

Weyl-von Neumann Theorem and Borel Complexity of Unitary Equivalence Modulo Compacts of Self-Adjoint Operators

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Abstract

Weyl-von Neumann Theorem asserts that two bounded self-adjoint operators A, B on a Hilbert space H are unitarily equivalent modulo compacts, i.e., $uAu^* + K = B$ for some unitary $u \in \mathcal{U}(H)$ and compact self-adjoint operator K , if and only if A and B have the same essential spectra: $\sigma_{\text{ess}}(A) = \sigma_{\text{ess}}(B)$. In this paper we consider to what extent the above Weyl-von Neumann's result can(not) be extended to unbounded operators using descriptive set theory. We show that if H is separable infinite-dimensional, this equivalence relation for bounded self-adjoint operators is smooth, while the same equivalence relation for general self-adjoint operators contains a dense G_δ -orbit but does not admit classification by countable structures. On the other hand, apparently related equivalence relation $A \sim B \Leftrightarrow \exists u \in \mathcal{U}(H) [u(A - i)^{-1}u^* - (B - i)^{-1} \text{ is compact}]$, is shown to be smooth.

Keywords. Weyl-von Neumann Theorem, Self-adjoint operators, Turbulence.

1 Introduction

The celebrated Weyl-von Neumann Theorem [Wey09, vN35] asserts that any bounded self-adjoint operator can be turned into a diagonalizable operator with arbitrarily small compact perturbations. More precisely:

Theorem 1.1 (Weyl-von Neumann). *Let A be a (not necessarily bounded) self-adjoint operator on a separable Hilbert space H and $\varepsilon > 0$, there exists a compact operator K with $\|K\| < \varepsilon$, such that $A + K$ is of the form*

$$A + K = \sum_{n=1}^{\infty} a_n \langle \xi_n, \cdot \rangle \xi_n,$$

where $a_n \in \mathbb{R}$ and $\{\xi_n\}_{n=1}^{\infty}$ is a CONS for H .

Weyl obtained Theorem 1.1 for bounded operators without norm estimates on K , and the present form of the theorem was obtained by von Neumann. Moreover, he also proved that K can be chosen to be of Hilbert-Schmidt class (in fact K can be chosen to be of Schatten p -class for any $p > 1$ by [Kur58], but $p = 1$ is impossible by [Kat57, Ros57]. See [Con99, RS81, AG61] for details). Berg [Ber71] generalized Theorem 1.1 to (unbounded) normal operators.

On the other hand, Weyl [Wey10] proved that the essential spectra of a self-adjoint operator is invariant under compact perturbations. Here, the essential spectra $\sigma_{\text{ess}}(A)$ of a self-adjoint operator A is the set of all λ in the spectral set $\sigma(A)$ of A which is either an eigenvalue of infinite multiplicity or an accumulation point in $\sigma(A)$. Based on Theorem 1.1, von Neumann showed ((1) \Rightarrow (2) below) that up to unitary conjugation, the converse to Weyl's compact perturbation Theorem holds:

Theorem 1.2 (Weyl-von Neumann). *Let A, B be bounded self-adjoint operators on H . Then the following conditions are equivalent:*

- (1) $\sigma_{\text{ess}}(A) = \sigma_{\text{ess}}(B)$.
- (2) *A and B are unitarily equivalent modulo compacts. More precisely, there exists a compact self-adjoint operator K on H and a unitary operator u on H , such that*

$$uAu^* + K = B.$$

Theorem 1.2 states that the essential spectra is a complete invariant for the classification problem of all bounded self-adjoint operators up to unitary equivalence modulo compacts. On the other hand, Theorem 1.1 and Weyl's Theorem 1.2 (2) \Rightarrow (1) above also hold for unbounded self-adjoint operators. It is therefore of interest to know whether Theorem 1.2 holds true for general unbounded self-adjoint operators. However, a simple example (Example 3.2) clarifies that von Neumann's Theorem 1.2 (1) \Rightarrow (2) cannot be generalized verbatim for unbounded operators. Moreover, further examples (Examples 3.3 and 3.5) show that it seems impossible to find a reasonable complete invariant characterizing this equivalence which is assigned to each self-adjoint operators in a constructible way.

It is the purpose of the present paper to show that there is a sharp contrast between the complexity of the above classification problem for bounded operators and that for unbounded operators by descriptive set theoretical method, especially the turbulence theorem established by Hjorth [Hjo00]. More precisely, we prove the following: let H be a separable infinite-dimensional Hilbert space, and $\text{SA}(H)$ be the Polish space of all (possibly unbounded) self-adjoint operators equipped with the strong resolvent topology (SRT, see §2.1). Then the set $\mathbb{B}(H)_{\text{sa}}$ of bounded self-adjoint operators on H is a Borel subset of $\text{SA}(H)$ (Lemma 3.11). Consider the semidirect product Polish group $G = \mathbb{K}(H)_{\text{sa}} \rtimes \mathcal{U}(H)$, where $\mathbb{K}(H)_{\text{sa}}$ is the additive Polish group of compact self-adjoint operators with the norm topology, and we equip the unitary group $\mathcal{U}(H)$ of H with the strong operator topology. The action of $\mathcal{U}(H)$ on $\mathbb{K}(H)_{\text{sa}}$ is given by conjugation. Then we consider the orbit equivalence relation $E_G^{\text{SA}(H)}$ of the G -action on $\text{SA}(H)$ given by $(K, u) \cdot A := uAu^* + K$ ($u \in \mathcal{U}(H)$, $K \in \mathbb{K}(H)_{\text{sa}}$). Since $\mathbb{B}(H)_{\text{sa}}$ is a G -invariant Borel subset, we may consider the restricted equivalence relation $E_G^{\mathbb{B}(H)_{\text{sa}}}$ as well. Therefore, the difference of the complexity of the above classification for bounded vs unbounded operators should be understood as the difference of the complexities of $E_G^{\text{SA}(H)}$ and $E_G^{\mathbb{B}(H)_{\text{sa}}}$. In this respect, let us now state our main theorem:

Theorem 1.3. *Denote by $\mathcal{F}(\mathbb{R})$ the Effros Borel space of closed subsets of \mathbb{R} . The following statements hold:*

- (1) $\text{SA}(H) \ni A \mapsto \sigma_{\text{ess}}(A) \in \mathcal{F}(\mathbb{R})$ is Borel. In particular, $E_G^{\mathbb{B}(H)_{\text{sa}}}$ is smooth.
- (2) There exists a dense G_δ orbit of the G -action on $\text{SA}(H)$. In particular, the action is not generically turbulent.
- (3) $E_G^{\text{SA}(H)}$ does not admit classification by countable structures.

Proofs of (1), (2) and (3) are given in Theorem 3.15, Theorem 3.17 and Theorem 3.33, respectively.

Remark 1.4. (Added April 10, 2014) After the paper was submitted, we were informed from Alexander Kechris that the Borelness of the map $\sigma_{\text{ess}}(\cdot)$ has been proved for the case of bounded operators on a Banach space in [LPS05]. We would like to thank him and the anonymous referee for the communications.

Regarding (3), we prove more precisely that the subspace $\text{EES}(H) = \{A \in \text{SA}(H); \sigma_{\text{ess}}(A) = \emptyset\}$, equipped with the norm resolvent topology (NRT, see §3.4.2.1) is shown to be a Polish G -space (with respect to the restricted action), and the G -action on $\text{EES}(H)$ is generically turbulent (Theorem 3.32). Since $A \mapsto \sigma_{\text{ess}}(A)$ is constant ($= \emptyset$) on $\text{EES}(H)$, this shows that the essential spectra is very far from a complete invariant even in this small subspace of $\text{SA}(H)$. Since NRT is stronger than SRT, this shows that $E_G^{\text{EES}(H)}$ is Borel reducible (in fact continuously embeddable) to $E_G^{\text{SA}(H)}$, whence (3) holds by Hjorth turbulence Theorem [Hjo00]. On the other hand, there is a related equivalence relation: define an equivalence relation $E_{\text{u.c.res}}^{\text{SA}(H)}$ on $\text{SA}(H)$ by

$$AE_{\text{u.c.res}}^{\text{SA}(H)}B \Leftrightarrow \exists u \in \mathcal{U}(H) [u(A - i)^{-1}u^* - (B - i)^{-1} \in \mathbb{K}(H)].$$

$E_G^{\text{SA}(H)}$ is stronger than $E_{\text{u.c.res}}^{\text{SA}(H)}$ in the sense that $E_G^{\text{SA}(H)} \subset E_{\text{u.c.res}}^{\text{SA}(H)}$ (Lemma 3.39), and $E_{\text{u.c.res}}^{\text{SA}(H)}$ restricted to $\mathbb{B}(H)_{\text{sa}}$ agrees with $E_G^{\mathbb{B}(H)_{\text{sa}}}$ (Lemma 3.40). Therefore $E_{\text{u.c.res}}^{\text{SA}(H)}$ is considered to be another extension of $E_G^{\mathbb{B}(H)_{\text{sa}}}$ to $\text{SA}(H)$. We show that unlike $E_G^{\text{SA}(H)}$, $E_{\text{u.c.res}}^{\text{SA}(H)}$ is actually smooth (Theorem 3.41), although the essential spectra cannot be a complete invariant (Example 3.42). In the last section, we give some comments on other equivalence relations related to unbounded self-adjoint operators as well as some questions.

The connection between descriptive set theory and other areas of mathematics such as ergodic theory or operator algebra theory have been proved to be very fruitful (see e.g. [FTT, KS01, KLP10, ST08], and references therein). However, apart from the pioneering work of Simon [Sim95] (see also [CN98]) for a special class of self-adjoint operators, apparently no descriptive study has been carried out for the space

SA(H). We hope that the present work not only shows the usefulness of the descriptive set theoretical viewpoint but also verifies that the theory of (unbounded) self-adjoint operators gives us rich examples of interesting equivalence relations.

2 Preliminaries

2.1 Operator Theory

Here we recall basic notions from spectral theory. Details can be found e.g. in [RS81, Sch10]. Let H be a separable infinite-dimensional Hilbert space. The group of unitary operators on H is denoted $\mathcal{U}(H)$. We denote $\mathbb{B}(H)$ (resp. $\mathbb{B}(H)_{\text{sa}}$) the space of all bounded (resp. bounded self-adjoint) operators on H , and $\mathbb{K}(H)$ (resp. $\mathbb{K}(H)_{\text{sa}}$) the space of all compact (resp. compact self-adjoint) operators on H . The convergence of bounded operators with respect to the strong operator topology (SOT for short) is denoted $x_n \xrightarrow{\text{SOT}} x$ or $x_n \rightarrow x$ (SOT). $(\mathcal{U}(H), \text{SOT})$ is a Polish group. The *domain* (resp. *range*) of a linear operator A is denoted $\text{dom}(A)$ (resp. $\text{Ran}(A)$).

Definition 2.1. We define $\text{SA}(H)$ to be the space of all (possibly unbounded) self-adjoint operators on H . The *strong resolvent topology* (SRT for short) is the weakest topology on $\text{SA}(H)$ for which $\text{SA}(H) \ni A \mapsto (A - i)^{-1}\xi \in H$ is continuous for every $\xi \in H$.

In other words, a sequence $\{A_n\}_{n=1}^{\infty}$ in $\text{SA}(H)$ converges to $A \in \text{SA}(H)$ in SRT, if and only if $\text{SOT} - \lim_{n \rightarrow \infty} (A_n - i)^{-1} = (A - i)^{-1}$. $\text{SA}(H)$ equipped with SRT is Polish: this is probably known, so we only indicate how to define a suitable metric: fix a CONS $\{\xi_n\}_{n=1}^{\infty}$ for H , and define a metric d on $\text{SA}(H)$ by

$$d(A, B) := \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{2^{n+m}} \sup_{t \in [-m, m]} \|e^{itA}\xi_n - e^{itB}\xi_n\|, \quad A, B \in \text{SA}(H).$$

Proposition 2.2. d is a complete metric on $\text{SA}(H)$ compatible with SRT, and $\text{SA}(H)$ is separable with respect to SRT. Consequently, $\text{SA}(H)$ is a Polish space.

The proof of Proposition 2.2 is done by the use of next Lemma (see [RS81, Theorem VIII.21] for the proof) together with a standard argument.

Lemma 2.3 (Trotter). *Let $\{A_k\}_{k=1}^{\infty} \subset \text{SA}(H)$. Then A_k converges to $A \in \text{SA}(H)$ in SRT, if and only if for each $\xi \in H$ and each compact subset K of \mathbb{R} , $\sup_{t \in K} \|e^{itA_k}\xi - e^{itA}\xi\|$ tends to 0.*

For $A \in \text{SA}(H)$, the *spectra* (resp. *point spectra*) of A is denoted $\sigma(A)$ (resp. $\sigma_p(A)$). The *essential spectra* of A , denoted $\sigma_{\text{ess}}(A)$, is the set of all $\lambda \in \sigma(A)$ which is either (i) an eigenvalue of A of infinite multiplicity or (ii) an accumulation point in $\sigma(A)$. Its complement $\sigma_d(A) := \sigma(A) \setminus \sigma_{\text{ess}}(A)$ is called the *discrete spectra*, which is the set of all isolated eigenvalues of finite multiplicity. The spectral measure of A is denoted $E_A(\cdot)$, and we write the spectral resolution of A as $A = \int_{\mathbb{R}} \lambda dE_A(\lambda)$. A is called *diagonalizable* if there exists a CONS $\{\xi_n\}_{n=1}^{\infty}$ consisting of eigenvectors of A . Let $a_n \in \mathbb{R}$ be the eigenvalue of A corresponding to ξ_n ($n \in \mathbb{N}$). In this case, the spectral resolution of A is written as

$$A = \sum_{n=1}^{\infty} a_n \langle \xi_n, \cdot \rangle \xi_n = \sum_{n=1}^{\infty} a_n e_n,$$

where e_n is the projection onto $\mathbb{C}\xi_n$ ($n \in \mathbb{N}$). Finally, we will also need results about operator ranges (see [Dix49-1, Dix49-2, FW71]):

Definition 2.4. We say that a subspace $\mathcal{R} \subset H$ is an *operator range* in H , if $\mathcal{R} = \text{Ran}(T)$ for some $T \in \mathbb{B}(H)$. We may choose T to be self-adjoint with $0 \leq T \leq 1$. If we put $H_n := E_T((2^{-n-1}, 2^{-n}])H$ ($n = 0, 1, \dots$), then H_n are pairwise orthogonal closed subspaces of H with $H = \bigoplus_{n=0}^{\infty} H_n$ (by the density of \mathcal{R}). We call $\{H_n\}_{n=0}^{\infty}$ the *associated subspaces for T* (see [FW71, §3] for details).

In this paper, we only consider dense operator ranges. We use the following result on the unitary equivalence of operator ranges. The essential idea of the next result, which is one of the key ingredients in the classification of operator ranges, is due to Köthe [Kö36], and it is formulated in this form by Fillmore-Williams [FW71, Theorem 3.3].

Theorem 2.5 (Köthe, Fillmore-Williams). *Let \mathcal{R}, \mathcal{S} be dense orator ranges in H with associated closed subspaces $\{H_n\}_{n=0}^\infty$ and $\{K_n\}_{n=0}^\infty$, respectively. Then there exists $u \in \mathcal{U}(H)$ such that $u\mathcal{R} = \mathcal{S}$, if and only if the following condition is satisfied: there exists $k \geq 0$ such that for each $n, l \geq 0$,*

$$\begin{aligned} \dim(H_n \oplus \cdots \oplus H_{n+l}) &\leq \dim(K_{n-k} \oplus \cdots \oplus K_{n+l+k}) \\ \dim(K_n \oplus \cdots \oplus K_{n+l}) &\leq \dim(H_{n-k} \oplus \cdots \oplus H_{n+l+k}), \end{aligned}$$

where $H_m = K_m = \{0\}$ for $m < 0$.

2.2 Borel Equivalence Relations and Hjorth Turbulence Theorem

Here we recall basic notions from (classical) descriptive set theory. The details can be found e.g., in [Gao09, Hjo00, Kec96]. Let E (resp. F) be an equivalence relation on a standard Borel space X (resp. Y). We say that E is *Borel reducible* to F , in symbols $E \leq_B F$, if there is a Borel map $f: X \rightarrow Y$ such that $x_1 E x_2 \Leftrightarrow f(x_1) F f(x_2)$ holds for every $x_1, x_2 \in X$. We say that E is *smooth*, if E is Borel reducible to the identity relation id_Z on some Polish space Z .

The notion of classification by countable structures lies at a higher level of complexity than smoothness. In order to avoid introducing concepts from logic, let us informally give its definition. We refer the reader to [Hjo00, §2] for the details.

Definition 2.6. We say that an equivalence relation E admits *classification by countable structures*, if there exists a countable language L such that E is Borel reducible to the isomorphism relation on the space X_L of countable L -structures induced by the logic action of the group S_∞ of all permutations of \mathbb{N} .

Hjorth's notion of turbulence provides us with a convenient criterion for finding an obstruction of a given equivalence relation to be classifiable by countable structures. Below we use a category quantifier \forall^* . Suppose that we are given a Polish space X and for each point $x \in X$ a proposition $P(x)$. We say that $P(x)$ holds for generic $x \in X$, denoted $\forall^* x [P(x)]$, if $\{x \in X; P(x)\}$ is comeager in X .

Definition 2.7. Let G be a Polish group and X a Polish G -space.

- (1) Let $x \in X$. For an open neighborhoods U of x in X and V of 1 in G , the *local U - V orbit* of x , denoted $\mathcal{O}(x, U, V)$, is the set of all $y \in U$ for which there exist $l \in \mathbb{N}$, $x = x_0, x_1, \dots, x_l = y \in U$, and $g_0, \dots, g_{l-1} \in V$, such that $x_{i+1} = g_i \cdot x_i$ for all $0 \leq i \leq l-1$.
- (2) The action α is *turbulent* at $x \in X$ if the local orbits $\mathcal{O}(x, U, V)$ of x are somewhere dense (i.e., its closure has nonempty interior) for every open $U \subset X$ and $V \subset G$ with $x \in U$ and $1 \in V$.
- (3) The action α is said to be *generically turbulent* if it satisfies (a) there is a dense orbit, (b) every orbit is meager, and (c) $\forall^* x \in X$ [The action is turbulent at x].

We use an apparently weaker but equivalent notion of weak generic turbulence:

Definition 2.8. Let G be a Polish group and X a Polish G -space. We say that the action is *weakly generically turbulent*, if

- (a) Every orbit is meager.
- (b) $\forall^* x \in X \forall^* y \in X \forall (\emptyset \neq U \overset{\text{open}}{\subset} X (1 \in) \forall V \overset{\text{open}}{\subset} G [x \in U \Rightarrow \overline{\mathcal{O}(x, U, V)} \cap [y]_G \neq \emptyset]$.

Next theorem is called Hjorth turbulence Theorem. Proof can be found in [Hjo00].

Theorem 2.9 (Hjorth). *Let G be a Polish group and X a Polish G -space with every orbit meager and some orbit dense. Then the following statements are equivalent:*

- (i) X is weakly generically turbulent.
- (ii) X is generically turbulent.
- (iii) For any Borel S_∞ -space Y , E_G^X is generically $E_{S_\infty}^Y$ -ergodic.

It follows from Theorem 2.9 that orbit equivalence relations of generically turbulent actions do not admit classification by countable structures.

3 Weyl-von Neumann Equivalence Relation $E_G^{\text{SA}(H)}$

3.1 Impossibility of von Neumann's Theorem for Unbounded Self-Adjoint Operators

Let H be a separable infinite-dimensional Hilbert space. von Neumann's Theorem ((1) \Rightarrow (2) of Theorem 1.2) asserts that bounded self-adjoint operators $A, B \in \mathbb{B}(H)_{\text{sa}}$ with the same essential spectra $\sigma_{\text{ess}}(A) = \sigma_{\text{ess}}(B)$ are unitarily equivalent modulo compacts, i.e., $B = uAu^* + K$ for some $u \in \mathcal{U}(H)$ and $K \in \mathbb{K}(H)$. In this section we consider the situation for unbounded self-adjoint operators (note that Weyl's Theorem (2) \Rightarrow (1) of Theorem 1.2 holds in full generality):

Question 3.1. Let $A, B \in \text{SA}(H)$ be such that $\sigma_{\text{ess}}(A) = \sigma_{\text{ess}}(B)$. Are there $u \in \mathcal{U}(H)$ and $K \in \mathbb{K}(H)_{\text{sa}}$ such that $uAu^* + K = B$?

The answer to the question is negative, as the following simple example shows:

Example 3.2. Let H_0 be a separable infinite-dimensional Hilbert space, and let $H = H_0 \oplus H_0$. Fix a CONS $\{\xi_n\}_{n=1}^{\infty}$ for H_0 , and let $A_0 := \sum_{n=1}^{\infty} n \langle \xi_n, \cdot \rangle \xi_n \in \text{SA}(H_0)$, and define $A, B \in \text{SA}(H)$ by $A := A_0 \oplus 0$, $B := 0 \oplus 0$. Then $\sigma_{\text{ess}}(A) = \sigma_{\text{ess}}(B) = \{0\}$, and since A is unbounded, so is $uAu^* + K$ for any $u \in \mathcal{U}(H)$ and $K \in \text{SA}(H)$. Thus $uAu^* + K \neq B$.

It is now clear why von Neumann's Theorem fails to hold for unbounded self-adjoint operators: if A, B are unitarily equivalent modulo compacts, then their domains $\text{dom}(A)$ and $\text{dom}(B)$ must be unitarily equivalent, i.e., $u \cdot \text{dom}(A) = \text{dom}(B)$ for some $u \in \mathcal{U}(H)$. In fact there are a lot of unbounded self-adjoint operators with the same essential spectra but have non-unitarily equivalent domains.

Example 3.3. Let $\{\xi_n\}_{n=0}^{\infty}$ be a fixed CONS for H . Let e_n be the projection of H onto $\mathbb{C}\xi_n$. Define $\{A_t\}_{t \in (0,1)} \subset \text{SA}(H)$ by $A_t = \sum_{n=1}^{\infty} 2^{(n^t)} e_n$ ($0 < t < 1$). We show that $\{A_t\}_{t \in (0,1)}$ is a family of self-adjoint operators with $\sigma_{\text{ess}}(A_t) = \emptyset$ ($0 < t < 1$) such that $\text{dom}(A_t)$ and $\text{dom}(A_s)$ are not unitarily equivalent for $0 < t \neq s < 1$. The first assertion is clear, since $2^{n^t} \xrightarrow{n \rightarrow \infty} \infty$. For each $0 < t < 1$, the domain of A_t is $\text{dom}(A_t) = \text{Ran}(A_t^{-1})$, where $A_t^{-1} = \sum_{n=1}^{\infty} 2^{-n^t} e_n$. Therefore the associated subspaces for A_t^{-1} are

$$\begin{aligned} H_n^{(t)} &:= E_{A_t^{-1}}((2^{-n-1}, 2^{-n}])H = \overline{\text{span}}\{\xi_k; 2^{-n-1} < 2^{-k^t} \leq 2^{-n}\} \\ &= \overline{\text{span}}\{\xi_k; n^{\frac{1}{t}} \leq k < (n+1)^{\frac{1}{t}}\}. \end{aligned}$$

Let $0 < t < s < 1$. Then

$$\begin{aligned} \dim(H_n^{(t)}) &\geq [(n+1)^{\frac{1}{t}}] - ([n^{\frac{1}{t}}] + 1) \geq (n+1)^{\frac{1}{t}} - n^{\frac{1}{t}} - 2, \\ \dim(H_n^{(s)}) &\leq (n+1)^{\frac{1}{s}} - n^{\frac{1}{s}}. \end{aligned}$$

Therefore for given $k, l \in \mathbb{N}$, and $n > k$, it holds that

$$\begin{aligned} \dim(H_{n-k}^{(s)} \oplus \cdots \oplus H_{n+l+k}^{(s)}) &\leq \sum_{m=n-k}^{n+l+k} \{(m+1)^{\frac{1}{s}} - m^{\frac{1}{s}}\} = (n+l+k)^{\frac{1}{s}} - (n-k)^{\frac{1}{s}} \\ &\stackrel{(*)}{\leq} (l+2k)s^{-1}(n+l+k)^{s^{-1}-1}, \\ \dim(H_n^{(t)} \oplus \cdots \oplus H_{n+l}^{(t)}) &\geq \sum_{m=n}^{n+l} \{(m+1)^{\frac{1}{t}} - m^{\frac{1}{t}} - 2\} = (n+l)^{\frac{1}{t}} - n^{\frac{1}{t}} - 2(l+1) \\ &\stackrel{(*)}{\geq} lt^{-1}n^{t^{-1}-1} - 2(l+1), \end{aligned}$$

where we used the mean value Theorem in (*). Since $t^{-1} > s^{-1} > 1$, it holds that

$$\lim_{n \rightarrow \infty} \frac{lt^{-1}n^{t^{-1}-1} - 2(l+1)}{(l+2k)s^{-1}(n+l+k)^{s^{-1}-1}} = \infty,$$

which in particular shows that $\dim(H_n^{(t)} \oplus \cdots \oplus H_{n+l}^{(t)}) > \dim(H_{n-k}^{(s)} \oplus \cdots \oplus H_{n+l+k}^{(s)})$ for large n . Since k, l are arbitrary, $\text{dom}(A_t)$ and $\text{dom}(A_s)$ are not unitarily equivalent by Theorem 2.5.

Remark 3.4. We remark that the classification of dense operator ranges up to unitary equivalence is completed by Lassner-Timmermann [LT76].

We next show that unitary equivalence of the domains is still insufficient. Namely we construct another continuous family $\{B_t\}_{t \in [0,1]}$ in $\text{SA}(H)$ with the same domain and the essential spectra, yet no two of them are unitarily equivalent modulo compacts.

Example 3.5. Let $\{\xi_n\}_{n=1}^\infty$ and $\{e_n\}_{n=1}^\infty$ be as in Example 3.3. Fix a bijection $\langle \cdot, \cdot \rangle: \mathbb{N}^2 \rightarrow \mathbb{N}$ given by $\langle k, m \rangle := 2^{k-1}(2m-1)$, $m, k \in \mathbb{N}$, and define a family $\{B_t\}_{t \in [0,1]} \subset \text{SA}(H)$ by

$$B_t := \sum_{n=1}^{\infty} \lambda_n^{(t)} e_n, \quad \lambda_{\langle k, m \rangle}^{(t)} := k + \frac{t}{m+2}, \quad t \in [0,1], \quad k, m \in \mathbb{N}.$$

It is easy to see that $\text{dom}(B_s) = \text{dom}(B_t)$, and $\sigma_{\text{ess}}(B_s) = \sigma_{\text{ess}}(B_t) = \mathbb{N}$ ($s, t \in [0,1]$). Let $0 \leq s < t \leq 1$. We then show that there are no $u \in \mathcal{U}(H)$ and $K \in \mathbb{K}(H)_{\text{sa}}$ satisfying $uB_t u^* + K = B_s$. Suppose by contradiction that there exist such u and K , and put $\eta_n := u\xi_n$ ($n \in \mathbb{N}$). Then $f_n := u e_n u^*$ is a projection onto $\mathbb{C}\eta_n$, and

$$\sum_{k,m=1}^{\infty} \left(k + \frac{t}{m+2} \right) f_{\langle k, m \rangle} + K = \sum_{k,m=1}^{\infty} \left(k + \frac{s}{m+2} \right) e_{\langle k, m \rangle}.$$

Apply the above equality to the vector $\eta_{\langle k, 1 \rangle}$ ($k \in \mathbb{N}$) to obtain

$$\left(k + \frac{t}{3} \right) \eta_{\langle k, 1 \rangle} + K \eta_{\langle k, 1 \rangle} = \sum_{l,m=1}^{\infty} \left(l + \frac{s}{m+2} \right) e_{\langle l, m \rangle} \eta_{\langle k, 1 \rangle}.$$

Since K is compact and $\langle k, 1 \rangle = 2^{k-1}$, $\eta_{\langle k, 1 \rangle} \xrightarrow{k \rightarrow \infty} 0$ weakly, we have $\|K \eta_{\langle k, 1 \rangle}\| \xrightarrow{k \rightarrow \infty} 0$. For $k, l, m \in \mathbb{N}$, let $a(k, l, m) := |k - l + \frac{t}{3} - \frac{s}{m+2}|$. Then $a(k, l, m) \geq |k - l| - |\frac{t}{3} - \frac{s}{m+2}| \geq 1 - \frac{2}{3} = \frac{1}{3}$ if $k \neq l$, while $a(k, k, m) = \frac{t}{3} - \frac{s}{m+2} \geq \frac{t-s}{3}$. Therefore $a(k, l, m) \geq \delta(t, s) := \frac{1}{3}(t-s) > 0$, ($k, l, m \in \mathbb{N}$). From this, we have

$$\begin{aligned} \|K \eta_{\langle k, 1 \rangle}\|^2 &= \sum_{l,m=1}^{\infty} \left| k - l + \frac{t}{3} - \frac{s}{m+2} \right|^2 \|e_{\langle l, m \rangle} \eta_{\langle k, 1 \rangle}\|^2 \\ &\geq \delta(t, s)^2 \sum_{l,m=1}^{\infty} \|e_{\langle l, m \rangle} \eta_{\langle k, 1 \rangle}\|^2 = \delta(t, s)^2. \end{aligned}$$

This is a contradiction to $\lim_{k \rightarrow \infty} \|K \eta_{\langle k, 1 \rangle}\| = 0$.

Taking all the above examples into account, it seems unlikely that there exists a complete invariant for the von Neumann type classification problem for $\text{SA}(H)$, such that the assignment of the invariant is constructible in some sense.

3.2 Orbit Equivalence Relation $E_G^{\text{SA}(H)}$

To consider the complexity of the classification problem of self-adjoint operators up to unitary equivalence modulo compact perturbations we use $\text{SA}(H)$ as parameter Polish space and regard the equivalence as orbit equivalence of a Polish group action:

Definition 3.6. (1) We define the Polish group G to be the semidirect product $\mathbb{K}(H)_{\text{sa}} \rtimes \mathcal{U}(H)$, where $\mathbb{K}(H)_{\text{sa}}$ is the additive Polish group of compact self-adjoint operators with the norm topology, and we equip $\mathcal{U}(H)$ with SOT. The action of $\mathcal{U}(H)$ on $\mathbb{K}(H)_{\text{sa}}$ is given by conjugation: $u \cdot K := u K u^*$, $u \in \mathcal{U}(H)$, $K \in \mathbb{K}(H)_{\text{sa}}$.

(2) We define the action $\alpha: G \times \text{SA}(H) \rightarrow \text{SA}(H)$ by

$$(K, u) \cdot A := u A u^* + K, \quad A \in \text{SA}(H), \quad u \in \mathcal{U}(H), \quad K \in \mathbb{K}(H)_{\text{sa}}.$$

It is easy to see that α is indeed an action. Therefore the classification problem in consideration is nothing but the study of the Borel complexity of the orbit equivalence relation $E_G^{\text{SA}(H)}$.

Definition 3.7. We call $E_G^{\text{SA}(H)}$ the *Weyl-von Neumann equivalence relation*.

Next we show that $\text{SA}(H)$ is a Polish G -space.

Proposition 3.8. *The action $\alpha: G \curvearrowright \text{SA}(H)$ is continuous.*

We first show the continuity of the $\mathbb{B}(H)_{\text{sa}}$ -action, where we equip the additive group $\mathbb{B}(H)_{\text{sa}}$ with the norm topology.

Proposition 3.9. *The action $\alpha_0: \mathbb{B}(H)_{\text{sa}} \curvearrowright \text{SA}(H)$ given by $(K, A) \mapsto A + K$ is continuous.*

The key point in the proof of Proposition 3.9 is the next lemma, which was communicated to us by Asao Arai. We are grateful to him for allowing us to include his proof.

Lemma 3.10 (Arai). *Let $K \in \mathbb{B}(H)_{\text{sa}}$ and let $A_n, A \in \text{SA}(H)$ ($n \in \mathbb{N}$) be such that $A_n \xrightarrow{\text{SRT}} A$. Then $A_n + K \xrightarrow{\text{SRT}} A + K$ holds.*

Proof. For any $z \in \mathbb{C} \setminus \mathbb{R}$, we have

$$\|K(A_n - z)^{-1}\| \leq \frac{\|K\|}{|\text{Im } z|}, \quad \|K(A - z)^{-1}\| \leq \frac{\|K\|}{|\text{Im } z|}.$$

Therefore it holds that if $\|K\| < |\text{Im } z|$,

$$\begin{aligned} (A_n + K - z)^{-1} &= \sum_{k=0}^{\infty} (-1)^k (A_n - z)^{-1} (K(A_n - z)^{-1})^k, \\ (A + K - z)^{-1} &= \sum_{k=0}^{\infty} (-1)^k (A - z)^{-1} (K(A - z)^{-1})^k. \end{aligned}$$

Therefore for arbitrary $\xi \in H$, we have

$$\begin{aligned} &\|(A_n + K - z)^{-1}\xi - (A + K - z)^{-1}\xi\| \\ &= \sum_{k=0}^{\infty} \|\{(A_n - z)^{-1}(K(A_n - z)^{-1})^k - (A - z)^{-1}(K(A - z)^{-1})^k\}\xi\|. \end{aligned} \quad (1)$$

Since $A_n \xrightarrow{n \rightarrow \infty} A$ (SRT), $K(A_n - z)^{-1} \xrightarrow{n \rightarrow \infty} K(A - z)^{-1}$ (SOT) holds. This implies that for each $k \geq 0$, $(A_n - z)^{-1}(K(A_n - z)^{-1})^k \xrightarrow{n \rightarrow \infty} (A - z)^{-1}(K(A - z)^{-1})^k$ (SOT). Therefore each term in (1) tends to 0 as $n \rightarrow \infty$. Furthermore, we see that

$$\|\{(A_n - z)^{-1}(K(A_n - z)^{-1})^k - (A - z)^{-1}(K(A - z)^{-1})^k\}\xi\| \leq 2|\text{Im } z|^{-1} \left(\frac{\|K\|}{|\text{Im } z|} \right)^k \|\xi\|, \quad (2)$$

and since $\sum_{k=0}^{\infty} (\|K\|/|\text{Im } z|)^k < \infty$, we have for $\|K\| < |\text{Im } z|$ that $(A_n + K - z)^{-1} \xrightarrow{k \rightarrow \infty} (A + K - z)^{-1}$ (SOT). Therefore by [RS81, Theorem VIII.19], $A_n + K \xrightarrow{n \rightarrow \infty} A + K$ (SRT) holds. \square

Proof of Proposition 3.9. Let $\{A_n\}_{n=1}^{\infty}$ (resp. $\{K_n\}_{n=1}^{\infty}$) be a sequence in $\text{SA}(H)$ (resp. in $\mathbb{B}(H)_{\text{sa}}$) converging to $A \in \text{SA}(H)$ (resp. to $K \in \mathbb{B}(H)_{\text{sa}}$). For any $\xi \in H$, we have

$$\begin{aligned} &\|(A_n + K_n - i)^{-1}\xi - (A + K - i)^{-1}\xi\| \\ &\leq \|\{(A_n + K_n - i)^{-1} - (A_n + K - i)^{-1}\}\xi\| + \|\{(A_n + K - i)^{-1} - (A + K - i)^{-1}\}\xi\|. \end{aligned} \quad (3)$$

By the resolvent identity [Sch10, §2.2, (2.4)], the first term in (3) is estimated as

$$\begin{aligned} \|\{(A_n + K_n - i)^{-1} - (A_n + K - i)^{-1}\}\xi\| &\leq \|(A_n + K_n - i)^{-1}(K_n - K)(A_n + K - i)^{-1}\xi\| \\ &\leq \|K_n - K\| \cdot \|\xi\| \xrightarrow{n \rightarrow \infty} 0. \end{aligned}$$

The second term in (3) also tends to 0 by Lemma 3.10. Therefore $A_n + K_n \xrightarrow{n \rightarrow \infty} A + K$ (SRT) holds. \square

Proof of Proposition 3.8. Assume that $A_n \in \text{SA}(H)$ (resp. $(K_n, u_n) \in G$) converges to $A \in \text{SA}(H)$ (resp. $(K, u) \in G$). Then $u_n A_n u_n^* \xrightarrow{n \rightarrow \infty} u A u^*$ (SRT), because the joint SOT-continuity of operator product on bounded sets shows that

$$(u_n A_n u_n^* - i) = u_n (A_n - i)^{-1} u_n^* \xrightarrow{n \rightarrow \infty} u (A - i)^{-1} u^* = (u A u^* - i)^{-1} \text{ (SOT)}.$$

Therefore by Proposition 3.9, we have $u_n A_n u_n^* + K_n \xrightarrow{n \rightarrow \infty} u A u^* + K$ (SRT). \square

3.3 Smoothness: Bounded Case

Recall that the Effros Borel structure on the space $\mathcal{F}(\mathbb{R})$ of all closed subsets of \mathbb{R} is the σ -algebra generated by sets of the form $\{F \in \mathcal{F}(\mathbb{R}); F \cap U \neq \emptyset\}$, where U is an open subset of \mathbb{R} . In this section, we show that Weyl-von Neumann equivalence relation restricted on $\mathbb{B}(H)_{\text{sa}}$ is smooth by showing that $\text{SA}(H) \ni A \mapsto \sigma_{\text{ess}}(A) \in \mathcal{F}(\mathbb{R})$ is Borel.

Lemma 3.11. $\mathbb{B}(H)_{\text{sa}}$ is an F_σ subset of $\text{SA}(H)$. In particular, it is Borel.

Proof. Let $F_n := \{A \in \mathbb{B}(H)_{\text{sa}}; \|A\| \leq n\}$ ($n \in \mathbb{N}$). Then $\mathbb{B}(H)_{\text{sa}} = \bigcup_{n=1}^{\infty} F_n$. It is straightforward to show that each F_n is SRT-closed. Thus $\mathbb{B}(H)_{\text{sa}}$ is F_σ in $\text{SA}(H)$. \square

By Lemma 3.11, $\mathbb{B}(H)_{\text{sa}}$ is a standard Borel space with respect to the subspace Borel structure. Since $\mathbb{B}(H)_{\text{sa}}$ is G -invariant, we may consider the restricted action of G on $\mathbb{B}(H)_{\text{sa}}$ and its orbit equivalence relation $E_G^{\mathbb{B}(H)_{\text{sa}}}$.

Theorem 3.12. $E_G^{\mathbb{B}(H)_{\text{sa}}}$ is a smooth equivalence relation.

Lemma 3.13. The map $\text{SA}(H) \ni A \mapsto \sigma(A) \in \mathcal{F}(\mathbb{R})$ is Borel.

Proof. It clearly suffices to show that for any $a, b \in \mathbb{R}$ ($a < b$), the set $\mathcal{U} = \{A \in \text{SA}(H); \sigma(A) \cap (a, b) = \emptyset\}$ is Borel. But it is well-known that \mathcal{U} is in fact SRT-closed (see e.g., [Sim95, Lemma 1.6]). \square

Next we show the Borelness of $A \mapsto \sigma_{\text{ess}}(A)$. Note however that $\mathcal{V} = \{A \in \text{SA}(H); \sigma_{\text{ess}}(A) \cap (a, b) = \emptyset\}$ is neither open nor closed. In fact $A \mapsto \sigma_{\text{ess}}(A)$ behaves quite discontinuously (with respect to any compatible Polish topology on $\mathcal{F}(\mathbb{R})$):

Proposition 3.14. Let $K, L \in \mathcal{F}(\mathbb{R})$ be nonempty. Then there exists $\{A_n\}_{n=1}^{\infty} \subset \text{SA}(H)$ and $A \in \text{SA}(H)$ with the property that

$$\sigma_{\text{ess}}(A_n) = K \quad (n \in \mathbb{N}), \quad \sigma_{\text{ess}}(A) = L, \quad A_n \xrightarrow{n \rightarrow \infty} A \quad \text{in } \text{SA}(H).$$

Proof. For each $k \in \mathbb{N}$, let H_k be a separable infinite-dimensional Hilbert space with CONS $\{\xi_{k,i}\}_{i=1}^{\infty}$, and let $e_{k,i}$ be the projection of H_k onto $\mathbb{C}\xi_{k,i}$ ($k, i \in \mathbb{N}$). Set $H = \bigoplus_{n=1}^{\infty} H_k$, and let $\{\lambda_n\}_{n=1}^{\infty}$ (resp. $\{\mu_n\}_{n=1}^{\infty}$) be a dense subset of K (resp. L). For each $n \in \mathbb{N}$, define $A_{n,k} \in \text{SA}(H_k)$ by

$$A_{n,k} := \begin{cases} \mu_k \sum_{i=1}^n e_{k,i} + \lambda_k \sum_{i=n+1}^{\infty} e_{k,i} & (1 \leq k \leq n) \\ \lambda_k \mathbf{1}_{H_k} & (k > n). \end{cases}$$

It is then straightforward to see that $A := \bigoplus_{k=1}^{\infty} \mu_k \mathbf{1}_{H_k}$, and $A_n := \bigoplus_{k=1}^{\infty} A_{n,k}$ ($n \in \mathbb{N}$) does the job. \square

Theorem 3.15. The map $\Phi: \text{SA}(H) \ni A \mapsto \sigma_{\text{ess}}(A) \in \mathcal{F}(\mathbb{R})$ is Borel.

Lemma 3.16. Let $A \in \text{SA}(H)$, and let \mathcal{K} be a norm-dense subset of $\mathbb{K}(H)_{\text{sa}}$. Then the following equality holds:

$$\sigma_{\text{ess}}(A) = \bigcap_{K \in \mathcal{K}} \sigma(A + K).$$

Proof. By Weyl's Theorem, the essential spectra are invariant under compact perturbations (Theorem 1.2 (2) \Rightarrow (1)). Therefore

$$\sigma_{\text{ess}}(A) = \bigcap_{K \in \mathbb{K}(H)_{\text{sa}}} \sigma_{\text{ess}}(A + K) \subset \bigcap_{K \in \mathbb{K}(H)_{\text{sa}}} \sigma(A + K) \subset \bigcap_{K \in \mathcal{K}} \sigma(A + K).$$

To prove the opposite inclusion, we show that $\sigma_{\text{d}}(A) \cap \bigcap_{K \in \mathcal{K}} \sigma(A + K) = \emptyset$. If $\sigma_{\text{d}}(A) = \emptyset$, this is obvious, so we assume that $\sigma_{\text{d}}(A) \neq \emptyset$. Let $E_A(\cdot)$ be the spectral measure of A , and let $\lambda \in \sigma_{\text{d}}(A)$. Then by the definition of the discrete spectra, there exists $\delta > 0$ such that $E_A((\lambda - \delta, \lambda + \delta)) = E_A(\{\lambda\})$ has rank $n \in \mathbb{N}$. Put $K := E_A(\{\lambda\}) \in \mathbb{K}(H)_{\text{sa}}$ (which is of finite rank). Then

$$A + K - \lambda = E_A(\{\lambda\}) + (A - \lambda)E_A(\mathbb{R} \setminus (\lambda - \delta, \lambda + \delta)).$$

This shows that $A + K - \lambda$ has the bounded inverse $(A + K - \lambda)^{-1}$. Choose by density an element $K_0 \in \mathcal{K}$ such that $\|K - K_0\| < 1/\|(A + K - \lambda)^{-1}\|$. Then $\|(A + K - \lambda)^{-1}(K_0 - K)\| < 1$, and $1 + (A + K - \lambda)^{-1}(K_0 - K)$ is invertible in $\mathbb{B}(H)$. It holds that

$$\begin{aligned} A + K_0 - \lambda &= A + K - \lambda + (K_0 - K) \\ &= (A + K - \lambda)\{1 + (A + K - \lambda)^{-1}(K_0 - K)\} \end{aligned}$$

has the bounded inverse $(A + K_0 - \lambda)^{-1} = \{1 + (A + K - \lambda)^{-1}(K_0 - K)\}^{-1}(A + K - \lambda)^{-1}$. Therefore $\lambda \notin \sigma(A + K_0)$, and we obtain

$$\sigma_d(A) \subset \mathbb{R} \setminus \bigcap_{K \in \mathcal{K}} \sigma(A + K).$$

This finishes the proof. \square

Proof of Theorem 3.15. Let $\mathcal{K} = \{K_n\}_{n=1}^\infty$ be a countable norm-dense subset of $\mathbb{K}(H)_{\text{sa}}$. By Lemma 3.16, we have

$$\sigma_{\text{ess}}(A) = \bigcap_{n=1}^\infty \sigma(A + K_n), \quad A \in \text{SA}(H). \quad (4)$$

By Lemma 3.13, the map $\text{SA}(H) \ni A \mapsto \sigma(A) \in \mathcal{F}(\mathbb{R})$ is Borel. To show that $\Phi : A \mapsto \sigma_{\text{ess}}(A)$ is Borel, we have only to show that the set $\mathcal{B} := \{A \in \text{SA}(H); \sigma_{\text{ess}}(A) \cap [a, b] \neq \emptyset\}$ is Borel for every closed interval $[a, b]$ in \mathbb{R} (since every open set in a metrizable space is F_σ). Now we use the following equivalence:

$$\bigcap_{n=1}^\infty \sigma(A + K_n) \cap [a, b] \neq \emptyset \Leftrightarrow \bigcap_{n=1}^N \sigma(A + K_n) \cap [a, b] \neq \emptyset \quad \text{for all } N \in \mathbb{N}. \quad (5)$$

Indeed, \Rightarrow is obvious, and \Leftarrow follows from the finite intersection property of the compact set $[a, b]$. Then for each $A \in \text{SA}(H)$, we deduce by (4) and (5) that

$$\begin{aligned} \mathcal{B} &= \left\{ A \in \text{SA}(H); \bigcap_{n=1}^\infty \sigma(A + K_n) \cap [a, b] \neq \emptyset \right\} \\ &= \bigcap_{k=1}^\infty \underbrace{\left\{ A \in \text{SA}(H); \bigcap_{n=1}^k \sigma(A + K_n) \cap [a, b] \neq \emptyset \right\}}_{=: \mathcal{B}_k} \end{aligned}$$

Therefore it is enough to show that \mathcal{B}_k is Borel for each $k \in \mathbb{N}$. Recall that since \mathbb{R} is σ -compact, Christensen's Theorem [Chr71] asserts that the intersection map $I_2 : \mathcal{F}(\mathbb{R}) \times \mathcal{F}(\mathbb{R}) \ni (K_1, K_2) \mapsto K_1 \cap K_2 \in \mathcal{F}(\mathbb{R})$ is Borel. By inductive argument, the k -fold intersection map $I_k : \mathcal{F}(\mathbb{R})^k \ni (K_1, \dots, K_k) \mapsto \bigcap_{i=1}^k K_i \in \mathcal{F}(\mathbb{R})$ is Borel. Since the addition map $\tau_n : \text{SA}(H) \ni A \mapsto A + K_n \in \text{SA}(H)$ is a homeomorphism for each $n \in \mathbb{N}$, the Borelness of $\Phi_0 : \text{SA}(H) \ni A \mapsto \sigma(A) \in \mathcal{F}(\mathbb{R})$ implies that the map

$$\Psi_k := I_k \circ (\Phi_0 \circ \tau_1 \times \dots \times \Phi_0 \circ \tau_k) : \text{SA}(H) \ni A \mapsto \bigcap_{i=1}^k \sigma(A + K_i) \in \mathcal{F}(\mathbb{R})$$

is Borel. Thus $\mathcal{B}_k = \Psi_k^{-1}(\{K \in \mathcal{F}(\mathbb{R}); K \cap [a, b] \neq \emptyset\})$ is Borel, so $\Phi : A \mapsto \sigma_{\text{ess}}(A)$ is Borel. \square

Proof of Theorem 3.12. Let $A, B \in \mathbb{B}(H)_{\text{sa}}$. By Weyl-von Neumann Theorem 1.2, $AE_G^{\mathbb{B}(H)_{\text{sa}}} B$ if and only if $\sigma_{\text{ess}}(A) = \sigma_{\text{ess}}(B)$. Therefore $\Phi : \text{SA}(H) \ni A \mapsto \sigma_{\text{ess}}(A) \in \mathcal{F}(\mathbb{R})$ restricted to $\mathbb{B}(H)_{\text{sa}}$ is a Borel reduction of $E_G^{\mathbb{B}(H)_{\text{sa}}}$ to $\text{id}_{\mathcal{F}(\mathbb{R})}$. \square

3.4 Non-classification: Unbounded Case

We have shown that $E_G^{\mathbb{B}(H)_{\text{sa}}}$ is smooth (Theorem 3.12), therefore bounded self-adjoint operators are concretely classifiable up to Weyl-von Neumann equivalence by their essential spectra. In this section, we show that the situation for unbounded operators is rather different: we show that the G -action on $\text{SA}(H)$ has a dense G_δ orbit $\text{SA}_{\text{full}}(H) := \{A \in \text{SA}(H); \sigma_{\text{ess}}(A) = \mathbb{R}\}$ (Theorem 3.17), whence the action is not generically turbulent. On the other hand we also show that it is unclassifiable by countable structures by showing that $E_G^Y \leq_B E_G^{\text{SA}(H)}$ for a generically turbulent G -action on a Polish space Y (Theorem 3.32). In

fact we choose $Y = \text{EES}(H) = \{A \in \text{SA}(H); \sigma_{\text{ess}}(A) = \emptyset\}$, which is a small part of $\text{SA}(H)$. Y equipped with the norm resolvent topology is Polish (Proposition 3.25), and the G -action on Y is just the restriction of the original action to Y . Note that $A \mapsto \sigma_{\text{ess}}(A)$ is constant on Y , and therefore $\sigma_{\text{ess}}(\cdot)$ is very far from a complete invariant for $E_G^{\text{SA}(H)}$.

3.4.1 $E_G^{\text{SA}(H)}$ is Not Generically Turbulent

We show that the G -action on $\text{SA}(H)$ is not generically turbulent, by showing that there exists a comeager G -orbit. More precisely:

Theorem 3.17. *The following statements hold:*

- (1) *The set $\text{SA}_{\text{full}}(H) := \{A \in \text{SA}(H); \sigma_{\text{ess}}(A) = \mathbb{R}\}$ is a dense G_δ subset of $\text{SA}(H)$.*
- (2) *If $A, B \in \text{SA}(H)$ satisfy $\sigma_{\text{ess}}(A) = \sigma_{\text{ess}}(B) = \mathbb{R}$, then $AE_G^{\text{SA}(H)} B$. Consequently, $\text{SA}_{\text{full}}(H)$ is a dense G_δ -orbit of the G -action.*

In particular, the G -action on $\text{SA}(H)$ is not generically turbulent.

Note that the above theorem shows that (1) \Rightarrow (2) of Theorem 1.2 holds true for $\text{SA}_{\text{full}}(H)$ (in fact von Neumann's proof itself works verbatim, as we see below), even though elements in $\text{SA}_{\text{full}}(H)$ are highly unbounded. The proof of Theorem 3.17 (1) is strongly inspired by the work of B. Simon [Sim95]. We start from the next result.

Proposition 3.18. *Let $\lambda \in \mathbb{R}$. Then the set $\{A \in \text{SA}(H); \lambda \in \sigma_{\text{ess}}(A)\}$ is a dense G_δ set in $\text{SA}(H)$.*

The next two lemmata are elementary and we omit the proofs.

Lemma 3.19. *Let $\{a_n\}_{n=1}^\infty \subset \mathbb{B}(H)$ be a sequence converging strongly to $a \in \mathbb{B}(H)$. If $\text{rank}(a_n) \leq k$ ($n \in \mathbb{N}$) holds for some fixed $k \in \mathbb{N}$, then $\text{rank}(a) \leq k$ holds.*

Lemma 3.20. *Let $A \in \text{SA}(H)$, (a, b) be an open interval in \mathbb{R} , and let $k \in \mathbb{N}$. Then $\text{rank}E_A((a, b)) \leq k$ holds if and only if for every continuous real valued function f on \mathbb{R} with $\text{supp}(f) \subset (a, b)$, $\text{rank}f(A) \leq k$ holds.*

Proof of Proposition 3.18. We express the complement of $\{A \in \text{SA}(H); \lambda \in \sigma_{\text{ess}}(A)\}$ as follows:

$$\begin{aligned} \{A \in \text{SA}(H); \lambda \notin \sigma_{\text{ess}}(A)\} &= \bigcup_{\varepsilon > 0} \{A \in \text{SA}(H); E_A((\lambda - \varepsilon, \lambda + \varepsilon)) \text{ is of finite rank}\} \\ &= \bigcup_{n=1}^\infty \underbrace{\bigcup_{k=1}^\infty \{A \in \text{SA}(H); \text{rank}E_A((\lambda - \frac{1}{n}, \lambda + \frac{1}{n})) \leq k\}}_{=: S_{n,k}} \end{aligned}$$

We show that $S_{n,k}$ is closed in $\text{SA}(H)$. Suppose $\{A_m\}_{m=1}^\infty \subset S_{n,k}$ converges to $A \in \text{SA}(H)$. Then let f be a continuous function with $\text{supp}(f) \subset (\lambda - \frac{1}{n}, \lambda + \frac{1}{n})$. Then as $A_m \xrightarrow{m \rightarrow \infty} A$ (SRT), we have $f(A_m) \xrightarrow{m \rightarrow \infty} f(A)$ (SOT) by [RS81, Theorem VIII.20 (b)]. By Lemma 3.20, $\text{rank}f(A_m) \leq k$ for each $m \in \mathbb{N}$. Therefore by Lemma 3.19, $\text{rank}f(A) \leq k$. Since this holds for arbitrary such f , Lemma 3.20 shows that $\text{rank}E_A((\lambda - \frac{1}{n}, \lambda + \frac{1}{n})) \leq k$. Therefore $A \in S_{n,k}$. This shows that $\{A \in \text{SA}(H); \lambda \notin \sigma_{\text{ess}}(A)\}$ is F_σ , so its complement $\{A \in \text{SA}(H); \lambda \in \sigma_{\text{ess}}(A)\}$ is G_δ . Next we show the density. Let $A \in \text{SA}(H)$, and let \mathcal{V} be an open neighborhood of A . Then by Weyl-von Neumann Theorem 1.1, we may find $A_0 \in \mathcal{V}$ of the form $A_0 = \sum_{n=1}^\infty \lambda_n e_n$, where $\{e_n\}_{n=1}^\infty$ is a mutually orthogonal projections with sum equal to 1, and $\lambda_n \in \mathbb{R}$. Then put $A_k := \sum_{n=1}^k \lambda_n e_n + \lambda \sum_{n=k+1}^\infty e_n$. It is straightforward to see that $\lambda \in \sigma_{\text{ess}}(A_k)$. and $A_k \xrightarrow{k \rightarrow \infty} A_0 \in \mathcal{V}$ (SOT). Since \mathcal{V} is arbitrary, $\{A \in \text{SA}(H); \lambda \in \sigma_{\text{ess}}(A)\}$ is a dense G_δ set. \square

Proof of Theorem 3.17 (1). For each $q \in \mathbb{Q}$, the set $G_q := \{A \in \text{SA}(H); q \in \sigma_{\text{ess}}(A)\}$ is a dense G_δ set in $\text{SA}(H)$ by Proposition 3.18. Therefore as $\sigma_{\text{ess}}(A)$ is a closed subset in \mathbb{R} , $\{A \in \text{SA}(H); \sigma_{\text{ess}}(A) = \mathbb{R}\} = \bigcap_{q \in \mathbb{Q}} G_q$ is also a dense G_δ set. \square

To finish the proof of Theorem 3.17 (2), we need the following variant of a well-known argument used in the proof of Theorem 1.2, (1) \Rightarrow (2).

Lemma 3.21 ([AG61]). *Let $\{\lambda_n\}_{n=1}^\infty, \{\mu_n\}_{n=1}^\infty$ be sequences of real numbers with the same set of accumulation point M . If both $\{\lambda_n\}_{n=1}^\infty, \{\mu_n\}_{n=1}^\infty$ have only finitely many isolated points, then there exists a permutation π of \mathbb{N} such that $\lim_{n \rightarrow \infty} (\lambda_n - \mu_{\pi(n)}) = 0$ holds.*

Proof. The setting as well as the proof is almost the same as the one in [AG61, §94], so we only explain the difference of the present setting from [AG61, §94]. For $k \in \mathbb{N}$, define

$$\varepsilon_k := \inf_{t \in M} |\lambda_k - t| + \frac{1}{k}, \quad \eta_k := \inf_{t \in M} |\mu_k - t| + \frac{1}{k}.$$

Since there are only finitely many isolated points in $\{\lambda_n\}_{n=1}^\infty, \{\mu_n\}_{n=1}^\infty$, all but finitely many members of $\{\mu_n\}_{n=1}^\infty, \{\lambda_n\}_{n=1}^\infty$ belong to M . Therefore $\varepsilon_k \rightarrow 0, \eta_k \rightarrow 0$ ($k \rightarrow \infty$). Now the rest of the proof is the same as the one in [AG61, §94], so we omit the proof. \square

Remark 3.22. Note that Lemma 3.21 does not hold in general without assuming some conditions on isolated points of M : consider the sequences $\{\lambda_n^{(t)}\}_{n=1}^\infty$ ($t \in [0, 1]$) in Example 3.3. We show that if $0 \leq s < t \leq 1$, then there is no permutation π of \mathbb{N} satisfying $\lim_{n \rightarrow \infty} (\lambda_{\pi(n)}^{(s)} - \lambda_n^{(t)}) = 0$, although both sequences have accumulation points $M = \mathbb{N}$. Assume by contradiction that such π exists. Then there exists $k_0 \in \mathbb{N}$ such that $|\lambda_{\pi(k)}^{(s)} - \lambda_k^{(t)}| < \frac{t-s}{8}$ for all $k \geq k_0$. Since $k \leq 2^k \leq 2^k(2m-1) \stackrel{\text{def}}{=} \langle k+1, m \rangle$ ($k, m \in \mathbb{N}$), this implies that

$$|\lambda_{\pi(\langle k, m \rangle)}^{(s)} - \lambda_{\langle k, m \rangle}^{(t)}| < \frac{t-s}{8}, \quad (k \geq k_0 + 1, m \in \mathbb{N}). \quad (6)$$

On the other hand, if $k, k', m, m' \in \mathbb{N}$ with $k \neq k', k, k' \geq k_0 + 1$, then

$$\begin{aligned} |\lambda_{\langle k', m' \rangle}^{(s)} - \lambda_{\langle k, m \rangle}^{(t)}| &= \left| k' + \frac{s}{m'+2} - k - \frac{t}{m+2} \right| \geq |k' - k| - \left| \frac{s}{m'+2} - \frac{t}{m+2} \right| \\ &\geq 1 - \frac{2}{3} > \frac{t-s}{8}. \end{aligned}$$

Therefore by (6), for each $k \geq k_0 + 1$ and $m \in \mathbb{N}$, there exists $\varphi_k(m) \in \mathbb{N}$ such that

$$\lambda_{\pi(\langle k, m \rangle)}^{(s)} = \lambda_{\langle k, \varphi_k(m) \rangle}^{(s)} \quad (k \geq k_0 + 1, m \in \mathbb{N}).$$

However, by (6) (for $k = k_0 + 1, m = 1$) it follows that

$$\frac{t-s}{8} > |\lambda_{\langle k_0+1, \varphi_{k_0+1}(1) \rangle}^{(s)} - \lambda_{\langle k_0+1, 1 \rangle}^{(t)}| = \left| \frac{s}{\varphi_{k_0+1}(1)+2} - \frac{t}{3} \right| \geq \frac{t-s}{3},$$

which is a contradiction. This completes the proof.

Proof of Theorem 3.17 (2). The proof goes exactly the same as von Neumann's proof: by Weyl-von Neumann Theorem 1.1, $[A]_G, [B]_G$ contain diagonalizable operators with essential spectra \mathbb{R} . Therefore we may assume that A, B are of the form $A = \sum_{n=1}^\infty a_n \langle \xi_n, \cdot \rangle \xi_n, B = \sum_{n=1}^\infty b_n \langle \eta_n, \cdot \rangle \eta_n$, where $\{a_n\}_{n=1}^\infty, \{b_n\}_{n=1}^\infty$ are real sequences. Since $\sigma_{\text{ess}}(A) = \sigma_{\text{ess}}(B) = \mathbb{R}$ and there are at most countably many isolated eigenvalues, this implies that the set of accumulation points of $\{a_n\}_{n=1}^\infty$ and $\{b_n\}_{n=1}^\infty$ are both \mathbb{R} . By Lemma 3.21, there exists a permutation π of \mathbb{N} such that $\lim_{k \rightarrow \infty} (a_{\pi(k)} - b_k) = 0$. Define $u \in \mathcal{U}(H)$ by $u\xi_k := \eta_{\pi^{-1}(k)}, k \in \mathbb{N}$. Then $uAu^* = \sum_{n=1}^\infty a_{\pi(n)} \langle \eta_n, \cdot \rangle \eta_n$, and define $K := \sum_{n=1}^\infty (b_n - a_{\pi(n)}) \langle \eta_n, \cdot \rangle \eta_n \in \mathbb{K}(H)_{\text{sa}}$. It holds that $uAu^* + K = B$. \square

3.4.2 $E_G^{\text{EES}(H)}$ is Generically Turbulent

As explained in the introduction to §3.4, we now study the restricted action of G on a subset $\text{EES}(H) = \{A \in \text{SA}(H); \sigma_{\text{ess}}(A) = \emptyset\}$ equipped with a new Polish topology.

3.4.2.1 Norm Resolvent Topology and Polish Space $\text{EES}(H)$

Let $\text{EES}(H) := \{A \in \text{SA}(H); \sigma_{\text{ess}}(A) = \emptyset\}$ (EES stands for Empty Essential Spectrum).

Definition 3.23. The *norm resolvent topology* (NRT) on $\text{SA}(H)$ is the weakest topology for which the map $\text{SA}(H) \ni A \mapsto (A - i)^{-1} \in \mathbb{B}(H)$ is norm-continuous.

Note that $(\text{SA}(H), \text{NRT})$ is not separable, whence not Polish. On the contrary, we show that

Proposition 3.24. $(\text{EES}(H), \text{NRT})$ is a Polish G -space with respect to the restriction β of the action $\alpha: G \curvearrowright \text{SA}(H)$ to $\text{EES}(H)$.

We first show

Proposition 3.25. $(\text{EES}(H), \text{NRT})$ is a Polish space.

We need preparations. The first lemma is well-known and second one is elementary.

Lemma 3.26. Let $A \in \text{SA}(H)$. Then $\sigma_{\text{ess}}(A) = \emptyset$ if and only if $(A - i)^{-1} \in \mathbb{K}(H)$.

Lemma 3.27. Let $x \in \mathbb{B}(H)$ be normal. Then there exists $A \in \text{SA}(H)$ such that $x = (A - i)^{-1}$ holds, if and only if both $\text{Ran}(x)$ and $\text{Ran}(x^*)$ are dense in H , and $(x^{-1} + i)^* = x^{-1} + i$.

Lemma 3.28. Let \mathcal{D} be a subspace of $\mathbb{K}(H)$ consisting of those normal elements x such that $\text{Ran}(x)$ and $\text{Ran}(x^*)$ are both dense in H . Then \mathcal{D} is a G_δ subset of $\mathbb{K}(H)$ with respect to the norm topology. In particular, \mathcal{D} is Polish.

Proof. It is clear that $\mathcal{D}_1 := \{x \in \mathbb{K}(H); xx^* = x^*x\}$ is closed. Let $\{\xi_n\}_{n=1}^\infty$ be a dense subset of H . Then it is easy to see that

$$\text{Ran}(x) \text{ is dense} \Leftrightarrow \forall k \in \mathbb{N} \forall l \in \mathbb{N} \exists m \in \mathbb{N} [\|x\xi_m - \xi_l\| < \frac{1}{k}].$$

Therefore

$$\mathcal{D}_2 := \{x \in \mathbb{K}(H); \text{Ran}(x) \text{ is dense}\} = \bigcap_{k=1}^\infty \bigcap_{l=1}^\infty \underbrace{\bigcup_{m=1}^\infty \{x \in \mathbb{K}(H); \|x\xi_m - \xi_l\| < \frac{1}{k}\}}_{\text{open}},$$

which is G_δ in $\mathbb{K}(H)$. Similarly, $\mathcal{D}_3 := \{x \in \mathbb{K}(H); \text{Ran}(x^*)\}$ is G_δ in $\mathbb{K}(H)$, and so is $\mathcal{D} = \mathcal{D}_1 \cap \mathcal{D}_2 \cap \mathcal{D}_3$. \square

Proof of Proposition 3.25. Let $\varphi: (\text{ESS}(H), \text{NRT}) \rightarrow (\mathbb{K}(H), \|\cdot\|)$ be a map given by $\varphi(A) = (A - i)^{-1}$, $(A \in \text{SA}(H))$. By the definition of NRT and the injectivity, φ is a homeomorphism of $\text{ESS}(H)$ onto its range. We see that

$$\varphi(\text{EES}(H)) = \mathcal{D}_0 := \{x \in \mathcal{D}; x^{-1} + i \in \text{SA}(H)\},$$

where \mathcal{D} is as in Lemma 3.28. Indeed, if $x = \varphi(A) \in \varphi(\text{EES}(H))$, then x is compact by Lemma 3.26. Moreover, $\text{Ran}(x)$ and $\text{Ran}(x^*)$ are dense in H , and $x^{-1} + i \in \text{SA}(H)$ by Lemma 3.27. This shows that $x \in \mathcal{D}_0$. Conversely, if $x \in \mathcal{D}$ is such that $A := x^{-1} + i \in \text{SA}(H)$, then by Lemma 3.26, $A \in \text{EES}(H)$, and $\varphi(A) = x$. This shows that $\varphi(\text{EES}(H)) = \mathcal{D}_0$.

We next show that \mathcal{D}_0 is closed in \mathcal{D} . Once this is proved, Lemma 3.28 implies that \mathcal{D}_0 is also Polish, and so is $\text{EES}(H)$. Let $\{x_n\}_{n=1}^\infty$ be a sequence in \mathcal{D}_0 converging in norm to $x \in \mathcal{D}$. Put $A_n := x_n^{-1} + i \in \text{SA}(H)$. Then $x_n = (A_n - i)^{-1} \xrightarrow{\|\cdot\|} x$, $x_n^* = (A_n + i)^{-1} \xrightarrow{\|\cdot\|} x^*$. Since $x \in \mathcal{D}$, x and x^* has dense ranges, whence by Kato-Trotter Theorem [RS81, Theorem VIII.22], there exists $A \in \text{SA}(H)$ such that $(A_n - i)^{-1} \xrightarrow{\text{SOT}} (A - i)^{-1}$. Since $(A_n - i)^{-1} \xrightarrow{\text{SOT}} x$ also, we have $x = (A - i)^{-1}$ and $x^{-1} + i = A \in \text{EES}(H)$. Therefore \mathcal{D}_0 is closed in \mathcal{D} . This finishes the proof. \square

We now show that $\text{EES}(H)$ is a Polish G -space.

Proposition 3.29. The action $\beta: G \times \text{EES}(H) \rightarrow \text{EES}(H)$ is continuous.

We need preparations. The proof of the next lemma is almost identical to that of Proposition 3.9 (one may use the joint norm-continuity of the operator product to get NRT-version of Lemma 3.10).

Lemma 3.30. Let $A_n, A \in \text{SA}(H)$ and let $K_n, K \in \mathbb{B}(H)_{\text{sa}}$ ($n \in \mathbb{N}$). If $A_n \xrightarrow{\text{NRT}} A$ and $\|K_n - K\| \xrightarrow{n \rightarrow \infty} 0$, then $A_n + K_n \xrightarrow{\text{NRT}} A + K$ holds.

The next lemma is known in operator theory.

Lemma 3.31. Let $x_n, x \in \mathbb{K}(H)$ and $u_n, u \in \mathcal{U}(H)$ ($n \in \mathbb{N}$) be such that $\|x_n - x\| \xrightarrow{n \rightarrow \infty} 0$ and $u_n \xrightarrow{n \rightarrow \infty} u$ (SOT). Then we have $\|u_n x_n - u x\| \xrightarrow{n \rightarrow \infty} 0$.

Proof of Proposition 3.29. Let $u_n, u \in \mathcal{U}(H)$, $K_n, K \in \mathbb{K}(H)_{\text{sa}}$, and $A_n, A \in \text{EES}(H)$ ($n \in \mathbb{N}$) be such that $u_n \xrightarrow{\text{SOT}} u$, $A_n \xrightarrow{\text{NRT}} A$ and $K_n \xrightarrow{\|\cdot\|} K$, respectively. We show that $u_n A_n u_n^* + K_n \xrightarrow{\text{NRT}} u A u^* + K$. By Lemma 3.30, it suffices to prove that $u_n A_n u_n^* \xrightarrow{\text{NRT}} u A u^*$. We compute the resolvent as follows:

$$\begin{aligned} \|(u_n A_n u_n^* - i)^{-1} - (u A u^* - i)^{-1}\| &= \|u_n (A_n - i)^{-1} u_n^* - u (A - i)^{-1} u^*\| \\ &\leq \|\{u_n (A_n - i)^{-1} - u (A - i)^{-1}\} u_n^*\| + \|u \{(A - i)^{-1} u^* - (A - i)^{-1} u_n^*\}\| \\ &= \|u_n (A_n - i)^{-1} - u (A - i)^{-1}\| + \|u (A + i)^{-1} - u_n (A + i)^{-1}\|. \end{aligned} \quad (7)$$

Since $(A_n - i)^{-1}$, $(A \pm i)^{-1}$ are compact (Lemma 3.26), the assumptions on u_n and A_n implies that (7) converges to 0 by Lemma 3.31. Therefore $u_n A_n u_n^* \xrightarrow{\text{NRT}} u A u^*$. This finishes the proof. \square

3.4.2.2 Generic Turbulence

Finally, we show the generic turbulence of $G \curvearrowright \text{EES}(H)$.

Theorem 3.32. *The restricted action β of G on $\text{EES}(H)$ is generically turbulent.*

Before we prove Theorem 3.32, let us state an immediate consequence:

Theorem 3.33. *$E_G^{\text{SA}(H)}$ does not admit classification by countable structures.*

Proof. By Theorem 3.32, $E_G^{\text{EES}(H)}$ is generically turbulent, and since NRT is stronger than SRT, we see that $E_G^{\text{EES}(H)}$ is Borel reducible (in fact continuously embeddable) to $E_G^{\text{SA}(H)}$ by the inclusion map $\iota: (\text{EES}(H), \text{NRT}) \rightarrow (\text{SA}(H), \text{SRT})$. \square

We now show that the action β is weakly generically turbulent and use Theorem 2.9.

Proposition 3.34. *For any $A \in \text{EES}(H)$, the orbit $[A]_G$ is NRT-dense and meager in $\text{EES}(H)$.*

For the proof, we use an easy lemma.

Lemma 3.35. *Let $A \in \text{SA}(H)$, $\lambda \in \sigma(A)$, $K \in \mathbb{K}(H)_{\text{sa}}$ and $c > \|K\|$. Then $\sigma(A + K) \cap [\lambda - c, \lambda + c] \neq \emptyset$.*

Proof. Suppose by contradiction that $B := A + K$ satisfies $\sigma(B) \cap [\lambda - c, \lambda + c] = \emptyset$. Then for $\mu \in \sigma(B)$, $|\mu - \lambda| \geq c$, and hence $\|(B - \lambda)^{-1}\| \leq c^{-1}$. It follows that

$$A - \lambda = B - K - \lambda = (B - \lambda)(1 - (B - \lambda)^{-1}K).$$

Since $\|(B - \lambda)^{-1}K\| \leq c^{-1}\|K\| < 1$, $1 - (B - \lambda)^{-1}K$ is invertible with bounded inverse, whence $A - \lambda$ also has the bounded inverse $(A - \lambda)^{-1} = (1 - (B - \lambda)^{-1}K)^{-1}(B - \lambda)^{-1}$, which contradicts $\lambda \in \sigma(A)$. \square

Proof of Proposition 3.34. First we show that the orbit $[A]_G$ is dense in $\text{EES}(H)$. Let $B \in \text{EES}(H)$. Then there exists CONS $\{\xi_n\}_{n=1}^\infty$ (resp. $\{\eta_n\}_{n=1}^\infty$) for H and a real sequence $\{a_n\}_{n=1}^\infty$ (resp. $\{b_n\}_{n=1}^\infty$) such that $A = \sum_{n=1}^\infty a_n \langle \xi_n, \cdot \rangle \xi_n$ and $B = \sum_{n=1}^\infty b_n \langle \eta_n, \cdot \rangle \eta_n$. Find $u \in \mathcal{U}(H)$ such that $u \xi_n = \eta_n$ ($n \in \mathbb{N}$). Put $K_N := \sum_{n=1}^N (b_n - a_n) \langle \eta_n, \cdot \rangle \eta_n$ ($N \in \mathbb{N}$). Then

$$u A u^* + K_N = \sum_{n=1}^N b_n \langle \eta_n, \cdot \rangle \eta_n + \sum_{n=N+1}^\infty a_n \langle \eta_n, \cdot \rangle \eta_n.$$

Since $A, B \in \text{EES}(H)$, $|a_n|, |b_n| \rightarrow \infty$ as $n \rightarrow \infty$. Therefore

$$\|(u A u^* + K_N - i)^{-1} - (B - i)^{-1}\| = \sup_{n \geq N+1} \left| \frac{1}{a_n - i} - \frac{1}{b_n - i} \right| \xrightarrow{N \rightarrow \infty} 0.$$

Therefore B is in the NRT-closure of $[A]_G$. Thus every orbit is dense.

Next we show that $[A]_G$ is meager. Let $0 \neq K \in \mathbb{K}(H)_{\text{sa}}$. Then choose $q \in \mathbb{Q} \cap (\|K\|, \infty)$. By Lemma 3.35, we have $\sigma(A + K) \cap [\lambda - q, \lambda + q] \neq \emptyset$ for each $\lambda \in \sigma(A) = \sigma_p(A)$. Thus we have (note that since H is separable, $\sigma_p(A)$ is at most countable)

$$[A]_G \subset \bigcup_{q \in \mathbb{Q}_{>0}} \bigcap_{\lambda \in \sigma_p(A)} \underbrace{\{B \in \text{EES}(H); \sigma_p(B) \cap [\lambda - q, \lambda + q] \neq \emptyset\}}_{=: S_{q,\lambda}}. \quad (8)$$

We show that the right hand side of (8) is meager. This is done in two steps:

Step 1. $S_{q,\lambda}$ is NRT-closed for each $q \in \mathbb{Q}_{>0}, \lambda \in \sigma_p(A)$.

Let $S_{q,\lambda} \ni B_n \xrightarrow{n \rightarrow \infty} B \in \text{EES}(H)$ (NRT). Assume that $\sigma_p(B) \cap [\lambda - q, \lambda + q] = \emptyset$. Therefore $\lambda \pm q \notin \sigma(B)$. Since $\mathbb{C} \setminus \sigma(B)$ is open, there exists $\varepsilon > 0$ such that $[\lambda - q - \varepsilon, \lambda + q + \varepsilon] \cap \sigma(B) = \emptyset$. By [RS81, Theorem VIII.23], $P_n := E_{B_n}((\lambda - q - \frac{\varepsilon}{2}, \lambda + q + \frac{\varepsilon}{2}))$ converges to $E_B((\lambda - q - \frac{\varepsilon}{2}, \lambda + q + \frac{\varepsilon}{2})) = 0$ in norm. Since P_n ($n \in \mathbb{N}$) is a projection, this shows that there exists $n_0 \geq 1$ such that $P_n = 0$ ($n \geq n_0$). This means in particular that $\sigma_p(B_{n_0}) \cap [\lambda - q, \lambda + q] = \emptyset$, a contradiction. Therefore $S_{q,\lambda}$ is NRT-closed.

Step 2. $\mathcal{S}_q := \bigcap_{\lambda \in \sigma_p(A)} S_{q,\lambda}$ is a (closed) nowhere-dense subset of $\text{EES}(H)$.

Assume by contradiction that there exists $B \in \mathcal{S}_q$ and $\varepsilon > 0$ such that \mathcal{S}_q contains an open neighborhood $\{C \in \text{EES}(H); \|(B - i)^{-1} - (C - i)^{-1}\| < \varepsilon\}$ of B . Let $A = \sum_{n=1}^{\infty} a_n \langle \xi_n, \cdot \rangle \xi_n$, $B = \sum_{n=1}^{\infty} b_n \langle \eta_n, \cdot \rangle \eta_n$, where $\{\xi_n\}_{n=1}^{\infty}, \{\eta_n\}_{n=1}^{\infty}$ are CONSs for H , and $|a_n|, |b_n| \nearrow \infty$. Since $|b_n| \nearrow \infty$, there is $n_0 \in \mathbb{N}$ such that $(|b_n|^2 + 1)^{-1/2} < \varepsilon/2$ for $n > n_0$. Since $|a_n| \nearrow \infty$, there is $n_1 \in \mathbb{N}$ such that $|a_{n_1}| > q$ and $|b_{n_0}| < |a_{n_1}| - q$ holds. Then we may also find $n_2 \in \mathbb{N}$ such that $|a_{n_1}| + q < |b_{n_2}|$ and $n_2 > n_0$ hold. Now define $C \in \text{EES}(H)$ by

$$C := \sum_{n=1}^{\infty} c_n \langle \eta_n, \cdot \rangle \eta_n, \quad c_n := \begin{cases} b_n & (1 \leq n \leq n_0) \\ b_{n_2 + (n - n_0)} & (n > n_0), \end{cases}.$$

By construction, we have

$$|c_n| \leq |b_{n_0}| < |a_{n_1}| - q \quad (n \leq n_0), \quad |c_n| \geq |b_{n_2}| > |a_{n_1}| + q \quad (n > n_0). \quad (9)$$

We compute

$$\|(C - i)^{-1} - (B - i)^{-1}\| \leq \sup_{n \geq n_0+1} \left(\frac{1}{\sqrt{|b_n|^2 + 1}} + \frac{1}{\sqrt{|b_{n_2 + (n - n_0)}|^2 + 1}} \right) < \varepsilon,$$

which shows by our assumption that $C \in \mathcal{S}_q$. However, we have $\sigma(C) \cap [a_{n_1} - q, a_{n_1} + q] = \emptyset$ by (9), which is a contradiction. Therefore \mathcal{S}_q is nowhere-dense. By Step 1 and Step 2, we have shown that $[A]_G$ is meager. \square

Finally, we show that the action of G on $\text{EES}(H)$ satisfies condition (b) of Definition 2.8. We need the following two elementary but useful lemmata.

Lemma 3.36. *Let $a, b \in \mathbb{R}$ and let $0 \leq s \leq 1$. If $ab \geq -1$, then*

$$\left| \frac{1}{(1-s)a + sb - i} - \frac{1}{a - i} \right| \leq \left| \frac{1}{b - i} - \frac{1}{a - i} \right|.$$

Lemma 3.37. $\text{EES}_{\pm\infty}(H) := \{A \in \text{EES}(H); \inf \sigma(A) = -\infty, \sup \sigma(A) = \infty\}$ is a G -invariant dense G_δ subset of $\text{EES}(H)$.

Proof of Theorem 3.32. By Theorem 2.9, it is enough to show that the action is weakly generically turbulent. We have shown that all orbits are dense and meager (Proposition 3.34). Therefore we have only to prove (b) in Definition 2.8. Let $A, B \in \text{EES}_{\pm\infty}(H)$, and let U be an open neighborhood of A in $\text{EES}(H)$, V be an open neighborhood of 1 in G . We may and do assume that U, V are of the form $U = \{C \in \text{EES}(H); \|(A - i)^{-1} - (C - i)^{-1}\| < \delta\}$, and $V = W_1 \times W_2$, where $W_1 = \{K \in \mathbb{K}(H)_{\text{sa}}; \|K\| < r\}$ and W_2 is an open neighborhood of 1 in $\mathcal{U}(H)$. We prove that $\mathcal{O}(A, U, V) \cap [B]_G \neq \emptyset$, which shows (b) because by Lemma 3.37, $\text{EES}_{\pm\infty}(H)$ is comeager in $\text{EES}(H)$. Let $A = \sum_{n=1}^{\infty} a_n \langle \xi_n, \cdot \rangle \xi_n$, $B = \sum_{n=1}^{\infty} b_n \langle \eta_n, \cdot \rangle \eta_n$ be the spectral resolutions of A, B respectively. Define $v \in \mathcal{U}(H)$ by $v\eta_n := \xi_n$ ($n \in \mathbb{N}$). Then

$$B_1 := vBv^* = \sum_{n=1}^{\infty} b_n \langle \xi_n, \cdot \rangle \xi_n \in [B]_G.$$

Let $I_A := \{n \in \mathbb{N}; a_n \geq 0\}$, $J_A := \{n \in \mathbb{N}; a_n < 0\}$ and define $I_B, J_B \subset \mathbb{N}$ analogously. By assumption, all I_A, J_A, I_B, J_B are infinite, so write

$$I_A = \{n_1 < n_2 < \dots\}, \quad J_A = \{n'_1 < n'_2 < \dots\} \\ I_B = \{m_1 < m_2 < \dots\}, \quad J_B = \{m'_1 < m'_2 < \dots\}.$$

Define a permutation π of \mathbb{N} by $\pi(n_k) := m_k$, $\pi(n'_k) = m'_k$, and define $u_\pi \in \mathcal{U}(H)$ by $u_\pi \xi_n := \xi_{\pi^{-1}(n)}$ ($n \in \mathbb{N}$). Then for each $k \in \mathbb{N}$, $u_\pi B_1 u_\pi^* \xi_{n_k} = b_{m_k} \xi_{n_k}$, $u_\pi B_1 u_\pi^* \xi_{n'_k} = b_{m'_k} \xi_{n'_k}$, and

$$B_2 := u_\pi B_1 u_\pi^* = \sum_{k=1}^{\infty} b_{m_k} \langle \xi_{n_k}, \cdot \rangle \xi_{n_k} + \sum_{k=1}^{\infty} b_{m'_k} \langle \xi_{n'_k}, \cdot \rangle \xi_{n'_k} \in [B]_G.$$

Then by the choice of I_A, J_A, I_B, J_B , we now have $a_{n_k} b_{m_k} \geq 0$, $a_{n'_k} b_{m'_k} \geq 0$, so that if we write $B_2 = \sum_{n=1}^{\infty} \tilde{b}_n \langle \xi_n, \cdot \rangle \xi_n$, we have $a_n \tilde{b}_n \geq 0$ ($n \in \mathbb{N}$). Next, let $K_N := \sum_{n=1}^N (-\tilde{b}_n + a_n) \langle \xi_n, \cdot \rangle \xi_n \in \mathbb{K}(H)_{\text{sa}}$, and consider $C_N := B_2 + K_N \in [B]_G$. Then as $|\tilde{b}_n|, |a_n| \rightarrow \infty$ ($n \rightarrow \infty$), we have

$$\|(C_N - i)^{-1} - (A - i)^{-1}\| = \sup_{n \geq N+1} \left| \frac{1}{\tilde{b}_n - i} - \frac{1}{a_n - i} \right| \xrightarrow{N \rightarrow \infty} 0,$$

so that there exists $N \in \mathbb{N}$ for which $\|(C_N - i)^{-1} - (A - i)^{-1}\| < \delta$. In particular, $C_N \in U$.

Claim. $C_N \in \overline{\mathcal{O}(A, U, V)} \cap [B]_G$.

The proof of the claim would conclude that (b) holds. To show that $C_N \in \overline{\mathcal{O}(A, U, V)}$, define for each $p \geq N + 1$ an operator

$$C_{N,p} := \sum_{n=1}^N a_n \langle \xi_n, \cdot \rangle \xi_n + \sum_{n=N+1}^p \tilde{b}_n \langle \xi_n, \cdot \rangle \xi_n + \sum_{n=p+1}^{\infty} a_n \langle \xi_n, \cdot \rangle \xi_n.$$

Then we have

$$\|(C_{N,p} - i)^{-1} - (A - i)^{-1}\| = \sup_{N+1 \leq n \leq p} \left| \frac{1}{\tilde{b}_n - i} - \frac{1}{a_n - i} \right| \leq \|(C_N - i)^{-1} - (A - i)^{-1}\| < \delta,$$

so $C_{N,p} \in U$ ($p \geq N + 1$) holds. Moreover, we see that

$$\|(C_N - i)^{-1} - (C_{N,p} - i)^{-1}\| = \sup_{n \geq p+1} \left| \frac{1}{\tilde{b}_n - i} - \frac{1}{a_n - i} \right| \xrightarrow{p \rightarrow \infty} 0.$$

We now show that $C_{N,p} \in \mathcal{O}(A, U, V)$, which implies $C_N \in \overline{\mathcal{O}(A, U, V)}$. Put $m_p := \max_{N+1 \leq n \leq p} |\tilde{b}_n - a_n|$, and choose $L \in \mathbb{N}$ so that $m_p < rL$. Define $K := \sum_{n=N+1}^p L^{-1} (\tilde{b}_n - a_n) \langle \xi_n, \cdot \rangle \xi_n \in \mathbb{K}(H)_{\text{sa}}$. Then $\|K\| = \frac{m_p}{L} < r$, whence $K \in W_1$. Therefore $g = (K, 1) \in V$. For each $0 \leq j \leq L$, define $A_j := A + jK = g^j \cdot A$. In particular, $A_0 = A, A_L = C_{N,p}$. Now by $a_n \tilde{b}_n \geq 0$ and Lemma 3.36, we have

$$\begin{aligned} \|(A_j - i)^{-1} - (A - i)^{-1}\| &= \sup_{N+1 \leq n \leq p} \left| \frac{1}{a_n + \frac{j}{L} (\tilde{b}_n - a_n) - i} - \frac{1}{a_n - i} \right| \\ &\leq \sup_{N+1 \leq n \leq p} \left| \frac{1}{\tilde{b}_n - i} - \frac{1}{a_n - i} \right| < \delta. \end{aligned}$$

Therefore $A_j \in U$ for each $0 \leq j \leq L$, whence $C_{N,p} \in \mathcal{O}(A, U, V)$ and the claim is proved. This shows that the action is weakly generically turbulent, so it is generically turbulent. \square

3.5 $E_{\text{u.c.res}}^{\text{SA}(H)}$ is Smooth

In the last part of this section, we consider another Borel equivalence relation related to $E_G^{\text{SA}(H)}$.

Definition 3.38. We define an equivalence relation $E_{\text{u.c.res}}^{\text{SA}(H)}$ (“unitary equivalence modulo compact difference of resolvents”) on $\text{SA}(H)$ by $A E_{\text{u.c.res}}^{\text{SA}(H)} B$ if and only if there exists $u \in \mathcal{U}(H)$ such that $u(A - i)^{-1} u^* - (B - i)^{-1} \in \mathbb{K}(H)$.

It is easy to see that $E_{\text{u.c.res}}^{\text{SA}(H)}$ is an equivalence relation. Note that Weyl-von Neumann equivalence relation $E_G^{\text{SA}(H)}$ is “stronger” than $E_{\text{u.c.res}}^{\text{SA}(H)}$:

Lemma 3.39. $E_G^{\text{SA}(H)} \subset E_{\text{u.c.res}}^{\text{SA}(H)}$ holds.

Proof. Let $A, B \in \text{SA}(H)$ be such that $AE_G^{\text{SA}(H)} B$. Then there exist $u \in \mathcal{U}(H)$ and $K \in \mathbb{K}(H)_{\text{sa}}$ such that $B = uAu^* + K$. Then by the resolvent identity [Sch10, §2.2, (2.4)]

$$\begin{aligned} (B - i)^{-1} - u(A - i)^{-1}u^* &= (B - i)^{-1} - (uAu^* - i)^{-1} \\ &= (B - i)^{-1}(uAu^* - B)(uAu^* - i)^{-1} \\ &= -(B - i)^{-1}K(uAu^* - i)^{-1} \in \mathbb{K}(H), \end{aligned}$$

whence $AE_{\text{u.c.res}}^{\text{SA}(H)} B$. □

It turns out that the restriction of $E_{\text{u.c.res}}^{\text{SA}(H)}$ to the F_σ subset $\mathbb{B}(H)_{\text{sa}}$ coincides with $E_G^{\mathbb{B}(H)_{\text{sa}}}$.

Lemma 3.40. *The restriction of $E_{\text{u.c.res}}^{\text{SA}(H)}$ to $\mathbb{B}(H)_{\text{sa}}$ coincides with $E_G^{\mathbb{B}(H)_{\text{sa}}}$.*

Proof. Let $A, B \in \mathbb{B}(H)_{\text{sa}}$. If $AE_G^{\mathbb{B}(H)_{\text{sa}}} B$, then $AE_{\text{u.c.res}}^{\text{SA}(H)} B$ by Lemma 3.39. Conversely, assume that $AE_{\text{u.c.res}}^{\text{SA}(H)} B$ holds. Then there exists $u \in \mathcal{U}(H)$ such that

$$(B - i)^{-1} - (uAu^* - i)^{-1} = (B - i)^{-1}(uAu^* - B)(uAu^* - i)^{-1} \in \mathbb{K}(H).$$

Let $K := (B - i)^{-1} - (uAu^* - i)^{-1}$. Then because A, B are bounded and self-adjoint, we have

$$B - uAu^* = -(B - i)K(uAu^* - i) \in \mathbb{K}(H)_{\text{sa}}.$$

This shows that $AE_G^{\mathbb{B}(H)_{\text{sa}}} B$. □

Therefore $E_{\text{u.c.res}}^{\text{SA}(H)}$ is considered to be another generalization of the smooth equivalence relation $E_G^{\mathbb{B}(H)_{\text{sa}}}$ to general self-adjoint operators. We have seen that $E_G^{\text{SA}(H)}$ is unclassifiable by countable structure. However, it turns out that apparently similar equivalence relation $E_{\text{u.c.res}}^{\text{SA}(H)}$ is actually smooth:

Theorem 3.41. *$E_{\text{u.c.res}}^{\text{SA}(H)}$ is a smooth equivalence relation.*

Before going to the proof, note that the essential spectra is not a complete invariant for $E_{\text{u.c.res}}^{\text{SA}(H)}$:

Example 3.42. Consider $H = H_0 \oplus H_0$ where H_0 is a separable infinite-dimensional Hilbert space, and let $A_0 \in \text{EES}(H_0)$. Then $A := A_0 \oplus 0, B := 0 \oplus 0 \in \text{SA}(H)$ satisfy $\sigma_{\text{ess}}(A) = \sigma_{\text{ess}}(B) = \{0\}$, but for any $u \in \mathcal{U}(H)$,

$$(A - i)^{-1} - u(B - i)^{-1}u^* = [(A_0 - i1_{H_0})^{-1} - i1_{H_0}] \oplus 0 \notin \mathbb{K}(H),$$

because $(A_0 - i1_{H_0})^{-1} \in \mathbb{K}(H)_{\text{sa}}$ (Lemma 3.26) and $1_{H_0} \notin \mathbb{K}(H_0)$. Therefore $(A, B) \notin E_{\text{u.c.res}}^{\text{SA}(H)}$.

Note that in Example 3.42, A is unbounded, while B is bounded. It turns out that if we add to $\sigma_{\text{ess}}(\cdot)$ the additional information of boundedness/unboundedness of the operator, then it becomes a complete invariant for $E_{\text{u.c.res}}^{\text{SA}(H)}$.

Definition 3.43. For each $A \in \text{SA}(H)$, we define $\bar{\sigma}_{\text{ess}}(A) \in \mathcal{F}(\mathbb{R}) \times \{0, 1\}$ by

$$\bar{\sigma}_{\text{ess}}(A) := \begin{cases} (\sigma_{\text{ess}}(A), 0) & (A \text{ is bounded}) \\ (\sigma_{\text{ess}}(A), 1) & (A \text{ is unbounded}) \end{cases}.$$

$\bar{\sigma}_{\text{ess}}(A)$ is something like a compactification of $\sigma_{\text{ess}}(A)$. Note that since $\mathbb{B}(H)_{\text{sa}}$ is a Borel subset of $\text{SA}(H)$ (Lemma 3.11), the map $\bar{\sigma}_{\text{ess}}: \text{SA}(H) \rightarrow \mathcal{F}(\mathbb{R}) \times \{0, 1\}$ is Borel by the Borelness of $\sigma_{\text{ess}}(\cdot)$ (Theorem 3.15). Now Theorem 3.41 is proved by the next Proposition:

Proposition 3.44. *$\bar{\sigma}_{\text{ess}}$ is a Borel reduction of $E_{\text{u.c.res}}^{\text{SA}(H)}$ to $\text{id}_{\mathcal{F}(\mathbb{R}) \times \{0, 1\}}$. In particular, $E_{\text{u.c.res}}^{\text{SA}(H)}$ is smooth.*

We recall a result due to Weyl ((i) \Leftrightarrow (ii), see [Sch10, Proposition 8.11]) and its variant (iii).

Lemma 3.45 (Weyl's criterion). *Let $A \in \text{SA}(H)$ and $\lambda \in \mathbb{R}$. The following conditions are equivalent:*

- (i) $\lambda \in \sigma_{\text{ess}}(A)$.
- (ii) *There exists a sequence $\{\xi_n\}_{n=1}^\infty \subset \text{dom}(A)$ of unit vectors which converges weakly to 0, such that $\lim_{n \rightarrow \infty} \|A\xi_n - \lambda\xi_n\| = 0$.*

(iii) *There exists a sequence $\{\xi_n\}_{n=1}^\infty$ of unit vectors in H which converges weakly to 0, such that $\|(A - i)^{-1}\xi_n - (\lambda - i)^{-1}\xi_n\| \xrightarrow{n \rightarrow \infty} 0$.*

We also use Weyl's criterion for bounded normal operators. Recall that the essential spectra $\sigma_{\text{ess}}(x)$ for a bounded normal operator $x \in \mathbb{B}(H)$ is defined in the same way as the case of self-adjoint operators: $\sigma_{\text{ess}}(x) = \sigma(x) \setminus \sigma_{\text{d}}(x)$, where $\sigma_{\text{d}}(x)$ is the set of all eigenvalues of x of finite multiplicity. The next lemma can be proved by the same argument as Weyl's criterion (i) \Leftrightarrow (ii) above:

Lemma 3.46 (Weyl's criterion for normal operators). *Let $x \in \mathbb{B}(H)$ be a normal operator, and let $\lambda \in \mathbb{C}$. Then $\lambda \in \sigma_{\text{ess}}(x)$ if and only if there exists a sequence $\{\xi_n\}_{n=1}^\infty$ of unit vectors in H converging weakly to 0, such that $\|x\xi_n - \lambda\xi_n\| \xrightarrow{n \rightarrow \infty} 0$.*

By Lemma 3.45 and Lemma 3.46, we have:

Corollary 3.47. *Let $A \in \text{SA}(H)$. Then A is bounded if and only if $0 \notin \sigma_{\text{ess}}((A - i)^{-1})$. Moreover, it holds that*

$$\sigma_{\text{ess}}((A - i)^{-1}) = \begin{cases} \{(\lambda - i)^{-1}; \lambda \in \sigma_{\text{ess}}(A)\} & (A \text{ is bounded}) \\ \{(\lambda - i)^{-1}; \lambda \in \sigma_{\text{ess}}(A)\} \cup \{0\} & (A \text{ is unbounded}) \end{cases}$$

Lemma 3.48. *Let $A, B \in \text{SA}(H)$ be such that $AE_{\text{u.c.res}}^{\text{SA}(H)} B$. Then $\bar{\sigma}_{\text{ess}}(A) = \bar{\sigma}_{\text{ess}}(B)$.*

Proof. By assumption, there exists $u \in \mathcal{U}(H)$ such that $u(A - i)^{-1}u^* - (B - i)^{-1} \in \mathbb{K}(H)$. Since the essential spectra of a bounded normal operators is invariant under compact perturbations (see e.g., [Con90, Propositions 4.2 and 4.6]), $\sigma_{\text{ess}}((A - i)^{-1}) = \sigma_{\text{ess}}(u(A - i)^{-1}u^*) = \sigma_{\text{ess}}((B - i)^{-1})$. Therefore by Corollary 3.47, $\sigma_{\text{ess}}(A) = \sigma_{\text{ess}}(B)$ and A is bounded if and only if so is B . \square

Finally, Proposition 3.44 follows from the following Berg's generalization (see [Con99, Theorem 39.8] and [Ber71]) of Theorem 1.2, which is usually called the Weyl-von Neumann-Berg Theorem.

Theorem 3.49 (Weyl-von Neumann-Berg). *Let A, B be bounded normal operators on H . Then $\sigma_{\text{ess}}(A) = \sigma_{\text{ess}}(B)$ if and only if there exists $u \in \mathcal{U}(H)$ and $K \in \mathbb{K}(H)$ such that $uAu^* + K = B$.*

Proof of Proposition 3.44. We already know that $\bar{\sigma}_{\text{ess}}$ is Borel. Therefore we have only to show that $AE_{\text{u.c.res}}^{\text{SA}(H)} B \Leftrightarrow \bar{\sigma}_{\text{ess}}(A) = \bar{\sigma}_{\text{ess}}(B)$ holds. (\Rightarrow) holds by Lemma 3.48. To show (\Leftarrow) , assume that $\bar{\sigma}_{\text{ess}}(A) = \bar{\sigma}_{\text{ess}}(B)$ holds. If A, B are bounded, then by Corollary 3.47, we have $\sigma_{\text{ess}}((A - i)^{-1}) = \{(\lambda - i)^{-1}; \lambda \in \sigma_{\text{ess}}(A)\} = \sigma_{\text{ess}}((B - i)^{-1})$, whence by Theorem 3.49, there exists $u \in \mathcal{U}(H)$ such that $(B - i)^{-1} - u(A - i)^{-1}u^* \in \mathbb{K}(H)$ holds. Hence $AE_{\text{u.c.res}}^{\text{SA}(H)} B$. If both A, B are unbounded, then again by Corollary 3.47, $\sigma_{\text{ess}}((A - i)^{-1}) = \sigma_{\text{ess}}((B - i)^{-1})$ holds, whence $AE_{\text{u.c.res}}^{\text{SA}(H)} B$ by the same argument. \square

4 Concluding Remarks and Questions

In this paper we have studied various equivalence relations on $\text{SA}(H)$. In this last section let us pose some questions regarding Weyl-von Neumann equivalence and some comments about other equivalence relations related to self-adjoint operators. First of all we do not know if the Weyl-von Neumann equivalence relation is Borel.

Question 4.1. Is $E_G^{\text{SA}(H)}$ Borel?

Note that $E_G^{\mathbb{B}(H)_{\text{sa}}}$ is Borel (because it is smooth), and the action of the subgroup $K(H)_{\text{sa}}$ of G generates a Borel equivalence relation $E_{\mathbb{K}(H)_{\text{sa}}}^{\text{SA}(H)}$ (because the action is free) which is easily seen to be generically turbulent. The G -action, however, is very far from free. We have seen that the essential spectra is not a complete invariant for $E_G^{\text{SA}(H)}$, but still $\sigma_{\text{ess}}(A) = \sigma_{\text{ess}}(B) = \mathbb{R} \Rightarrow AE_G^{\text{SA}(H)} B$ holds despite their nature of unboundedness.

Question 4.2. Which $M \in \mathcal{F}(\mathbb{R})$ has the property that $\sigma_{\text{ess}}(A) = \sigma_{\text{ess}}(B) = M \Rightarrow AE_G^{\text{SA}(H)} B$?

Note that this is related to the following question posed to us by Uffe Haagerup and Todor Tsankov:

Question 4.3 (Haagerup, Tsankov). Consider the action of the semidirect product group $G' = c_0 \rtimes S_\infty$ where S_∞ acts on the real Banach space $c_0 = c_0(\mathbb{N}, \mathbb{R})$ by permutation. G' acts on $\mathbb{R}^{\mathbb{N}}$ naturally by

$$((a_n)_{n=1}^\infty, \sigma) \cdot (x_n)_{n=1}^\infty := (x_{\sigma^{-1}(n)} + a_n)_{n=1}^\infty, \quad (a_n)_{n=1}^\infty \in c_0, \sigma \in S_\infty, (x_n)_{n=1}^\infty \in \mathbb{R}^{\mathbb{N}}.$$

Is $E_G^{\text{SA}(H)} \leq_B E_{G'}^{\mathbb{R}^{\mathbb{N}}}$?

Finally, let us remark that different way of perturbing self-adjoint operators may give rise to distinct equivalence relations: let $1 \leq p < \infty$, and let $S^p(H)_{\text{sa}}$ be the additive Polish group of self-adjoint Schatten p -class operators on H equipped with Schatten p -norm. $S^p(H)_{\text{sa}}$ acts on $\text{SA}(H)$ by addition, and we may consider an action of $G_p := S^p(H)_{\text{sa}} \rtimes \mathcal{U}(H)$ on $\text{SA}(H)$ analogous to $G \curvearrowright \text{SA}(H)$.

It is especially of interest to know whether one of $E_{G_1}^{\text{SA}(H)}$ and $E_G^{\text{SA}(H)}$ is Borel reducible to the other (note that by Kato-Rosenblum Theorem [Kat57, Ros57], trace class perturbation is rather different from other Schatten class or compact perturbations). Note also that the orbit equivalence relation of $S^p(H)_{\text{sa}}$ -action on $\text{SA}(H)$ can be thought of as a non-commutative version of the ℓ^p -action on $\mathbb{R}^{\mathbb{N}}$ studied by Dougherty-Hjorth [DH99].

There are also many other interesting equivalence relations involving the structure of unbounded self-adjoint operators. Let us state some more results we have obtained after the first version of the paper has been written. Since the proofs of the results stated below are not short, they will appear elsewhere. As we have observed, one major difference between $E_G^{\mathbb{B}(H)_{\text{sa}}}$ and $E_G^{\text{SA}(H)}$ comes from the complexity of determining when two operators $A, B \in \text{SA}(H)$ have unitarily equivalent domains. In this respect, let us define two equivalence relations $E_{\text{dom}}^{\text{SA}(H)}$ and $E_{\text{dom},u}^{\text{SA}(H)}$ by

$$\begin{aligned} AE_{\text{dom}}^{\text{SA}(H)} B &\Leftrightarrow \text{dom}(A) = \text{dom}(B), \\ AE_{\text{dom},u}^{\text{SA}(H)} B &\Leftrightarrow \exists u \in \mathcal{U}(H) [u \cdot \text{dom}(A) = \text{dom}(B)]. \end{aligned}$$

It is expected from our present work that they are rather pathological equivalence relations. It turns out that both $E_{\text{dom}}^{\text{SA}(H)}$ and $E_{\text{dom},u}^{\text{SA}(H)}$ are Borel, and moreover we have $E_{\text{dom},u}^{\text{SA}(H)} \leq_B E_{\text{dom}}^{\text{SA}(H)}$. In fact much more can be proved: $E_{\text{dom}}^{\text{SA}(H)}$ is F_σ , and universal for K_σ equivalence relation, and in particular is not Borel reducible to any orbit equivalence relation of a Polish group action.

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