On the Stokes semigroup in some non-Helmholtz domains

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Abstract. This paper shows that L^p-Helmholtz decomposition is not necessary to establish the analyticity of the Stokes semigroup in $C_{0,\sigma}$, the L^∞-closure of the space of all compactly supported smooth solenoidal vector fields. In fact, in a sector-like domain for which the L^p-Helmholtz decomposition does not hold, the analyticity of the Stokes semigroup in $C_{0,\sigma}$ is proved.

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1. Introduction

This paper is concerned with the Stokes semigroup; i.e., the solution operator $S(t): v_0 \to v(\ ,t)$ of the initial-boundary problem for the Stokes system

$$v_t - \Delta v + \nabla q = 0, \quad \operatorname{div} v = 0 \quad \text{in} \quad \Omega \times (0,T)$$

under zero Dirichlet boundary condition with initial condition $v|_{t=0} = v_0$, where Ω is a domain in $\mathbf{R}^n$ with $n \geq 2$. It is well known that $S(t)$ forms an analytic semigroup in $L^p_\sigma(\Omega)$ $(1 < p < \infty)$ for various kind of domains Ω including smoothly bounded domains [13], [18], where $L^p_\sigma(\Omega)$ is the L^p-closure of $C^\infty_{c,\sigma}(\Omega)$, the space of all solenoidal vector fields with compact support in Ω. In fact, the analyticity of $S(t)$ in $L^p_\sigma(\Omega)$ holds for any uniformly C^2-domain

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Ω provided that $L^p(\Omega)$ admits a topological direct sum decomposition, called the Helmholtz decomposition:

$$L^p(\Omega) = L^p_\sigma(\Omega) \oplus G^p(\Omega),\ G^p(\Omega) = \{\nabla q \in L^p(\Omega) \mid q \in L^p_{loc}(\Omega)\}$$

This is recently proved in [12], where the maximal regularity in $L^p_\sigma(\Omega)$ is also established. The Helmholtz decomposition holds for any domain if $p = 2$ and for various kind of domains like bounded or exterior domains with smooth boundary for $1 < p < \infty$ [11]. However, for any $p > 2$ there is an improper smooth sector-like domain such that the L^p-Helmholtz decomposition fails to satisfy [8], [15]. To be more precise, let S_θ denote $S_\theta = \{x = (x_1, x_2) \mid |\arg x| < \theta/2\}$, which is a sector in the plane $\mathbf{R}^2$ with opening angle $0 < \theta < 2\pi$. We say that a planar domain Ω is a sector-like domain with opening angle θ if $\Omega \backslash D_R = S_\theta \backslash D_R$ for some $R > 0$ (up to rotation and translation), where D_R is an open disk of radius R centered at the origin (figure 1).

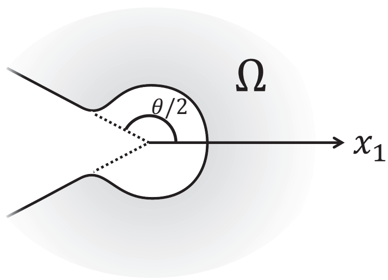

Figure 1

According to [15, Example 2, Fig. 5], the L^p-Helmholtz decomposition fails for a sector-like domain when $p > q'_\theta$ or $p < q_\theta$ with $q_\theta = 2/(1 + \pi/\theta)$ and $1/q_\theta + 1/q'_\theta = 1$ even if it is smooth. (For $p \in (q_\theta, q'_\theta)$ the L^p-Helmholtz decomposition holds [15].) Note that if the opening angle is larger than π, there always exists $p > 2$ such that the L^p-Helmholtz decomposition fails.

The goal of this paper is to prove that the Stokes semigroup forms an analytic semigroup in $C_{0,\sigma}(\Omega)$ for a C^3 sector-like domain Ω for which the L^p-Helmholtz decomposition may fail. This shows that the existence of the L^p-Helmholtz decomposition may not be necessary for the analyticity of $S(t)$ in $C_{0,\sigma}(\Omega)$ although it is convenient to establish [3]. Note that the analyticity of $S(t)$ in $L^p_\sigma(\Omega)$ is not sufficient to guarantee its analyticity in $C_{0,\sigma}(\Omega)$ as shown in [19]. In fact, in [19] $S(t)$ is not analytic in $C_{0,\sigma}(\Omega)$ when Ω is an infinite layer domain in $\mathbf{R}^n$ $(n \geq 3)$ while it is analytic in $L^p_\sigma(\Omega)$ $(1 < p < \infty)$; see e.g. [5], [6], [7]. (For a cylindrical domain (or an infinite cylinder) it is shown in [4] that $S(t)$ is analytic in $C_{0,\sigma}(\Omega)$ which is also analytic in $L^p_\sigma(\Omega)$ [10].)

Theorem 1.1. *Let Ω be a C^3 sector-like domain. Then $S(t)$ is a C_0-analytic semigroup in $C_{0,\sigma}(\Omega)$, the L^∞-closure of $C^\infty_{c,\sigma}(\Omega)$. (Moreover, $t\|\nabla^2 S(t)v_0\|_\infty / \|v_0\|_\infty$ is bounded in $(0,T)$ where $\|\cdot\|_\infty$ denotes the supremum norm in Ω.)*

Interpolating L^∞-result (Theorem 1.1) and L^2-result yields that the Stokes semigroup $S(t)$ is a C_0-analytic semigroup in a complex interpolated space $X_p = \left[L^2_\sigma(\Omega), C_{0,\sigma}(\Omega)\right]_\theta, 2/p = 1-\theta$. However, it is not clear that this space X_p (continuously embedded in $L^p_\sigma(\Omega)$) agrees with $L^p_\sigma(\Omega)$.

By a recent result of [1] (see also [2], [3]) to show Theorem 1.1 it suffices to establish the next theorem which will be proved in the rest of this paper. Since we invoked the Hölder theory in [1], we need $C^{2,\gamma}$ regularity to apply results in [1]. This is a reason why we assume C^3 in Theorem 1.1 although it is not optimal at all. Note that it turns out that C^2 regularity is enough to prove the analyticity as in [3, Remark 1.5 (ii)].

Theorem 1.2. *Let Ω be a C^2 sector-like domain. Then Ω is admissible in the sense of* [1, Definition 2.3].

2. Uniqueness for the Neumann problem

We consider the uniqueness of the homogeneous Neumann problem

$$\Delta u = 0 \quad \text{in} \quad \Omega, \quad \frac{\partial u}{\partial n_\Omega} = 0 \quad \text{on} \quad \partial\Omega, \tag{2.1}$$

where n_Ω is the unit exterior normal vector field of $\partial\Omega$.

Lemma 2.1. *Let Ω be a C^2 sector-like domain. Let $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ be a solution of* (2.1) *satisfying*

$$\|d_\Omega \nabla u\|_\infty < \infty \tag{2.2}$$

where $d_\Omega(x) = \inf_{y\in\partial\Omega}|x-y|$. Then u is a constant function.

Lemma 2.2. *Let $\Omega = S_\theta$. Let $u \in C^2(\Omega) \cap C^1\left(\overline{\Omega}\setminus\{0\}\right)$ be a solution of* (2.1) *(except $x=0$) satisfying* (2.2). *Assume that for some $R>0$*

$$F(R) := \int_{\Gamma_R\cap\Omega} \frac{\partial u}{\partial r} d\mathcal{H}^1 = 0 \tag{2.3}$$

where $\Gamma_R = \partial D_R$ and $\partial/\partial r$ is the radial derivative. Then u is a constant function.

Remark 2.3. The no flux condition (2.3) is necessary in Lemma 2.2. In fact $u = \log|x|$ solves (2.1) with (2.2) since $d_\Omega(x) = |x|\sin\left(\min\left((\theta/2-\varphi), \pi/2\right)\right)$ for $\varphi = \arg x > 0$.

A key step for the proof of both Lemmas is to show boundedness of a solution.

Lemma 2.4 (Boundedness). *For $R>0$ let $u \in C^2(S_\theta \backslash D_R) \cap C^1\left(\overline{S}_\theta \backslash D_R\right)$ satisfy*

$$\Delta u = 0 \quad in \quad S_\theta \backslash D_R, \quad \frac{\partial u}{\partial n_\Omega} = 0 \quad on \quad (\partial S_\theta) \backslash D_R. \tag{2.4}$$

Assume that $\|d_{S_\theta} \nabla u\|_\infty < \infty$ and $F(R_1) = 0$ for some $R_1 > R$. Then u is bounded in $S_\theta \backslash D_{R+\delta}$ for any $\delta > 0$.

Proof of Lemma 2.4. We may assume $R=1$ by dilation. We use polar coordinates $x_1 = e^s \cos\varphi$, $x_2 = e^s \sin\varphi$ so that $S_\theta \backslash D_R$ is transformed to a region $\{(s,\varphi) \mid s \geq 0, |\varphi| < \theta/2\}$. The transformed dependent variable is denoted by U, i.e. $U(s,\varphi) = u(x_1,x_2)$. Then U solves

$$\Delta U = 0 \text{ in } \mathbf{R}_+ \times \left(-\frac{\theta}{2}, \frac{\theta}{2}\right), \ \frac{\partial U}{\partial \varphi} = 0 \text{ on } \mathbf{R}_+ \times \{\pm\theta\} = \Gamma \ \ (\mathbf{R}_+ = (0,\infty)) \tag{2.5}$$

and satisfies

$$|\nabla U| \leq C'/d(\varphi), \quad d(\varphi) = \operatorname{dist}((s,\varphi),\Gamma) \tag{2.6}$$

since $d_{S_\theta} \nabla u$ is bounded. Since (2.4) holds, integration by parts shows that the flux $F(R^*)$ is independent of R^*. Thus $F(R^*) = 0$ holds for all $R^* > 1$, which yields $F(e^s) = dE(s)/ds = 0$ for all $s > 0$ with $E(s) = \int_{-\theta/2}^{\theta/2} U(s,\varphi)d\varphi$ since $r\partial/\partial r = \partial/\partial s$. Thus $E(s)$ is a constant c independent of $s > 0$. We may assume $c = 0$ by subtracting c from U. By integration of $\partial_\varphi U$ with respect to φ variable (2.6) implies that $U(s,\varphi)$ blows up at most logarithmically at $\pm\theta/2$. By the uniform estimate (2.6) we observe that

$$\sup_{S_0>0} \|U : L^q((S_0, S_0+1) \times (-\theta/2, \theta/2))\| < \infty$$

for any $q > 1$ (cf. [3]). By a standard elliptic regularity theory this implies that U is bounded in $(\delta,\infty) \times (-\theta/2,\theta/2)$; see e.g. appendix of our companion paper [4]. □

Proof of Lemma 2.2. As in the proof of Lemma 2.4 we use the polar coordinates (s,φ). We observe that U satisfies (2.5) in $\mathbf{R} \times (-\theta/2,\theta/2)$. By Lemma 2.4 we observe that U is bounded for $s > 1$. A similar argument implies that U is also bounded for $s < -1$. Moreover, we may assume $E(s) = 0$ for all $s > 0$.

We shall prove that $U \equiv 0$ by the strong maximum principle [17, Ch. 2, Sec. 3]. Assume that $U \not\equiv 0$. Then we may assume that $\sup U > 0$ by considering $-U$ if necessary. This supremum is not attained in $\mathbf{R} \times [-\theta/2,\theta/2]$. Indeed, if it is attained in the interior, then the strong maximum principle implies that $U \equiv \sup U > 0$ which contradicts the property $E(s) = 0$. If the maximum is taken on the boundary, $U \equiv \sup U$ since otherwise the Hopf lemma [17, Ch. 2, Sec. 3, Thm. 7] implies $\partial U/\partial\varphi > 0$ at that point which contradicts the Neumann condition. This again contradicts the property $E(s) = 0$.

We may assume that there is a sequence $z_m = (s_m, \varphi_m)$ such that $U(z_m) \to \sup U$ and $|s_m| \to \infty$ as $m \to \infty$. We may assume that $s_m \to \infty$ since the case $s_m \to -\infty$ can be treated similarly. We may assume that

$\varphi_m \to \varphi_*$ for some $\varphi_* \in [-\theta/2, \theta/2]$ by taking a subsequence. We shift U and define $U_m(z) := U(s + s_m, \varphi)$ for $z = (s, \varphi)$ and observe that $\{U_m\}$ is a bounded sequence of solutions of (2.5) in $\mathbf{R} \times [-\theta/2, \theta/2]$. By Weyl's type lemma all derivatives are bounded so the Arzela-Ascoli theorem implies that U_m converges to some solution V of (2.5) in $\mathbf{R} \times [-\theta/2, \theta/2]$ with its first derivatives locally uniformly in $\mathbf{R} \times [-\theta/2, \theta/2]$ by taking a subsequence. Then V satisfies $\int_{-\theta/2}^{\theta/2} V(s,\varphi)d\varphi = 0$ for all $s > 0$ and $V(0, \varphi_*) = \max V = \sup U > 0$. As before the strong maximum principle and the Hopf lemma implies that $V \equiv \sup U > 0$ which contradicts $\int_{-\theta/2}^{\theta/2} V(s,\varphi)d\varphi = 0$. We thus conclude that $U \equiv 0$. □

Proof of Lemma 2.1. By the assumption $\Omega \backslash D_{R_0} = S_\theta \backslash D_{R_0}$ for some $R_0 > 0$. By (2.1) we observe that $F(R) = 0$ for all $R > R_0$. By Lemma 2.4, we see that u is bounded in $\overline{\Omega}$. As in the proof of Lemma 2.4 we use the polar coordinates (s, φ). We observe that U satisfies (2.5) in $(\log R_0, \infty) \times (-\theta/2, \theta/2)$. By the no flux condition $F(R) = 0$ we may assume that $E(s) = 0$ for $s \in (\log R_0, \infty)$ by adding a constant.

As in the proof of Lemma 2.2 one is able to prove that $u \equiv 0$ by a minor modification. □

3. Weighted L^∞ estimates for the Neumann problem

We are interested in a priori estimates for a weak solution of the Neumann problem. Let Ω be a C^2-domain in $\mathbf{R}^n$ $(n \geq 1)$. Let $g \in L^1_{loc}(\partial\Omega)$ be a tangential vector field, i.e. $g \cdot n_\Omega = 0$ on $\partial\Omega$. We say that $u \in L^1_{loc}(\overline{\Omega})$ is a weak solution of

$$\Delta u = 0 \quad \text{in} \quad \Omega, \quad \frac{\partial u}{\partial n_\Omega} = \operatorname{div}_{\partial\Omega} g \tag{3.1}$$

if u satisfies

$$\int_\Omega u \Delta\varphi \, dx = \int_{\partial\Omega} \nabla_{\partial\Omega}\varphi \cdot g \, d\mathcal{H}^1 \tag{3.2}$$

for all $\varphi \in C^2_c(\overline{\Omega})$ with $\frac{\partial\varphi}{\partial n_\Omega} = 0$ on $\partial\Omega$, where $\operatorname{div}_{\partial\Omega} = \nabla_{\partial\Omega}\cdot$ denotes the surface divergence [4], where $\nabla_{\partial\Omega} = P_{\partial\Omega}\nabla$ and $P_{\partial\Omega}$ is the tangential projection, i.e., $P_{\partial\Omega} = I - n_\Omega \otimes n_\Omega$. This definition is essentially given in [1] and is the same as in [2]. Note that the tangential gradient $\nabla_{\partial\Omega}$ can be replaced by ∇ since g is tangential. The main feature of this definition is that u can be unbounded near $\partial\Omega$. Such a notion of weak solutions are elaborated by [16] to include the case that the Neumann data contains Dirac measure.

Lemma 3.1. *Let Ω be a C^2 sector-like domain in $\mathbf{R}^2$. Then there exists a constant C independent of $R \geq 1$ such that the estimate*

$$\|d_\Omega(x)\nabla u\|_\infty \leq C \|g\|_\infty \tag{3.3}$$

holds for all weak solution $u \in L^1_{loc}(\overline{\Omega}_R)$ of (3.1) in $\Omega_R = D_{2R} \cap \Omega$ with $g \in L^\infty(\partial\Omega_R)$ satisfying $g \cdot n_\Omega = 0$ on $(\partial\Omega_R)\backslash\Omega$ and $g = 0$ on $\partial\Omega_R \cap \Omega$ provided that $\|d_\Omega \nabla u\|_\infty < \infty$.

Proof of Lemma 3.1. Although the proof is similar to [4, Lemma 2.5], we give it for completeness. As in [1], [2], we argue by contradiction. Suppose that (3.3) were false. Then there would exist $\{u_m, g_m, R_m\}_{m=1}^{\infty}$ satisfying

$$1 = \|d_\Omega \nabla u_m\|_{L^\infty(\Omega_{R_m})} > m\|g_m\|_{L^\infty(\partial\Omega\cap D_{2R_m})} \tag{3.4}$$

such that $u_m \in L^1_{loc}(\overline{\Omega}_{R_m})$ is a weak solution of (3.1) in Ω_R with $g_m \in L^\infty(\partial\Omega_{R_m})$ satisfying $g_m \cdot n_\Omega = 0$ on $\partial\Omega \cap D_{2R}$ and $g_m = 0$ on $\partial D_{2R} \cap \Omega$. We take $x_m \in \overline{\Omega}_{R_m}$ such that

$$|d_\Omega(x_m)\nabla u_m(x_m)| > 1/2. \tag{3.5}$$

We may assume that $u_m(x_m) = 0$ by adding a constant.

There are two cases depending on the behavior of $\{x_m\}_{m=1}^{\infty}$.

Case 1. There exists a subsequence still denoted by $\{x_m\}$ which converges to $\hat{x} \in \overline{\Omega}$ as $m \to \infty$.

Case 2. The sequence $\{x_m\}$ tends to infinity, i.e. $|x_m| \to \infty$.

We discuss Case 1 which is divided into two cases, (a) $\hat{x} \in \Omega$ and (b) $\hat{x} \in \partial\Omega$. We may assume that $R_m \to R \in [1, \infty]$ by taking a subsequence. In the case (a) by (3.4) it is easy to prove that $\{u_m\}$ converges to a weak solution of (3.1) in $\Omega^\infty = \Omega_R$ if $R < \infty$ and $= \Omega$ if $R = \infty$ with $g = 0$ by taking a subsequence. Moreover, the convergence is locally uniform with its derivatives in Ω^∞ so that $u(\hat{x}) = 0$, since $\{u_m\}$ is harmonic and bounded in $L^q_{loc}(\overline{\Omega})$ for all $q \geq 1$ by (3.4) and $u_m(x_m) = 0$. If R is finite, then the elliptic regularity [4, Appendix A] implies that $u \in C^\infty(\Omega^\infty) \cap C^1(\overline{\Omega^\infty})$. Although there are two corner points in $\Omega^\infty \cap \{|x| = 2R\}$, one can show that u is smooth up to these points by reflection in s of (s, φ)-variable in the proof of Lemma 2.4 since the Neumann data at $|x| = 2R$ is zero. The uniqueness (up to constant) of the Neumann problem in this domain is easy to prove as in Lemma 2.1 by the strong maximum principle. We thus conclude that $u \equiv 0$. However, by (3.5) we have $|d_\Omega(\hat{x})\nabla u(\hat{x})| \geq 1/2$, which yields a contradiction. If $R = \infty$, then we apply Lemma 2.1 to conclude $u \equiv 0$, which yields a contradiction. The case (b) can be treated as in [1] by rescaling u_m as

$$v_m(x) = u_m(x_m + d_m x) \tag{3.6}$$

with $d_m = d_\Omega(x_m)$. (We only need C^2-regularity of Ω as in [4] in this step.) We apply the uniqueness result in a half space [1, Lemma 2.9] to get a contradiction. If R is finite, then there might be a chance that the rescaled limit space (obtained as a limit of $\Omega_m = \{x \mid x_m + d_m x \in \Omega_{R_m}\}$) is not a half space but a quadrant type space $\{x_2 > 0, x_1 < R\}$. In this case we extend a solution by even reflection outside $x_1 = R$ and reduce the problem in the half space.

We next study Case 2. We rescale u_m as

$$w_m(x) = u_m(|x_m|x) \tag{3.7}$$

and set $y_m = x_m/|x_m|$, $H_m = R_m/|x_m|$. We may assume that $H_m \to H \in [1, \infty]$. Then $\{w_m\}$ converges to a weak solution w of (3.1) in $\Omega^\infty = S_\theta \cap D_{2H}$ with $g = 0$ by taking a subsequence. We have to divide the case depending on $y_m \to \hat{y} \in \Omega^\infty$ and $\hat{y} \in \partial\Omega^\infty$. The second case can be handled by rescaling w_m of (3.7) by (3.6) and reduce the problem to the uniqueness in the half space. The first case is more involved. The limit w of $\{w_m\}$ must satisfy

$$|\nabla w(x)| \leq C/d_{S_\theta}(x), \quad x \in \Omega^\infty = S_\theta \cap D_{2H}. \tag{3.8}$$

One would like to apply the uniqueness in Ω^∞ with (3.8). In the case $H < \infty$, the uniqueness result like Lemma 2.2 can be proved since the no flux condition (2.3) is automatically fulfilled. The case $H = \infty$ needs to prove the no flux condition (2.3). We introduce a cut-off function η_k $(k = 1, 2, \ldots)$ defined by $\eta_k(x) = \eta\left(k\left(|x| - 1/2\right) + 1/2\right)$ where $\eta \in C^2[0, \infty]$ satisfies $\eta(s) = 0$ for $0 \leq s \leq 1/2$ and $\eta(s) = 1$ for $s \geq 1$ with $0 \leq \eta \leq 1$ and $\eta' \geq 0$. Since w_m is a weak solution of (3.1) in $\Omega^m = \Omega_{R_m}/|x_m|$ with $g = \tilde{g}_m,\ \tilde{g}_m(x) = g_m\left(|x_m|x\right)$, we observe that

$$\int_{\Omega_m \cap D^c_{1/2}} w_m \Delta\eta_k dx = \int_{\partial\Omega_m \cap D_{2H_m}} \nabla_{\partial\Omega}\, \eta_k \cdot \tilde{g}_m d\mathcal{H}^1.$$

Since $\|\tilde{g}_m\|_\infty = \|g_m\|_\infty \leq 1/m$ by (3.4), sending $m \to \infty$ implies

$$\int_{S_\theta \cap D^c_{1/2}} w \Delta\eta_k dx = 0.$$

Integrating by parts yields

$$k \int_{1/2}^{1/2+1/k} \left(\int_{\Gamma_r \cap S_\theta} \frac{\partial w}{\partial r}\, d\mathcal{H}^1 \right) \eta'\left(k(r - 1/2) + 1/2\right) r dr = 0.$$

Sending $k \to \infty$ yields the no flux condition

$$\int_{\Gamma_{1/2} \cap S_\theta} \frac{\partial w}{\partial r}\, d\mathcal{H}^1 = 0. \tag{3.9}$$

By (3.9) one is able to apply Lemma 2.2 with $w(\hat{y}) = 0$ to conclude that $w \equiv 0$ while (3.5) implies $|d_{S_\theta}(\hat{y})\nabla w(\hat{y})| > 1/2$ which is a contradiction. □

Theorem 3.2. *Let Ω be a C^2 sector-like domain in $\mathbf{R}^2$. Then there exists a constant C such that* (3.3) *holds for all weak solution $u \in L^1_{loc}(\overline{\Omega})$ of* (3.1) *with $\nabla u \in L^2(\Omega)$ and $g \in L^\infty(\partial\Omega)$ with $g \cdot n_\Omega = 0$ on $\partial\Omega$.*

As in the proof [2, Proposition 2.6] that strictly admissibility implies the admissibility, Theorem 3.2 implies Theorem 1.2.

Remark 3.3. (i) The estimate (3.3) is very similar to saying that Ω is strictly admissible in the sense of [2]. However, there is an important difference. In Theorem 3.2, we restrict u such that ∇u is globally square integrable. So Theorem 3.2 does not assert that Ω is strictly admissible.
(ii) To show admissibility in [1] we invoked C^3-regularity of a domain. This is because we have used C^3-regularity to prove the uniqueness of the Neumann problem as well as the flattening procedure as in the proof of handling case (b) below. However, in the present paper uniqueness results in Section 2 require only C^2-regularity. If one examines carefully as in [4], the flattening procedure requires only C^2-regularity so do Lemma 3.1 and Theorem 3.2.

Proof of Theorem 3.2. Let $R_0 > 0$ such that $\Omega\backslash D_{R_0} = S_\theta \backslash D_{R_0}$. For $R > R_0$ let w_R be a solution of the Neumann problem

$$\Delta w_R = 0 \text{ in } \Omega_R, \quad \frac{\partial w_R}{\partial n_\Omega} = 0 \text{ on } (\partial\Omega_R)\backslash\Omega, \quad \frac{\partial w}{\partial r} = \frac{\partial u}{\partial r} \text{ on } \partial\Omega_R \cap \Omega.$$

Since $\nabla u \in L^2(\Omega)$ so that $\frac{\partial u}{\partial r} \in L^2(\partial\Omega_R \cap \Omega)$ for almost all $R > R_0$ by the Lax-Milgram theorem, this problem admits a solution w_R (unique up to constant) with $\nabla w_R \in L^2(\Omega_R)$ for almost all $R > R_0$. We shall consider such R in the sequel.

We set $u_R = u - w_R$ and observe that

$$\|d_\Omega \nabla u_R\|_{L^\infty(\Omega_R)} \leq C\|g\|_\infty \tag{3.10}$$

by Lemma 3.1 since $\nabla u_R \in L^2(\Omega_R)$ implies $d_\Omega|\nabla u_R| \in L^\infty(\Omega_R)$ for a harmonic u_R by two dimensionality; see [1, Remark 2.4 (ii)].

If we prove that $\nabla w_R \to 0$ in $L^2(\Omega)$, then the desired estimate follows from (3.10) by the lower semicontinuity of $\|d_\Omega \nabla u\|_\infty$ with respect to L^2-convergence $\nabla u_R \to \nabla u$.

It remains to prove that $\nabla w_R \to 0$ in $L^2(\Omega)$ as $R \to \infty$ by taking a subsequence. It is convenient to introduce (s, φ) coordinates as in the proof of Lemma 2.4. We observe that

$$\int_{S_\theta\backslash D_{R_0}} |\nabla f|^2 dx_1 dx_2 = \int_W |\nabla_{s,\varphi} \tilde{f}|^2 ds\, d\varphi, \quad W = (\log R_0, \infty) \times (-\theta/2, \theta/2),$$

where $\tilde{f}(s,\varphi) = f(e^s \cos\varphi, e^s \sin\varphi)$. By definition we have

$$\int_{\Omega_R} |\nabla w_R|^2 dx_1 dx_2 = \int_{|x|=2R} \frac{\partial w_R}{\partial r} w_R d\mathcal{H}^1 = \int_{|x|=2R} \frac{\partial u}{\partial r} w_R d\mathcal{H}^1.$$

We use (s, φ) coordinates to get

$$\int_{\Omega_R} |\nabla w_R|^2 dx_1 dx_2 = \int_{-\theta}^{\theta} e^{-s} \frac{\partial \tilde{u}}{\partial s} \tilde{w}_R\, e^s \bigg|_{s=\log 2R} d\varphi = \int_{-\theta}^{\theta} \frac{\partial \tilde{u}}{\partial s} \tilde{w}_R \bigg|_{s=\log 2R} d\varphi. \tag{3.11}$$

Since w_R satisfies the no flux condition, we may assume that $\int_{-\theta/2}^{\theta/2} \tilde{w}_R d\varphi = 0$ at $s = \log 2R$. By the Poincaré inequality and the trace theorem [9] we have

$$\int_{-\theta/2}^{\theta/2} |\tilde{w}_R|^2 \, d\varphi \leq C \int_{W_R} |\nabla_{s,\varphi} \tilde{w}_R|^2 \, d\varphi\, ds, \quad W_R = (\log R_0, \log 2R) \times (-\theta/2, \theta/2)$$

with C independent of R. By the Hölder inequality (3.11) now yields

$$\int_{\Omega_R} |\nabla w_R|^2 dx_1 dx_2 \leq \left(\int_{-\theta/2}^{\theta/2} \left| \frac{\partial \tilde{u}}{\partial s} \right|^2 d\varphi \bigg|_{s=\log 2R} \right)^{1/2} \left(C \int_{\Omega_R} |\nabla w_R|^2 dx_1 dx_2 \right)^{1/2}.$$

This implies

$$\int_{\Omega_R} |\nabla w_R|^2 dx_1 dx_2 \leq C \int_{-\theta/2}^{\theta/2} \left| \frac{\partial \tilde{u}}{\partial s} \right|^2 d\varphi \bigg|_{s=\log 2R}. \tag{3.12}$$

Since $\nabla u \in L^2(\Omega)$ so that $\nabla_{s,\varphi}\tilde{u} \in L^2(W)$, the right-hand side of (3.12) tends to zero as $R \to \infty$ by taking a suitable subsequence. Thus (3.12) implies that $\nabla w_R \to 0$ in $L^2(\Omega)$ (by interpreting $\nabla w_R = 0$ outside Ω_R) for a subsequence $R \to \infty$. □

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