mathcal{E}_v^d(\boldsymbol{\gamma}_T)(t).$$

Hence we have shown the following theorem.

Theorem 5.2 *Let $(\boldsymbol{\gamma}_T, \boldsymbol{\gamma}_T^d) : I \to \Delta_5$ be a timelike Legendrian immersion, then $\mathcal{E}_v^d(\boldsymbol{\gamma}_T)(t) = \mathcal{E}_v^d(\boldsymbol{\gamma}_T^d)(t)$ for all $t \in I$.*

Remark 5.3 *By Proposition 4.28, if we take $\theta = \pi/2$, then we can also prove $\mathcal{E}_v^d(\boldsymbol{\gamma}_T)(t) = \mathcal{E}_v^d(\boldsymbol{\gamma}_T^d)(t)$ for all $t \in I$.*

Theorem 5.4 *Suppose that $(\boldsymbol{\gamma}_h, \boldsymbol{\gamma}_h^d) : I \to \Delta_1$ is a spacelike Legendrian immersion with the spacelike hyperbolic Legendrian curvature (m_h, n_h). Then we have the following assertions for all $t \in I$:*

(i) *If $m_h^2(t) > n_h^2(t)$, then $\boldsymbol{\gamma}_h^s$ is a timelike front in S_1^2 and $\mathcal{E}_v^d(\boldsymbol{\gamma}_h^s)(t) = \mathcal{E}_v^d(\mathcal{E}_v^d(\boldsymbol{\gamma}_h))(t)$.*

(ii) *If $m_h^2(t) < n_h^2(t)$, then $\boldsymbol{\gamma}_h^s$ is a spacelike front in S_1^2. Moreover,*

  (a) *if $(n_h^2(t) - m_h^2(t))^3 > (\dot{m}_h(t)n_h(t) - m_h(t)\dot{n}_h(t))^2$, then $\mathcal{E}_v^h(\boldsymbol{\gamma}_h^s)(t) = \mathcal{E}_v^h(\mathcal{E}_v^h(\boldsymbol{\gamma}_h))(t)$.*

  (b) *if $(n_h^2(t) - m_h^2(t))^3 < (\dot{m}_h(t)n_h(t) - m_h(t)\dot{n}_h(t))^2$, then $\mathcal{E}_v^d(\boldsymbol{\gamma}_h^s)(t) = \mathcal{E}_v^d(\mathcal{E}_v^h(\boldsymbol{\gamma}_h))(t)$.*

Proof. According to Proposition 4.4, if $m_h^2(t) > n_h^2(t)$, then $(\mathcal{E}_v^d(\boldsymbol{\gamma}_h)(t), \boldsymbol{\gamma}_h^s(t)) \in \Delta_5$ is a timelike Legendrian immersion. Therefore, $\boldsymbol{\gamma}_h^s$ is a timelike front in S_1^2. Moreover, it follows from Theorem 5.2, we have $\mathcal{E}_v^d(\boldsymbol{\gamma}_h^s)(t) = \mathcal{E}_v^d(\mathcal{E}_v^d(\boldsymbol{\gamma}_h))(t)$.

On the other hand, If $m_h^2(t) < n_h^2(t)$, then $(\mathcal{E}_v^h(\boldsymbol{\gamma}_h)(t), \boldsymbol{\gamma}_h^s(t)) \in \Delta_1$ is a spacelike Legendrian immersion. Therefore, $\boldsymbol{\gamma}_h^s$ is a spacelike front in S_1^2. Moreover, it follows from Proposition 4.4 and Theorem 5.1, we have $\mathcal{E}_v^h(\boldsymbol{\gamma}_h^s)(t) = \mathcal{E}_v^h(\mathcal{E}_v^h(\boldsymbol{\gamma}_h))(t)$ and $\mathcal{E}_v^d(\boldsymbol{\gamma}_h^s)(t) = \mathcal{E}_v^d(\mathcal{E}_v^h(\boldsymbol{\gamma}_h))(t)$. □

Remark 5.5 *Since $\mathcal{E}_v^d(\boldsymbol{\gamma}_h) : I \to S_1^2$ is a timelike front in S_1^2, $\mathcal{E}_v^h(\mathcal{E}_v^d(\boldsymbol{\gamma}_h))$ does not exist.*

By Δ_1-duality, Theorems 5.1 and 5.4, we have the following corollary.

Corollary 5.6 *Suppose that $(\boldsymbol{\gamma}_d^h, \boldsymbol{\gamma}_d) : I \to \Delta_1$ is a spacelike Legendrian immersion with the spacelike de Sitter Legendrian curvature (m_d, n_d). Then we have the following assertions for all $t \in I$:*

(i) *If $m_d^2(t) > n_d^2(t)$, then $\boldsymbol{\gamma}_d^s$ is a spacelike front in S_1^2. Moreover,*

  (a) *if $(m_d^2(t) - n_d^2(t))^3 > (\dot{m}_d(t)n_d(t) - m_d(t)\dot{n}_d(t))^2$, then $\mathcal{E}_v^h(\boldsymbol{\gamma}_d^s)(t) = \mathcal{E}_v^h(\mathcal{E}_v^h(\boldsymbol{\gamma}_d))(t)$.*

  (b) *if $(m_d^2(t) - n_d^2(t))^3 < (\dot{m}_d(t)n_d(t) - m_d(t)\dot{n}_d(t))^2$, then $\mathcal{E}_v^d(\boldsymbol{\gamma}_d^s)(t) = \mathcal{E}_v^d(\mathcal{E}_v^h(\boldsymbol{\gamma}_d))(t)$*

(ii) *If $m_d^2(t) < n_d^2(t)$, then $\boldsymbol{\gamma}_d^s$ is a timelike front in S_1^2 and $\mathcal{E}_v^d(\boldsymbol{\gamma}_d^s)(t) = \mathcal{E}_v^d(\mathcal{E}_v^d(\boldsymbol{\gamma}_d))(t)$.*

By almost the same arguments as Theorem 5.4, we have the following result.

Theorem 5.7 *If $(\boldsymbol{\gamma}_T, \boldsymbol{\gamma}_T^d) : I \to \Delta_5$ is a timelike Legendrian immersion with the timelike de Sitter Legendrian curvature (m_T, n_T), then $\boldsymbol{\gamma}_T^h$ is a spacelike front in H_1^2. Moreover, we have the following assertions for all $t \in I$,*

(a) *if $(n_T^2(t) + m_T^2(t))^3 > (\dot{m}_T(t)n_T(t) - m_T(t)\dot{n}_T(t))^2$, then $\mathcal{E}_v^h(\boldsymbol{\gamma}_T^h)(t) = \mathcal{E}_v^h(\mathcal{E}_v^d(\boldsymbol{\gamma}_T))(t)$.*

(b) *if $(n_T^2(t) + m_T^2(t))^3 < (\dot{m}_T(t)n_T(t) - m_T(t)\dot{n}_T(t))^2$, then $\mathcal{E}_v^d(\boldsymbol{\gamma}_T^h)(t) = \mathcal{E}_v^d(\mathcal{E}_v^d(\boldsymbol{\gamma}_T))(t)$.*

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