Proceedings of the 39th Sapporo Symposium on Partial Differential Equations

Edited by
S. Ei, Y. Giga, S. Jimbo, H. Kubo, T. Ozawa, T. Sakajo
H. Takaoka, Y. Tonegawa, and K. Tsutaya

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PREFACE

This volume is intended as the proceedings of Sapporo Symposium on Partial Differential Equations, held on August 25 through August 27 in 2014 at Faculty of Science, Hokkaido University.

Sapporo Symposium on PDE has been held annually to present the latest developments on PDE with a broad spectrum of interests not limited to the methods of a particular school. Professor Taira Shirota started the symposium more than 35 years ago. Professor Kôji Kubota and late Professor Rentaro Agemi made a large contribution to its organization for many years.

We always thank their significant contribution to the progress of the Sapporo Symposium on PDE.

S. Ei, Y. Giga, S. Jimbo, H. Kubo, T. Ozawa, T. Sakajo
H. Takaoka, Y. Tonegawa, and K. Tsutaya

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Flow monotonicity and Strichartz inequalities

The 39th Sapporo Symposium on Partial Differential Equations
（第３９回偏微分方程式論札幌シンポジウム）

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Program Committee: S. Ei, Y. Giga, S. Jimbo, H. Kubo, T. Ozawa, T. Sakajo
H. Takaoka, Y. Tonegawa, and K. Tsutaya

Period (期間) August 25, 2014 - August 27, 2014

Venue (場所) Rm 203 and Rm 201, Faculty of Science Building #5, Hokkaido University
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Weakly coupled systems of the infinity Laplace equations

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Sharp well-posedness for the Chern-Simons-Dirac system in one dimension

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Generalized eigenvalue problems for (p, q)-Laplacian

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α-Gauss curvature flows with flat sides

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14:30-14:45	*
14:45-15:45	Neal Bez (Saitama University) Flow monotonicity and Strichartz inequalities
15:45-	Closing

* Free discussion with speakers

STABILITY AND MINIMALITY FOR A NONLOCAL ISOPERIMETRIC PROBLEM

N.FUSCO

1. The model. Properties of local minimizers

Diblock copolymers are extensively studied materials, used to engineer nanostructures thanks to their peculiar properties and rich pattern formation. The resulting patterns depend on the chemical bounds between the two different polymers, say A and B, and on the relative lengths of each block. Some of the most commonly observed structures are schematized in Figure 1 and it has been observed, see e.g. [24], that they closely approximate periodic surfaces with constant mean curvature. A well established theory used in the modeling of microphase separation for A/B

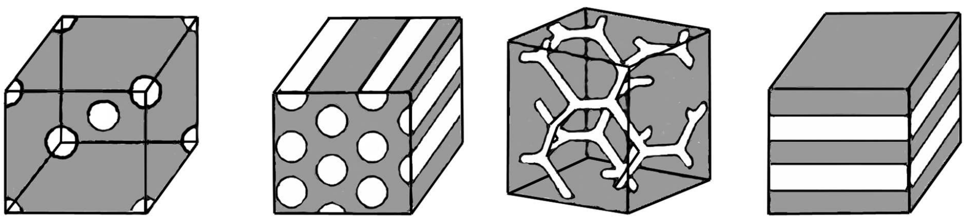

Figure 1. From left to right spherical spots, cylinders, gyroids and lamellae.

diblock copolymer melts is based on the energy first proposed by *Ohta-Kawasaki*, see [16]:

$$\mathcal{E}_\varepsilon(u) := \varepsilon \int_\Omega |\nabla u|^2\, dx + \frac{1}{\varepsilon}\int_\Omega (u^2-1)^2\, dx + \gamma_0 \int_\Omega\int_\Omega G(x,y)\big(u(x)-m\big)\big(u(y)-m\big)\, dx\, dy\ , \quad (1.1)$$

where u is an $H^1(\Omega)$ phase parameter describing the density distribution of the components ($u=-1$ stands for phase A, $u=+1$ for phase B), $m = \fint_\Omega u$ is the difference of the phases' volume fractions and G is the Green's function for $-\Delta$. The parameter $\gamma_0 \geq 0$ is characteristic of the material. Note that the first two terms in (1.1) correspond to the so called *Modica-Mortola functional* and approximate the perimeter of the interface as ε tends to 0. These terms clearly drive the system toward a raw partition in few sets of pure phases with minimal interface area, whereas the Green's term favors a finely intertwined distribution of the materials.

Since ε is a small parameter, from the point of view of mathematical analysis it is more convenient to consider the variational limit of the energy (1.1), which is given by

$$\mathcal{E}(u) := \frac{1}{2}|Du|(\Omega) + \gamma\int_\Omega\int_\Omega G(x,y)\big(u(x)-m\big)\big(u(y)-m\big)\, dx\, dy\ ,$$

where now u is a function of bounded variation in Ω with values ± 1, $|Du|(\Omega)$ is the total variation of u in Ω, and $\gamma = 3\gamma_0/16 \geq 0$. Writing

$$E = \{x \in \Omega : u(x) = 1\}\ ,$$

so that $u = \chi_E - \chi_{\Omega\setminus E}$, this energy may be rewritten in a useful geometric fashion as

$$J(E) = P_\Omega(E) + \gamma\int_\Omega\int_\Omega G(x,y)\big(u(x)-m\big)\big(u(y)-m\big)\, dx\, dy\ , \quad (1.2)$$

where $P_\Omega(E)$ is the perimeter of E in Ω.

A challenging mathematical problem is to prove that global minimizers of (1.2) are periodic: this is known to be true in one dimension, see e.g. [15], but still open in higher dimensions, where only partial results are known, see e.g. [2, 22]. A more reasonable task is to exhibit a class of periodic solutions which are local minimizers of the approximating and limit energies (1.1) and (1.2), rather than investigating general properties of global minimizers: this is the direction taken, among others, by Ren and Wei and by Choksi and Sternberg. The first authors in a series of papers [17, 18, 19, 20, 21] construct several examples of lamellar, spherical and cylindrical critical configurations and find conditions under which they are stable, i.e., their second variation is positive definite. The main contribution in [3] is the computation of the second variation for general critical configurations of (1.2). However, all these papers leave open the basic question whether the positivity of the second variation implies local minimality.

In order to discuss this question we start by considering the periodic case, where $\Omega = \mathbb{T}^N$ is the N-dimensional flat torus of unit volume. In this case $G(x,y)$ is the solution of

$$-\Delta_y G(x,y) = \delta_x - 1 \quad \text{in } \mathbb{T}^N, \qquad \int_{\mathbb{T}^N} G(x,y)\,dy = 0\,,$$

where δ_x denotes the Dirac measure supported at x. Set $u_E = \chi_E - \chi_{\Omega\setminus E}$, and denote by $m = \fint_{\mathbb{T}^N} u_E \in (-1,1)$ the fixed volume fraction of the two phases and by v_E the unique solution to

$$-\Delta v_E = u_E - m \quad \text{in } \mathbb{T}^N \qquad \int_{\mathbb{T}^N} v_E\,dx = 0\,. \tag{1.3}$$

Note that

$$\begin{aligned}\int_{\mathbb{T}^N} |\nabla v_E|^2\,dx &= -\int_{\mathbb{T}^N} v_E \Delta v_E\,dx = \int_{\mathbb{T}^N} v_E(u_E - m)\,dx = \int_{\mathbb{T}^N} v_E u_E\,dx \\ &= \int_{\mathbb{T}^N}\int_{\mathbb{T}^N} G(x,y)u_E(x)u_E(y)\,dxdy = \int_{\mathbb{T}^N}\int_{\mathbb{T}^N} G(x,y)\big(u_E(x)-m\big)\big(u_E(y)-m\big)\,dx\,dy\,.\end{aligned} \tag{1.4}$$

Therefore we may further rewrite the functional in (1.2) as

$$J(E) = P_{\mathbb{T}^N}(E) + \gamma\int_{\mathbb{T}^N} |\nabla v_E|^2\,dx\,. \tag{1.5}$$

A C^2 minimizer of $J(E)$ under a volume constraint satisfies the *Euler-Lagrange equation*

$$H_{\partial E}(x) + 4\gamma v_E(x) = \lambda \qquad \text{for all } x \in \partial E, \tag{1.6}$$

where $H_{\partial E}(x)$ denotes the sum of the principal curvatures of ∂E at x and the number λ is a constant Lagrange multiplier associated to the volume constraint $\fint_{\mathbb{T}^N} u_E\,dx = m$.

In the following, a C^2 solution E of equation (1.6) will be called a *regular critical point.* Note that when $\gamma = 0$ any periodic constant mean curvature smooth set E is a regular critical point.

In order to present the main result proved in [1] we need to introduce a suitable notion of local minimality. Since our energy functional is invariant under translations, it is convenient to define the distance between two subsets of $\mathbb{T}^N$ modulo translations in the following way:

$$d(E,F) := \min_x |E \triangle (x+F)|\,. \tag{1.7}$$

Accordingly, we may give the following

Definition 1.1. We say that a set $E \subset \mathbb{T}^N$ of finite perimeter is a *local minimizer* for the functional (1.5) if there esists $\delta > 0$ such that

$$J(F) \geq J(E)$$

for all $F \subset \mathbb{T}^N$ with $|E| = |F|$ and $d(E,F) \leq \delta$.

Note that given any set $E \subset \mathbb{T}^N$ by standard elliptic regularity $v_E \in W^{2,p}(\mathbb{T}^N)$ for all $p \geq 1$. Moreover it is easily checked that there exists $C = C(N) > 0$ such that if E, $F \subset \mathbb{T}^N$ are measurable, then

$$\left| \int_{\mathbb{T}^N} |\nabla v_E|^2 \, dx - \int_{\mathbb{T}^N} |\nabla v_F|^2 \, dx \right| \leq C |E \triangle F| \,, \tag{1.8}$$

where v_E and v_F are defined as in (1.3).

An important tool to get the regularity of local minimizers is the following result that is essentially proved in [6] (see also [1, Proposition 2.7]).

Proposition 1.2. *Let E be a local minimizer for the functional* (1.5) *and let $\delta > 0$ be as in Definition 1.1. There exists $\lambda > 0$ such that E solves the following penalized minimization problem:*

$$\min \Big\{ J(F) + \lambda \big||F| - |E|\big| : \, F \subset \mathbb{T}^N \,, \, d(E, F) \leq \frac{\delta}{2} \Big\} \,.$$

As a consequence of this result and of inequality (1.8) it is then easy to show that if E is a local minimizer of J according to Definition 1.1, then E is *almost minimizer* of the perimeter, i.e., there exist $\omega, r_0 > 0$ such that

$$P_{\mathbb{T}^N}(E) \leq P_T(F) + \omega r^n$$

for all $F \subset \mathbb{T}^N$ such that $E \Delta F \subset\subset B_r(x_0)$ for some $x_0 \in \mathbb{T}^N$ and $0 < r < r_0$. At this point, using the regularity theory for (almost) minimizers of the perimeter it is not too hard to show that the following result holds (see [12, Proposition 2.2.]).

Theorem 1.3. *Let E be a local minimizer for* (1.5). *There exists a closed set $\Sigma \subset \partial E$ such that $\partial E \setminus \Sigma$ is a C^∞ manifold. Moreover, the Hausdorff dimension of the singular set satisfies* $\dim_{\mathcal{H}}(\Sigma) \leq N - 8$.

2. Second variation and local minimality

We are now going to present the main result of [1] which states that any regular critical point of J with positive second variation is a local minimizer. To this aim, given a set $E \subset \mathbb{T}^N$ of class C^2 and a C^2-vector field $X : \mathbb{T}^N \to \mathbb{T}^N$ a C^2-vector field, we consider the associated flow $\Phi : \mathbb{T}^N \times (-1, 1) \to \mathbb{T}^N$ defined by $\dfrac{\partial \Phi}{\partial t} = X(\Phi)$, $\Phi(x, 0) = x$. We define the *second variation of J at E* with respect to the flow Φ to be the value

$$\frac{d^2}{dt^2} J(E_t) \Big|_{t=0} \,,$$

where $E_t := \Phi(\cdot, t)(E)$. Throughout the section, when no confusion is possible, we shall omit the indication of E, writing v instead of v_E and ν instead of ν^E, the exterior unit normal to the boundary of E. Before stating the representation formula for the second variation, we fix some notation. Given a vector X, its tangential part on ∂E is defined as $X_\tau := X - (X \cdot \nu)\nu$. In particular, we will denote by D_τ the tangential gradient operator given by $D_\tau \varphi := (D\varphi)_\tau$. We also recall that the second fundamental form $B_{\partial E}$ of ∂E is given by $D_\tau \nu$ and that the square $|B_{\partial E}|^2$ of its Euclidean norm coincides with the the sum of the squares of the principal curvatures of ∂E.

Theorem 2.1. *If E, X, and Φ are as above, we have*

$$\begin{aligned}\frac{d^2}{dt^2}J(E_t)\Big|_{t=0} &= \int_{\partial E}\Big(|D_\tau(X\cdot\nu)|^2 - |B_{\partial E}|^2(X\cdot\nu)^2\Big)\,d\mathcal{H}^{N-1}\\ &\quad + 8\gamma\int_{\partial E}\int_{\partial E} G(x,y)\big((X\cdot\nu)(x)\big)\big((X\cdot\nu)(y)\big)d\mathcal{H}^{N-1}(x)\,d\mathcal{H}^{N-1}(y)\\ &\quad + 4\gamma\int_{\partial E}\partial_\nu v(X\cdot\nu)^2\,d\mathcal{H}^{N-1} - \int_{\partial E}(4\gamma v + H_{\partial E})\,\mathrm{div}_\tau\big(X_\tau(X\cdot\nu)\big)\,d\mathcal{H}^{N-1}\\ &\quad + \int_{\partial E}(4\gamma v + H_{\partial E})(\mathrm{div}\,X)(X\cdot\nu)\,d\mathcal{H}^{N-1}\,.\end{aligned}\tag{2.1}$$

In the case of a critical set E the computation of the second variation was carried out in [3]. The novelty here is that the above result, proved in [1, Theorem 3.1], deals with a general regular set. This explains the presence of the last two terms in the formula.

Remark 2.2. Notice that if E is also critical, from (1.6) it follows that

$$\int_{\partial E}(4\gamma v + H_{\partial E})\,\mathrm{div}_\tau\big(X_\tau(X\cdot\nu)\big)\,d\mathcal{H}^{N-1} = 0\,.$$

Moreover, if in addition

$$|\Phi(\cdot,t)(E)| = |E| \qquad \text{for all } t\in[0,1],\tag{2.2}$$

then it can be shown (see [3, (2.30)]) that

$$0 = \frac{d^2}{dt^2}|E_t|\Big|_{t=0} = \int_{\partial E}(\mathrm{div}\,X)(X\cdot\nu)\,d\mathcal{H}^{N-1}\,.$$

Hence, again from (1.6), we have

$$\begin{aligned}\frac{d^2}{dt^2}J(E_t)\Big|_{t=0} &= \int_{\partial E}\Big(|D_\tau(X\cdot\nu)|^2 - |B_{\partial E}|^2(X\cdot\nu)^2\Big)\,d\mathcal{H}^{N-1}\\ &\quad + 8\gamma\int_{\partial E}\int_{\partial E} G(x,y)\big((X\cdot\nu)(x)\big)\big((X\cdot\nu)(y)\big)d\mathcal{H}^{N-1}(x)\,d\mathcal{H}^{N-1}(y)\\ &\quad + 4\gamma\int_{\partial E}\partial_\nu v(X\cdot\nu)^2\,d\mathcal{H}^{N-1}\,.\end{aligned}$$

Note that this formula coincides exactly with the one given in [3, (2.20)], where it was obtained using a particular family of asymptotically volume preserving diffeomorphisms.

The previous remark motivates the following definition. Given a C^2 open set $E\subset\mathbb{T}^N$ we denote by $\widetilde{H}^1(\partial E)$ the set of all functions $\varphi\in H^1(\partial E)$ such that $\int_{\partial E}\varphi\,d\mathcal{H}^{N-1}=0$, endowed with the norm $\|\nabla\varphi\|_{L^2(\partial E)}$. To E we then associate the quadratic form $\partial^2 J(E):\widetilde{H}^1(\partial E)\to\mathbb{R}$ defined as

$$\begin{aligned}\partial^2 J(E)[\varphi] &= \int_{\partial E}\big(|D_\tau\varphi|^2 - |B_{\partial E}|^2\varphi^2\big)\,d\mathcal{H}^{N-1}\\ &\quad + 8\gamma\int_{\partial E}\int_{\partial E}G(x,y)\varphi(x)\varphi(y)d\mathcal{H}^{N-1}(x)\,d\mathcal{H}^{N-1}(y) + 4\gamma\int_{\partial E}\partial_\nu v\varphi^2\,d\mathcal{H}^{N-1}\,.\end{aligned}\tag{2.3}$$

If E is a regular critical set and the flow Φ satisfies (2.2), then

$$\frac{d|E_t|}{dt}\Big|_{t=0} = \int_{\partial E}X\cdot\nu\,d\mathcal{H}^{N-1} = 0\,.$$

Hence, $\partial^2 J(E)[X\cdot\nu]$ coincides with the second variation of J at E with respect to Φ.

Notice that, setting $\mu := \varphi\mathcal{H}^{N-1}\lfloor\partial E$, the nonlocal term

$$\int_{\partial E}\int_{\partial E}G(x,y)\varphi(x)\varphi(y)d\mathcal{H}^{N-1}(x)\,d\mathcal{H}^{N-1}(y)$$

can be rewritten as

$$\int_{\mathbb{T}^N}\int_{\mathbb{T}^N} G(x,y)d\mu(x)\,d\mu(y) = \int_{\mathbb{T}^N} |\nabla z|^2\,dx\,, \tag{2.4}$$

where $z \in H^1(\mathbb{T}^N)$ is a weak solution to the equation

$$-\Delta z = \mu \qquad \text{in } \mathbb{T}^N\,.$$

Thus the nonlocal term (2.4) is equivalent to the square of the H^{-1}-norm of the measure μ.

The following corollary is a simple consequence of the definition of local minimality.

Corollary 2.3. *Let E be a regular local minimizer of J according to* Definition 1.1.*Then*

$$\partial^2 J(E)[\varphi] \geq 0 \qquad\qquad \textit{for all } \varphi \in \widetilde{H}^1(\partial E).$$

Before stating our main result, a further important remark is in order. If $E \subset \mathbb{T}^N$ is of class C^2 and $\Phi(x,t) = x + t\eta e_i$ for some $\eta \in \mathbb{R}$ and some element e_i of the canonical basis in $\mathbb{R}^N$, we clearly have $J\big(\Phi(\cdot,t)(E)\big) = J(E)$, by the translation invariance of J. Hence,

$$\frac{d^2}{dt^2} J(E_t)_{\big|t=0} = \partial^2 J(E)[\eta\nu_i] = 0\,.$$

In view of this it is convenient to introduce the subspace $T(\partial E) \subset \widetilde{H}^1(\partial E)$ generated by the functions ν_i, $i = 1,\dots,N$. Note that we can then write

$$\widetilde{H}^1(\partial E) = T^\perp(\partial E) \otimes T(\partial E)\,, \tag{2.5}$$

where

$$T^\perp(\partial E) := \Big\{\varphi \in \widetilde{H}^1(\partial E) : \int_{\partial E} \varphi\nu_i\, d\mathcal{H}^{N-1} = 0\,, i = 1,\dots,N\Big\}$$

is the orthogonal set, in the L^2-sense, to the space of infinitesimal translations $T(\partial E)$.

In view of this remark it is then natural to give the following definition.

Definition 2.4. In the following we say that the functional J has *positive second variation* at the critical set E if

$$\partial^2 J(E)[\varphi] > 0 \qquad \text{for all } \varphi \in \widetilde{H}^1(\partial E) \setminus T(\partial E)$$

or, equivalently, for all $\varphi \in T^\perp(\partial E) \setminus \{0\}$.

The first step in proving that a regular critical point with positive second variation is a local minimizer is to consider the simpler situation when the boundary of the competing set F can be written as a graph of a $W^{2,p}$ function over the boundary of E with a sufficiently small norm. The precise statement is given by the next theorem.

Theorem 2.5. *Let $p > \max\{2, N-1\}$ and let E be a regular critical set for J with positive second variation. There exist $\delta > 0$, $C_0 > 0$ such that*

$$J(F) \geq J(E) + C_0\big(d(E,F)\big)^2\,,$$

whenever $F \subset \mathbb{T}^N$ satisifes $|F| = |E|$ and $\partial F = \{x+\psi(x)\nu(x) : x \in \partial E\}$ for some $\|\psi\|_{W^{2,p}(\partial E)} \leq \delta$.

We now briefly describe the strategy of the proof of this theorem. The idea is to construct suitable volume-preserving flows connecting the critical set E to a given close competitor F and to analyze carefully the continuity properties of the quadratic form $\partial^2 J$ along the flow (see Theorem 2.6). A technical difficulty in this analysis comes from the translation invariance, since we have to avoid the degenerate directions at all times. This issue is dealt with in Lemma 2.7, where it is shown that given any set F sufficiently $W^{2,p}$-close to E, one can always find a translation of F such that the function describing the boundary of the new set has small component in $T^\perp(\partial E)$.

Theorem 2.6. *Let $E \subset \mathbb{T}^N$ be a set of class C^3 and let $p > N-1$. For all $\varepsilon > 0$ there exist a tubular neighborhood $\mathcal{N}(\partial E)$ and two positive constants δ, C with the following properties. If $\psi \in C^2(\partial E)$ and $\|\psi\|_{W^{2,p}(\partial E)} \leq \delta$ then there exists a field $X \in C^2$ with* $\operatorname{div} X = 0$ *in $\mathcal{N}(\partial E)$ such that*

$$\|X - \psi\nu\|_{L^2(\partial E)} \leq \varepsilon \|\psi\|_{L^2(\partial E)} .$$

Moreover, the associated flow

$$\Phi(x,0) = x\,, \qquad \frac{\partial \Phi}{\partial t} = X(\Phi)$$

satisfies $\partial E_1 = \{x + \psi(x)\nu : \in \partial E\}$, and for every $t \in [0,1]$

$$\|\Phi(\cdot,t) - \mathrm{Id}\|_{W^{2,p}(\partial E)} \leq C\|\psi\|_{W^{2,p}(\partial E)}\,,$$

where Id *denotes the identity map. If in addition E_1 has the same volume as E, then for every t we have $|E_t| = |E|$ and*

$$\int_{\partial E_t} X \cdot \nu^{E_t}\, d\mathcal{H}^{N-1} = 0\,.$$

Next lemma says that when considering a sufficiently close competitor F we may assume that its translational component is as small as we wish. This property is crucial to ensure that the flow connecting E to F provided by Theorem 2.6 has the additional property that E_t has a small translational component for all $t \in (0,1)$.

Lemma 2.7. *Let $E \subset \mathbb{T}^N$ be of class C^3 and let $p > N-1$. For any $\delta > 0$ there exist η_0, $C > 0$ such that if $F \subset \mathbb{T}^N$ satisfies $\partial F = \{x + \psi(x)\nu(x) : x \in \partial E\}$ for some $\psi \in C^2(\partial E)$ with $\|\psi\|_{W^{2,p}(\partial E)} \leq \eta_0$, then there exist $\sigma \in \mathbb{R}^N$ and $\varphi \in W^{2,p}(\partial E)$ with the properties that*

$$|\sigma| \leq C\|\psi\|_{W^{2,p}(\partial E)}, \qquad \|\varphi\|_{W^{2,p}(\partial E)} \leq C\|\psi\|_{W^{2,p}(\partial E)}$$

and

$$\partial F - \sigma = \{x + \varphi(x)\nu(x) : x \in \partial E\}, \qquad \Big|\int_{\partial E} \varphi\nu\, d\mathcal{H}^{N-1}\Big| \leq \delta\|\varphi\|_{L^2(\partial E)}\,.$$

With Theorem 2.5 at hands the next step is to show that any $W^{2,p}$-local minimizer is in fact an L^1-local minimizer. This is done by a contradiction argument: we assume that there exists a sequence E_h of sets such that $|E_h| = |E|$, and $E_h \to E$ in L^1, but inequality (2.6) fails along the sequence. Then, following an idea used in [9] for a two dimensional problem related to epitaxial growth, we replace the sequence E_h with a new sequence F_h of minimizers of suitable penalized problems, tailored in such a way that (2.6) still fails. Using regularity techniques we then show that in fact the sets F_h have uniformly bounded curvatures and converge to E strongly in $W^{2,p}$, thus contradicting the $W^{2,p}$-local minimality of E. A penalization approach via regularity has been recently used also in [4] to prove the quantitative isoperimetric inequality in the Euclidean case. However, our method is quite different and seems more suited to deal with local minimizers. Note also that in the proof of the result below, the positivity of the second variation is only needed to say that E is a $W^{2,p}$-local minimizer and no use of second variation whatsoever is made in the proof. The local minimality result proved in [1] then reads as follows.

Theorem 2.8. *Let $E \subset \mathbb{T}^N$ be a regular critical set of J such that*

$$\partial^2 J(E)[\varphi] > 0 \qquad \text{for all } \varphi \in T^\perp(\partial E) \setminus \{0\}.$$

Then, there exist δ, $C > 0$ such that

$$J(F) \geq J(E) + C(d(E,F))^2 \tag{2.6}$$

for all $F \subset \mathbb{T}^N$, with $|F| = |E|$ and $\alpha(E,F) < \delta$.

It is important to remark that Theorem 2.8, besides proving strict local minimality, contains a quantitative estimate of the deviation from minimality for sets close to E in L^1. This can be viewed as a quantitative isoperimetric inequality for the nonlocal perimeter (1.2), in the spirit of the recent results proved in [8], see also [7, 4]. Indeed, since our result holds also when $\gamma = 0$, we cover the important case of local minimizers of the area functional under periodicity conditions.

Corollary 2.9. *Let $E \subset \mathbb{T}^N$ be a regular set whose boundary has constant mean curvature and such that*

$$\int_{\partial E} \big(|D_\tau \varphi|^2 - |B_{\partial E}|^2 \varphi^2\big)\, d\mathcal{H}^{N-1} > 0 \qquad \text{for all } \varphi \in T^\perp(\partial E) \setminus \{0\}.$$

Then, there exist δ, $C > 0$ such that

$$P_{\mathbb{T}^N}(F) \geq P_{\mathbb{T}^N}(E) + C(\alpha(E,F))^2$$

for all $F \subset \mathbb{T}^N$, with $|F| = |E|$ and $\alpha(E,F) < \delta$.

Previous related investigations were carried out by B. White [25], who proved that the strict positivity of the second variation implies local minimality with respect to small L^∞-perturbations. His result was recently extended by F. Morgan and A. Ros in [13], where they show that strictly stable constant mean curvature hypersurfaces are area minimizing with respect to small L^1-perturbations, up to dimension $N = 7$. Our corollary removes the restriction $N \leq 7$ and improves their result in a quantitative fashion.

Note that Corollary 2.9 applied to the unit ball E and with $\mathbb{T}^N$ replaced by $c\mathbb{T}^N$ for $c > 0$ sufficiently large, yields the quantitative isoperimetric inequality in the standard Euclidean case for bounded open sets F with small asymmetry index $d(E,F)$. This fact, in view of Lemma 5.1 in [8], implies the quantitative isoperimetric inequality for all sets, thus leading to an alternative proof based on the second variation.

3. Applications of the local minimality criterion and further developments

A first application of Theorem 2.8 deals with lamellar configurations. To this aim, for a given volume fraction $m \in (-1,1)$ we denote by u_L the one-strip lamellar configuration corresponding to the set $L := \mathbb{T}^{N-1} \times [0, \frac{m+1}{2}]$ and by $\mathcal{L}_m$ the collection of all sets which may be obtained from L by translations and relabeling of coordinates.

Theorem 3.1. *Assume that L is the unique, up to translations and relabeling of coordinates, global minimizer of the periodic isoperimetric problem. Then the same set is also the unique global minimizer of the non local functional* (1.5)*, provided γ is sufficiently small.*

In the two-dimensional case it has been proved in [11] that if $|m| < 1 - \frac{2}{\pi}$, then the lamellar sets of $\mathcal{L}_m$ are the unique global minimizers of the periodic isoperimetric problem in $\mathbb{T}^2$. Therefore, from the above theorem one immediately gets the following result, first proved in [23].

Corollary 3.2. *Let $N = 2$. Fix any m such that $|m| < 1 - \frac{2}{\pi}$. Then for small $\gamma > 0$, any solution of*

$$\min\left\{ P_{\mathbb{T}^2}(E) + \gamma \int_{\mathbb{T}^2} |\nabla v_E|^2\, dx :\, E \subset \mathbb{T}^2,\, |E| = \frac{m+1}{2} \right\}$$

belongs to $\mathcal{L}_m$, that is, it is lamellar.

The corollary above holds only for $N = 2$, where the minimality range of lamellar sets is completely determined. For $N = 3$, to the best of our knowledge the global (with uniqueness) minimality of $\mathcal{L}_m$ is known only in the case $m = 0$ (see [10]). In the following result we show the result still holds for m sufficiently close to 0.

Theorem 3.3. *There exists $\varepsilon > 0$ such that if $m \in (-\varepsilon, \varepsilon)$ the lamellar sets in $\mathcal{L}_m$ are the unique solutions to the corresponding periodic isoperimetric problem in $\mathbb{T}^3$.*

As before we have the following corollary.

Corollary 3.4. *Let $N = 3$. There exists $m_0 > 0$ and γ_0 such that for $|m| < m_0$ such that any solution of*

$$\min\Big\{P_{\mathbb{T}^3}(E) + \gamma \int_{\mathbb{T}^3} |\nabla v_E|^2\, dx : \, E \subset \mathbb{T}^3, \, |E| = \frac{m+1}{2}\Big\}$$

belongs to $\mathcal{L}_m$, provided that $\gamma \le \gamma_0$.

We also mention as a consequence of Theorem 2.8 that in any dimension and for any $\gamma > 0$ lamellar configurations are local minimizers, provided that the number of strips is sufficiently large. To this aim, given $m \in (-1, 1)$ and an integer $k > 1$, we set $L_k := \mathbb{T}^{N-1} \times \cup_{i=1}^{k} [\frac{i-1}{k}, \frac{i-1}{k} + \frac{m+1}{2k}]$ and denote by $\mathcal{L}_{m,k}$ the collection of all sets which may be obtained from L_k by translations and relabeling of coordinates.

Proposition 3.5. *Fix $m \in (-1, 1)$ and $\gamma > 0$. Then there exists an integer k_0 such that for $k \ge k_0$ all sets in $\mathcal{L}_{m,k}$ are isolated local minimizers of* (1.5)*, according to Definition 1.1.*

We now state a result that links Theorem 2.8 with the existence of local minimizers for the Ohta-Kawasaki energy (1.1). Fix $m \in (-1, 1)$. We say that a function $u \in H^1(\mathbb{T}^N)$ is an *isolated local minimizer* for the functional $\mathcal{E}_\varepsilon$ with prescribed volume m, if $\fint_{\mathbb{T}^N} u\, dx = m$ and there exists $\delta > 0$ such that

$$\mathcal{E}_\varepsilon(u) < \mathcal{E}_\varepsilon(w) \qquad \text{for all } w \in H^1(\mathbb{T}^N) \text{ with } \fint_{\mathbb{T}^N} w\, dx = m, \quad 0 < \min_\tau \|u - w(\cdot + \tau)\|_{L^1(\mathbb{T}^N)} \le \delta\,.$$

Since it is well-known that the functionals $\mathcal{E}_\varepsilon$ only Γ-converge in L^1 to the sharp interface energy J, the L^1-local minimality result proved in Theorem 2.8 allows to show:

Theorem 3.6. *Let E be a regular critical set for the functional J with positive second variation and $u = \chi_E - \chi_{\mathbb{T}^N \setminus E}$. Then there exist $\varepsilon_0 > 0$ and a family $\{u_\varepsilon\}_{\varepsilon < \varepsilon_0}$ of isolated local minimizers of $\mathcal{E}_\varepsilon$ with prescribed volume $m = \fint_{\mathbb{T}^N} u\, dx$ such that $u_\varepsilon \to u$ in $L^1(\mathbb{T}^N)$ as $\varepsilon \to 0$.*

Note that it can be also shown that the radius δ in the local minimality condition is uniform throughout the family $\{u_\varepsilon\}_{\varepsilon < \varepsilon_0}$ and depends only on the local minimality radius of the set E appearing in Definition 1.1.

A variant of our result, which is important in the applications, is the Neumann problem. As before we consider the functional

$$J_N(E) := P_\Omega(E) + \gamma \int_\Omega |\nabla v_E|^2\, dx \tag{3.1}$$

and the function

$$u_E = \chi_E - \chi_{\Omega \setminus E}\,, \qquad m = \fint_\Omega u_E\, dx\,,$$

but the condition on v_E is now

$$\begin{cases} -\Delta v_E = u_E - m & \text{in } \Omega \\ \displaystyle\int_\Omega v_E\, dx = 0\,, \quad \frac{\partial v_E}{\partial \nu} = 0\,, & \text{on } \partial\Omega\,. \end{cases} \tag{3.2}$$

As in (1.4) we have

$$\int_\Omega |\nabla v_E|^2\, dx = \int_\Omega \int_\Omega G(x,y) u_E(x) u_E(y)\, dx dy\,,$$

where G is the solution of

$$\begin{cases} -\Delta_y G(x,y) = \delta_x - \frac{1}{|\Omega|} & \text{in } \Omega \\ \displaystyle\int_\Omega G(x,y)\,dy = 0\,, \quad \nabla_y G(x,y)\cdot\nu(y) = 0\,, & \text{if } y \in \partial\Omega\,. \end{cases}$$

As in the periodic case, if E is a sufficiently smooth (local) minimizer of the functional (3.1), then it satisfies the Euler-Lagrange equation

$$H_{\partial E}(x) + 4\gamma v_E(x) = \lambda \qquad \text{for all } x \in \partial E \cap \Omega,$$

and moreover ∂E must meet $\partial\Omega$ ortoghonally (if at all), see [3, Remark 2.8].

Note that, unlike in the periodic case, the functional J_N is not translation invariant, therefore we don't need to consider the distance d defined in (1.7). Provided that ∂E does not meet $\partial\Omega$, the formula (2.1) for the second variation and the regularity result stated in Theorem 1.3 for the periodic case still holds in the Neumann case, without any change in the proof (see [1, Section 6]).

If $\partial E \cap \partial\Omega \neq \emptyset$, the situation is more delicate. First, one has to ensure that the flow Φ associated to the vector field $X : \overline{\Omega} \to \overline{\Omega}$ does not leave $\overline{\Omega}$, secondly the formula for the second variation is more complicate since it contains an extra integral on $\partial E \cap \partial\Omega$. On the other hand, since the problem is not translation invariant, the spaces $T(\partial E)$, $T^\perp(\partial E)$, and the decomposition (2.5) are no longer needed. Therefore, we say that J_N has *positive second variation* at the critical set E if

$$\partial^2 J_N(E)[\varphi] > 0 \qquad \text{for all } \varphi \in \widetilde{H}^1(\partial E) \setminus \{0\},$$

where $\partial^2 J_N$ is given in [12, Proposition 4.1]. The following result is contained in [12, Theorem 1.1].

Theorem 3.7. *Let $E \subset \Omega$ be a regular critical set with positive second variation. Then there exist C, $\delta > 0$ such that*

$$J_N(F) \geq J_N(E) + C|E \triangle F|^2\,,$$

for all $F \subset \Omega$, with $|F| = |E|$ and $|E \triangle F| < \delta$.

We conclude by mentioning two interesting global minimality results. The first one is proved in [14], and deals with thin rectangles $\Omega_\varepsilon = (0,\varepsilon) \times (0,1)$. Fix the mass constraint $m = 0$ and an integer k and denote by $E_k \subset \Omega_\varepsilon$ the lamellar configuration corresponding to k horizontal stripes of equal length (see the precise definition given in [14, (3.5) and (3.6)]. Then, if

$$\frac{12k^2(k-1)^2}{2k-1} < \gamma < \frac{12k^2(k+1)^2}{2k+1}\,,$$

for all sufficiently large integer j the lamellar configuration E_k is the unique global minimizer of J_N in Ω_{ε_j}, with $\varepsilon_j = 1/j$.

The second result is proved in [5] where the authors consider the minimum problem

$$\min\Big\{ P(E) + \gamma \int_\Omega |\nabla v_E|^2\,dx :\ E \subset \Omega,\ |E| = |B_r| \Big\}\,, \tag{3.3}$$

where Ω is a bounded open set in $\mathbb{R}^N$ of class C^2, v_E is defined as in (3.2) and $r > 0$ is fixed. Roughly speaking, they show that if the fixed mass is sufficiently small then the unique minimizer of (3.3) is an almost spherical single droplet. Here we state some of the results proved therein.

Theorem 3.8. *Let Ω be a bounded open set with C^2 boundary. There exist $\delta_0, r_0 > 0$ such that the following holds. Assume $r \leq r_0$ and*

$$\gamma r^3 |\log r| < \delta_0 \ \textit{ if } N = 2 \quad \textit{or} \quad \gamma r^3 < \delta_0 \ \textit{ if } N \geq 3\,.$$

Then, every minimizer E_r of the problem (3.3) satisfies the following properties:

(i) E_r *is a convex set and there exists* $x_r \in \Omega$ *and* $\varphi_r : \mathbb{S}^{N-1} \to \mathbb{R}$ *such that* $\|\varphi\|_{C^1} \lesssim \gamma r^{N+3}$ *and* $\partial E_r = \{x_r + (r + \varphi(y))y : \ y \in \mathbb{S}^{N-1}\}$*;*
(ii) E_r *is an exact ball if and only if the domain* Ω *is itself a ball, i.e., up to translations* $\Omega = B_R$ *for some* $R > 0$*, in which case* $E_r = B_r$ *is the unique minimizer.*

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A coupled surface-bulk convection-diffusion equations with an application to drop dynamics with soluble surfactant

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1 Introduction

Many problems in biological, physical and material sciences involve solving partial differential equations in complex domains or deformable interfaces. In particular, the underlying material quantities in the bulk domain may couple with the one in the interface through adsorption and desorption processes. Meanwhile, the concentration of surface quantities might change the physical behavior of the interface through the modifications of interfacial forces. For instance, surfactant molecules typically consisting of a hydrophilic head and a hydrophobic tail may adsorb and desorb between bulk fluids and the interface so that the interfacial tension can be reduced. Meanwhile, this non-uniform distribution of surfactant molecules produce extra force (Marangoni force) along the tangential direction to affect the dynamics. In practice, the surfactant may be soluble only to some portion of bulk domain enclosed by the interface where the interface and the soluble region are evolving simultaneously. In order to simulate this problem, we have to introduce two surfactant concentrations in the system; namely, the surface concentration along the interface, and the volume concentration in the bulk region. Thus, one need to solve a coupled system of surface-bulk convection-diffusion equations [7, 12, 10].

Another example comes from cell biology applications where proteins inside the cell can diffuse and bind to the membrane whereas membrane-

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bound proteins can dissociate and diffuse to the inner cytoplasm [8]. To simulate this problem, one need to solve a coupled system of surface-volume reaction-diffusion equations. Many other examples in physical and biological systems that have the similar adsorption or desorption mechanisms in the dynamics can be found in the reference [10]. In [4], we have successfully developed a mass conservative scheme for convection-diffusion equation on moving interface and applied to simulate the interfacial flows with insoluble surfactant [4, 5, 6]. A recent work of Khatri and Tornberg [3] used segment projection method to represent the interface and solve the surfactant equation. More up-to-dated numerical methods for solving Navier-Stokes flows with insoluble surfactant can be found in [3] as well.

In this paper, we summarize our previous work to soluble surfactant case published in [1]. However, as a very first step, we need to develop a numerical scheme for solving coupled surface-bulk convection-diffusion equations. There are at least three major numerical issues from our point of view. (1) How to handle the adsorption and desorption between the interface and the bulk accurately? (2) How to maintain the total surfactant mass conserved during the evolution? (3) Since the surfactant might be soluble to only one of buck fluid, how to avoid the surfactant being present in other bulk regions via either convection or diffusion mechanism? Here, we formulate the coupled surface-bulk convection-diffusion equations in the immersed boundary framework so that the adsorption and desorption processes can be termed as a singular source in the bulk equation. Moreover, by using the indicator function, we can embed the bulk equation into the whole computational domain so that regular Eulerian finite difference scheme can be applied without handling the complicated moving irregular domain. We develop a new conservative scheme for solving the coupled bulk-surface concentration equations which the total surfactant mass can be conserved exactly in discrete sense. By introducing the indicator function and solving the bulk equation in the regular computational domain, one can avoid evaluating the surfactant flux across the interface due to adsorption and desorption processes.

2 A coupled surface-bulk concentration model

As in [10], we consider the same coupled bulk-surface material (or surfactant) concentration model in which the adsorption and desorption can be occurred on the moving deformable interface. Consider a domain Ω in R^2 and there is an interface Σ, which is a simple closed curve immersed in Ω. The interior of the interface is Ω_0, and the exterior is Ω_1 so that $\Omega = \Omega_0 \cup \Omega_1$, see the illustration of these domains in Figure 1. The interface is represented by a Lagrangian form $\boldsymbol{X}(\alpha,t), 0 \le \alpha \le L_b$, where α is the Lagrangian material coordinate attached to the interface which is not necessarily to be the arc-length parameter. The unit tangent vector of the interface can be written as $\boldsymbol{\tau} = \frac{\partial \boldsymbol{X}}{\partial \alpha} / \left| \frac{\partial \boldsymbol{X}}{\partial \alpha} \right|$; thus, the unit outward normal vector $\boldsymbol{n}$ pointing into Ω_1 can be defined accordingly. In addition, the interface Σ is moving with a given velocity field $\boldsymbol{u} = (u, v)$ in Ω; that is,

$$\frac{\partial \boldsymbol{X}(\alpha,t)}{\partial t} = \boldsymbol{U}(\alpha,t) = \int_{\Omega} \boldsymbol{u}(\boldsymbol{x},t)\delta(\boldsymbol{x} - \boldsymbol{X}(\alpha,t))\, d\boldsymbol{x}, \tag{1}$$

where $\delta(\boldsymbol{x}) = \delta(x)\,\delta(y)$ is the two-dimensional Dirac delta function. We use the above usual delta function formulation in the immersed boundary method [9] to represent the interpolation of the velocity field into the interface. Here we assume the velocity field is incompressible ($\nabla \cdot \boldsymbol{u} = 0$) in Ω and no flow boundary condition ($\boldsymbol{u} \cdot \boldsymbol{n}_1 = 0$) is imposed on $\partial\Omega = \partial\Omega_1$. Notice that, in later section, the velocity field can be obtained by solving the Navier-Stokes equations.

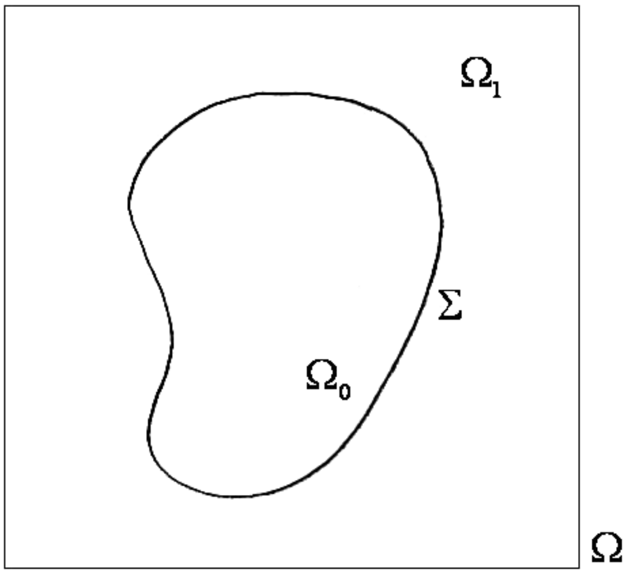

Figure 1: Illustration of domains.

It is assumed that the surfactant exists on the interface as a monolayer and is adsorbed from or desorbed into the bulk fluid in Ω_1; that is, the surfactant is soluble in the exterior bulk Ω_1 but not in the interior one

Ω_0. Therefore, we have to introduce two surfactant concentrations in the system; namely, the surface concentration $\Gamma(\alpha, t)$ along the interface Σ, and the bulk concentration $C(x, y, t)$ in the region Ω_1. By taking the adsorption and desorption of bulk surfactant into account, the dimensionless surface concentration equation can be modified as

$$\frac{\partial \Gamma}{\partial t} + (\nabla_s \cdot \boldsymbol{u})\,\Gamma = \frac{1}{Pe_s}\,\nabla_s^2 \Gamma + (S_a/\lambda) C_s (1 - \Gamma) - S_d \Gamma, \tag{2}$$

where $\nabla_s = (\boldsymbol{I} - \boldsymbol{n} \otimes \boldsymbol{n})\nabla$ and $\nabla_s^2 = \nabla_s \cdot \nabla_s$ are the surface gradient and surface Laplacian operators, respectively. The dimensionless number Pe_s is the surface Peclet number, S_a and S_d are the absorption and desorption Stanton number, respectively, and λ is the dimensionless adsorption depth. Those parameters are defined as

$$Pe_s = U_\infty R/D_s, \quad S_a = k_a/U_\infty, \quad S_d = k_d R/U_\infty, \quad \lambda = \Gamma_\infty/(C_\infty R)$$

where $R, U_\infty, \Gamma_\infty, C_\infty$ are the reference values for the length, flow velocity, the surface and bulk concentration, and k_a, k_d are the absorption and desorption coefficients. C_s is the bulk surfactant concentration adjacent to the interface which can be defined later. The above non-dimensionalization process can be found in [12, 7, 4]. Notice that, as in [4], the interface is tracked in Lagrangian manner and the surface concentration is defined at the material point, so the time derivative in Eq. (2) has the meaning of the material derivative naturally.

The dimensionless bulk concentration in the exterior region Ω_1 [7, 10, 11, 12] can be written as

$$\frac{\partial C}{\partial t} + \boldsymbol{u} \cdot \nabla C = \frac{1}{Pe} \nabla^2 C \tag{3}$$

$$\frac{1}{\lambda Pe} \frac{\partial C}{\partial \boldsymbol{n}}\Big|_\Sigma = (S_a/\lambda) C_s (1 - \Gamma) - S_d \Gamma \qquad \frac{\partial C}{\partial \boldsymbol{n}_1}\Big|_{\partial \Omega_1} = 0, \tag{4}$$

where Pe is the Peclet number, $\boldsymbol{n}$ is the unit normal vector on Σ pointing into Ω_1 and $\boldsymbol{n}_1$ is the unit outward normal to the boundary $\partial\Omega_1 = \partial\Omega$.

Eqs. (2)-(4) describe the present coupled surface-bulk concentration equations. Since the fluid is incompressible and no flow velocity boundary condition is imposed on $\partial\Omega_1$, one can conclude that the total surfactant mass (the surfactant mass on the interface Σ and the mass in the bulk region Ω_1) must be conserved. The conservation property can be proved easily as follows.

3 Navier-Stokes flow with soluble surfactant

Consider an incompressible flow problem consisting of two-phase fluids in a fixed two-dimensional square domain $\Omega = \Omega_0 \cup \Omega_1$ where an interface Σ separates Ω_0 from Ω_1 as illustrated in Figure 1. As in previous section, it is assumed that the surfactant exists on the interface as a monolayer and is adsorbed from or desorbed to the bulk fluid in Ω_1; that is, the surfactant is soluble in the exterior bulk but not in the interior one. The interface is contaminated by the surfactant so that the distribution of the surfactant changes the surface tension accordingly. In order to formulate the problem using the immersed boundary approach, we simply treat the interface as an immersed boundary that exerts force to the fluids and moves with local fluid velocity. For simplicity, we assume equal viscosity and density for both fluids, and neglect the gravity. Certainly, the present Navier-Stokes solver can be replaced by the one with different density and viscosity ratios.

As in [4], the non-dimensional Navier-Stokes flow in the usual immersed boundary formulation can be written as

$$\begin{aligned}
\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p &= \frac{1}{Re}\,\nabla^2 \boldsymbol{u} + \frac{\boldsymbol{f}}{Re\; Ca}, &(5)\\
\nabla\cdot\boldsymbol{u} &= 0, &(6)\\
\boldsymbol{f}(\boldsymbol{x},t) &= \int_\Sigma \boldsymbol{F}(\alpha,t)\,\delta(\boldsymbol{x}-\boldsymbol{X}(\alpha,t))\,d\alpha, &(7)\\
\frac{\partial \boldsymbol{X}(\alpha,t)}{\partial t} = \boldsymbol{U}(\alpha,t) &= \int_\Omega \boldsymbol{u}(\boldsymbol{x},t)\,\delta(\boldsymbol{x}-\boldsymbol{X}(\alpha,t))d\boldsymbol{x}, &(8)\\
\boldsymbol{F}(\alpha,t) &= \frac{\partial}{\partial\alpha}(\sigma(\alpha,t)\boldsymbol{\tau}(\alpha,t)), &(9)
\end{aligned}$$

where $\boldsymbol{u}$ is the fluid velocity and p is the pressure. The dimensionless numbers are the Reynolds number ($Re = \rho U_\infty R/\mu$) describing the ratio between the inertial force and the viscous force, and the Capillary number ($Ca = \mu U_\infty/\sigma_\infty$) describing the strength of the surface tension. The presence of surfactant will reduce the surface tension of the interface by the Langmuir equation of state

$$\sigma = 1 + El\ln(1-\Gamma), \tag{10}$$

where σ is the surface tension, and El is the elasticity number measuring the sensitivity of the surface tension to the surfactant concentration. Since the surfactant is soluble in Ω_1, we need to solve the coupled surface-bulk concentration equations (2)-(4) to close the system.

4 Numerical method

In the following, we describe how to march one time step for the solutions. At the beginning of each time step, the interface position, the fluid velocity, the surface and bulk concentrations must be given. The numerical algorithm is as follows.

1. Compute the surface tension and unit tangent on the interface.
2. Distribute the interfacial force from the Lagrangian markers into the fluid.
3. Solve the Navier-Stokes equations by the projection method.
4. Interpolate the new velocity on the fluid lattice points onto the marker points and move the marker points to new positions.
5. Compute the indicator function.
6. Compute the surface surfactant concentration using the scheme in [4].
7. Compute the bulk surfactant concentration using the scheme in [1].

Note that, the detailed numerical implementation of first four steps is quite standard in immersed boundary method and can be found in any related literature. Here, we just use the same solver as in our previous work [4]. The last four steps are exactly the same four steps shown in [1].

5 Numerical result: A drop under shear flow

Figure 2 shows the evolutionary interface positions of clean (denoted by dash-dotted line) and soluble surfactant (denoted by solid line) cases for a drop under shear flow based on the results of grid number $N = 256$. The clean drop bears no surfactant along the interface throughout the evolution so no bulk and surface surfactant equations are needed to be solved and the surface tension remains to be a constant $\sigma = 1$. (Note that, we use the clean drop as a comparison simply because of zero initial surface concentration is chosen in present setting.) Due to shear stresses, both drops will be elongated and gradually aligned with the flow directions. For the soluble case, the interface will start to absorb the bulk surfactant so the bulk concentration decreases while the surface concentration increases in the beginning, see Figure 3 in detail. Later, both absorption and desorption processes become more balanced so the bulk and surface concentrations become quite steady. As expected, the largest surface concentration appears to occur at the drop tips after the drop aligned with the flow. The drop with soluble surfactant has smaller surface tension than the clean drop so the deformation tends to be larger. One can see from Figure 2 that the clean drop approaches to a steady state shape after $T = 9.0$ while the soluble surfactant drop continues to deform slightly afterwards. Our numerical results are physically reasonable and qualitatively consistent with those obtained in other literature such as in [11].

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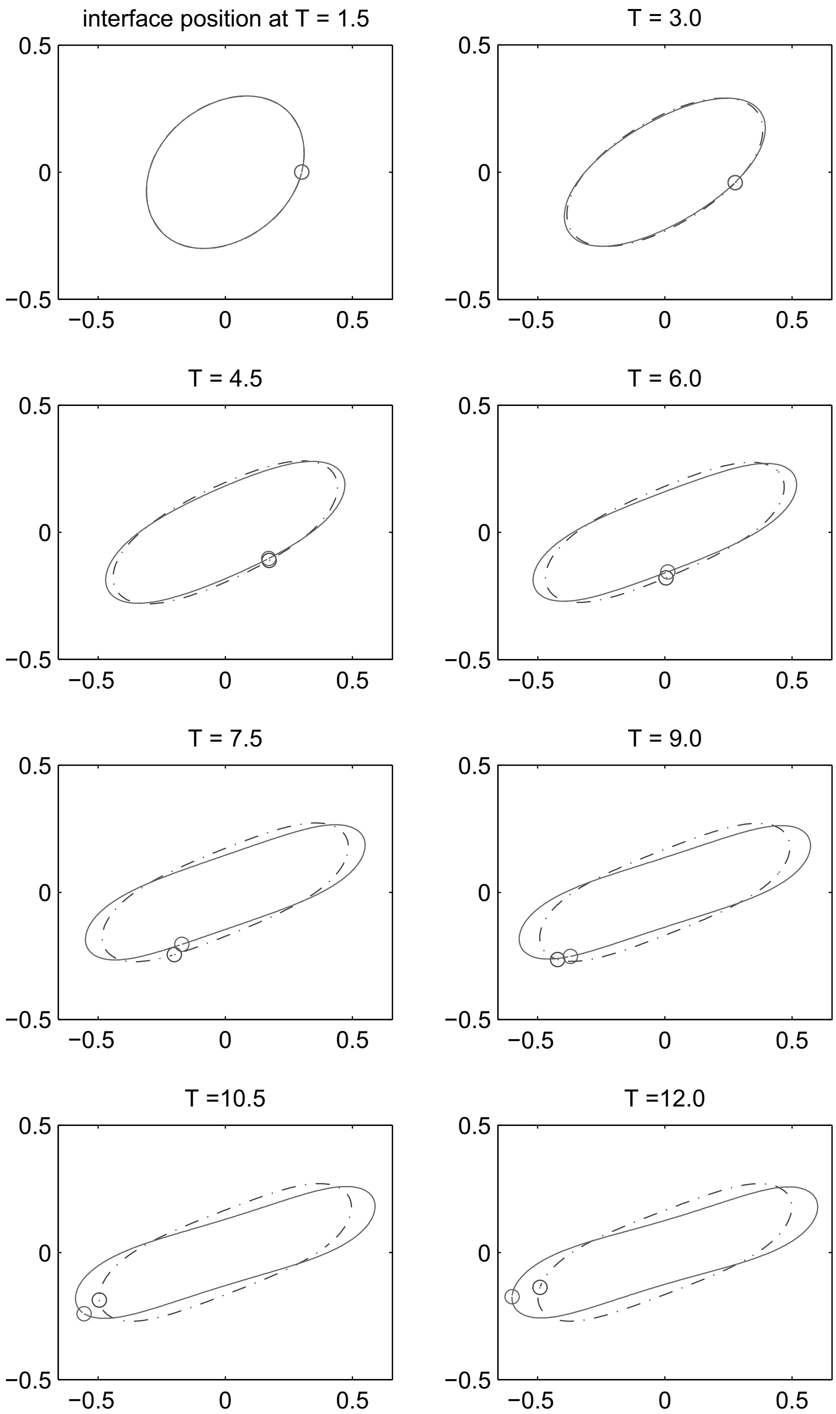

Figure 2: The interface positions of clean (denoted by "-.") and soluble (denoted by "-") interfacial flows for a drop under shear flow.

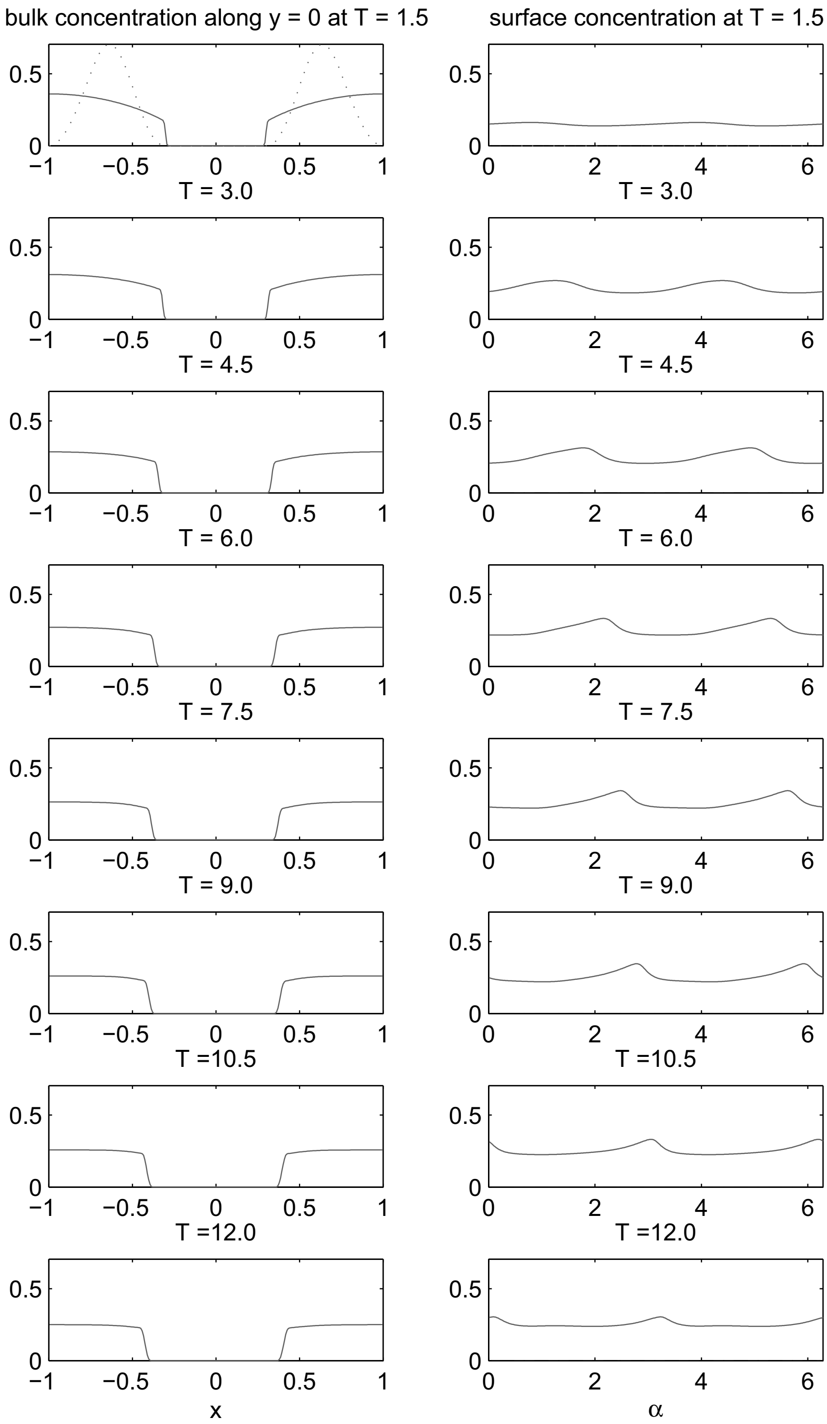

Figure 3: Left column: The evolutionary bulk concentration along $y = 0$ for soluble case. Right column: The evolutionary surface concentration for soluble (denoted by "-") case. The surface concentrations along the interface are plotted in counter-clockwise direction starting from the point marked by "o" in Figure 2.

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On the 2 phase problem including the phase transition

Yoshihiro Shibata *

Main topic of my talk is a local well-posedness of the compressible and incompressible phase transition problem with nearly flat interface in the maximal L_p-L_q framework. The plan of my talk is the following.

Contents

1 Modeling

Following the J. Pruess idea in [5], we discuss the modeling of two phase problem. Let Ω be a domain in the N dimensional Euclidean space $\mathbb{R}^N$ ($N \geq 2$) with boundary Γ_0. Let Ω_- be a subdomain of Ω with boundary Γ. We assume that $\Gamma = \partial\Omega_- \subset \Omega$ and that $\Gamma_0 \cap \Gamma = \emptyset$. Set $\Omega_+ = \Omega - \overline{\Omega_-}$. Let $\varphi = \varphi(\xi, t) = (\varphi_1(\xi, t), \ldots, \varphi_N(\xi, t))$ be a function defined on the closure of Ω for each time variable $t \in (0, T)$, $\xi = (\xi_1, \ldots, \xi_N)$ being the reference coordinate system. We assume that the map $\xi \to \varphi(\xi, t)$ is one to one for each $t \in (0, T)$ [†]. Set $(\partial_t\varphi)(\xi, t) = \mathbf{v}(x, t)$ with $x = \varphi(\xi, t)$. Set

$$\Omega_\pm(t) = \{x = \varphi(\xi, t) \mid \xi \in \Omega_\pm\}, \quad \Gamma(t) = \{x = \varphi(\xi, t) \mid \xi \in \Gamma\}, \quad \Gamma_0(t) = \{x = \varphi(\xi, t) \mid \xi \in \Gamma_0\}$$

and $\dot\Omega(t) = \Omega_-(t) \cup \Omega_+(t)$. Let $\mathbf{n}_{\Gamma(t)}$ be the unit outer normal to $\Gamma(t)$ pointed from $\Omega_-(t)$ to $\Omega_+(t)$ and let $\mathbf{n}_{\Gamma_0(t)}$ the unit outer normal to $\Gamma_0(t)$. Set

$$[[v]] = v_- - v_+ \quad \text{(the jump of } v \text{ accross } \Gamma(t))$$

for any v defined on $\dot\Omega(t)$. Here and hereafter, we write $v_\pm = v|_{\Omega_\pm(t)}$. Moreover, given $v_\pm$ defined on $\Omega_\pm(t)$, we define v by $v(x) = v_\pm(x)$ for $x \in \Omega_\pm(t)$. Let $H_\Gamma = -\mathrm{div}\,_\Gamma \mathbf{n}_\Gamma$ be the mean curvature of $\Gamma(t)$.

Remark 1. Let Σ be a hypersurface in $\mathbb{R}^N$ defined by $x = \psi(\theta)$ with $\theta \in \Theta \subset \mathbb{R}^{N-1}$. The Laplace Beltrami operator Δ_Σ on Σ is defined by

$$\Delta_\Sigma f = \frac{1}{\sqrt{g}}\partial_i(\sqrt{g}g^{ij}\partial_j f)$$

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†Since this chapter is concerned with the modeling, we do not care the regularity of boundary and the map φ. Moreover, we do not mention any integrability of functions regorously. These are formulated mathematically in sections 2 and 3.

Here, f is a function defined on Σ and $\partial_i f = \partial_i(f \circ \psi)$. Moreover, $\tau_i = \partial_i \psi$, $g_{ij} = \tau_i \cdot \tau_j$, $G = (g_{ij})$, $g = \det G$, and $G^{-1} = (g^{ij})$. Let $\mathbf{n}_\Sigma$ be the unit outer normal to Σ. We know that

$$\frac{1}{\sqrt{g}}\partial_i(\sqrt{g}g^{ij}\tau_j) = H_\Sigma \mathbf{n}_\Sigma, \tag{1.1}$$

$$\Delta_\Sigma \psi = H_\Sigma \mathbf{n}_\Sigma. \tag{1.2}$$

In our modeling, the following well-known formula plays a fundamental role.

Reynolds transport theorem

$$\frac{d}{dt}\int_{\dot\Omega(t)} f\,dx = \int_{\dot\Omega(t)} \partial_t f\,dx + \int_{\Gamma(t)} [[f]]\mathbf{v}\cdot\mathbf{n}_{\Gamma(t)}\,d\sigma + \int_{\Gamma_0(t)} f\mathbf{v}\cdot\mathbf{n}_{\Gamma_0(t)}\,d\sigma,$$

where $d\sigma$ means the surface elements not only of $\Gamma(t)$ but also of $\Gamma_0(t)$.

In fact, let $J(\xi,t)$ be the Jacobian of the transformation: $x = \varphi(\xi,t)$. We have $\partial_t J(\xi,t) = (\operatorname{div}_x \mathbf{v})J(\xi,t)$, so that

$$\begin{aligned}
\frac{d}{dt}\int_{\Omega_-(t)} f\,dx &= \frac{d}{dt}\int_{\Omega_-} f(\varphi(\xi,t),t)J(\xi,t)\,d\xi = \int_{\Omega_-}(\partial_t f + \mathbf{v}\cdot\nabla_x f + (\operatorname{div}_x\mathbf{v})f)J(\xi,t)\,d\xi \\
&= \int_{\Omega_-(t)}\{\partial_t f + \operatorname{div}_x(\mathbf{v}f)\}\,dx = \int_{\Omega_-(t)}\partial_t f\,dx + \int_{\Gamma(t)} f(\mathbf{v}\cdot\mathbf{n}_{\Gamma(t)})\,d\sigma.
\end{aligned}$$

Analogously,

$$\begin{aligned}
\frac{d}{dt}\int_{\Omega_+(t)} f\,dx &= \frac{d}{dt}\int_{\Omega_+} f(\varphi(\xi,t),t)J(\xi,t)\,d\xi = \int_{\Omega_+}(\partial_t f + \mathbf{v}\cdot\nabla_x f + (\operatorname{div}_x\mathbf{v})f)J(\xi,t)\,d\xi \\
&= \int_{\Omega_+(t)}\{\partial_t f + \operatorname{div}_x(\mathbf{v}f)\}\,dx = \int_{\Omega_+(t)}\partial_t f\,dx - \int_{\Gamma(t)} f(\mathbf{v}\cdot\mathbf{n}_{\Gamma(t)})\,d\sigma + \int_{\Gamma_0(t)} f(\mathbf{v}\cdot\mathbf{n}_{\Gamma_0(t)})\,d\sigma.
\end{aligned}$$

Combining these formulas, we have the Reynolds transport theorem. $\square$

In this orientation, we have

$$\frac{d}{dt}|\Gamma(t)| = -\int_{\Gamma(t)} H_\Gamma \mathbf{v}\cdot\mathbf{n}_\Gamma\,d\sigma \tag{1.3}$$

In fact,

$$\partial_t \det G = (\det G)\operatorname{tr}(\dot G G^{-1}) = 2(\det G)g^{ij}\dot\tau_i\tau_j = 2(\det G)g^{ij}\partial_i\dot\psi\tau_j.$$

Thus, recalling (1.1), we have

$$\begin{aligned}
\frac{d}{dt}|\Gamma(t)| &= \frac{d}{dt}\int_\Theta \sqrt{g}\,d\theta = \int_\Theta \frac{1}{2\sqrt{g}}\partial_t \det G\,d\theta = \int_\Theta \frac{1}{2\sqrt{g}}2(\det G)g^{ij}\partial_i\dot\psi\tau_j\,d\sigma \\
&= -\int_\Theta \partial_i(\sqrt{g}g^{ij}\tau_j)\dot\psi\,d\theta = -\int_\Theta \frac{1}{\sqrt{g}}\partial_i(\sqrt{g}g^{ij}\tau_j)\dot\psi\sqrt{g}\,d\theta = -\int_{\Gamma(t)} H_\Gamma\mathbf{v}\cdot\mathbf{n}_\Gamma\,d\sigma.
\end{aligned}$$

In the following, we use the following notation:

- $\rho : \dot\Omega(t) \to \overline{\mathbb{R}_+} = [0,\infty)$ is the mass field,
- $\mathbf{u} : \dot\Omega(t) \to \mathbb{R}^N$ the velocity field,
- $\pi : \dot\Omega(t) \to \mathbb{R}$ the pressure field,
- $\mathbf{T} : \dot\Omega(t) \to \{A \in GL_N(\mathbb{R}) \mid {}^T A = A\}$ the stress tensor field,
- $\mathbf{D} = \frac{1}{2}({}^T\nabla\mathbf{u} + \nabla\mathbf{u}) : \dot\Omega(t) \to \{A \in GL_N(\mathbb{R}) \mid {}^T A = A\}$ the strain tensor field,

- $\theta : \dot{\Omega}(t) \to \overline{\mathbb{R}_+}$ the thermal field,
- $e : \dot{\Omega}(t) \to \overline{\mathbb{R}_+}$ the internal energy,
- $\mathbf{q} : \dot{\Omega}(t) \to \mathbb{R}^N$ the heat flux,
- $\mathbf{f} : \dot{\Omega}(t) \to \mathbb{R}^N$ the external force,
- $r : \Omega \to \mathbb{R}$ the heat supply.

For our modeling, we use the following Navier-Stokes-Fourier system of equations: for $x \in \dot{\Omega}(t)$

$$\partial_t\rho + \mathrm{div}\,(\rho\mathbf{u}) = 0 \qquad \textbf{conservation of mass}; \tag{1.4}$$

$$\partial_t(\rho\mathbf{u}) + \mathrm{div}\,(\rho\mathbf{u}\otimes\mathbf{u}) - \mathrm{div}\,\mathbf{T} = \rho\mathbf{f} \qquad \textbf{conservation of momentum}; \tag{1.5}$$

$$\partial_t(\frac{\rho}{2}|\mathbf{u}|^2 + \rho e) + \mathrm{div}\,((\frac{\rho}{2}|\mathbf{u}|^2 + \rho e)\mathbf{u}) - \mathrm{div}\,(\mathbf{T}\mathbf{u} - \mathbf{q}) = \rho\mathbf{f}\cdot\mathbf{u} + \rho r \qquad \textbf{conservation of energy}. \tag{1.6}$$

Here, for any $\mathbf{u} = (u_1, \ldots, u_N)$, $\mathbf{u}\otimes\mathbf{u}$ is the $N\times N$ matrix whose (i,j) component is u_iu_j, and for any $\mathbf{w} = (w_1, \ldots, w_N)$ and $N\times N$ matrix valued function $\mathbf{S} = (S_{ij})$ their divergence $\mathrm{div}\,\mathbf{w}$ and $\mathrm{div}\,\mathbf{S}$ are defined by

$$\mathrm{div}\,\mathbf{w} = \sum_{j=1}^N \partial_j w_j, \ \ \mathrm{div}\,\mathbf{S} = (\sum_{j=1}^N \partial_j S_{1j}, \ldots, \sum_{j=1}^N \partial_j S_{Nj}).$$

From now on, we are concerend with the jump condition on $\Gamma(t)$ and boundary condition on $\Gamma_0(t)$. In the following we assume that $\mathbf{v} \neq \mathbf{u}$ on $\Gamma(t)$, but $\mathbf{v} = \mathbf{u}$ on $\Gamma_0(t)$.

First, we consider the mass conservation:

$$\frac{d}{dt}\int_{\dot{\Omega}(t)} \rho\,dx = 0. \tag{1.7}$$

By (1.4) and the Reynolds transport theorem, we have

$$\begin{aligned}\frac{d}{dt}\int_{\dot{\Omega}(t)} \rho\,dx &= \int_{\dot{\Omega}(t)} \partial_t\rho\,dx + \int_{\Gamma(t)} [[\rho]]\mathbf{v}\cdot\mathbf{n}_{\Gamma(t)}\,d\sigma + \int_{\Gamma_0(t)} \rho\mathbf{u}\cdot\mathbf{n}_{\Gamma_0(t)}\,d\sigma \\ &= -\int_{\dot{\Omega}(t)} \mathrm{div}\,(\rho\mathbf{u})\,dx + \int_{\Gamma(t)} [[\rho]]\mathbf{v}\cdot\mathbf{n}_{\Gamma(t)}\,d\sigma + \int_{\Gamma_0(t)} \rho\mathbf{u}\cdot\mathbf{n}_{\Gamma_0(t)}\,d\sigma \\ &= -\int_{\Gamma(t)} [[\rho(\mathbf{u}-\mathbf{v})]]\cdot\mathbf{n}_{\Gamma(t)}\,d\sigma.\end{aligned}$$

Thus, to obtain (1.7), it is sufficient to assume that

$$[[\rho(\mathbf{u}-\mathbf{v})]]\cdot\mathbf{n}_{\Gamma(t)} = 0 \quad \text{on } \Gamma(t). \tag{1.8}$$

In this case, $\rho_2(\mathbf{u}_2 - \mathbf{v})\cdot\mathbf{n}_{\Gamma(t)} = \rho_1(\mathbf{u}_1 - \mathbf{v})\cdot\mathbf{n}_{\Gamma(t)}$ on $\Gamma(t)$, so that the phase flux $\jmath$ is defined by

$$\jmath = \rho_2(\mathbf{u}_2 - \mathbf{v})\cdot\mathbf{n}_{\Gamma(t)} = \rho_1(\mathbf{u}_1 - \mathbf{v})\cdot\mathbf{n}_{\Gamma(t)}. \tag{1.9}$$

- When $\jmath = 0$, $\mathbf{u}_2\cdot\mathbf{n}_\Gamma = \mathbf{u}_1\cdot\mathbf{n}_\Gamma$, namely $[[\mathbf{u}]]\cdot\mathbf{n}_\Gamma = 0$. This case is called without phase transition.
- When $\jmath \neq 0$, $[[\rho]] \neq 0$, $\frac{\jmath}{\rho_2} = \mathbf{u}_2\cdot\mathbf{n}_{\Gamma(t)} - \mathbf{v}\cdot\mathbf{n}_{\Gamma(t)}$, $\frac{\jmath}{\rho_1} = \mathbf{u}_1\cdot\mathbf{n}_{\Gamma(t)} - \mathbf{v}\cdot\mathbf{n}_{\Gamma(t)}$, so that

$$\jmath = \frac{[[\mathbf{u}]]\cdot\mathbf{n}_{\Gamma(t)}}{[[1/\rho]]}. \tag{1.10}$$

- When $\jmath \neq 0$ and $[[\rho]] = 0$, $\jmath$ can not be decided by the velocity field $\mathbf{u}$.

The case $\jmath \neq 0$ is called with phase transition.

Next, we consider the conservation of momentum:

$$\frac{d}{dt}\int_{\dot\Omega(t)} \rho\mathbf{u}\,dx = \int_{\dot\Omega(t)} \rho\mathbf{f}\,dx. \tag{1.11}$$

By (1.5) and the Reynolds transport theorem, we have

$$\begin{aligned}
&\frac{d}{dt}\int_{\dot\Omega(t)} \rho\mathbf{u}\,dx = \int_{\dot\Omega(t)} \partial_t(\rho\mathbf{u})\,dx + \int_{\Gamma(t)} [[\rho\mathbf{u}]]\mathbf{v}\cdot\mathbf{n}_{\Gamma(t)}\,d\sigma + \int_{\Gamma_0(t)} (\rho\mathbf{u})\mathbf{u}\cdot\mathbf{n}_{\Gamma_0(t)}\,d\sigma \\
&= \int_{\dot\Omega(t)} \rho\mathbf{f}\,dx - \int_{\dot\Omega(t)} \mathrm{div}\,(\rho\mathbf{u}\otimes\mathbf{u})\,dx + \int_{\dot\Omega(t)} \mathrm{div}\,\mathbf{T}\,dx + \int_{\Gamma(t)} [[\rho\mathbf{u}]]\mathbf{v}\cdot\mathbf{n}_{\Gamma(t)}\,d\sigma + \int_{\Gamma_0(t)} (\rho\mathbf{u})\mathbf{u}\cdot\mathbf{n}_{\Gamma_0(t)}\,d\sigma \\
&= \int_{\dot\Omega(t)} \rho\mathbf{f}\,dx - \int_{\Gamma(t)} ([[\rho\mathbf{u}\otimes(\mathbf{u}-\mathbf{v}) - [[\mathbf{T}]])\mathbf{n}_{\Gamma(t)}\,d\sigma + \int_{\Gamma_0(t)} \mathbf{T}\mathbf{n}_{\Gamma_0(t)}\,d\sigma.
\end{aligned}$$

Thus, in order that $\frac{d}{dt}\int_{\dot\Omega(t)} \rho\mathbf{u}\,dx = 0$ holds, it is sufficient to assume that

$$\begin{aligned}
[[\rho\mathbf{u}\otimes(\mathbf{u}-\mathbf{v}) - \mathbf{T}]]\mathbf{n}_{\Gamma(t)} &= \mathrm{div}_{\,\Gamma}\mathbf{T}_\Gamma \quad \text{on } \Gamma(t), \\
\mathbf{T}\mathbf{n}_{\Gamma_0(t)} &= 0 \quad \text{on } \Gamma_0(t)
\end{aligned} \tag{1.12}$$

Here, $\mathbf{T}_\Gamma$ and $\mathbf{T}_{\Gamma_0}$ are stress tensor fields on $\Gamma(t)$ and $\Gamma_0(t)$, respectively. We assume that $\mathrm{div}_{\,\Gamma_0}\mathbf{T}_{\Gamma_0} = 0$ on $\Gamma_0(t)$ and that $\mathrm{div}_{\,\Gamma}\mathbf{T}_\Gamma = -\sigma H_\Gamma \mathbf{n}_{\Gamma(t)}$, where σ is a non-negative constant describing the coefficient of surface tension.

We represent the interface condition (1.12) with the help of the phase flux as follows:

$$[[\rho\mathbf{u}(\mathbf{u}-\mathbf{v})]]\mathbf{n}_{\Gamma(t)} = \rho_1\mathbf{u}_1(\mathbf{u}_1-\mathbf{v})\cdot\mathbf{n}_{\Gamma(t)} - \rho_2\mathbf{u}_2(\mathbf{u}_2-\mathbf{v})\cdot\mathbf{n}_{\Gamma(t)} = \jmath[[\mathbf{u}]].$$

Moreover, by (1.4) we rewrite (1.5) as follows:

$$\partial_t(\rho\mathbf{u}) + \mathrm{div}\,(\rho\mathbf{u}\otimes\mathbf{u}) = \mathbf{u}(\partial_t\rho + \mathrm{div}\,(\rho\mathbf{u})) + \rho(\partial_t\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u}) = \rho(\partial_t\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u}).$$

Finally, we have

$$\begin{aligned}
\rho(\partial_t\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u}) - \mathrm{div}\,\mathbf{T} &= \rho\mathbf{f} \quad \text{in } \dot\Omega(t), \\
\jmath[[\mathbf{u}]] - [[\mathbf{T}\mathbf{n}_{\Gamma(t)}]] &= -\sigma H_\Gamma\mathbf{n}_{\Gamma(t)} \quad \text{on } \Gamma(t), \\
\mathbf{T}\mathbf{n}_{\Gamma_0(t)} &= 0 \quad \text{on } \Gamma_0(t).
\end{aligned} \tag{1.13}$$

Here and in the following, for any N-vector valued functions $\mathbf{w} = (w_1,\dots,w_N)$, $\mathbf{z} = (z_1,\dots,z_N)$ and scalor function f, we set $\mathbf{w}\cdot\nabla f = \sum_{j=1}^N w_j\partial_j f$ and $\mathbf{w}\cdot\nabla\mathbf{z}$ is the N vector function whose i th component is $\mathbf{w}\cdot\nabla z_i$.

Next, we consider the balance of energy. We look for a sufficient condition to obtain the conservation of energy:

$$\frac{d}{dt}\Big(\int_{\dot\Omega(t)} \Big(\frac{\rho}{2}|\mathbf{u}|^2 + \rho e\Big)\,dx + \sigma|\Gamma(t)|\Big) = \int_{\dot\Omega(t)} (\rho\mathbf{f}\cdot\mathbf{u} + \rho r)\,dx. \tag{1.14}$$

By (1.6) and the Reynolds transport theorem, we have

$$\begin{aligned}
&\frac{d}{dt}\int_{\dot\Omega(t)} \Big(\frac{\rho}{2}|\mathbf{u}|^2 + \rho e\Big)\,dx \\
&= \int_{\dot\Omega(t)} \partial_t\Big(\frac{\rho}{2}|\mathbf{u}|^2 + \rho e\Big)\,dx + \int_{\Gamma(t)} \Big[\Big[\frac{\rho}{2}|\mathbf{u}|^2 + \rho e\Big]\Big]\mathbf{v}\cdot\mathbf{n}_{\Gamma(t)}\,d\sigma + \int_{\Gamma_0(t)} \Big(\frac{\rho}{2}|\mathbf{u}|^2 + \rho e\Big)\mathbf{u}\cdot\mathbf{n}_{\Gamma_0(t)}\,d\sigma \\
&= \int_{\dot\Omega(t)} (\rho\mathbf{f}\cdot\mathbf{u} + \rho r)\,dx - \int_{\dot\Omega(t)} \mathrm{div}\,\Big(\Big(\frac{\rho}{2}|\mathbf{u}|^2 + \rho e\Big)\mathbf{u}\Big)\,dx + \int_{\dot\Omega(t)} \mathrm{div}\,(\mathbf{T}\mathbf{u} - \mathbf{q})\,dx \\
&\quad + \int_{\Gamma(t)} \Big[\Big[\frac{\rho}{2}|\mathbf{u}|^2 + \rho e\Big]\Big]\mathbf{v}\cdot\mathbf{n}_{\Gamma(t)}\,d\sigma + \int_{\Gamma_0(t)} \Big(\frac{\rho}{2}|\mathbf{u}|^2 + \rho e\Big)\mathbf{u}\cdot\mathbf{n}_{\Gamma_0(t)}\,d\sigma \\
&= \int_{\dot\Omega(t)} (\rho\mathbf{f}\cdot\mathbf{u} + \rho r)\,dx - \int_{\Gamma(t)} \Big[\Big[\Big(\frac{\rho}{2}|\mathbf{u}|^2 + \rho e\Big)(\mathbf{u}-\mathbf{v}) - (\mathbf{T}\mathbf{u} - \mathbf{q})\Big]\Big]\cdot\mathbf{n}_{\Gamma(t)}\,d\sigma + \int_{\Gamma_0(t)} (\mathbf{T}\mathbf{u} - \mathbf{q})\cdot\mathbf{n}_{\Gamma_0(t)}\,d\sigma.
\end{aligned}$$

On the other hand, by (1.3) we have

$$\frac{d}{dt}\Big(\int_{\dot\Omega(t)}\big(\frac{\rho}{2}|\mathbf{u}|^2+\rho e\big)\,dx+\sigma|\Gamma(t)|\Big)-\int_{\dot\Omega(t)}(\rho\mathbf{f}\cdot\mathbf{u}+\rho r)\,dx$$
$$=-\int_{\Gamma(t)}\Big([[(\frac{\rho}{2}|\mathbf{u}|^2+\rho e)(\mathbf{u}-\mathbf{v})-(\mathbf{Tu}-\mathbf{q})]]\cdot\mathbf{n}_{\Gamma(t)}+H_\Gamma\mathbf{v}\cdot\mathbf{n}_{\Gamma(t)}\Big)\,d\sigma+\int_{\Gamma_0(t)}(\mathbf{Tu}-\mathbf{q})\cdot\mathbf{n}_{\Gamma_0(t)}\,d\sigma.$$

Thus, in order to obtain (1.14), it is sufficient to assume that

$$\begin{aligned}[][[(\frac{\rho}{2}|\mathbf{u}|^2+\rho e)(\mathbf{u}-\mathbf{v})-(\mathbf{Tu}-\mathbf{q})]]\cdot\mathbf{n}_{\Gamma(t)}+\sigma H_\Gamma\mathbf{v}\cdot\mathbf{n}_{\Gamma(t)}&=0\quad\text{on }\Gamma(t),\\ (\mathbf{Tu}-\mathbf{q})\cdot\mathbf{n}_{\Gamma_0(t)}&=0\quad\text{on }\Gamma_0(t).\end{aligned}\tag{1.15}$$

Since $\mathbf{Tn}_{\Gamma_0(t)}=0$ on $\Gamma_0(t)$, we assume that $\mathbf{q}\cdot\mathbf{n}_{\Gamma_0(t)}=0$ on $\Gamma_0(t)$. Using (1.9) and (1.13), we have

$$\begin{aligned}&[[\frac{\rho}{2}|\mathbf{u}|^2(\mathbf{u}-\mathbf{v})]]\cdot\mathbf{n}_{\Gamma(t)}=\frac{\rho_1}{2}|\mathbf{u}_1|^2(\mathbf{u}_1-\mathbf{v})\cdot\mathbf{n}_{\Gamma(t)}-\frac{\rho_2}{2}|\mathbf{u}_2|^2(\mathbf{u}_2-\mathbf{v})\cdot\mathbf{n}_{\Gamma(t)}\\ &=\frac{\jmath}{2}|\mathbf{u}_1-\mathbf{v}+\mathbf{v}|^2-|\mathbf{u}_2-\mathbf{v}+\mathbf{v}|^2)=\frac{\jmath}{2}(|\mathbf{u}_1-\mathbf{v}|^2+2\mathbf{u}_1\cdot\mathbf{v}-|\mathbf{u}_2-\mathbf{v}|^2-2\mathbf{u}_2\cdot\mathbf{v})\\ &=\frac{\jmath}{2}([[|\mathbf{u}-\mathbf{v}|^2]]-2[[\mathbf{u}]]\cdot\mathbf{v})=\frac{\jmath}{2}[[|\mathbf{u}-\mathbf{v}|^2]]+\jmath[[\mathbf{u}]]\cdot\mathbf{v}\\ &=\frac{\jmath}{2}[[|\mathbf{u}-\mathbf{v}|^2]]+[[\mathbf{Tn}_{\Gamma(t)}]]\cdot\mathbf{v}-\sigma H_\Gamma\mathbf{v}\cdot\mathbf{n}_{\Gamma(t)}\end{aligned}$$

Since $[[\rho e(\mathbf{u}-\mathbf{v})]]\cdot\mathbf{n}_{\Gamma(t)}=\jmath[[e]]$, the first equation of (1.15) becomes:

$$-\sigma H_\Gamma\mathbf{v}\cdot\mathbf{n}_{\Gamma(t)}=\frac{\jmath}{2}[[|\mathbf{u}-\mathbf{v}|^2]]+[[\mathbf{Tv}]]\cdot\mathbf{n}_{\Gamma(t)}-\sigma H_\Gamma\mathbf{v}\cdot\mathbf{n}_{\Gamma(t)}+\jmath[[e]]-[[\mathbf{Tu}]]\cdot\mathbf{n}_{\Gamma(t)}+[[\mathbf{q}]]\cdot\mathbf{n}_{\Gamma(t)}$$

Thus, we have

$$\frac{\jmath}{2}[[|\mathbf{u}-\mathbf{v}|^2]]+\jmath[[e]]-[[\mathbf{T}(\mathbf{u}-\mathbf{v})]]\cdot\mathbf{n}_{\Gamma(t)}+[[\mathbf{q}]]\cdot\mathbf{n}_{\Gamma(t)}=0.$$

Moreover, using (1.4) and (1.5), we rewrite (1.6) as follows:

$$\begin{aligned}&\partial_t(\frac{\rho}{2}|\mathbf{u}|^2+\rho e)+\operatorname{div}((\frac{\rho}{2}|\mathbf{u}|^2+\rho e)\mathbf{u})-\operatorname{div}(\mathbf{Tu}-\mathbf{q})\\ &=(\frac{1}{2}|\mathbf{u}|^2+e)\partial_t\rho+\rho(\mathbf{u}\cdot\partial_t\mathbf{u}+\partial_t e)+(\frac{1}{2}|\mathbf{u}|^2+e)\operatorname{div}(\rho\mathbf{u})+\rho\mathbf{u}\cdot(\mathbf{u}\cdot\nabla\mathbf{u}+\nabla e)\\ &\quad-(\operatorname{div}\mathbf{T})\cdot\mathbf{u}-\mathbf{T}:\nabla\mathbf{u}+\operatorname{div}\mathbf{q}\\ &=(\frac{1}{2}|\mathbf{u}|^2+e)(\partial_t\rho+\operatorname{div}(\rho\mathbf{u}))+\mathbf{u}\cdot(\rho(\partial_t\mathbf{u}+\mathbf{u}\cdot\nabla\mathbf{u})-\operatorname{div}\mathbf{T})+\rho(\partial_t e+\mathbf{u}\cdot\nabla\mathbf{u})-\mathbf{T}:\nabla\mathbf{u}+\operatorname{div}\mathbf{q}.\end{aligned}$$

Here, we have set $\mathbf{T}:\nabla\mathbf{u}=\sum_{i,j=1}^N T_{ij}\partial_i u_j$. Thus, we have

$$\rho(\partial_t e+\mathbf{u}\cdot\nabla e)+\operatorname{div}\mathbf{q}-\mathbf{T}:\nabla\mathbf{u}=\rho r.$$

Summing up, we have obtained

$$\begin{aligned}\rho(\partial_t e+\mathbf{u}\cdot\nabla e)+\operatorname{div}\mathbf{q}-\mathbf{T}:\nabla\mathbf{u}&=\rho r\quad\text{in }\dot\Omega(t),\\ \frac{\jmath}{2}[[|\mathbf{u}-\mathbf{v}|^2]]+\jmath[[e]]-[[\mathbf{T}(\mathbf{u}-\mathbf{v})]]\cdot\mathbf{n}_{\Gamma(t)}+[[\mathbf{q}]]\cdot\mathbf{n}_{\Gamma(t)}&=0\quad\text{on }\Gamma(t),\\ \mathbf{q}\cdot\mathbf{n}_{\Gamma_0(t)}&=0\quad\text{on }\Gamma_0(t).\end{aligned}\tag{1.16}$$

The interface condition is still not enough. To find one more condition, we consider the entropy. For this purpose, we introduce

Constitutive Laws in the Phases

- Newton's law: The stress tensor $\mathbf{T}$ is given by

$$\mathbf{T} = 2\mu(\rho,\theta)\mathbf{D}(\mathbf{u}) + (\lambda(\rho,\theta) - \mu(\rho,\theta))\mathrm{div}\,\mathbf{u}\mathbf{I} - \pi\mathbf{I}$$

Here, $\mathbf{I}$ is the $N \times N$ identity matrix, μ and λ are in general C^∞ functions with respect to $(\rho,\theta) \in (0,\infty)\times(0,\infty)$ and we assume that

$$\mu > 0, \quad \lambda \geq \frac{N-2}{N}\mu. \tag{1.17}$$

But, to prove local well-posedness, it suffices to assume that $\mu > 0$ and $\lambda > 0$.

- Fourier's law: The heat flux $\mathbf{q}$ is given by

$$\mathbf{q} = -d(\rho,\theta)\nabla\theta. \tag{1.18}$$

Here, $d(\rho,\theta)$ is a positive C^∞ function with respect to $(\rho,\theta) \in (0,\infty)\times(0,\infty)$.

- the first law of thermodynamics: For the internal energy e and the entropy η for the unit mass,

$$de = \theta d\eta + \frac{\pi}{\rho^2}d\rho \tag{1.19}$$

- If we define the free energy ψ for the unit mass by

$$\psi = e - \theta\eta, \tag{1.20}$$

then, we have

$$d\psi = de - \theta d\eta - \eta d\theta = -\eta d\theta + \frac{\pi}{\rho^2}d\rho.$$

Thus,

$$\eta = -\frac{\partial\psi}{\partial\theta}, \quad \frac{\pi}{\rho^2} = \frac{\partial\psi}{\partial\rho} \tag{1.21}$$

- specific heat: $\kappa_v = \frac{\partial e}{\partial\theta}$ is obtained by

$$\kappa_v = \frac{\partial e}{\partial\theta} = \frac{\partial}{\partial\theta}(\psi + \theta\eta) = \frac{\partial\psi}{\partial\theta} + \eta + \theta\frac{\partial\eta}{\partial\theta} = -\theta\frac{\partial^2\psi}{\partial\theta^2}. \tag{1.22}$$

We assume that $e = e(\rho,\theta)$ and $\eta(\rho,\theta)$ are C^∞ functions with respect to $(\rho,\theta) \in (0,\infty)\times(0,\infty)$ and that κ_v is a positive C^∞ function with respect to $(\rho,\theta) \in (0,\infty)\times(0,\infty)$.

Next, we consider the law of entropy increase:

$$\frac{d\Phi}{dt} \geq 0 \tag{1.23}$$

with entropy: $\Phi = \int_{\dot\Omega(t)} \rho\eta\,dx$. By the first law of thermodynamics (1.19) and (1.4), we have

$$\begin{aligned}\partial_t e + \mathbf{u}\cdot\nabla e &= \frac{\partial e}{\partial\eta}(\partial_t\eta + \mathbf{u}\cdot\nabla\eta) + \frac{\partial e}{\partial\rho}(\partial_t\rho + \mathbf{u}\cdot\nabla\rho)\\ &= \theta(\partial_t\eta + \mathbf{u}\cdot\nabla\eta) - \frac{\pi}{\rho^2}\rho\mathrm{div}\,\mathbf{u} = \theta(\partial_t\eta + \mathbf{u}\cdot\nabla\eta) - \frac{\pi}{\rho}\mathrm{div}\,\mathbf{u}\end{aligned}$$

In addition, since

$$\mathbf{T} : \nabla\mathbf{u} = 2\mu|\mathbf{D}(\mathbf{u})|^2 + (\lambda - \mu)(\mathrm{div}\,\mathbf{u})^2 - \pi\mathrm{div}\,\mathbf{u}$$

by the first equation of (1.16) we have

$$\rho\theta(\partial_t\eta + \mathbf{u}\cdot\nabla\eta) - \mathrm{div}\,(d\nabla\theta) - (2\mu|\mathbf{D}(\mathbf{u})|^2 + (\lambda - \mu)(\mathrm{div}\,\mathbf{u})^2) = \rho r.$$

On the other hand, we have

$$\partial_t(\rho\eta) + \mathrm{div}\,(\rho\eta\mathbf{u}) = \eta(\partial_t\rho + \mathrm{div}\,(\rho\mathbf{u})) + \rho(\partial_t\eta + \mathbf{u}\cdot\nabla\eta) = \rho(\partial_t\eta + \mathbf{u}\cdot\nabla\eta).$$

In the following, we assume that $\theta > 0$. Combining these two equations, we have

$$\partial_t(\rho\eta) + \mathrm{div}\,(\rho\eta\mathbf{u}) = \frac{1}{\theta}\{\mathrm{div}\,(d\nabla\theta) + 2\mu|\mathbf{D}(\mathbf{u})|^2 + (\lambda-\mu)(\mathrm{div}\,\mathbf{u})^2 + \rho r\}. \tag{1.24}$$

When $r = 0$, we look for a sufficient condition to obtain (1.23). By the Reynolds transport theorem, (1.24) and the divergence theorem of Gauss, we have

$$\begin{aligned}
\frac{d}{dt}\Phi &= \frac{d}{dt}\int_{\dot\Omega(t)}\rho\eta\,dx = \int_{\dot\Omega(t)}\partial_t(\rho\eta)\,dx + \int_{\Gamma(t)}[[\rho\eta]]\mathbf{v}\cdot\mathbf{n}_{\Gamma(t)}\,d\sigma + \int_{\Gamma_0(t)}\rho\eta\mathbf{u}\cdot\mathbf{n}_{\Gamma_0(t)}\,d\sigma\\
&= -\int_{\dot\Omega(t)}\mathrm{div}\,(\rho\eta\mathbf{u})\,dx + \int_{\dot\Omega(t)}\frac{1}{\theta}\mathrm{div}\,(d\nabla\theta)\,dx + \int_{\dot\Omega(t)}(2\mu|\mathbf{D}(\mathbf{u})|^2 + (\lambda-\mu)(\mathrm{div}\,\mathbf{u})^2)\,dx\\
&\quad + \int_{\Gamma(t)}[[\rho\eta]]\mathbf{v}\cdot\mathbf{n}_{\Gamma(t)}\,d\sigma + \int_{\Gamma_0(t)}\rho\eta\mathbf{u}\cdot\mathbf{n}_{\Gamma_0(t)}\,d\sigma\\
&= -\int_{\Gamma(t)}[[(\rho\eta)(\mathbf{u}-\mathbf{v})]]\cdot\mathbf{n}_{\Gamma(t)}\,d\sigma + \int_{\Gamma(t)}[[\frac{d}{\theta}\nabla\theta]]\cdot\mathbf{n}_{\Gamma(t)}\,d\sigma + \int_{\Gamma_0(t)}\frac{1}{\theta}(d\nabla\theta\cdot\mathbf{n}_{\Gamma_0(t)})\,d\sigma\\
&\quad + \int_{\dot\Omega(t)}\{\frac{d|\nabla\theta|^2}{\theta^2} + 2\mu|\mathbf{D}(\mathbf{u})|^2 + (\lambda-\mu)(\mathrm{div}\,\mathbf{u})^2\}\,dx.
\end{aligned}$$

Thus, to obtain (1.23), first we have the following sufficient conditions:

$$\begin{aligned}
[[(\rho\eta)(\mathbf{u}-\mathbf{v}) - \frac{d}{\theta}\nabla\theta]]\cdot\mathbf{n}_{\Gamma(t)} &= 0 \quad \text{on } \Gamma(t),\\
d\nabla\theta\cdot\mathbf{n}_{\Gamma_0(t)} &= 0 \quad \text{on } \Gamma_0(t).
\end{aligned} \tag{1.25}$$

Moreover,

$$(\mathrm{div}\,\mathbf{u})^2 = \Big(\sum_{j=1}^N\partial_i u_i)\Big)^2 = \sum_{i,j=1}^N(\partial_i u_i)(\partial_j u_j) \le N\sum_{j=1}^N(\partial_j u_j)^2 = N\sum_{i=1}^N\mathbf{D}_{jj}(\mathbf{u}) \le N|\mathbf{D}(\mathbf{u})|^2. \tag{1.26}$$

Since $\lambda - \frac{N-2}{N}\mu \ge 0$ as follows from (1.17), we have

$$2\mu|\mathbf{D}(\mathbf{u})|^2 + (\lambda-\mu)(\mathrm{div}\,\mathbf{u})^2 \ge (\frac{2}{N}\mu + \lambda - \mu)(\mathrm{div}\,\mathbf{u})^2 = (\lambda - \frac{N-2}{N}\mu)(\mathrm{div}\,\mathbf{u})^2 \ge 0,$$

which is the reason why we need to assume that $\lambda \ge \frac{N-2}{N}\mu$ in (1.17).

Next, assuming that $[[\theta]] = 0$ and using (1.9), we rewrite the first condition of (1.25). We observe that

$$\begin{aligned}
0 &= [[(\rho\eta)(\mathbf{u}-\mathbf{v}) - \frac{d}{\theta}\nabla\theta]]\cdot\mathbf{n}_{\Gamma(t)}\\
&= (\rho_1\eta_1)(\mathbf{u}_1-\mathbf{v})\cdot\mathbf{n}_{\Gamma(t)} - (\rho_2\eta_2)(\mathbf{u}_2-\mathbf{v})\cdot\mathbf{n}_{\Gamma(t)} - \frac{(d_1\nabla\theta_1 - d_2\nabla\theta_2)\cdot\mathbf{n}_{\Gamma(t)}}{\theta}\\
&= \frac{\jmath(\eta_1-\eta_2)\theta - (d_1\nabla\theta_1 - d_2\nabla\theta_2)\cdot\mathbf{n}_{\Gamma(t)}}{\theta} = \frac{1}{\theta}(\jmath[[\theta\eta]] - [[d\nabla\theta]]\cdot\mathbf{n}_{\Gamma(t)})
\end{aligned}$$

Thus, we have

$$[[\theta]] = 0, \quad \jmath[[\theta\eta]] - [[d\nabla\theta]]\cdot\mathbf{n}_{\Gamma(t)} = 0 \quad \text{on } \Gamma(t) \tag{1.27}$$

the second formula of which is called the Stefan law. In particular, when $\jmath = 0$, this is the usual jump condition $[[d\nabla\theta]]\cdot\mathbf{n}_{\Gamma(t)} = 0$ on $\Gamma(t)$.

Finally, assuming that $\jmath \neq 0$ and $[[\rho]] \neq 0$, and using (1.27), we rewrite (1.16). Let τ_i $(i = 1, \ldots, N-1)$ be the tangent vectors of $\Gamma(t)$, and therefore we write $\mathbf{u}-\mathbf{v} = ((\mathbf{u}-\mathbf{v})\cdot\mathbf{n}_{\Gamma(t)})\mathbf{n}_{\Gamma(t)} + \sum_{i=1}^{N-1}((\mathbf{u}-\mathbf{v})\cdot\tau_i)\tau_i$. Using the orthogonality of $\{\tau_1, \ldots, \tau_{N-1}, \mathbf{n}_{\Gamma(t)}\}$, we have

$$|\mathbf{u}-\mathbf{v}|^2 = |(\mathbf{u}-\mathbf{v})\cdot\mathbf{n}_{\Gamma(t)}|^2 + \sum_{i=1}^{N-1}|(\mathbf{u}-\mathbf{v})\cdot\tau_i|^2.$$

We assume that

$$[[\mathbf{u}\cdot\tau_i]] = [[(\mathbf{u}-\mathbf{v})\cdot\tau_i]] = 0. \tag{1.28}$$

Then, by (1.9) we have

$$\begin{aligned}
[[\frac{1}{2}|\mathbf{u}-\mathbf{v}|^2]] &= \frac{1}{2}[[|(\mathbf{u}-\mathbf{v})\cdot\mathbf{n}_{\Gamma(t)}|^2]] = \frac{1}{2}(|(\mathbf{u}_1-\mathbf{v})\cdot\mathbf{n}_{\Gamma(t)}|^2 - |(\mathbf{u}_2-\mathbf{v})\cdot\mathbf{n}_{\Gamma(t)}|^2)\\
&= \frac{1}{2}((\frac{\jmath}{\rho_1})^2 - (\frac{\jmath}{\rho_2})^2) = \frac{\jmath^2}{2}[[\frac{1}{\rho^2}]],
\end{aligned}$$

which implies that

$$\frac{\jmath}{2}|\mathbf{u}-\mathbf{v}|^2]] = \jmath^3[[\frac{1}{2\rho}]].$$

Since $e = \psi + \theta\eta$, we have $[[e]] = [[\psi]] + [[\theta\eta]]$. Thus, using the Stefan law (1.27), we rewrite the jump condition in (1.16) as follows:

$$\begin{aligned}
0 &= \frac{\jmath}{2}[[|\mathbf{u}-\mathbf{v}|^2]] - [[\mathbf{u}-\mathbf{v})\mathbf{T}\mathbf{n}_{\Gamma(t)}]] + \jmath[[e]] - [[d\nabla\theta]]\cdot\mathbf{n}_{\Gamma(t)}\\
&= \jmath^3[[\frac{1}{2\rho^2}]] - [[(\mathbf{u}-\mathbf{v})\mathbf{T}\mathbf{n}_{\Gamma(t)}]] + \jmath[[\psi]] + \jmath[[\theta\eta]] - \jmath[[\theta\eta]]\\
&= \jmath^3[[\frac{1}{2\rho^2}]] - [[(\mathbf{u}-\mathbf{v})\mathbf{T}\mathbf{n}_{\Gamma(t)}]] + \jmath[[\psi]].
\end{aligned}$$

Moreover, the second term is rewritten as follows:

$$\begin{aligned}
[[(\mathbf{u}-\mathbf{v})\mathbf{T}\mathbf{n}_{\Gamma(t)}]] &= [[(\mathbf{u}-\mathbf{v})\cdot\mathbf{n}_{\Gamma(t)})\mathbf{n}_{\Gamma(t)}\mathbf{T}\mathbf{n}_{\Gamma(t)}]] + \sum_{i=1}^{N-1}[[((\mathbf{u}-\mathbf{v})\cdot\tau_i)\tau_i\mathbf{T}\mathbf{n}_{\Gamma(t)}]]\\
&= ((\mathbf{u}_1-\mathbf{v})\cdot\mathbf{n}_{\Gamma(t)})\mathbf{n}_{\Gamma(t)}\mathbf{T}_1\mathbf{n}_{\Gamma(t)} - ((\mathbf{u}_2-\mathbf{v})\cdot\mathbf{n}_{\Gamma(t)})\mathbf{n}_{\Gamma(t)}\mathbf{T}_2\mathbf{n}_{\Gamma(t)}\\
&\quad + \sum_{i=1}^{N-1}((\mathbf{u}_1-\mathbf{v})\cdot\tau_i)\tau_i\mathbf{T}_1\mathbf{n}_{\Gamma(t)} - ((\mathbf{u}_2-\mathbf{v})\cdot\tau_i)\tau_i\mathbf{T}_2\mathbf{n}_{\Gamma(t)}
\end{aligned}$$

By (1.9) we have

$$\begin{aligned}
((\mathbf{u}_1-\mathbf{v})\cdot\mathbf{n}_{\Gamma(t)})\mathbf{n}_{\Gamma(t)}\mathbf{T}_1\mathbf{n}_{\Gamma(t)} - ((\mathbf{u}_2-\mathbf{v})\cdot\mathbf{n}_{\Gamma(t)})\mathbf{n}_{\Gamma(t)}\mathbf{T}_2\mathbf{n}_{\Gamma(t)} &= \jmath(\frac{\mathbf{n}_{\Gamma(t)}\mathbf{T}_1\mathbf{n}_{\Gamma(t)}}{\rho_1} - \frac{\mathbf{n}_{\Gamma(t)}\mathbf{T}_2\mathbf{n}_{\Gamma(t)}}{\rho_2})\\
&= \jmath[[\frac{1}{\rho}\mathbf{n}_{\Gamma(t)}\mathbf{T}\mathbf{n}_{\Gamma(t)}]].
\end{aligned}$$

On the other hand, by (1.13) and (1.28), we have

$$\begin{aligned}
&((\mathbf{u}_1-\mathbf{v})\cdot\tau_i)\tau_i\mathbf{T}_1\mathbf{n}_{\Gamma(t)} - ((\mathbf{u}_2-\mathbf{v})\cdot\tau_i)\tau_i\mathbf{T}_2\mathbf{n}_{\Gamma(t)} = (\mathbf{u}_1-\mathbf{v})\cdot\tau_i)\tau_i\cdot[[\mathbf{T}\mathbf{n}_{\Gamma(t)}]]\\
&\quad = (\mathbf{u}_1-\mathbf{v})\cdot\tau_i)(\jmath\tau_i\cdot[[\mathbf{u}]] - \sigma H_\Gamma(\tau_i\cdot\mathbf{n}_{\Gamma(t)})) = 0.
\end{aligned}$$

Here, we have used the formula $[[\mathbf{u}]] = [[\mathbf{u}\cdot\mathbf{n}_{\Gamma(t)}]]\mathbf{n}_{\Gamma(t)}$ which follows from (1.28).

Summing up, we have obtained

$$0 = \jmath([[\psi]] + \jmath^2[[\frac{1}{2\rho^2}]] - [[\frac{1}{\rho}\mathbf{n}_{\Gamma(t)}\mathbf{T}\mathbf{n}_{\Gamma(t)}]]).$$

Since $\jmath \neq 0$, finally we arrive at the condition:

$$[[\psi]] + \jmath^2[[\frac{1}{2\rho^2}]] - [[\frac{1}{\rho}\mathbf{n}_{\Gamma(t)}\mathbf{T}\mathbf{n}_{\Gamma(t)}]] = 0 \quad \text{on } \Gamma(t). \tag{1.29}$$

This is called the generalized Gibbs-Thomson law.

Next, we calculate $V_\Gamma := \mathbf{v}\cdot\mathbf{n}_{\Gamma(t)}$. By (1.9) we have $\mathbf{v}\cdot\mathbf{n}_{\Gamma(t)} = \mathbf{u}_1\cdot\mathbf{n}_{\Gamma(t)} - \frac{j}{\rho_1}$. When $\jmath = 0$, we have $[[\mathbf{u}]] = 0$, so that $\mathbf{v}\cdot\mathbf{n}_{\Gamma(t)} = \mathbf{u}\cdot\mathbf{n}$.

When $\jmath \neq 0$ and $[[\rho]] \neq 0$ by (1.10) we have $\jmath = \frac{[[\mathbf{u}]]\cdot\mathbf{n}_{\Gamma(t)}}{[[1/\rho]]}$, so that

$$\begin{aligned}
\mathbf{v}\cdot\mathbf{n}_{\Gamma(t)} &= \mathbf{u}_1\cdot\mathbf{n}_{\Gamma(t)} - \frac{j}{\rho_1}\\
&= \mathbf{u}_1\cdot\mathbf{n}_{\Gamma(t)} - \frac{1}{\rho_1}\frac{\mathbf{u}_1\cdot\mathbf{n}_{\Gamma(t)} - \mathbf{u}_2\cdot\mathbf{n}_{\Gamma(t)}}{1/\rho_1 - 1/\rho_2} = \mathbf{u}_1\cdot\mathbf{n}_{\Gamma(t)} - \frac{\rho_2(\mathbf{u}_1\cdot\mathbf{n}_{\Gamma(t)} - \mathbf{u}_2\cdot\mathbf{n}_{\Gamma(t)})}{\rho_2 - \rho_1}\\
&= \frac{\rho_2\mathbf{u}_1\cdot\mathbf{n}_{\Gamma(t)} - \rho_1\mathbf{u}_1\cdot\mathbf{n}_{\Gamma(t)} - \rho_2\mathbf{u}_1\cdot\mathbf{n}_{\Gamma(t)} + \rho_2\mathbf{u}_2\cdot\mathbf{n}_{\Gamma(t)}}{\rho_2 - \rho_1}\\
&= \frac{[[\rho\mathbf{u}]]\cdot\mathbf{n}_{\Gamma(t)}}{[[\rho]]}.
\end{aligned}$$

Summing up, we have obtained

$$\begin{aligned}
V_\Gamma := \mathbf{v}\cdot\mathbf{n}_{\Gamma(t)} &= \mathbf{u}\cdot\mathbf{n}_{\Gamma(t)} \quad (\jmath = 0),\\
V_\Gamma := \mathbf{v}\cdot\mathbf{n}_{\Gamma(t)} &= \frac{[[\rho\mathbf{u}]]\cdot\mathbf{n}_{\Gamma(t)}}{[[\rho]]} \quad (\jmath \neq 0 \text{ and } [[\rho]] \neq 0).
\end{aligned} \tag{1.30}$$

Next, we consider the case where $\jmath \neq 0$ and $[[\rho]] = 0$. In this case, $[[\mathbf{u}]]\cdot\mathbf{n}_{\Gamma(t)} = 0$, which combined with (1.28) furnishes that $[[\mathbf{u}]] = 0$, so that (1.13) is written as follows:

$$[[\mathbf{T}\mathbf{n}_{\Gamma(t)}]] = \sigma H_\Gamma \mathbf{n}_{\Gamma(t)} \quad \text{on } \Gamma(t).$$

To derive (1.29), we assume that $[[\rho]] \neq 0$, so that we reconsider the second condition of (1.16). By $[[\mathbf{u}]] = 0$, $[[|\mathbf{u}-\mathbf{v}|^2]] = 0$. By the Stefan law (1.27), we have $\jmath[[e]] + [[\mathbf{q}]]\cdot\mathbf{n}_{\Gamma(t)} = \jmath[[\psi]]$. By (1.13) we have $[[\mathbf{T}\mathbf{n}_{\Gamma(t)}]] = \sigma H_\Gamma\mathbf{n}_{\Gamma(t)}$, so that

$$\begin{aligned}
[[(\mathbf{u}-\mathbf{v})\mathbf{T}\mathbf{n}_{\Gamma(t)}]] &= (\mathbf{u}_1 - \mathbf{v})\cdot\mathbf{T}_1\mathbf{n}_{\Gamma(t)} - (\mathbf{u}_2 - \mathbf{v})\cdot\mathbf{T}_1\mathbf{n}_{\Gamma(t)} = (\mathbf{u}_1 - \mathbf{v})\cdot[[\mathbf{T}\mathbf{n}_{\Gamma(t)}]] = (\mathbf{u}_1 - \mathbf{v})\cdot\sigma H_\Gamma\mathbf{n}_{\Gamma(t)}\\
&= \jmath\frac{\sigma}{\rho_1}H_\Gamma.
\end{aligned}$$

Dividing the above formula by $\jmath \neq 0$ and using (1.16), we have

$$[[\psi]] - \frac{\sigma}{\rho_1}H_\Gamma = 0. \quad \text{on } \Gamma(t). \tag{1.31}$$

Finally, using the facts $\frac{\partial e}{\partial\theta} = \kappa_v$ and $\partial_t\rho + \mathbf{u}\cdot\nabla\rho = -\rho\,\mathrm{div}\,\mathbf{u}$ we rewrite (1.6) as follows:

$$\begin{aligned}
\rho r + (2\mu|\mathbf{D}(\mathbf{u})|^2 + (\lambda-\mu)(\mathrm{div}\,\mathbf{u})^2) - \pi\,\mathrm{div}\,\mathbf{u} + \mathrm{div}\,(d\nabla\theta) &= \rho(\partial_t e + \mathbf{u}\cdot\nabla e)\\
&= \rho(\frac{\partial e}{\partial\theta}\partial_t\theta + \frac{\partial e}{\partial\rho}\partial_t\rho + \frac{\partial e}{\partial\theta}\mathbf{u}\cdot\nabla\theta + \frac{\partial e}{\partial\rho}\mathbf{u}\cdot\nabla\rho)\\
&= \rho\kappa(\partial_t\theta + \mathbf{u}\cdot\nabla\theta) - \rho^2\frac{\partial e}{\partial\rho}\mathrm{div}\,\mathbf{u}.
\end{aligned}$$

Thus, we have

$$\rho\kappa_v(\partial_t\theta + \mathbf{u}\cdot\nabla\theta) - \mathrm{div}\,(d\nabla\theta) - (2\mu|\mathbf{D}(\mathbf{u})|^2 + (\lambda-\mu)(\mathrm{div}\,\mathbf{u})^2) + (\pi - \rho^2\frac{\partial e}{\partial\rho})\mathrm{div}\,\mathbf{u} = \rho r. \tag{1.32}$$

Summing up, we have the following Model equations:

Equations in $\dot{\Omega}(t)$

$$
\begin{aligned}
&\partial_t \rho + \mathrm{div}\,(\rho \mathbf{u}) = 0, \\
&\rho(\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u}) - \mathrm{div}\,\mathbf{T} = \rho f, \\
&\rho\kappa_v(\partial_t\theta + \mathbf{u}\cdot\nabla\theta) - \mathrm{div}\,(d\nabla\theta) - (2\mu|\mathbf{D}(\mathbf{u})|^2 + (\lambda-\mu)(\mathrm{div}\,\mathbf{u})^2) + (\pi - \rho^2\frac{\partial e}{\partial \rho})\mathrm{div}\,\mathbf{u} = \rho r.
\end{aligned}
\tag{1.33}
$$

Boudary condition on $\Gamma_0(t)$

$$
\mathbf{T}\mathbf{n}_{\Gamma_0(t)} = 0, \quad d\nabla\theta\cdot\mathbf{n}_{\Gamma_0(t)} = 0 \quad \text{on } \Gamma_0(t). \tag{1.34}
$$

Interface condition on $\Gamma(t)$.

- When $\jmath = 0$,

$$
\begin{aligned}
&[[\mathbf{u}]] = 0, \quad [[\mathbf{T}\mathbf{n}_{\Gamma(t)}]] = \sigma H_\Gamma \mathbf{n}_{\Gamma(t)}, \\
&[[\theta]] = 0, \quad [[d\nabla\theta\cdot\mathbf{n}_{\Gamma(t)}]] = 0, \\
&V_\Gamma := \mathbf{v}\cdot\mathbf{n}_{\Gamma(t)} = \mathbf{u}\cdot\mathbf{n}_{\Gamma(t)}.
\end{aligned}
\tag{1.35}
$$

- When $\jmath \neq 0$ and $[[\rho]] \neq 0$,

$$
\begin{aligned}
&\mathcal{T}_{\mathbf{n}_{\Gamma(t)}}[[\mathbf{u}]] = 0, \quad \jmath[[\mathbf{u}]] - [[\mathbf{T}\mathbf{n}_\Gamma]] = -\sigma H_\Gamma \mathbf{n}_{\Gamma(t)}, \quad [[\psi]] + \jmath^2[[\frac{1}{2\rho^2}]] - [[\frac{1}{\rho}\mathbf{n}_{\Gamma(t)}\mathbf{T}\mathbf{n}_{\Gamma(t)}]] = 0, \\
&[[\theta]] = 0, \quad \jmath[[\theta\eta]] - [[d\nabla\theta\cdot\mathbf{n}_{\Gamma(t)}]] = 0, \\
&V_\Gamma := \mathbf{v}\cdot\mathbf{n}_{\Gamma(t)} = \frac{[[\rho\mathbf{u}]]\cdot\mathbf{n}_{\Gamma(t)}}{[[\rho]]}, \\
&\jmath = \frac{[[\mathbf{u}]]\cdot\mathbf{n}_{\Gamma(t)}}{[[1/\rho]]}.
\end{aligned}
\tag{1.36}
$$

  Here, $\mathcal{T}_{\mathbf{n}_{\Gamma(t)}}\mathbf{w} = \mathbf{w} - (\mathbf{w}\cdot\mathbf{n}_{\Gamma(t)})\mathbf{n}_{\Gamma(t)}$ for any N vector $\mathbf{w} = (w_1, \dots, w_N)$ (the tangential component of $\mathbf{w}$ along $\mathbf{n}_{\Gamma(t)}$).

- When $\jmath \neq 0$ and $\rho = \rho_1 = \rho_2$ (constants),

$$
\begin{aligned}
&[[\mathbf{u}]] = 0, \quad [[\mathbf{T}\mathbf{n}_\Gamma]] = \sigma H_\Gamma \mathbf{n}_{\Gamma(t)}, \\
&[[\theta]] = 0, \quad \jmath[[\theta\eta]] - [[d\nabla\theta\cdot\mathbf{n}_{\Gamma(t)}]] = 0, \quad \rho[[\psi]] - \sigma H_\Gamma = 0, \\
&V_\Gamma := \mathbf{v}\cdot\mathbf{n}_{\Gamma(t)} = \mathbf{u}\cdot\mathbf{n}_{\Gamma(t)} - j/\rho.
\end{aligned}
\tag{1.37}
$$

Remark 2. Assuming that $\Omega_- = \Omega$ and $\Omega_+ = \emptyset$, we have the one phase problem. In this case, as boundary conditions on $\Gamma_0(t)$, we have

$$
\mathbf{T}\mathbf{n}_{\Gamma_0(t)} = \sigma H_{\Gamma_0}\mathbf{n}_{\Gamma(t)}, \quad d\nabla\theta\cdot\mathbf{n}_{\Gamma_0(t)} = 0 \quad \text{on } \Gamma_0(t).
$$

2 Problem

In this talk, we consider the compressible-incompressible phase transition problem in $\mathbb{R}^N$ with nearly flat interface. Let $h_0(x')$ be a function with respect to $x' = (x_1, \dots, x_{N-1})$ and we set

$$
\begin{aligned}
\Omega_\pm &= \{x = (x_1, \dots, x_N) \in \mathbb{R}^N \mid \pm x_N > h_0(x') \quad \text{for } x' \in \mathbb{R}^{N-1}\}, \\
\Gamma &= \{x \in \mathbb{R}^N \mid x_N = h(x') \quad \text{for } x' \in \mathbb{R}^{N-1}\}.
\end{aligned}
$$

In this case, $\Omega = \mathbb{R}^N$ and $\Gamma_0 = \emptyset$. Let $h(x', t)$ be a unknown function and the time evolution of domains $\Omega_\pm$ and the surface Γ is given by

$$
\begin{aligned}
\Omega_\pm(t) &= \{x = (x_1, \dots, x_N) \in \mathbb{R}^N \mid \pm(x_N - h(x', t)) > 0 \quad \text{for } x' \in \mathbb{R}^{N-1}\}, \\
\Gamma(t) &= \{x \in \mathbb{R}^N \mid x_N = h(x', t) \quad \text{for } x' \in \mathbb{R}^{N-1}\}.
\end{aligned}
\tag{2.1}
$$

In this case, $\mathbf{n}_{\Gamma(t)} = (-\nabla' h, 1)/\sqrt{1+|\nabla' h|^2}$.

We consider the following problem:

$$\begin{cases} \rho_+(\partial_t \mathbf{u}_+ + \mathbf{u}_+\cdot\nabla\mathbf{u}_+) - \mathrm{Div}\,\mathbf{T}_+ = 0, \quad \partial_t\rho_+ + \mathrm{div}\,(\rho_+\mathbf{u}_+) = 0 & \text{in } \Omega_+(t), \\ \rho_+\kappa_+(\partial_t\theta_+ + \mathbf{u}_+\cdot\nabla\theta_+) - \mathrm{div}\,(d_+\nabla\theta_+) - \mathbf{T}_+ : \nabla\mathbf{u}_+ - \rho^2\frac{\partial e_+}{\partial\rho}\mathrm{div}\,\mathbf{u}_+ = 0 & \text{in } \Omega_+(t), \\ \rho_{*-}(\partial_t\mathbf{u}_- + \mathbf{u}_-\cdot\nabla\mathbf{u}_-) - \mathrm{Div}\,\mathbf{T}_- = 0, \quad \mathrm{div}\,\mathbf{u}_- = 0 & \text{in } \Omega_-(t), \\ \rho_{*-}\kappa_-(\partial_t\theta_- + \mathbf{u}_-\cdot\nabla\theta_-) - \mathrm{div}\,(d_-\nabla\theta_-) - \mathbf{T}_- : \nabla\mathbf{u}_- = 0 & \text{in } \Omega_-(t), \end{cases} \tag{2.2}$$

subject to the jump conditions: for $x \in \Gamma(t)$ and $t > 0$,

$$\begin{cases} [[\frac{1}{\rho}]]\jmath^2\mathbf{n}_\Gamma - [[\mathbf{T}\mathbf{n}_\Gamma]] = -\sigma H_\Gamma\mathbf{n}_\Gamma, & \mathcal{T}_{\mathbf{n}_{\Gamma(t)}}[[\mathbf{u}]] = 0, \\ \jmath[[\theta\eta]] - [[d\frac{\partial\theta}{\partial\mathbf{n}_\Gamma}]] = 0, & [[\theta]] = 0, \\ [[\psi]] + [[\frac{1}{2\rho^2}]]\jmath^2 - [[\frac{1}{\rho}\mathbf{n}_\Gamma\cdot\mathbf{T}\mathbf{n}_\Gamma]] = 0, & \partial_t h = \dfrac{[[\rho\mathbf{u}]]\cdot(-\nabla' h, 1)}{[[\rho]]}, \\ \quad \jmath = \dfrac{[[\rho\mathbf{u}]]\mathbf{n}_{\Gamma(t)}}{[[\rho]]}, \end{cases} \tag{2.3}$$

and the initial conditions:

$$\rho_+|_{t=0} = \rho_{*+} + \rho_{0+} \ \text{ in } \Omega_+, \ \ (\mathbf{u}_\pm, \theta_\pm)|_{t=0} = (\mathbf{u}_{0\pm}, \theta_* + \theta_{0\pm}) \ \text{ in } \Omega_\pm, \ \ h|_{t=0} = h_0 \ \text{ on } \mathbb{R}^{N-1}. \tag{2.4}$$

Here, $\rho_{*\pm}$ are positive constants describing the reference mass densities of $\Omega_\pm$, θ_* is a positive constant, σ is a positive constant describing the coefficient of the surface tension, $\mathbf{T}_\pm = \mathbf{S}_\pm - \pi_\pm\mathbf{I}$ with

$$\mathbf{S}_+ = \mathbf{S}_+(\mathbf{u}_+, \rho_+, \theta_+)) = \mu_+\mathbf{D}(\mathbf{u}_+) + (\lambda_+ - \mu_+)\mathrm{div}\,\mathbf{u}\mathbf{I}, \quad \mathbf{S}_- = \mathbf{S}_-(\mathbf{u}, \theta_-) = \mu_-\mathbf{D}(\mathbf{u}_-),$$

and $\mu_+ = \mu_+(\rho_+, \theta_+)$, $\lambda_+ = \lambda_+(\rho_+, \theta_+)$ and $\mu_- = \mu_-(\theta_-)$ are viscosity coefficient. Several quantities are derived from the specific free energy $\psi_+ = \psi_+(\rho, \theta)$ and $\psi_- = \psi_-(\theta)$ as follows:

- $e_\pm = \psi_\pm + \theta\eta_\pm$ the internal energy,
- $\eta_\pm = -\dfrac{\partial\psi_\pm}{\partial\theta}$ the entropy,
- $\kappa_\pm = -\theta\dfrac{\partial^2\psi_\pm}{\partial\theta^2}$ the heat capacity.

Here, $\psi_+(\theta, \rho)$ is a real valued C^∞ function with respect to $(\rho, \theta) \in (0,\infty)\times(0,\infty)$ and $\psi_-(\theta)$ a real valued C^∞ function with respect to $\theta \in (0,\infty)$. We assume that

$$\kappa_+(\rho,\theta) > 0 \ \text{ for any } (\rho,\theta) \in (0,\infty)\times(0,\infty), \quad \kappa_-(\theta) > 0 \ \text{ for any } \theta \in (0,\infty).$$

We also assume that π_+ is given by $\pi_+ = P_+(\rho,\theta)$, where P_+ is some C^∞ function with respect to $(\rho,\theta) \in (0,\infty)\times(0,\infty)$ such that $\dfrac{\partial P_+}{\partial\rho} > 0$ for any $(\rho,\theta) \in (0,\infty)\times(0,\infty)$. Finally, $d_+ = d_+(\rho,\theta)$, $\mu_+ = \mu_+(\rho,\theta)$, $\lambda_+ = \lambda_+(\rho,\theta)$ are positive C^∞ functions with respect to $(\rho,\theta) \in (0,\infty)\times(0,\infty)$, and $d_- = d_-(\theta)$ and $\mu_- = \mu_-(\theta)$ are positive C^∞ functions with respect to $\theta \in (0,\infty)$.

I will talk about the local wellposedness of problem problem (2.2), (2.3) and (2.4). To state our main result, we transform $\Gamma(t)$ to the flat interface. Set

$$\mathbb{R}^N_\pm = \{x = (x_1, \ldots, x_N) \in \mathbb{R}^N \mid \pm x_N > 0\}, \quad \mathbb{R}^N_0 = \{x \in \mathbb{R}^N \mid x_N = 0\}.$$

We transfer the problem given in domains $\Omega_\pm(t)$ to that in $\dot{\mathbb{R}}^N = \mathbb{R}^N_+ \cup \mathbb{R}^N_-$ with interface $\mathbb{R}^N_0$. Let $h(x', t)$ be a function appearing in the definition of $\Gamma(t)$ in (2.1). Let $H(x, t)$ be a solution to the equations:

$(1-\Delta)H = 0$ in $\mathbb{R}^N$ with $H|_{x_N=0} = h(x', t)$, where $\Delta H = \sum_{j=1}^N \partial_j^2 H$. To prove the local well-posedness, we assume that h_0 is small enough, so that we may assume that

$$1 + \frac{\partial}{\partial x_N} H(x,t) \geq \frac{1}{2} \quad \text{for any } x \in \mathbb{R}^N \text{ and } t \in (0,T). \tag{2.5}$$

If we consider the transformation:

$$y_N = x_N + H(x,t), \quad y_j = x_j \ \ (j = 1, \ldots, N-1), \tag{2.6}$$

then by (2.5) $\Omega_\pm(t)$ and $\Gamma(t)$ are transformed to $\mathbb{R}^N_\pm$ and $\mathbb{R}^N_0$, respectively, because $y_N = h(y', t)$ when $x_N = 0$ and $\frac{\partial y_N}{\partial x_N} = 1 + (\frac{\partial H}{\partial x_N})(x,t) \geq \frac{1}{2}$. Let $\mathbf{u}_\pm$, ρ_+, π_- and $\theta_\pm$ satisfy problem (2.2), (2.3) and (2.4). Set

$$\begin{aligned}
\hat{\mathbf{u}}_\pm(x,t) &= \mathbf{u}_\pm(x', x_N + H(x,t), t), \quad \hat\rho_+(x,t) = \rho_+(x', x_N + H(x,t), t) - \rho_{*+}, \\
\hat\pi_-(x,t) &= \pi_-(x', x_N + H(x,t), t) - \pi_{*-}, \quad \hat\theta_\pm(x,t) = \theta_\pm(x', x_N + H(x,t), t) - \theta_*, \\
\mu_{*+} &= \mu_+(\rho_{*+}, \theta_*), \quad \lambda_{*+} = \lambda_+(\rho_{*+}, \theta_*), \quad \mu_{*-} = \mu_-(\theta_*), \\
\kappa_{*+} &= \kappa_+(\rho_{*+}, \theta_*), \quad \kappa_{*-} = \kappa_-(\theta_*), \quad d_{*+} = d_+(\rho_{*+}, \theta_*), \quad d_{*-} = d_-(\theta_*), \\
\hat\mu_+ &= \mu_+(\hat\rho_+ + \rho_{*+}, \hat\theta_+ + \theta_*) - \mu_+(\rho_{*+}, \theta_*), \quad \hat\mu_- = \mu_-(\hat\theta_- + \theta_*) - \mu_-(\theta_*), \\
\hat\lambda_+ &= \lambda_+(\hat\rho_+ + \rho_{*+}, \hat\theta_+ + \theta_*) - \lambda_+(\rho_{*+}, \theta_*), \\
\hat\kappa_+ &= \kappa_+(\hat\rho_+ + \rho_{*+}, \hat\theta_+ + \theta_*) - \kappa_+(\rho_{*+}, \theta_*), \quad \hat\kappa_- = \kappa_-(\hat\theta_- + \theta_*) - \kappa_-(\theta_*), \\
\hat d_+ &= d_+(\hat\rho_+ + \rho_{*+}, \hat\theta_+ + \theta_*) - d_+(\rho_{*+}, \theta_*), \quad \hat d_- = d_-(\hat\theta_- + \theta_*) - d_-(\theta_*).
\end{aligned}$$

Setting $H_0 = \partial_t H$, $H_j = \partial_j H$ $(j = 1, \ldots, N)$, we have

$$\begin{aligned}
(\partial_t f)(x', x_N + H(x,t), t) &= \partial_t \hat f(x,t) - \frac{H_0}{1 + H_N} \partial_N \hat f(x,t), \\
(\partial_j f)(x', x_N + H(x,t), t) &= \partial_j \hat f(x,t) - \frac{H_j}{1 + H_N} \partial_N \hat f(x,t) \quad (j = 1, \ldots, N).
\end{aligned} \tag{2.7}$$

In the following, we set

$$K_j = \frac{H_j}{1 + H_N} \ \ (j = 0, 1, \ldots, N), \quad \mathbf{K} = (K_1, \ldots, K_N), \quad \mathbf{K}_0 = (K_0, \mathbf{K}).$$

By (2.7) we have

$$\nabla \pi_- = Q \nabla \hat\pi_- = \begin{pmatrix} 1 & 0 & \cdots & 0 & K_1 \\ \vdots & & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & K_{N-1} \\ 0 & 0 & \cdots & 0 & \frac{1}{1+H_N} \end{pmatrix} \begin{pmatrix} \partial_1 \hat\pi_- \\ \vdots \\ \\ \partial_N \hat\pi_- \end{pmatrix},$$

and Q^{-1} is given by

$$Q^{-1} = \begin{pmatrix} 1 & 0 & \cdots & 0 & -H_1 \\ \vdots & & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -H_{N-1} \\ 0 & 0 & \cdots & 0 & 1 + H_N \end{pmatrix} = \mathbf{I} + \mathbf{Q}_1 \quad \text{with } Q_1 = \begin{pmatrix} 0 & \cdots & 0 & -H_1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & -H_{N-1} \\ 0 & \cdots & 0 & H_N \end{pmatrix}.$$

By (2.7) we have

$$\begin{aligned}
\operatorname{div} \mathbf{u}_\pm &= \operatorname{div} \hat{\mathbf{u}}_\pm + V_{\mathrm{div}}(\hat{\mathbf{u}}_\pm, H) \\
&= \frac{1}{1 + H_N} \big\{ \operatorname{div} \hat{\mathbf{u}}_\pm - f_-(\hat{\mathbf{u}}_\pm, H) \big\} = \frac{1}{1 + H_N} \big\{ \operatorname{div} \hat{\mathbf{u}}_\pm - \operatorname{div} \mathbf{f}_-(\hat{\mathbf{u}}_\pm, H) \big\}
\end{aligned} \tag{2.8}$$

with

$$V_{\rm div}\,(\hat{\mathbf{u}}_\pm, H) = -\sum_{j=1}^N K_j\partial_N\hat{u}_{\pm j}, \quad f_-(\hat{\mathbf{u}}_\pm, H) = \sum_{j=1}^{N-1}(H_N\partial_j\hat{u}_{\pm j} - H_j\partial_N\hat{u}_{\pm j}),$$

$$\mathbf{f}_-(\hat{\mathbf{u}}_\pm, H) = -(H_N\hat{u}_{\pm 1}, \ldots, H_N\hat{u}_{\pm N-1}, -\sum_{j=1}^{N-1} H_j\hat{u}_{\pm j}).$$

For any $N\times N$ matrix of functions $\mathbf{G} = (\mathbf{g}_1, \ldots, \mathbf{g}_N)$, by (2.8) we have

$$\mathrm{Div}\,\mathbf{G} = \mathrm{Div}\,\hat{\mathbf{G}} + \mathbf{V}_{\rm Div}\,(\hat{\mathbf{G}}, H) \tag{2.9}$$

with $\mathbf{V}_{\rm Div}\,(\hat{\mathbf{G}}, H) = (V_{\rm div}\,(\hat{\mathbf{g}}_1, H), \ldots, V_{\rm div}\,(\hat{\mathbf{g}}_N, H))$. Moreover, we set

$$D_{ij}(\mathbf{u}_\pm) = D_{ij}(\hat{\mathbf{u}}_\pm) + V_{D_{ij}}(\hat{\mathbf{u}}_\pm, H), \;\; \mathbf{D}(\mathbf{u}_\pm) = \mathbf{D}(\hat{\mathbf{u}}_\pm) + \mathbf{V}_{\mathbf{D}}(\hat{\mathbf{u}}_\pm, H), \tag{2.10}$$

where $V_{D_{ij}}(\hat{\mathbf{u}}_\pm, H) = -(K_i\partial_N\hat{u}_{\pm j} + K_j\partial_N\hat{u}_{\pm i})$ and $\mathbf{D}(\hat{\mathbf{u}}_\pm, H)$ is the $N\times N$ matrix whose (i,j) component is $V_{D_{ij}}(\hat{\mathbf{u}}_\pm, H)$.

Under these preparations, we see easily that problem (2.2), (2.3) and (2.4) is transformed to the following problem:

$$\begin{cases} \partial_t\hat{\rho}_+ + \mathbf{v}_+\cdot\nabla\hat{\rho}_+ + \hat{\rho}_+(\mathrm{div}\,\hat{\mathbf{u}}_+ + V_{\rm div}\,(\hat{\mathbf{u}}_+, H)) = 0 & \\ \rho_{*+}\partial_t\hat{\mathbf{u}}_+ - \mathrm{Div}\,\mathbf{S}_{*+}(\hat{\mathbf{u}}_+) = \mathbf{F}_+ & \text{in } \mathbb{R}^N_+\times(0,T), \\ \rho_{*+}\kappa_{*+}\partial_t\hat{\theta}_+ - d_{*+}\Delta\hat{\theta}_+ = F_{\theta+} & \end{cases}$$

$$\begin{cases} \rho_{*-}\partial_t\hat{\mathbf{u}}_- - \mathrm{Div}\,\mathbf{S}_{*-}(\hat{\mathbf{u}}_-) + \nabla\hat{\pi}_- = \mathbf{F}_- & \\ \mathrm{div}\,\hat{\mathbf{u}}_- = f_- = \mathrm{div}\,\mathbf{f}_- & \text{in } \mathbb{R}^N_-\times(0,T), \\ \rho_{*-}\kappa_{*-}\partial_t\hat{\theta}_- - d_{*-}\Delta\hat{\theta}_- = F_{\theta-} & \end{cases}$$

$$\begin{cases} \mu*_-D_{iN}(\hat{\mathbf{u}}_-)|_- - \mu_{*+}D_{iN}(\hat{\mathbf{u}}_+)|_+ = G_i \quad (i = 1, \ldots, N-1) & \\ (\mu_{*-}D_{NN}(\hat{\mathbf{u}}_-) - \hat{\pi}_-)|_- - (\mu_{*+}D_{NN}(\hat{\mathbf{u}}_+) + (\lambda_{*+} - \mu_{*+})\mathrm{div}\,\hat{\mathbf{u}}_+)|_+ - \sigma\Delta' H = G_N & \\ \dfrac{(\mu_{*-}D_{NN}(\hat{\mathbf{u}}_-) - \hat{\pi}_-)|_-}{\rho_{*-}} - \dfrac{\mu_{*+}D_{NN}(\hat{\mathbf{u}}_+) + (\lambda_{*+} - \mu_{*+})\mathrm{div}\,\hat{\mathbf{u}}_+)|_+}{\rho_{*+}} = G_{N+1} & \\ \hat{u}_{-i}|_- - \hat{u}_{+i}|_+ = K_i \quad (i = 1, \ldots, N-1) & \text{on } \mathbb{R}^N_0\times(0,T), \\ \hat{\theta}_-|_- - \hat{\theta}_+|_+ = 0, \quad d_{*+}\partial_N\hat{\theta}_-|_- - d_{*+}\partial_N\hat{\theta}|_+ = G_\theta & \\ \partial_t H - \Big(\dfrac{\rho_{*-}}{\rho_{*-} - \rho_{*+}}\hat{u}_{-N}|_- - \dfrac{\rho_{*+}}{\rho_{*-} - \rho_{*+}}\hat{u}_{+N}|_+\Big) = G_h & \end{cases}$$

$$\hat{\rho}_+|_{t=0} = \hat{\rho}_{+0} \quad \text{in } \mathbb{R}^N_+, \quad (\hat{\mathbf{u}}_\pm, \hat{\theta}_\pm)|_{t=0} = (\hat{\mathbf{u}}_{0\pm}, \hat{\theta}_{0\pm}) \quad \text{in } \mathbb{R}^N_\pm, \quad H|_{t=0} = H_0 \quad \text{on } \mathbb{R}^N_0 \tag{2.11}$$

with

$$\begin{aligned} &\mathbf{v}_+ = (\hat{u}_{+1}, \ldots, \hat{u}_{+N-1}, \hat{u}_{+N} - K_0 - \sum_{j=1}^N K_j\hat{u}_{+j}), \\ &\mathbf{S}_{*+}(\mathbf{u}) = \mu_{*+}\mathbf{D}(\mathbf{u}) + (\lambda_{*+} - \mu_{*+})\mathrm{div}\,\mathbf{u}\mathbf{I}, \quad \mathbf{S}_{*-}(\mathbf{u}) = \mu_{*-}\mathbf{D}(\mathbf{u}), \\ &\hat{\rho}_{0+}(x) = \rho_{0+}(x', x_N + H_0(x)), \;\; \hat{\mathbf{u}}_{0\pm}(x) = \mathbf{u}_{0\pm}(x', x_N + H_0(x)), \;\; \hat{\theta}_{0\pm}(x) = \theta_{0\pm}(x', x_N + H_0(x)). \end{aligned} \tag{2.12}$$

Here, $f|_\pm(x_0) = \lim_{\substack{x\to x_0 \\ x\in\Omega_\pm(t)}} f(x)$ for $x_0 \in \mathbb{R}^N_0$. Moreover, the right-hand sides in (2.11) are defined by the following formulas:

$$\begin{aligned} \mathbf{F}_+ &= \mathbf{F}_+(\hat{\rho}_+, \hat{\mathbf{u}}_+, H) \\ &= -\hat{\rho}_+\{\partial_t\hat{\mathbf{u}}_+ - K_0\partial_N\hat{\mathbf{u}}_+ + \hat{\mathbf{u}}_+\cdot\nabla\hat{\mathbf{u}}_+ - (\hat{\mathbf{u}}_+\cdot\mathbf{K})\partial_N\hat{\mathbf{u}}_+\} \\ &\quad + \rho_{*+}\{K_0\partial_N\hat{\mathbf{u}}_+ - \hat{\mathbf{u}}_+\cdot\nabla\hat{\mathbf{u}}_+ + (\hat{\mathbf{u}}_+\cdot\mathbf{K})\partial_N\hat{\mathbf{u}}_+\} \end{aligned}$$

$$
\begin{aligned}
&+ \mathrm{Div}\,\{\hat\mu_+(\mathbf{D}(\hat{\mathbf{u}}_+) + \mathbf{V_D}(\hat{\mathbf{u}}_+, H)) + (\hat\lambda_+ - \hat\mu_+)(\mathrm{div}\,\hat{\mathbf{u}}_+ + V_{\mathrm{div}}\,(\hat{\mathbf{u}}_+, H))\mathbf{I}\}\\
&+ \mathbf{V}_{\mathrm{Div}}\,((\hat\mu_+ + \mu_{*+})(\mathbf{D}(\hat{\mathbf{u}}_+) + \mathbf{V_D}(\hat{\mathbf{u}}_+, H)) + (\hat\lambda_+ + \lambda_{*+} - (\hat\mu_+ + \mu_{*+}))(\mathrm{div}\,\hat{\mathbf{u}}_+ + V_{\mathrm{div}}\,(\hat{\mathbf{u}}_+, H))\mathbf{I})\\
&+ \mu_{*+}\mathrm{Div}\,\mathbf{V_D}(\hat{\mathbf{u}}_+, H) + (\lambda_{*+} - \mu_{*+})\nabla V_{\mathrm{div}}\,(\hat{\mathbf{u}}_+, H)\} - Q\nabla P_+(\hat\rho_+ + \rho_{*+}, \hat\theta_+ + \theta_*),
\end{aligned}
$$

$$
\begin{aligned}
F_{\theta+} &= F_{\theta+}(\hat\rho_+, \hat{\mathbf{u}}_+, \hat\theta_+, H)\\
&= -((\hat\rho_+ + \rho_{*+})(\hat\kappa_+ + \kappa_{*+}) - \rho_{*+}\kappa_{*+})(\partial_t\hat\theta_+ - K_0\partial_N\hat\theta_+ + \hat{\mathbf{u}}_+\cdot\nabla\hat\theta_+ - (\hat{\mathbf{u}}_+\cdot\mathbf{K})\partial_N\hat\theta_+)\\
&\quad + \rho_{*+}\kappa_{*+}(K_0\partial_N\hat\theta_+ - \hat{\mathbf{u}}_+\cdot\nabla\hat\theta_+ + (\hat{\mathbf{u}}_+\cdot\mathbf{K})\partial_N\hat\theta_+) + \sum_{j=1}^N\partial_j(\hat d_+(\partial_j\hat\theta_+ - K_j\partial_N\hat\theta_+))\\
&\quad - d_{*+}\sum_{j=1}^N\partial_j(K_j\partial_N\hat\theta_+) + \sum_{j=1}^N K_j\partial_N((\hat d_+ + d_{*+})(\partial_j\hat\theta_+ - K_j\partial_N\hat\theta_+))\\
&\quad + 2(\hat\mu_+ + \mu_{*+})|(\mathbf{D}(\hat{\mathbf{u}}_+) + \mathbf{V_D}(\hat{\mathbf{u}}_+, H)|^2 + (\hat\lambda_+ + \lambda_{*+} - (\hat\mu_+ + \mu_{*+}))(\mathrm{div}\,\hat{\mathbf{u}}_+ + V_{\mathrm{div}}\,(\hat{\mathbf{u}}_+, H))^2\\
&\quad + (P_+(\hat\rho_+ + \rho_{*+}, \hat\theta_+ + \theta_*) - (\hat\rho_+ + \rho_{*+})^2\frac{\partial e_+}{\partial\rho}(\hat\rho_+ + \rho_{*+}, \hat\theta_+ + \theta_*))(\mathrm{div}\,\hat{\mathbf{u}}_+ + V_{\mathrm{div}}\,(\hat{\mathbf{u}}_+, H)),
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{F}_- &= \mathbf{F}_-(\hat{\mathbf{u}}_-, H)\\
&= -\rho_{*-}\mathbf{Q_1}(\partial_t\hat{\mathbf{u}}_- - \mu_{*-}\mathrm{Div}\,\mathbf{D}(\hat{\mathbf{u}}_-)) - (I + \mathbf{Q}_1)((K_0\partial_N\hat{\mathbf{u}}_- - \hat{\mathbf{u}}_-\cdot\nabla\hat{\mathbf{u}}_- + (\hat{\mathbf{u}}_-\cdot\mathbf{K})\partial_N\hat{\mathbf{u}}_-)\\
&\quad + (I + \mathbf{Q}_1)\{\mathrm{Div}\,(\hat\mu_-(\mathbf{D}(\hat{\mathbf{u}}_-) + \mathbf{V_D}(\hat{\mathbf{u}}_-, H)) + \mathbf{V}_{\mathrm{Div}}\,((\hat\mu_- + \mu_{*-})(\mathrm{Div}\,\mathbf{D}(\hat{\mathbf{u}}_-) + \mathbf{V_D}(\hat{\mathbf{u}}_-, H)))\},
\end{aligned}
$$

$$
f_{=}f_-(\hat{\mathbf{u}}_-, H) = \sum_{j=1}^{N-1}\{H_N\partial_j\hat u_{-j} - H_j\partial_N\hat u_{-j}),
$$

$$
\mathbf{f}_- = \mathbf{f}_-(\hat{\mathbf{u}}_-, H) = -(H_N\hat u_{-1}, \ldots, H_N\hat u_{-\,N-1}, -\sum_{j=1}^{N-1}H_j\hat u_{-j}),
$$

$$
\begin{aligned}
F_{\theta-} &= F_{\theta-}(\hat{\mathbf{u}}_-, \hat\theta_-, H)\\
&= -\rho_{*-}\hat\kappa_-(\partial_t\hat\theta_- - K_0\partial_N\hat\theta_- + \hat{\mathbf{u}}_-\cdot\nabla\hat\theta_- - (\hat{\mathbf{u}}_-\cdot\mathbf{K})\partial_N\hat\theta_-)\\
&\quad + \rho_{*-}\kappa_{*-}(K_0\partial_N\hat\theta_- - \hat{\mathbf{u}}_-\cdot\nabla\hat\theta_- + (\hat{\mathbf{u}}_-\cdot\mathbf{K})\partial_N\hat\theta_-) + \sum_{j=1}^N\partial_j(\hat d_+(\partial_j\hat\theta_- - K_j\partial_N\hat\theta_-))\\
&\quad - d_{*-}\sum_{j=1}^N\partial_j(K_j\partial_N\hat\theta_-) + \sum_{j=1}^N K_j\partial_N((\hat d_- + d_{*-})(\partial_j\hat\theta_- - K_j\partial_N\hat\theta_-))\\
&\quad + 2(\hat\mu_- + \mu_{*-})|(\mathbf{D}(\hat{\mathbf{u}}_-) + \mathbf{V_D}(\hat{\mathbf{u}}_-, H)|^2,
\end{aligned}
$$

$$
\begin{aligned}
G_i &= G_i(\hat\rho_+, \hat{\mathbf{u}}_\pm, H)\\
&= -\{\hat\mu_-(D_{iN}(\hat{\mathbf{u}}_-) + V_{D_{iN}}(\hat{\mathbf{u}}_-, H))|_- - \hat\mu_+(D_{iN}(\hat{\mathbf{u}}_+) + V_{D_{iN}}(\hat{\mathbf{u}}_+, H))|_+\}\\
&\quad - (\mu_{*-}V_{D_{iN}}(\hat{\mathbf{u}}_-, H)|_- - \mu_{*+}V_{D_{iN}}(\hat{\mathbf{u}}_+, H)|_+)\\
&\quad + \sum_{j=1}^{N-1}(\partial_jH)\{(\hat\mu_- + \mu_{*-})(D_{ij}(\hat{\mathbf{u}}_-) + V_{D_{ij}}(\hat{\mathbf{u}}_-, H))|_- - (\hat\mu_+ + \mu_{*+})(D_{ij}(\hat{\mathbf{u}}_+) + V_{D_{ij}}(\hat{\mathbf{u}}_+, H))|_+\}\\
&\quad - \sum_{j=1}^{N-1}(\partial_iH)(\partial_jH)\{(\hat\mu_- + \mu_{*-})(D_{ij}(\hat{\mathbf{u}}_-) + V_{D_{ij}}(\hat{\mathbf{u}}_-))|_- - (\hat\mu_+ + \mu_{*+})(D_{ij}(\hat{\mathbf{u}}_+) + V_{D_{ij}}(\hat{\mathbf{u}}_+))|_+\},
\end{aligned}
$$

$$
\begin{aligned}
G_N &= G_N(\hat\rho_+, \hat{\mathbf{u}}_\pm, H)\\
&= -\{\hat\mu_-(D_{NN}(\hat{\mathbf{u}}_-) + V_{D_{NN}}(\hat{\mathbf{u}}_-, H))|_- - \hat\mu_+(D_{NN}(\hat{\mathbf{u}}_+) + V_{D_{NN}}(\hat{\mathbf{u}}_+, H))|_+\}\\
&\quad - (\mu_{*-}V_{D_{NN}}(\hat{\mathbf{u}}_-)|_- - \mu_{*+}V_{D_{NN}}(\hat{\mathbf{u}}_+, H)|_+) - (\hat\lambda_+ - \hat\mu_+)(\mathrm{div}\,\hat{\mathbf{u}}_+ + V_{\mathrm{div}}\,(\hat{\mathbf{u}}_+, H))|_+\\
&\quad + (\lambda_{*+} - \mu_{*+})V_{\mathrm{div}}\,(\hat{\mathbf{u}}_+, H)|_+ - (P_+(\hat\rho_+ + \rho_{*+}, \hat\theta_+ + \theta_* - P_+(\rho_{*+}, \theta_*))|_+
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{j=1}^{N-1}(\partial_j H)((\hat\mu_-+\mu_{*-})(D_{Nj}(\hat{\mathbf{u}}_-)+V_{D_{Nj}}(\hat{\mathbf{u}}_-,H))|_- -(\hat\mu_++\mu_{*+})(D_{Nj}(\hat{\mathbf{u}}_+)+V_{D_{Nj}}(\hat{\mathbf{u}}_+,H))|_+\\
&+\sigma\Big(\frac{1}{\sqrt{1+|\nabla' H|^2}}-1\Big)\Delta' H-\sigma\sum_{j,k=1}^{N-1}\frac{\partial_j H\partial_k H}{(1+|\nabla' H|^2)^{3/2}}\\
&+\Big(\frac{1}{\rho_{*-}}-\frac{1}{\hat\rho_+|_++\rho_{*+}}\Big)^{-1}(u_{-N}|_- -u_{+N}|_+)^2(1+|\nabla' H|^2),\\
&G_{N+1}=G_{N+1}(\hat\rho_+,\hat{\mathbf{u}}_\pm,\hat\theta_\pm,H)\\
&=-\frac{1}{\rho_{*-}}(\hat\mu_-D_{NN}(\hat{\mathbf{u}}_-)+V_{D_{NN}}(\hat{\mathbf{u}}_-,H))|_- -\frac{1}{\rho_{*+}}\mu_{*-}V_{D_{NN}}(\hat{\mathbf{u}}_-,H)|_-\\
&+\frac{1}{\hat\rho_++\rho_{*+}}[\hat\mu_+(D_{NN}(\hat{\mathbf{u}}_+)+V_{D_{NN}}(\hat{\mathbf{u}}_+,H))+(\hat\lambda_+-\hat\mu_+)(\operatorname{div}\hat{\mathbf{u}}_++V_{\rm div}(\hat{\mathbf{u}}_+,H))]|_+\\
&+\frac{1}{\hat\rho_++\rho_{*+}}(\mu_{*+}V_{D_{NN}}(\hat{\mathbf{u}}_+,H)+(\lambda_{*+}-\mu_{*+})\operatorname{div}\hat{\mathbf{u}}_+)|_+\\
&+\Big(\frac{1}{\hat\rho_++\rho_{*+}}-\frac{1}{\rho_{*+}}\Big)(\mu_{*+}D_{NN}(\hat{\mathbf{u}}_+)+(\lambda_{*+}-\mu_{*+})\operatorname{div}\hat{\mathbf{u}}_+)|_+\\
&-\Big(\frac{1}{\hat\rho_++\rho_{*+}}P_+(\hat\rho_++\rho_{*+},\hat\theta_++\theta_*)-\frac{1}{\rho_{*+}}P_+(\rho_{*+},\theta_*)\Big)|_+\\
&-\sum_{i,j=1}^{N-1}(\partial_i H)(\partial_j H)\Big(\frac{1}{\rho_{*-}}(\hat\mu_-+\mu_{*-})(D_{ij}(\hat{\mathbf{u}}_-)+V_{D_{ij}}(\hat{\mathbf{u}}_-,H)\Big)|_- +|\nabla' H|^2(\hat\pi_-+\pi_{*-})\\
&+\sum_{i,j=1}^{N-1}(\partial_i H)(\partial_j H)\Big(\frac{1}{\hat\rho_++\rho_{*+}}\{(\hat\mu_++\mu_{*+})(D_{ij}(\hat{\mathbf{u}}_+)+V_{D_{ij}}(\hat{\mathbf{u}}_+,H))|_+\\
&+|\nabla' H|^2\{(\hat\lambda_++\lambda_{*+}-(\hat\mu_++\mu_{*+}))(\operatorname{div}\hat{\mathbf{u}}_++V_{\rm div}(\hat{\mathbf{u}}_+,H))|_+ -P(\hat\rho_++\rho_{*+},\hat\theta_++\theta_*)|_+\}\\
&+\sum_{i=1}^{N-1}(\partial_i H)\Big[\frac{2}{\rho_{*-}}(\hat\mu_-+\mu_{*-})(D_{iN}(\hat{\mathbf{u}}_-)+V_{D_{iN}}(\hat{\mathbf{u}}_-,H))|_-\\
&-\frac{2}{\hat\rho_++\rho_{*+}}(\hat\mu_++\mu_{*+})(D_{iN}(\hat{\mathbf{u}}_+)+V_{D_{iN}}(\hat{\mathbf{u}}_-,H))|_+\Big]\\
&+(\psi_-(\hat\theta_-+\theta_*)-\psi_-(\theta_*))|_- -(\psi_+(\hat\rho_++\rho_{*+},\hat\theta_++\theta_*)-\psi_+(\rho_{*+},\theta_*))|_+\\
&+|\nabla' H|^2(\psi_-(\hat\theta_-+\theta_*)|_- -\psi_+(\hat\rho_++\rho_{*+},\hat\theta_++\theta_*)|_+)\\
&+(1+|\nabla' H|^2)^2\frac{1}{2}\Big(\frac{1}{\rho_{*-}}+\frac{1}{\hat\rho_++\rho_{*+}}\Big)\Big(\frac{1}{\rho_{*-}}-\frac{1}{\hat\rho_++\rho_{*+}}\Big)^{-1}(\hat u_{-N}|_- -\hat u_{+N}|_+)^2,\\
&K_i=K_i(\hat{\mathbf{u}}_\pm,H)=-(\partial_i H)(\hat u_{-N}|_- -\hat u_{+N}|_+),\\
&G_\theta=G_\theta(\hat\rho_+,\hat{\mathbf{u}}_\pm,\hat\theta_\pm,H)\\
&=(1+|\nabla' H|^2)(\hat u_{-N}|_- -\hat u_{+N}|_+)\Big(\frac{1}{\rho_{*-}}-\frac{1}{\hat\rho_+|_++\rho_{*+}}\Big)^{-1}\times\\
&\quad((\hat\theta_-+\theta_*)\eta_-(\hat\theta_-+\theta_*)|_- -(\hat\theta_++\theta_*)\eta_+(\hat\rho_++\rho_{*+},\hat\theta_++\theta_*)|_+)\\
&-(\hat d_-(\nabla\hat\theta_- -\mathbf{K}\partial_N\hat\theta_-)|_- -\hat d_+(\nabla\hat\theta_+-\mathbf{K}\partial_N\hat\theta_+)|_+)\cdot(-\nabla' H,1)\\
&+(d_{*-}\nabla'\hat\theta_-|_- -d_{*+}\nabla'\hat\theta_+|_+)\cdot\nabla' H+(d_{*-}\partial_N\hat\theta_-|_- -d_{*+}\partial_N\hat\theta_+|_+)\mathbf{K}\cdot(-\nabla' H,1),\\
&G_h=G_h(\hat\rho_+,\hat{\mathbf{u}}_\pm,H)\\
&=\rho_{-*}\Big(\frac{1}{\hat\rho_+|_++\rho_{*+}-\rho_{*-}}-\frac{\rho_{*+}}{\rho_{*-}-\rho_{*+}}\Big)\hat u_{-N}|_- -\Big(\frac{\hat\rho_++\rho_{*+}}{\hat\rho_++\rho_{*+}-\rho_{*-}}-\frac{\rho_{*+}}{\rho_{*+}-\rho_{*-}}\Big)\hat u_{+N}|_+\\
&-\frac{\hat\rho_++\rho_{*+}}{\hat\rho_++\rho_{*+}-\rho_{*+}}\sum_{j=1}^{N-1}(\partial_j H)\hat u_{+j}|_+ +\frac{\rho_{*-}}{\hat\rho_+|_++\rho_{*+}-\rho_{*-}}\sum_{j=1}^{N-1}(\partial_j H)\hat u_{-j}|_-.
\end{aligned}
$$

The phase flux $\jmath$ is eliminated by using the formula:

$$\jmath = (\hat{u}_{-N}|_- - \hat{u}_{+N}|_+)\Big(\frac{1}{\rho_{*-}} - \frac{1}{\hat\rho_+|_+ + \rho_{*+}}\Big)^{-1}\sqrt{1+|\nabla' H|^2} \quad \text{on } \mathbb{R}^N_0 \times (0,T).$$

Moreover, we use the formula:

$$H_\Gamma \mathbf{n}_{\Gamma(t)} = \Big\{\operatorname{div}'\Big(\frac{\nabla' H}{1+|\nabla' H|^2}\Big)\Big\}(-\nabla' H, 1)/\sqrt{1+|\nabla' H|^2} \quad \text{on } \mathbb{R}^N_0 \times (0,T)$$

where $\nabla' H = (\partial_1 H, \ldots, \partial_{N-1}H)$ and $\operatorname{div}'\mathbf{v}' = \sum_{j=1}^{N-1}\partial_j v_j$ for $\mathbf{v}' = (v_1, \ldots, v_{N-1})$.

The following theorem is my main result concerning the local well-posedness of problem (2.11).

Theorem 1. *Let $1 < p, q < \infty$ with $2/p + N/q < 1$. Assume that $\rho_{*\pm}$ and θ_* satisfy the condition* (**??**). *Then, given any positive time T, there exists an $\epsilon > 0$ such that problem* (2.11) *admits unique solutions $\hat\rho_+$, $\hat{\mathbf{u}}_\pm$ and $\hat\theta_\pm$ with*

$$\begin{aligned}
&\hat\rho_+ \in W^1_p((0,T), L_q(\mathbb{R}^N_+)) \cap L_p((0,T), W^1_q(\mathbb{R}^N_+)),\\
&(\hat{\mathbf{u}}_\pm, \hat\theta_\pm) \in W^1_p((0,T), L_q(\mathbb{R}^N_\pm)) \cap L_p((0,T), W^2_q(\mathbb{R}^N_\pm)),\\
&H \in W^1_p((0,T), W^2_q(\mathbb{R}^N)) \cap L_p((0,T), W^3_q(\mathbb{R}^N))
\end{aligned}$$

provided that the smallness condition:

$$\|\hat\rho_+\|_{W^1_q(\mathbb{R}^N_+)} + \sum_{\ell=\pm}\|(\hat{\mathbf{u}}_\ell, \hat\theta_\ell)\|_{B^{2(1-1/p)}_{q,p}(\mathbb{R}^N_\ell)} + \|H_0\|_{B^{3-1/p}_{q,p}(\mathbb{R}^N)} \le \epsilon$$

and compatibility condition:

$$\begin{aligned}
\operatorname{div}\hat{\mathbf{u}}_{-0} = f_-(\hat{\mathbf{u}}_{-0}, H_0) = \operatorname{div}\mathbf{f}_-(\hat{\mathbf{u}}_{-0}, H_0) &\quad \textit{in } \mathbb{R}^N_-,\\
\mu{*-}D_{iN}(\hat{\mathbf{u}}_{-0})|_- - \mu_{*+}D_{iN}(\hat{\mathbf{u}}_{+0})|_+ = G_i(\hat\rho_{0+}, \hat{\mathbf{u}}_{0\pm}, H_0) \quad (i=1,\ldots,N-1) &\quad \textit{on } \mathbb{R}^N_0,\\
\hat{u}_{-i}|_- - \hat{u}_{+i}|_+ = K_i(\hat{\mathbf{u}}_\pm, H_0) \quad (i=1,\ldots,N-1) &\quad \textit{on } \mathbb{R}^N_0,\\
\hat\theta_{0-}|_- - \hat\theta_{0+}|_+ = 0 &\quad \textit{on } \mathbb{R}^N_0,\\
d_{*+}\partial_N\hat\theta_{0-}|_- - d_{*+}\partial_N\hat\theta_{0+}|_+ = G_\theta(\hat\rho_{0+}, \hat{\mathbf{u}}_{0\pm}, \hat\theta_{0\pm}, H_0) &\quad \textit{on } \mathbb{R}^N_0,\\
\Big(\frac{1}{\rho_{*-}} - \frac{1}{\rho_{*+}}\Big)(\mu_{*+}D_{NN}(\hat{\mathbf{u}}_{0+}) + (\lambda_{*+} - \mu_{*+})\operatorname{div}\hat{\mathbf{u}}_{0+})|_+ = G_{N+1} - \frac{1}{\rho_{*-}}G_N - \frac{\sigma}{\rho_{*-}}\Delta' H_0 &\quad \textit{on } \mathbb{R}^N_0.
\end{aligned}$$

Remark 3. (1) The mathematical study of the compressible and incompressible two phase problem is quite few as far as the author knows. First Denisova [2] studied the evolution of the compressible and incompressible two phase flow with sharp interface without phase transition under some restriction on the viscosity coefficients. Recently, Kubo, Shibata and Soga [4] studied the same problem as in [2] without any restriction on viscosity coefficients in case of without surface tension and without phase transition. This abstract is the first manuscript to treat the compressible and incompressible two phase problem with phase transition. The incompressible and incompressible two phase problem with phase transition was studied by J. Pruess, Y. Shibata, S. Shimizu and G. Simonett [5, 6].

3 Maximal L_p-L_q regularity

In the following, we assume that $N < q < \infty$ in view of the Sobolev imbedding theorem: $\|v\|_{L_\infty(\Omega)} \le C\|v\|_{W^1_q(\Omega)}$ with $\Omega = \mathbb{R}^N_\pm$ and $\Omega = \mathbb{R}^N$. To solve problem (2.11), we use the maximal L_p-L_q regualrity for the parabolic equations. From this point of view, we represent $\hat\rho_+$ by the integration along the characteristic curve generated by $\mathbf{v}_+$[‡].

[‡]Tani [12] represented the mass density with the help of the velocity field to prove the local well-posedness of the Navier-Stokes equations describing the compressible viscous fluid flow (cf. also [11, 13]). It was also suggested by J. Prüss to the author to represnt $\hat\rho_+$ by $\hat{\mathbf{u}}_+$ and H with the help of the equation of balance of mass when the author visited Halle university in the early of April, 2014.

Let $\hat{\mathbf{w}}_+ = (\hat{w}_{+1}, \ldots, \hat{w}_{+N})$ be the Lions extension to $\mathbb{R}^N$ defined by

$$\hat{\mathbf{w}}_+(x,t) = \begin{cases} \hat{\mathbf{u}}_+(x,t) & \text{for } x_N > 0, \\ 3\hat{\mathbf{u}}_+(x', -x_N, t) - 2\hat{\mathbf{u}}_+(x', -2x_N, t) & \text{for } x_N < 0, \end{cases} \tag{3.1}$$

and in view of (2.12) we define $\mathbf{v}$ by $\mathbf{v} = (\hat{w}_{+1}, \ldots, \hat{w}_{+N-1}, \hat{w}_{+N} - K_0 - \sum_{j=1}^N K_j \hat{w}_{+j})$. Note that $\mathbf{v} = \mathbf{v}_+$ on $\mathbb{R}^N_+$. We assume that

$$\int_0^T \|\nabla \mathbf{v}(\cdot, t)\|_{L_\infty(\mathbb{R}^N)}\, dt \leq \epsilon_1 \tag{3.2}$$

with some small positive constant $\epsilon_1 > 0$. We use the usual fixed point argument to solve the nonlinear problem and in this argument we keep the situation where $\hat{\mathbf{u}}_+$ and H satisfy (3.2).

Let $\hat{\xi}$ be the solution to the Cauchy problem:

$$\frac{d}{dt}\hat{\xi}(\eta, t) = \mathbf{v}(\hat{\xi}(\eta, t), t), \quad \hat{\xi}(\eta, 0) = \eta \in \mathbb{R}^N.$$

According to Strömer [10], we choose $\epsilon_1 > 0$ so small that the map: $\eta \mapsto \xi$ is bijective on $\mathbb{R}^N$ for any $t \in [0, T]$. We denote its inverse map by $\hat{\eta} = \hat{\eta}(\xi, t)$. Setting $J(\eta, t) = \det(\frac{\partial \hat{\xi}}{\partial \eta})$, we have $\partial_t J(\eta, t) = (\operatorname{div} \mathbf{v})(\hat{\xi}(\eta, t), t) J(\eta, t)$. We look for $\hat{\rho}_+$ satisfying the equation:

$$\partial_t \hat{\rho}_+ + \mathbf{v} \cdot \nabla \hat{\rho}_+ + \hat{\rho}_+(\operatorname{div} \hat{\mathbf{w}}_+ + V_{\rm div}\,(\hat{\mathbf{w}}_+, H)) = 0 \quad \text{in } \mathbb{R}^N \times (0, T).$$

Since

$$\frac{d}{dt}(\hat{\rho}_+(\hat{\xi}(\eta, t), t) J(\eta, t)) = (\partial_t \hat{\rho}_+ + \operatorname{div}\,(\hat{\rho}_+ \hat{\mathbf{v}}))(\hat{\xi}(\eta, t), t) J(\eta, t) = g(\hat{\xi}(\eta, t), t) \rho_+(\hat{\xi}(\eta, t), t) J(\eta, t),$$

with $g = \operatorname{div} \mathbf{v} - \operatorname{div} \hat{\mathbf{w}}_+ - V_{\rm div}\,(\hat{\mathbf{w}}_+, H)$, we define $\hat{\rho}_+(\xi, t)$ by

$$\hat{\rho}_+(\xi, t) = (\rho_{*+} + \tilde{\rho}_{0+}(\eta)) J(\eta, t)^{-1} e^{\int_0^t g(\hat{\xi}(\eta, s), s)\, ds} \tag{3.3}$$

with $\eta = \hat{\eta}(\xi, t)$, where $\tilde{\rho}_{0+}(\eta)$ is the same Lions extension of $\hat{\rho}_{0+}(\xi)$ to $\mathbb{R}^N$ as in (3.1). Moreover, since J satisfies the equation: $\partial_t J(\eta, t) = (\operatorname{div} \mathbf{v})(\hat{\xi}(\eta, t), t) J(\eta, t)$ with $J(\eta, 0) = 1$, we have

$$J(\eta, t) = e^{\int_0^t (\operatorname{div} \mathbf{v})(\hat{\xi}(\eta, s), s)\, ds},$$

which is inserted into the formula of $\hat{\rho}_+$ in (3.3) furnishes finally that

$$\hat{\rho}_+(\xi, t) = (\rho_{*+} + \tilde{\rho}_{0+}(\eta)) e^{-\int_0^t (\operatorname{div} \hat{\mathbf{w}}_+ + V_{\rm div}\,(\hat{\mathbf{w}}_+, H))(\hat{\xi}(\eta, s), s)\, ds} \tag{3.4}$$

with $\eta = \hat{\eta}(\xi, t)$.

Inserting the formula of $\hat{\rho}_+$ given in (3.4) into the right-hand sides: $\mathbf{F}_+ = \mathbf{F}_+(\hat{\rho}_+, \hat{\mathbf{u}}_\pm, H)$, $F_{\theta+} = F_{\theta+}(\hat{\rho}_+, \hat{\mathbf{u}}_\pm, \hat{\theta}_+, H)$, $G_j = G_j(\hat{\rho}_+, \mathbf{u}_\pm, H)$ $(j = 1, \ldots, N+1)$ and $G_\theta = G_\theta(\hat{\rho}_+, \hat{\mathbf{u}}_\pm, \hat{\theta}_\pm, H)$ in (2.11), we have the interface problem for the parabolic equations. As the linearized problem, we have the decoupled two systems. One is the Stokes equation with interface condition:

$$\begin{aligned} &\rho_{*+}\partial_t \mathbf{u}_+ - \operatorname{Div} \mathbf{S}_{*+}(\mathbf{u}_+) = \mathbf{f}_+ && \text{in } \mathbb{R}^N_+ \times (0, T) \\ &\rho_{*-}\partial_t \mathbf{u}_- - \operatorname{Div} \mathbf{S}_{*-}(\mathbf{u}_-) + \nabla \pi_- = \mathbf{f}_-, \quad \operatorname{div} \mathbf{u}_- = f_{\rm div} = \operatorname{div} \mathbf{f}_{\rm div} && \text{in } \mathbb{R}^N_- \times (0, T) \end{aligned} \tag{3.5}$$

subject to the interface condition: for $x \in \mathbb{R}^N_0$ and $t \in (0, T)$

$$\begin{aligned} &\mu_{*-} D_{iN}(\mathbf{u}_-)|_- - \mu_{*+} D_{iN}(\mathbf{u}_+)|_+ = g_i \quad (i = 1, \ldots, N-1), \\ &(\mu_{*-} D_{NN}(\mathbf{u}_-) - \pi_-)|_- - (\mu_{*+} D_{NN}(\mathbf{u}_+) + (\lambda_{*+} - \mu_{*+}) \operatorname{div} \mathbf{u}_+)|_+ - \sigma \Delta' H = g_N, \\ &\frac{1}{\rho_{*-}}(\mu_{*-} D_{NN}(\mathbf{u}_-) - \pi_-)|_- - \frac{1}{\rho_{*+}}(\mu_{*+} D_{NN}(\mathbf{u}_+) + (\lambda_{*+} - \mu_{*+}) \operatorname{div} \mathbf{u}_+)|_+ = g_{N+1}, \\ &u_{-i}|_- - u_{+i}|_+ = h_i \quad (i = 1, \ldots, N-1), \\ &\partial_t H - \Big(\frac{\rho_{*-}}{\rho_{*-} - \rho_{*+}} u_{-N} - \frac{\rho_{*+}}{\rho_{*-} - \rho_{*+}} u_{+N}\Big) = d \end{aligned} \tag{3.6}$$

and the initial condition:

$$\mathbf{u}_\pm|_{t=0} = \mathbf{u}_{0\pm} \quad \text{in } \mathbb{R}^N_\pm, \quad H|_{t=0} = H_0 \quad \text{in } \mathbb{R}^N. \tag{3.7}$$

Another is the heat equations with interface condition:

$$\begin{aligned} \rho_{*+}\kappa_{*+}\partial_t\theta_+ - d_{*+}\Delta\theta_+ &= \tilde f_+ \quad \text{in } \mathbb{R}^N_+ \times (0,T) \\ \rho_{*-}\kappa_{*-}\partial_t\theta_- - d_{*-}\Delta\theta_- &= \tilde f_- \quad \text{in } \mathbb{R}^N_- \times (0,T) \end{aligned} \tag{3.8}$$

subject to the interface condition: for $x \in \mathbb{R}\ N_0$ and $t \in (0,T)$

$$\theta_-|_- - \theta_+|_+ = 0, \quad d_{*+}\partial_N\theta_-|_- - d_{*+}\partial_N\theta_+|_+ = \tilde g \tag{3.9}$$

and the initial condition:

$$\theta_\pm|_{t=0} = \theta_{0\pm} \quad \text{on } \mathbb{R}^N_\pm. \tag{3.10}$$

Note that the interface condition (3.6) is equivalent to the following interface condition:

$$\begin{aligned}
&\mu_{*-}D_{iN}(\mathbf{u}_-)|_- - \mu_{*+}D_{iN}(\mathbf{u}_+)|_+ = g_i \quad (i = 1, \dots, N-1), \\
&(\mu_{*-}D_{NN}(\mathbf{u}_-) - \pi_-)|_- = \frac{\rho_{*-}}{\rho_{*-} - \rho_{*+}}(\sigma\Delta' H + g_N - \rho_{*+}g_{N+1}), \\
&(\mu_{*+}D_{NN}(\mathbf{u}_+) + (\lambda_{*+} - \mu_{*+})\operatorname{div}\mathbf{u}_+)|_+ = \frac{\rho_{*+}}{\rho_{*-} - \rho_{*+}}(\sigma\Delta' H + g_N - \rho_{*-}g_{N+1}), \\
&u_{-i}|_- - u_{+i}|_+ = h_i \quad (i = 1, \dots, N-1), \\
&\partial_t H - \Big(\frac{\rho_{*-}}{\rho_{*-} - \rho_{*+}}u_{-N} - \frac{\rho_{*+}}{\rho_{*-} - \rho_{*+}}u_{+N}\Big) = d.
\end{aligned} \tag{3.11}$$

We have the following theorem about the maximal L_p-L_q regularity for problem (3.5), (3.6), (3.7).

Theorem 2. *Let $1 < p, q < \infty$ and $0 < T < \infty$. Assume that $\rho_{*-} \neq \rho_{*+}$. Then, given right-hand sides of* (3.5) *and* (3.6)

$$\begin{aligned}
&\mathbf{f}_\pm \in L_p((0,T), L_q(\mathbb{R}^N_\pm)), \quad f_{\rm div} \in L_p((0,T), W^1_q(\mathbb{R}^N_-)), \quad \mathbf{f}_{\rm div} \in W^1_p((0,T), L_q(\mathbb{R}^N_-)) \\
&g_i \in L_p((0,T), W^1_q(\mathbb{R}^N)) \cap W^1_p((0,T), W^{-1}_q(\mathbb{R}^N)) \quad (i = 1, \dots, N+1), \\
&h_j \in L_p((0,T), W^2_q(\mathbb{R}^N)) \cap W^1_p((0,T), L_p(\mathbb{R}^N)) \quad (j = 1, \dots, N-1), \quad d \in L_p((0,T), W^2_q(\mathbb{R}^N)),
\end{aligned}$$

and initial data $\mathbf{u}_{0\pm} \in B^{2(1-1/p)}_{q,p}(\mathbb{R}^N_\pm)$ and $H_0 \in B^{3-1/p}_{q,p}(\mathbb{R}^N)$ satisfying the compatibility conditions:

$$\begin{aligned}
&\operatorname{div}\mathbf{u}_{0-} = f_-|_{t=0} = \operatorname{div}\mathbf{f}_{\rm div}|_{t=0} && \textit{in } \mathbb{R}^N_-, \\
&\mu_{*-}D_{iN}(\mathbf{u}_{0-})|_- - \mu_{*+}D_{iN}(\mathbf{u}_{0+})|_+ = g_i|_{t=0} \quad (i = 1, \dots, N-1) && \textit{on } \mathbb{R}^N_0, \\
&(\mu_{*+}D_{NN}(\hat{\mathbf{u}}_{0+}) + (\lambda_{*+} - \mu_{*+})\operatorname{div}\hat{\mathbf{u}}_{0+})|_+ \\
&\quad = \frac{\rho_{*+}}{\rho_{*-} - \rho_{*+}}(\sigma\Delta' H_0 + g_N|_{t=0} - \rho_{*-}g_{N+1}|_{t=0}) && \textit{on } \mathbb{R}^N_0, \\
&u_{0-i}|_- - u_{0+i}|_+ = h_i|_{t=0} \quad (i = 1, \dots, N-1) && \textit{on } \mathbb{R}^N_0.
\end{aligned}$$

then, problem (3.5), (3.6), (3.7) *admits unique solutions $\mathbf{u}_\pm$ and H with*

$$\begin{aligned}
&\mathbf{u}_\pm \in L_p((0,T), W^2_q(\mathbb{R}^N_\pm)) \cap W^1_p((0,T), L_q(\mathbb{R}^N_\pm)), \\
&H \in L_p((0,T), W^3_q(\mathbb{R}^N)) \cap W^1_p((0,T), W^2_q(\mathbb{R}^N))
\end{aligned}$$

possessing the estimates:

$$\sum_{\ell=\pm}\{\|\mathbf{u}_\ell\|_{L_p((0,t), W^2_q(\mathbb{R}^N_\ell))} + \|\partial_t\mathbf{u}_\ell\|_{L_p((0,t), L_q(\mathbb{R}^N_\ell))}\} + \|\partial_t H\|_{L_p((0,t), W^2_q(\mathbb{R}^N))} + \|H\|_{L_p((0,t), W^3_q(\mathbb{R}^N))}$$

$$
\begin{aligned}
&\le C^{\gamma t}\{\sum_{\ell=\pm}(\|\mathbf{u}_{0\ell}\|_{B^{2(1-1/p)}_{q,p}(\mathbb{R}^N_\ell)} + \|\mathbf{f}_\ell\|_{L_p((0,t),L_q(\mathbb{R}^N_\ell))}) + \|f_{\rm div}\|_{L_p((0,t),W^1_q(\mathbb{R}^N_-)} \\
&\quad + \|\mathbf{f}_{\rm div}\|_{L_p((0,T),L_q(\mathbb{R}^N_-))} + \sum_{i=1}^{N+1}(\|g_i\|_{L_p((0,t),W^1_q(\mathbb{R}^N))} + \|\partial_t g_i\|_{L_p((0,t),W^{-1}_q(\mathbb{R}^N))}) \\
&\qquad + \sum_{j=1}^{N-1}(\|h_j\|_{L_p((0,t),W^2_q(\mathbb{R}^N))} + \|\partial_t h_j\|_{L_p((0,t),L_q(\mathbb{R}^N))}) + \|d\|_{L_p((0,t),W^2_q(\mathbb{R}^N))}\}
\end{aligned}
$$

for any $t \in (0,T)$ with some positive constants C and γ independent of t and T.

And also, we have the following theorem about the maximal L_p-L_q regularity for problem (3.8), (3.9), (3.10).

Theorem 3. *Let $1 < p, q < \infty$ and $0 < T < \infty$. Then, given right-hand sides of* (3.8) *and* (3.9)*:*

$$
\tilde f_\pm \in L_p((0,T),L_q(\mathbb{R}^N_\pm)), \quad \tilde g \in L_p((0,T),W^1_q(\mathbb{R}^N)) \cap W^1_p((0,T),W^{-1}_q(\mathbb{R}^N))
$$

and initial data $\theta_{0\pm}$ for (3.8) *satisfying the compatibility condition:*

$$
[[\theta_0]] = 0, \quad d_{*-}\partial_N\theta_{0-}|_- - d_{*-}\partial_N\theta_{0+}|_+ = \tilde g|_{t=0} \quad \text{on } \mathbb{R}^N_0,
$$

problem (3.8) *and* (3.9) *admits unique solutions $\theta_\pm$ with*

$$
\theta_\pm \in L_p((0,T),W^2_q(\mathbb{R}^N_\pm)) \cap W^1_p((0,T),L_q(\mathbb{R}^N_\pm))
$$

satisfying the estimate:

$$
\begin{aligned}
&\sum_{\ell=\pm}\{\|\theta_\ell\|_{L_p((0,t),W^2_q(\mathbb{R}^N_\ell))} + \|\partial_t\theta_\ell\|_{L_p((0,t),L_q(\mathbb{R}^N_\ell))}\} \\
&\le C^{\gamma t}\{\sum_{\ell=\pm}(\|\theta_{0\ell}\|_{B^{2(1-1/p)}_{q,p}(\mathbb{R}^N_\ell)} + \|\tilde f_\ell\|_{L_p((0,t),L_q(\mathbb{R}^N_\ell))}) + \|\tilde g\|_{L_p((0,t),W^1_q(\mathbb{R}^N))} + \|\partial_t\tilde g\|_{L_p((0,t),W^{-1}_q(\mathbb{R}^N))})\}
\end{aligned}
$$

for any $t \in (0,T)$ with some positive constants C and γ independent of t and T.

Remark 4. The proof of Theorem 3 is found in [3], but we can prove it by using the same argument as in the proof of Theorem 2.

4 $\mathcal{R}$-bounded solution operators

To prove Theorem 2, we consider the following generalized resolvent problem:

$$
\begin{aligned}
&\rho_{*+}\lambda\mathbf{u}_+ - \mathrm{Div}\,\mathbf{S}_{*+}(\mathbf{u}_+) = \mathbf{f}_+ && \text{in } \mathbb{R}^N_+ \\
&\rho_{*-}\lambda\mathbf{u}_- - \mathrm{Div}\,\mathbf{S}_{*-}(\mathbf{u}_-) + \nabla\pi_- = \mathbf{f}_-, \quad \mathrm{div}\,\mathbf{u}_- = f_{\rm div} = \mathrm{div}\,\mathbf{f}_{\rm div} && \text{in } \mathbb{R}^N_-
\end{aligned}
\tag{4.1}
$$

subject to the interface condition: for $x \in \mathbb{R}^N_0$

$$
\begin{aligned}
&\mu_{*-}D_{iN}(\mathbf{u}_-)|_- - \mu_{*+}D_{iN}(\mathbf{u}_+)|_+ = g_i \quad (i = 1,\dots,N-1), \\
&(\mu_{*-}D_{NN}(\mathbf{u}_-) - \pi_-)|_- - (\mu_{*+}D_{NN}(\mathbf{u}_+) + (\lambda_{*+} - \mu_{*+})\mathrm{div}\,\mathbf{u}_+)|_+ - \sigma\Delta' H = g_N, \\
&\frac{1}{\rho_{*-}}(\mu_{*-}D_{NN}(\mathbf{u}_-) - \pi_-)|_- - \frac{1}{\rho_{*+}}(\mu_{*+}D_{NN}(\mathbf{u}_+) + (\lambda_{*+} - \mu_{*+})\mathrm{div}\,\mathbf{u}_+)|_+ = g_{N+1}, \\
&u_{-i}|_- - u_{+i}|_+ = h_i \quad (i = 1,\dots,N-1), \\
&\lambda H - \Big(\frac{\rho_{*-}}{\rho_{*-} - \rho_{*+}}u_{-N} - \frac{\rho_{*+}}{\rho_{*-} - \rho_{*+}}u_{+N}\Big) = d,
\end{aligned}
\tag{4.2}
$$

which is corresponding to the time dependent problem (3.5), (3.6), (3.7).

Before stating the main result of this section, we first introduce the definition of $\mathcal{R}$-boundedness and the operator valued Fourier multiplier theorem due to Weis [14].

Definition 4.1. A family of operators $\mathcal{T} \subset \mathcal{L}(X, Y)$ is called $\mathcal{R}$-bounded on $\mathcal{L}(X, Y)$, if there exist constants $C > 0$ and $p \in [1, \infty)$ such that for any $n \in \mathbb{N}$, $\{T_j\}_{j=1}^n \subset \mathcal{T}$, $\{f_j\}_{j=1}^n \subset X$ and sequences $\{r_j(u)\}_{j=1}^n$ of independent, symmetric, $\{-1, 1\}$-valued random variables on $[0, 1]$ there holds the inequality:

$$\Big\{\int_0^1 \|\sum_{j=1}^n r_j(u) T_j f_j\|_Y^p \, du\Big\}^{\frac{1}{p}} \le C\Big\{\int_0^1 \|\sum_{j=1}^n r_j(u) f_j\|_X^p \, du\Big\}^{\frac{1}{p}}.$$

The smallest such C is called $\mathcal{R}$-bound of $\mathcal{T}$, which is denoted by $\mathcal{R}_{\mathcal{L}(X,Y)}(\mathcal{T})$. Here and in the following, $\mathcal{L}(X, Y)$ denotes the set of all bounded linear operators from X into Y.

Let $\mathcal{D}(\mathbb{R}, X)$ and $\mathcal{S}(\mathbb{R}, X)$ be the set of all X valued C^∞ functions having compact supports and the Schwartz space of rapidly decreasing X valued functions, respectively, while $\mathcal{S}'(\mathbb{R}, X) = \mathcal{L}(\mathcal{S}(\mathbb{R}, \mathbb{C}), X)$. Given $M \in L_{1,\mathrm{loc}}(\mathbb{R} \setminus \{0\}, X)$, we define the operator $T_M : \mathcal{F}^{-1}\mathcal{D}(\mathbb{R}, X) \to \mathcal{S}'(\mathbb{R}, Y)$ by

$$T_M \phi = \mathcal{F}^{-1}[M\mathcal{F}[\phi]], \quad (\mathcal{F}[\phi] \in \mathcal{D}(\mathbb{R}, X)), \tag{4.3}$$

The following theorem is obtained by Weis [14].

Theorem 4. *Let X and Y be two UMD Banach spaces and $1 < p < \infty$. Let M be a function in $C^1(\mathbb{R} \setminus \{0\}, \mathcal{L}(X, Y))$ such that*

$$\mathcal{R}_{\mathcal{L}(X,Y)}(\{(\tau\frac{d}{d\tau})^\ell M(\tau) \mid \tau \in \mathbb{R} \setminus \{0\}) \le \kappa < \infty \quad (\ell = 0, 1)$$

with some constant κ. Then, the operator T_M defined in (4.3) is extended to a bounded linear operator from $L_p(\mathbb{R}, X)$ into $L_p(\mathbb{R}, Y)$. Moreover, denoting this extension by T_M, we have

$$\|T_M\|_{\mathcal{L}(L_p(\mathbb{R},X), L_p(\mathbb{R},Y))} \le C\kappa$$

for some positive constant C depending on p, X and Y.

Remark 5. For the definition of UMD space, we refer to a book due to Amann [1]. For $1 < q < \infty$, Lebesgue space $L_q(\Omega)$ and Sobolev space $W_q^m(\Omega)$ are both UMD spaces.

Theorem 5. *Let $1 < q < \infty$ and $0 < \epsilon < \pi/2$. Set*

$$\begin{aligned}
&\Sigma_\epsilon = \{\lambda = \gamma + i\tau \in \mathbb{C} \setminus \{0\} \mid |\arg\lambda| \le \pi - \epsilon\}, \quad \Sigma_{\epsilon,\lambda_0} = \{\lambda \in \Sigma_\epsilon \mid |\lambda| \ge \lambda_0\} \ (\lambda_0 > 0),\\
&X_q = \{(\mathbf{f}_+, \mathbf{f}_-, f_{\mathrm{div}}, \mathbf{f}_{\mathrm{div}}, \mathbf{g}, \mathbf{h}, d) \mid \mathbf{f}_+ \in L_q(\mathbb{R}^N_+), \ \ \mathbf{f}_-, \mathbf{f}_{\mathrm{div}} \in L_q(\mathbb{R}^N_-), \ \ f_{\mathrm{div}} \in W_q^1(\mathbb{R}^N),\\
&\qquad \mathbf{g} = (g_1, \ldots, g_{N+1}) \in W_q^1(\mathbb{R}^N), \ \ \mathbf{h} = (h_1, \ldots, h_{N-1}) \in W_q^2(\mathbb{R}^N), \ \ d \in W_q^2(\mathbb{R}^N)\},\\
&\mathcal{X}_q = \{F = (F_{+1}, F_{-1}, F_{-2}, F_{-3}., F_{-4}, F_1, F_2, F_3, F_4, F_5, F_6) \mid F_{\pm 1} \in L_q(\mathbb{R}^N_\pm),\\
&\qquad F_{-2}, F_{-3}, F_{-4} \in L_q(\mathbb{R}^N_-), \ \ F_1, F_2, F_3, F_4, F_5 \in L_q(\mathbb{R}^N), \ \ F_6 \in W_q^2(\mathbb{R}^N)\}.
\end{aligned}$$

Then, there exist a constant $\lambda_0 > 0$ and operator families $\mathcal{A}_\pm(\lambda) \in \mathrm{Hol}\,(\Sigma_{\epsilon,\lambda_0}, \mathcal{L}(X_q, W_q^2(\mathbb{R}^N_\pm)))$, $\mathcal{P}_- \in \mathrm{Hol}\,(\Sigma_{\epsilon,\lambda_0}, \mathcal{L}(X_q, \hat{W}_q^1(\mathbb{R}^N_-)))$, $\mathcal{H}(\lambda) \in \mathrm{Hol}\,(\Sigma_{\epsilon,\lambda_0}, \mathcal{L}(X_q, W_q^3(\mathbb{R}^N)))$ such that for any $\lambda \in \Sigma_{\epsilon,\lambda_0}$ and $\mathbf{F} = (\mathbf{f}_+, \mathbf{f}_-, f_{\mathrm{div}}, \mathbf{f}_{\mathrm{div}}, \mathbf{g}, \mathbf{h}, d) \in X_q$, $\mathbf{u}_\pm = \mathcal{A}_\pm(\lambda)\mathbf{F}_\lambda$, $\pi_- = \mathcal{P}_-(\lambda)\mathbf{F}_\lambda$ and $H = \mathcal{H}(\lambda)\mathbf{F}_\lambda$ are unique solutions of problem (4.1) and (4.2) and we have

$$\begin{aligned}
&\mathcal{R}_{\mathcal{L}(\mathcal{X}_q, L_q(\mathbb{R}^N_\pm))}(\{(\tau\partial_\tau)^\ell G^1_\lambda \mathcal{A}_\pm(\lambda) \mid \lambda \in \Sigma_{\epsilon,\lambda_0}\}) \le c \quad (\ell = 0, 1),\\
&\mathcal{R}_{\mathcal{L}(\mathcal{X}_q, L_q(\mathbb{R}^N_\pm))}(\{(\tau\partial_\tau)^\ell \nabla\mathcal{P}_-(\lambda) \mid \lambda \in \Sigma_{\epsilon,\lambda_0}\}) \le c \quad (\ell = 0, 1),\\
&\mathcal{R}_{\mathcal{L}(\mathcal{X}_q, W_q^2(\mathbb{R}^N_\pm))}(\{(\tau\partial_\tau)^\ell G^2_\lambda \mathcal{H}(\lambda) \mid \lambda \in \Sigma_{\epsilon,\lambda_0}\}) \le c \quad (\ell = 0, 1)
\end{aligned}$$

with some constant c. Here, $G^1_\lambda\mathcal{A}_\pm(\lambda) = (\lambda\mathcal{A}_\pm(\lambda), \lambda^{1/2}\nabla\mathcal{A}_\pm(\lambda), \nabla^2\mathcal{A}_\pm(\lambda))$, $G^2_\lambda\mathcal{H}(\lambda) = (\lambda\mathcal{H}(\lambda), \nabla\mathcal{H}(\lambda))$, $\mathbf{F}_\lambda = (F_+, F_-, \lambda^{1/2}f_{\mathrm{div}}, \nabla f_{\mathrm{div}}, \lambda\mathbf{f}_{\mathrm{div}}, \lambda^{1/2}\mathbf{g}, \nabla\mathbf{g}, \lambda\mathbf{h}, \lambda^{1/2}\nabla\mathbf{h}, \nabla^2\mathbf{h}, d)$, $\hat{W}_q^1(\mathbb{R}^N) = \{\pi_- \in L_{q,\mathrm{loc}}(\mathbb{R}^N) \mid \nabla\pi_- \in L_q(\mathbb{R}^N)\}$, and $\mathrm{Hol}\,(U, X)$ denotes the set of all holomorphic functions defined on U with their values in X.

5 Sketch of proof of Theorem 2 with the help of $\mathcal{R}$ bounded solution operators

In this section, we consider problem (3.5), (3.6) and (3.7). First, we construct $\mathbf{v}_\pm$ and h such that

- $\mathbf{v}_\pm \in W^1_p((0,\infty), L_q(\mathbb{R}^N_\pm)) \cap L_p((0,\infty), W^2_q(\mathbb{R}^N_\pm))$, $h \in W^1_p((0,\infty), W^2_q(\mathbb{R}^N)) \cap L_p((0,\infty), W^3_q(\mathbb{R}^N))$;
- $\mathbf{v}_\pm|_{t=0} = \mathbf{u}_{0\pm}$ in $\mathbb{R}^N_\pm$, $\quad h|_{t=0} = H_0$ in $\mathbb{R}^N$;
- $\|\mathbf{v}_\pm\|_{L_p((0,\infty),W^2_q(\mathbb{R}^N_\pm))} + \|\partial_t \mathbf{v}_\pm\|_{L_p((0,\infty),L_q(\mathbb{R}^N_\pm))} \leq C\|\mathbf{u}_{0\pm}\|_{B^{2(1-1/p)}_{q,p}(\mathbb{R}^N_\pm)}$,
  $\|h\|_{L_p((0,\infty),W^3_q(\mathbb{R}^N))} + \|\partial_t h\|_{L_p((0,\infty),W^2_q(\mathbb{R}^N))} \leq C\|H_{0\pm}\|_{B^{3-1/p}_{q,p}(\mathbb{R}^N_\pm)}$.

Using $\mathbf{v}_\pm$ and h, we transfer problem (3.5), (3.6) and (3.7) to the case where $\mathbf{u}_{0\pm} = 0$ and $H_0 = 0$. Moreover, by the compatibility condition, we may assume that $\mathbf{f}_{\rm div}|_{t=0} = 0$, $g_i|_{t=0} = 0$ $(i = 1, \ldots, N+1)$, $h_i|_{t=0} = 0$ $(i = 1, \ldots, N-1)$ and $\sigma\Delta' H_0 + g_N|_{t=0} - \rho_{*-} g_{N+1}|_{t=0} = 0$.

Let $\mathcal{L}$ and $\mathcal{L}^{-1}$ be the Laplace transform with respect to t and its inverse transform. The operator $\Lambda^{1/2}_\gamma$ is defined by

$$\Lambda^{1/2}_\gamma f = \mathcal{L}^{-1}[\lambda^{1/2}\mathcal{L}[f]] \quad \text{with } \lambda = \gamma + i\tau.$$

Second, we consider the zero initial data case. The right members: $\mathbf{f}_\pm$, $f_{\rm div}$, $\mathbf{f}_{\rm div}$, g_i, h_i and d are extended by 0 with respect to t to $(-\infty, 0)$ and we denote such zero extension by $\mathbf{f}_{\pm 0}$, $f_{\rm div\,0}$, $\mathbf{f}_{\rm div\,0}$, $\mathbf{g}_0 = (g_{10}, \ldots, g_{N+10})$, $\mathbf{h}_0 = (h_{10}, \ldots, h_{N-10})$ and d_0. Applying the Laplace transform, we have

$$\begin{aligned}
&\rho_{*+}\lambda\hat{\mathbf{u}}_+ - \mathrm{Div}\,\mathbf{S}_{*+}(\hat{\mathbf{u}}_+) = \hat{\mathbf{f}}_{+0} && \text{in } \mathbb{R}^N_+\\
&\rho_{*-}\lambda\hat{\mathbf{u}}_- - \mathrm{Div}\,\mathbf{S}_{*-}(\hat{\mathbf{u}}_-) + \nabla\hat{\pi}_- = \hat{\mathbf{f}}_{-0}, \quad \mathrm{div}\,\hat{\mathbf{u}}_- = \hat{f}_{\rm div\,0} = \mathrm{div}\,\hat{\mathbf{f}}_{\rm div\,0} && \text{in } \mathbb{R}^N_-
\end{aligned} \tag{5.1}$$

subject to the interface condition: for $x \in \mathbb{R}^N_0$

$$\begin{aligned}
&\mu_{*-}D_{iN}(\hat{\mathbf{u}}_-)|_- - \mu_{*+}D_{iN}(\hat{\mathbf{u}}_+)|_+ = \hat{g}_{i0} \quad (i = 1, \ldots, N-1),\\
&(\mu_{*-}D_{NN}(\hat{\mathbf{u}}_-) - \hat{\pi}_-)|_- - (\mu_{*+}D_{NN}(\hat{\mathbf{u}}_+) + (\lambda_{*+} - \mu_{*+})\mathrm{div}\,\hat{\mathbf{u}}_+)|_+ - \sigma\Delta'\hat{H} = \hat{g}_{N0},\\
&\frac{1}{\rho_{*-}}(\mu_{*-}D_{NN}(\hat{\mathbf{u}}_-) - \hat{\pi}_-)|_- - \frac{1}{\rho_{*+}}(\mu_{*+}D_{NN}(\hat{\mathbf{u}}_+) + (\lambda_{*+} - \mu_{*+})\mathrm{div}\,\hat{\mathbf{u}}_+)|_+ = \hat{g}_{N+10},\\
&\hat{u}_{-i}|_- - \hat{u}_{+i}|_+ = \hat{h}_{i0} \quad (i = 1, \ldots, N-1),\\
&\lambda\hat{H} - \Big(\frac{\rho_{*-}}{\rho_{*-} - \rho_{*+}}\hat{u}_{-N} - \frac{\rho_{*+}}{\rho_{*-} - \rho_{*+}}\hat{u}_{+N}\Big) = \hat{d}_0.
\end{aligned} \tag{5.2}$$

By Theorem 5, we have $\hat{\mathbf{u}}_\pm(\lambda) = \mathcal{A}_\pm(\lambda)\mathbf{F}^0_\lambda$, $\pi_-(\lambda) = \mathcal{P}_-(\lambda)\mathbf{F}^0_\lambda$ and $\hat{H}(\lambda) = \mathcal{H}(\lambda)\mathbf{F}^0_\lambda$, where

$$\mathbf{F}^0_\lambda = (\hat{\mathbf{f}}_{+0}, \hat{\mathbf{f}}_{0-}, \lambda^{1/2}\hat{f}_{\rm div\,0}, \nabla\hat{f}_{\rm div\,0}, \lambda\hat{\mathbf{f}}_{\rm div\,0}, \lambda^{1/2}\hat{\mathbf{g}}_0, \nabla\hat{\mathbf{g}}_0, \lambda\hat{\mathbf{h}}_0, \lambda^{1/2}\nabla\hat{\mathbf{h}}_0, \nabla^2\hat{\mathbf{h}}_0, \hat{d}_0).$$

We set $\mathbf{u}_\pm(\cdot, t) = \mathcal{L}^{-1}[\hat{\mathbf{u}}_\pm(\cdot,\lambda)](t)$, $\pi_-(\cdot,t) = \mathcal{L}^{-1}[\hat{\pi}_-(\cdot,\lambda)]$ and $H(\cdot,t) = \mathcal{L}^{-1}[\hat{H}(\cdot,\lambda)](t)$. Note that $(\partial_t, \Lambda^{1/2}_\gamma\nabla, \nabla^2)\mathbf{u}_\pm = \mathcal{L}^{-1}[G^1_\lambda\hat{\mathbf{u}}_\pm(\cdot,\lambda)](t)$, $\nabla\pi_- = \mathcal{L}^{-1}[\nabla\hat{\pi}_-(\cdot,\lambda)](t)$ and $(\partial_t, \nabla)H = \mathcal{L}^{-1}[G^2_\lambda\hat{H}(\cdot,\lambda)](t)$. By Theorem 5 and Theorem 4, we have

$$\begin{aligned}
&\sum_{\ell=\pm}\|e^{\gamma t}(\partial_t\mathbf{u}_\ell, \Lambda^{1/2}_\gamma\nabla\mathbf{u}_\ell, \nabla^2\mathbf{u}_\ell)\|_{L_p(\mathbb{R},L_q(\mathbb{R}^N_\ell))} + \|e^{\gamma t}\nabla\pi_-\|_{L_p(\mathbb{R},L_q(\mathbb{R}^N_-))} + \|e^{\gamma t}(\partial_t H, \nabla H)\|_{L_p(\mathbb{R},W^2_q(\mathbb{R}^N))}\\
&\leq C\{\|e^{\gamma t}\mathbf{f}_{+0}\|_{L_p(\mathbb{R},L_q(\mathbb{R}^N_+))} + \|e^{\gamma t}\mathbf{f}_{-0}\|_{L_p(\mathbb{R},L_q(\mathbb{R}^N_-))} + \|e^{\gamma t}(\Lambda^{1/2}_\gamma f_{\rm div\,0}, \nabla f_{\rm div\,0})\|_{L_p(\mathbb{R},L_q(\mathbb{R}^N_-))}\\
&\quad + \|e^{\gamma t}\partial_t\mathbf{f}_{\rm div\,0}\|_{L_p(\mathbb{R},L_q(\mathbb{R}^N_-))} + \sum_{j=1}^{N+1}\|e^{\gamma t}(\Lambda^{1/2}_\gamma g_{j0}, \nabla g_{j0})\|_{L_p(\mathbb{R},L_q(\mathbb{R}^N))}\\
&\quad + \sum_{j=1}^{N-1}\|e^{\gamma t}(\partial_t h_{j0}, \Lambda^{1/2}\nabla h_{j0}, \nabla^2 h_{j0})\|_{L_p(\mathbb{R},L_q(\mathbb{R}^N))} + \|e^{\gamma t}d_0\|_{L_p(\mathbb{R},W^2_q(\mathbb{R}^N))}\}.
\end{aligned}$$

for any $\gamma \geq \gamma_0$ with some constants $\gamma_0 > 0$ and $C > 0$. And, $\mathbf{f}_{\pm 0}$, $f_{\mathrm{div}\,0}$, $\mathbf{f}_{\mathrm{div}\,0}$, g_{j0}, h_{j0} and d_0 vanish for $t < 0$, so that we can show that $\mathbf{u}_\pm$, π_- and H also vanish for $t < 0$. Moreover, we use the following facts:

$$\begin{aligned}
C_1\|e^{\gamma t}(\partial_t f, \Lambda_\gamma^{1/2}\nabla f, \nabla^2 f)\|_{L_p(\mathbb{R},L_q(\Omega))} &\leq \|e^{\gamma t}\partial_t f\|_{L_p(\mathbb{R},L_q(\Omega))} + \|e^{\gamma t}f\|_{L_p(\mathbb{R},W^2_q(\Omega))}\\
&\leq C_2\|e^{\gamma t}(\partial_t f, \Lambda_\gamma^{1/2}\nabla f, \nabla^2 f)\|_{L_p(\mathbb{R},L_q(\Omega))},\\
\|e^{\gamma t}\Lambda_\gamma^{1/2} f\|_{L_p(\mathbb{R},L_q(\mathbb{R}^N))} &\leq C\{\|e^{\gamma t}\partial_t f\|_{L_p(\mathbb{R},W^{-1}_q(\mathbb{R}^N))} + \|e^{\gamma t}f\|_{L_p(\mathbb{R},W^1_q(\mathbb{R}^N))}\}.
\end{aligned}$$

with some positive constants C_1, C_2 and C, where $\Omega = \mathbb{R}^N_\pm$ and $\Omega = \mathbb{R}^N$. In this way, we can prove Theorem 2 for $T = \infty$. When T is finite, we use the cut-off procedure with respect to time variable.

The detailed proof concerning the incompressible one phase problem is found in Shibata [7, 8, 9].

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Quasi-convex Hamilton-Jacobi equations on networks

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Abstract

In this talk, I will present results obtained in collaboration with Régis Monneau in [3]. They are concerned with Hamilton-Jacobi equations on networks. The Hamiltonians are quasi-convex with respect to the gradient variable and can be discontinuous with respect to the space variable at vertices of the network. We explain how general jonction conditions reduce, in this setting, to jonction conditions of optimal control type and we prove a general comparison principle between sub-linear sub- and supersolutions.

Keywords: Hamilton-Jacobi equations, networks, quasi-convex Hamiltonians, discontinuous Hamiltonians, flux-limited solutions, comparison principle, vertex test function, optimal control, discontinuous running cost.

1 The simplest network: a junction

The simplest network is made of one vertex and a finite number of infinite edges; it is referred to as a *junction*. For the sake of clarity, Hamiltonians are assumed to be constant with respect to the space variable on each edge.

A junction can be viewed as the set of N distinct copies ($N \geq 1$) of the half-line which are glued at the origin. For $i = 1, ..., N$, each branch J_i is assumed to be isometric to $[0, +\infty)$ and

$$J = \bigcup_{i=1,...,N} J_i \qquad \text{with} \quad J_i \cap J_j = \{0\} \quad \text{for} \quad i \neq j \tag{1.1}$$

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where the origin 0 is called the *junction point*. For points $x, y \in J$, $d(x,y)$ denotes the geodesic distance on J defined as

$$d(x,y) = \begin{cases} |x-y| & \text{if } x, y \text{ belong to the same branch,} \\ |x|+|y| & \text{if } x, y \text{ belong to different branches.} \end{cases}$$

For a smooth real-valued function u defined on J, $\partial_i u(x)$ denotes the (spatial) derivative of u at $x \in J_i$ and the "gradient" of u is defined as follows,

$$u_x(x) := \begin{cases} \partial_i u(x) & \text{if} \quad x \in J_i^* := J_i \setminus \{0\}, \\ (\partial_1 u(0), ..., \partial_N u(0)) & \text{if} \quad x = 0. \end{cases} \tag{1.2}$$

With such a notation in hand, we consider the following Hamilton-Jacobi equation on the junction J

$$\begin{cases} u_t + H_i(u_x) = 0 & \text{for} \quad t \in (0,+\infty) & \text{and} \quad x \in J_i^*, \\ u_t + F(u_x) = 0 & \text{for} \quad t \in (0,+\infty) & \text{and} \quad x = 0 \end{cases} \tag{1.3}$$

subject to the initial condition

$$u(0,x) = u_0(x) \quad \text{for} \quad x \in J. \tag{1.4}$$

Structure condition on the Hamiltonians: there exist numbers $p_i^0 \in \mathbb{R}$ such that for each $i = 1, ..., N$,

$$\begin{cases} \textbf{(Continuity)} & H_i \in C(\mathbb{R}) \\ \textbf{(Quasi-convexity)} & \begin{cases} H_i \text{ nonincreasing in } (-\infty, p_i^0] \\ H_i \text{ nondecreasing in } [p_i^0, +\infty) \end{cases} \\ \textbf{(Coercivity)} & \lim_{|q|\to+\infty} H_i(q) = +\infty. \end{cases} \tag{1.5}$$

Condition on the *junction function* $F : \mathbb{R}^N \to \mathbb{R}$:

$$F \text{ is continuous and non-increasing with respect to all variables.} \tag{1.6}$$

2 Relevant junction conditions

Given a *flux limiter* $A \in \mathbb{R} \cup \{-\infty\}$, the A-limited flux through the junction point is defined for $p = (p_1, \ldots, p_N)$ as

$$F_A(p) = \max\left(A, \max_{i=1,\ldots,N} H_i^-(p_i)\right) \tag{2.1}$$

for some given $A \in \mathbb{R} \cup \{-\infty\}$ where H_i^- is the nonincreasing part of H_i defined by

$$H_i^-(q) = \begin{cases} H_i(q) & \text{if} \quad q \le p_i^0, \\ H_i(p_i^0) & \text{if} \quad q > p_i^0. \end{cases}$$

We now consider the following important special case of (1.3),

$$\begin{cases} u_t + H_i(u_x) = 0 & \text{for} \quad t \in (0,+\infty) & \text{and} \quad x \in J_i^*, \\ u_t + F_A(u_x) = 0 & \text{for} \quad t \in (0,+\infty) & \text{and} \quad x = 0. \end{cases} \tag{2.2}$$

3 Main results

Theorem 3.1 (Comparison principle on a junction)**.** *Assume that the Hamiltonians satisfy* (1.5)*, the junction function satisfies* (1.6) *and that the initial datum u_0 is uniformly continuous. Then for all (relaxed) sub-solution u and (relaxed) super-solution v of* (1.3)-(1.4) *satisfying for some $T > 0$ and $C_T > 0$ that for all $(t, x) \in [0, T) \times J$,*

$$u(t, x) \le C_T(1 + d(0, x)), \quad v(t, x) \ge -C_T(1 + d(0, x)),$$

we have

$$u \le v \quad \textit{in} \quad [0, T) \times J.$$

Theorem 3.2 (General junction conditions reduce to flux-limited ones)**.** *Assume that the Hamiltonians satisfy* (1.5) *and that the junction function satisfies* (1.6) *and that the initial datum u_0 is uniformly continuous. Then there exists $A_F \in \mathbb{R}$ such that any relaxed viscosity solution of* (1.3) *is in fact a viscosity solution of* (2.2) *with $A = A_F$.*

Theorem 3.3 (Existence and uniqueness on a junction)**.** *Assume that the Hamiltonians satisfy* (1.5)*, that F satisfies* (1.6) *and that the initial datum u_0 is uniformly continuous. Then there exists a unique (relaxed) viscosity solution u of* (1.3)*,* (1.4) *such that for every $T > 0$, there exists a constant $C_T > 0$ such that*

$$|u(t, x) - u_0(x)| \le C_T \quad \textit{for all} \quad (t, x) \in [0, T) \times J.$$

4 Related works and perspectives

The general theory developed in [3] opens many perspectives and will be further developed in forthcoming works.

For example, with such a comparison principle at our disposal, it is now possible to get various homogenization results. A first one is described in [3] about a periodic equation posed on a network generated by $\epsilon\mathbb{Z}^d$. A second one was obtained even more recently in [2]. An example of applications of this result is the case where a periodic Hamiltonian $H(x, p)$ is perturbed by a compactly supported function of the space variable $f(x)$, say. Such a situation is considered in lectures by Lions at Collège de France [4] Rescaling the solution, the expected effective Hamilton-Jacobi equation is supplemented with a junction condition which keeps memory of the compact perturbation.

We would also like to mention that the extension of our results to a higher dimensional setting (in the spirit of [1]) is now reachable for quasi-convex Hamiltonians and will be achieved soon in a future work.

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WEAKLY COUPLED SYSTEMS OF THE INFINITY LAPLACE EQUATIONS

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1. Introduction

The talk is based on the results in [14] of the recent collaboration with Hung V. Tran. We consider the Dirichlet problem for the weakly coupled systems of the infinity Laplace equations:

$$\begin{cases} -\Delta_\infty u_i + \displaystyle\sum_{j=1}^m c_{ij}(u_i - u_j) = 0 & \text{in } U \text{ for } i = 1, \dots, m \\ u_i = g_i & \text{on } \partial U \text{ for } i = 1, \dots, m, \end{cases} \tag{1.1}$$

where U is a bounded domain with a smooth boundary in $\mathbb{R}^n$, and $(c_{ij})_{i,j=1}^m$ is a given constant matrix which describes the generator of an irreducible continuous-time Markov chain with m states satisfying

$$c_{ij} > 0 \text{ for } i \neq j, \quad \text{and} \quad \sum_{j=1}^m c_{ij} = 0,$$

and $g_i \in C(\partial U)$ are given functions for $i = 1, \dots, m$. Here u_i are unknown functions and the operator Δ_∞ is the so-called *game infinity Laplacian*, i.e., for a smooth function f,

$$\Delta_\infty f := \frac{\mathrm{tr}\left(Df \otimes Df D^2 f\right)}{|Df|^2} = \frac{\sum_{i,j=1}^n f_{x_i} f_{x_j} f_{x_i x_j}}{|Df|^2}.$$

The study of the infinity Laplacian began with pioneer works by Aronsson [2, 3] to understand a so-called *absolutely minimizing Lipschitz* function. More precisely, the equation arises in the L^∞ calculus of variations as the Euler–Lagrange equation for properly interpreted minimizers of all of energy functionals $u \mapsto \|Du\|_{L^\infty(V)}$ for all open sets $V \subseteq U$. Aronsson achieved existence results and pointed out that we cannot expect the classical solutions in general. However, he could not prove uniqueness and stability results. It turned out that the theory of viscosity solution is an appropriate instrument for the study of infinity Laplacian. Jensen [11] gave fundamental results on the comparison principle and hence uniqueness of the single infinity Laplace equation in the viscosity solution sense, and generated considerable interest in the theory. Nowadays, there are a great number of works related to the infinity Laplace equation.

In the talk, I present (i) Derivation, (ii) Characterization of solutions by comparison with "generalized cones" for systems. If time permitted, I want to discuss the application of comparison with "generalized cones", which is a property of blow up limits.

2. Derivation

Peres, Schramm, Sheffield, and Wilson [15] showed that the infinity Laplace equation arises in the study of certain two-player, zero-sum stochastic games. They introduced a random-turn game called *ε-tug-of-war*, in which two players try to move a token in an open set U toward a favorable spot on the boundary ∂U corresponding to a given payoff function g on ∂U. Inspired by this work, we derive the system of the infinity Laplace equation (1.1).

Let $U \subset \mathbb{R}^n$ be a bounded domain with smooth boundary, which is the place where the game is played by two persons, player I and player II. Suppose that there are m modes: mode 1, …, mode m, and m corresponding the number of given functions $g_i \in C(\partial U)$ for $i = 1, \ldots, m$. We call g_i the payoff function on the boundary of U corresponding to mode i for $1 \le i \le m$. We consider the following two-player, zero-sum game.

Fix a number $\varepsilon > 0$, a token $x_0 := x \in U$, and a mode $m_0 := i \in \{1, \ldots, m\}$. Suppose that both players start the game at position $x_0 = x$ and mode $m_0 = i$, and have the same position and mode all the time. At each time step $t_k := \varepsilon^2 k$ for $k \in \mathbb{N}$, the players toss a fair coin and the winner of the toss is allowed to choose a next token $x_k \in \overline{B}(x_{k-1}, \varepsilon) \cap \overline{U}$, and the mode is switched from m_{k-1} to mode $m_k = j$ for any $j \in \{1, \ldots, m\}$ with the probability which is determined by a piecewise-deterministic Markov process introduced by Davis [8]. The change from modes to modes with the starting point $m_0 = i$ is determined by a continuous-time Markov chain on $[0, \infty)$: $\nu(0) = i$, and for $\Delta s > 0$,

$$\mathbb{P}\big(\nu(s + \Delta s) = j \mid \nu(s) = i\big) = \frac{c_{ij}}{2}\Delta s + o(\Delta s) \quad \text{as } \Delta s \to 0 \text{ for } i \ne j, \tag{2.1}$$

where $o : [0, \infty) \to [0, \infty)$ is a function satisfying $o(r)/r \to 0$ as $r \to 0$. After k steps, if $x_k \in U$ then the game moves to step $k + 1$. Otherwise, if $x_k \in \partial U$ then the game ends and player II pays the payoff $g_{m_k}(x_k)$ to player I as they are at mode $m_k = \nu(t_k)$. Notice that the change of modes is determined solely by the Markov chain (2.1), and is not determined by the two players. In particular, $\nu(t_k)$ can take any value in $\{1, \ldots, m\}$ with probability determined by (2.1). The expected payoff is

$$\mathbb{E}_i\big[g_{\nu(t_k)}(x_k)\big].$$

A *strategy* for a player is a way of choosing the players' next move as a function of all previous information (played moves, all known coin tosses and known states.) It is a map from the set of partially played games to moves (or in the case of a random strategy, a probability distribution on moves.) Usually, one would think of a good strategy as being Markovian, i.e., as a map from the current state to the next move. However, in some settings, it is also useful to allow more general strategies that take into account the history.

We consider the *value* which the players get. Of course player I wants to maximize the expected payoff, while player II wants to minimize it in this tug-of-war game. Let S_I and S_{II} be the strategies of player I and player II, respectively, and then we define the cost functions by

$$J_i^{\varepsilon}(S_I, S_{II})(x) := \begin{cases} \mathbb{E}_{S_I, S_{II}} \mathbb{E}_i\big[g_{\nu(t_k)}(x_k)\big] & \text{if the game terminates with probability one,} \\ -\infty & \text{otherwise,} \end{cases}$$

where x and i are the starting point and mode of the game. The value of the game for player I is then defined as

$$u_i^{\varepsilon, I}(x) := \sup_{S_I} \inf_{S_{II}} J_i^{\varepsilon}(S_I, S_{II})(x).$$

In the talk, I show that the limit of $u_i^{\varepsilon,I}$ as $\varepsilon \to 0$ satisfies the system (1.1).

3. Characterization of solutions

Henceforth we only consider the simple system with two equations and we assume $c_{12} = c_{12} = 1, c_{11} = c_{22} = -1$.

For the single infinite Laplace equation

$$-\Delta_\infty u = 0 \quad \text{in } U, \tag{3.1}$$

Crandall, Evans and Gariepy [7] realized that comparison with cones characterizes subsolutions and supersolutions of (3.1), and nowadays it is well-known that this plays important roles in the establishment of regularity results of solutions of (3.1). See [16, 9, 10]. In this section, we derive "generalized cones" for systems and establish comparison with "generalized cones".

We first present one way to find the class of particular solutions of (3.1), and that cones are solutions of (3.1) everywhere except the vertices. Let us find radially symmetric solution u of (3.1), i.e.

$$u(x) = \eta(|x|),$$

where $\eta : [0,\infty) \to \mathbb{R}$ is some smooth function. We calculate, for $x \neq 0$,

$$\begin{aligned} Du(x) &= \eta'(|x|) \cdot \frac{x}{|x|}, \\ D^2u(x) &= \eta''(|x|) \cdot \frac{x \otimes x}{|x|^2} + \eta'(|x|) \cdot \Big(I - \frac{x \otimes x}{|x|^2} \Big) \frac{1}{|x|^2}. \end{aligned}$$

Plug these into (3.1) to get that

$$-\eta''(r) = 0,$$

which implies that $\eta(r) = ar + b$ for any $a, b \in \mathbb{R}$. From these calculations, we establish that the cones

$$u(x) = a|x - x_0| + b \quad \text{for any } x_0 \in \mathbb{R}^n, \text{ and } a, b \in \mathbb{R} \tag{3.2}$$

are solutions of (3.1) in $U \setminus \{x_0\}$.

Following the idea above, we first find particular solutions of (1.1) in the form of cones' like. We consider u_i radially symmetric of the form

$$u_i(x) = \eta_i(|x|),$$

where $\eta_i : [0,\infty) \to \mathbb{R}$ are smooth functions for $i = 1, 2$. Assume that (u_1, u_2) is a solution of (1.1) in $\mathbb{R}^n \setminus \{0\}$. Then (η_1, η_2) satisfies

$$\begin{cases} -\eta_1'' + \eta_1 - \eta_2 = 0 & \text{in } (0,\infty), \\ -\eta_2'' + \eta_2 - \eta_1 = 0 & \text{in } (0,\infty). \end{cases} \tag{3.3}$$

Solving this system of ordinary differential equations with arbitrary initial data at 0, we get that, for $s > 0$,

$$\begin{cases} \eta_1(s) = C_1 e^{\sqrt{2}s} + C_2 e^{-\sqrt{2}s} + as + b, \\ \eta_2(s) = -C_1 e^{\sqrt{2}s} - C_2 e^{-\sqrt{2}s} + as + b, \end{cases}$$

where C_1, C_2, a, b are arbitrary constants.

We can then easily check that the pair (ψ_1, ψ_2) defined by

$$\begin{cases} \psi_1(x) := C_1 e^{\sqrt{2}|x-x_0|} + C_2 e^{-\sqrt{2}|x-x_0|} + a|x - x_0| + b, \\ \psi_2(x) := -C_1 e^{\sqrt{2}|x-x_0|} - C_2 e^{-\sqrt{2}|x-x_0|} + a|x - x_0| + b, \end{cases} \tag{3.4}$$

is a solution of (1.1) in $\mathbb{R}^n \setminus \{x_0\}$ for any $x_0 \in \mathbb{R}^n, C_1, C_2, a, b \in \mathbb{R}$. We call (ψ_1, ψ_2) a pair of *"generalized cones"*.

We introduce the notion of comparison with "generalized cones" following the single case.

Definition 1 (Comparison with "Generalized Cones")**.** (i) *A pair $(u_1, u_2) \in C(\overline{U})^2$ enjoys comparison with "generalized cones" from above in U if (u_1, u_2) satisfies that for any $x_0 \in U$ and $r > 0$ such that $\overline{B}(x_0, r) \subset U$,*

$$\text{if } u_i \le \psi_i \text{ on } \partial B(x_0, r) \cup \{x_0\} \text{ for } i = 1, 2, \text{ then } u_i \le \psi_i \text{ on } \overline{B}(x_0, r) \text{ for } i = 1, 2,$$

for any choices of $C_1, C_2, a, b \in \mathbb{R}$.
(ii) *A pair $(u_1, u_2) \in C(\overline{U})^2$ enjoys comparison with "generalized cones" from below in U if (u_1, u_2) satisfies that for any $x_0 \in U$ and $r > 0$ such that $\overline{B}(x_0, r) \subset U$,*

$$\text{if } u_i \ge \psi_i \text{ on } \partial B(x_0, r) \cup \{x_0\} \text{ for } i = 1, 2, \text{ then } u_i \ge \psi_i \text{ on } \overline{B}(x_0, r) \text{ for } i = 1, 2,$$

for any choices of $C_1, C_2, a, b \in \mathbb{R}$.

In the case of the single equation (3.1), to characterize subsolutions by using comparison with cone, one could choose in (3.2)

$$a := \frac{\max_{|y-x_0|=r} u(y) - u(x_0)}{r}, \quad b := u(x_0).$$

For comparison with "generalized cones" for systems, we need to appropriately choose C_1, C_2, a, b in (3.4). In order to do so, we introduce the following notations. For $x_0 \in U$, $r > 0$ such that $\overline{B}(x_0, r) \subset U$, we set

$$\begin{aligned}
M_i(x_0, r) &:= \max_{|y-x_0|=r} u_i(y), \\
C_1(x_0, r) &:= \frac{-(u_1(x_0) - u_2(x_0))e^{-\sqrt{2}r}}{2(e^{\sqrt{2}r} - e^{-\sqrt{2}r})} + \frac{M_1(x_0, r) - M_2(x_0, r)}{2(e^{\sqrt{2}r} - e^{-\sqrt{2}r})}, \\
C_2(x_0, r) &:= \frac{(u_1(x_0) - u_2(x_0))e^{\sqrt{2}r}}{2(e^{\sqrt{2}r} - e^{-\sqrt{2}r})} - \frac{M_1(x_0, r) - M_2(x_0, r)}{2(e^{\sqrt{2}r} - e^{-\sqrt{2}r})}, \\
a(x_0, r) &:= \frac{M_1(x_0, r) + M_2(x_0, r) - (u_1(x_0) + u_2(x_0))}{2r}, \\
b(x_0) &:= \frac{u_1(x_0) + u_2(x_0)}{2}.
\end{aligned}$$

Here is one of the main theorems of [14]:

Theorem 3.1 (Characterization of Subsolutions and Supersolution of (1.1))**.**
Let $(u_1, u_2) \in C(\overline{U})^2$. The pair (u_1, u_2) is a viscosity subsolution (*resp., supersolution*) *of* (1.1) *if and only if (u_1, u_2) satisfies comparison with "generalized cones" from above* (*resp., below*)*.*

4. Linearity of blow-up limits

Take $x_0 \in U$ and $R > 0$ such that $\overline{B}(x_0, R) \subset U$. For each $r > 0$ sufficiently small, set

$$v_i^r(x) = \frac{u_i(x_0 + rx) - u_i(x_0)}{r}, \quad \text{for } |x| \le \frac{R}{r}, \ i = 1, 2.$$

Clearly $\{v_i^r\}$ is precompact in $C(B(0, R))$. Thus for any sequence $\{r_j\}_{j\in\mathbb{N}}$ with $r_j \to 0$ as $j \to \infty$, we can pass to a subsequence if necessary and get $v_i^{r_j} \to v_i \in \mathrm{Lip}\,(\mathbb{R}^n)$ locally uniformly in $\mathbb{R}^n$ as $j \to \infty$. We call v_i a *blow-up limit* of u_i. We now prove that all of blow-up limits v_i are affine. Notice that (v_1, v_2) here really depends on the subsequence we take. In general, a pair (v_1, v_2) of blow-up limits depends on the choice of subsequences and it might not be unique.

Let us recall the literature on regularity results for the single infinity Laplace equation here. Note first that in all of these papers, the result on affine blow-up limits [7] plays an important role. Savin [16] showed that this blow-up limit is unique and achieved C^1 regularity for solutions in case $n = 2$. Evans and Savin [9] then established $C^{1,\alpha}$ regularity for solutions in this setting. The proofs in [16, 9] depend highly on the geometry of the 2-dimensional space and cannot be extended to the case with $n \geq 3$. Recently, Evans and Smart [10] used the nonlinear adjoint method to prove that this blow-up limit is unique, which yields the differentiability everywhere of solutions for all $n \geq 2$. The questions on C^1 and $C^{1,\alpha}$ regularity, however, are still open for $n \geq 3$.

Here is another main result of [14].

Theorem 4.1. *All of blow up limits of u_i are affine for $i = 1, 2$.*

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Sharp well-posedness for the Chern-Simons-Dirac system in one dimension

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1 Introduction

This talk is based on a joint work with Professor Shuji Machihara (Saitama University).

We consider the Cauchy problem for the Chern-Simons-Dirac system in one spatial dimension:

$$\begin{cases} i\gamma^0(\partial_t - iA_0)\psi + i\gamma^1(\partial_x - iA_1)\psi = m\psi, & (t,x) \in \mathbb{R}^{1+1}, \\ \partial_t A_1 - \partial_x A_0 = \psi^\dagger \gamma^0 \psi, & (t,x) \in \mathbb{R}^{1+1}, \\ \partial_t A_0 - \partial_x A_1 = 0, & (t,x) \in \mathbb{R}^{1+1}, \\ \psi(0,x) = \psi_0(x),\ \mathbb{A}(0,x) = \mathbb{A}_0(x), & x \in \mathbb{R}, \end{cases} \tag{1.1}$$

where the spinor $\psi = \binom{\psi_1}{\psi_2}$ is a $\mathbb{C}^2$-valued unknown function, the gauge components A_0 and A_1 of the gauge $\mathbb{A} = \binom{A_0}{A_1}$ are real valued unknown functions, and $\psi_0 = \binom{\psi_{0,1}}{\psi_{0,2}}$, $\mathbb{A}_0 = \binom{A_{0,0}}{A_{0,1}}$ are given $\mathbb{C}^2$ and $\mathbb{R}^2$ valued functions, respectively, and $m \geq 0$ is a constant. We use $\psi^\dagger$ to denote the conjugate transpose of ψ. The Dirac matrices are definded as follows:

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The Chern-Simons action was proposed as an alternative gauge field theory to the standard Maxwell theory of electrodynamics on Minkowski space $\mathbb{R}^{1+2}$ ([5]). This $\mathbb{R}^{1+1}$ model system (1.1) was introduced by Huh [6].

We state two important mathematical properties for the Chern-Simons-Dirac system. Firstly, the solutions to the Chern-Simons-Dirac system satisfy the conservation of charge, i.e., we have $\|\psi(t)\|_{L^2} = \|\psi_0\|_{L^2}$ for any $t > 0$. Secondly, the Chern-Simons-Dirac system with $m = 0$ is invariant under the scale transformation

$$\psi(t,x) \to \lambda\psi(\lambda t, \lambda x), \quad \mathbb{A}(t,x) \to \lambda\mathbb{A}(\lambda t, \lambda x) \tag{1.2}$$

for any $\lambda > 0$. We study the well-posedness for the solution in the standard inhomogeneous Sobolev spaces

$$(\psi(t), \mathbb{A}(t)) \in H^s(\mathbb{R}) \times H^r(\mathbb{R}) \tag{1.3}$$

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with some indices $(s,r) \in \mathbb{R}^2$. This scaling test implies the homogeneous Sobolev space $\dot{H}^{-1/2}(\mathbb{R}) \times \dot{H}^{-1/2}(\mathbb{R})$ is the scale invariant space, and we call $s = r = -1/2$ in (1.3) the critical regularity for the problem (1.1). In many cases of other problems, it is difficult to obtain the well-posedness in spaces below the critical regularities of the each problems.

We introduce some known results on this problem. Huh [6] showed local in time well-posedness of the Cauchy problem (1.1) in $L^2(\mathbb{R}) \times L^2(\mathbb{R})$ and global in time well-posedness in $H^1(\mathbb{R}) \times H^1(\mathbb{R})$. Huh observed the null structure of the Chern-Simons-Dirac system and used the charge conservation law to extend the local solution to global one. Bournaveas, Candy, and Machihara [1] showed local in time well-posedness in $H^s(\mathbb{R}) \times H^r(\mathbb{R})$ with $-1/2 < r \leq s \leq r+1$. It is an almost critical result since r and s can be close to $-1/2$. They also showed global in time well-posedness under the additional condition $s \geq 0$. However, they did not put any answer on the problem with the critical regularity $s = r = -1/2$. With regards to this point, we remark that Machihara and Ogawa [8] obtained global in time well-posedness in $L^1(\mathbb{R}) \times L^1(\mathbb{R})$ which is also invariant under the scale transformation (1.2).

The following is our well-posed result.

Theorem 1. *Let (s,r) satisfy*

$$(-1/2 < s < 1/2 \text{ and } r = -1/2) \text{ or } (s = 0 \text{ and } -1 \leq r \leq -1/2), \tag{1.4}$$

then the Cauchy problem (1.1) is time locally well-posed in $H^s(\mathbb{R}) \times H^r(\mathbb{R})$.

Remark 1. Theorem 1 says that the well-posedness holds in the scale critical regularity for $A_\pm$. Furthermore, the Sobolev regularity for $A_\pm$ crosses over the scale critical regularity if the spinor belongs to the charge class $L^2(\mathbb{R})$.

In order to obtain well-posedness of (1.1) with such low regularities, we extract the worst part of nonlinearity, which appears in the massless case $m = 0$. The worst part and remaining parts can be handled separately and estimates of remaining parts similar to the argument in [1].

We remark that Theorem 1 does not follow from a standard iteration argument from the following fact:

Theorem 2. *Let $s \in \mathbb{R}$ and $r \leq -1/2$. Then the flow map $(\psi_0, \mathbb{A}_0) \mapsto (\psi, \mathbb{A})$: $H^s(\mathbb{R}) \times H^r(\mathbb{R}) \to C([-T,T]; H^s(\mathbb{R}) \times H^r(\mathbb{R}))$ of the Cauchy problem (1.1) is not locally uniformly continuous.*

However Theorem 2 does not imply the ill-posedness, it precludes proofs of the well-posedness by the contraction argument. If the contraction argument would work, the flow map would be C^∞ and so locally uniformly continuous.

Thanks to the charge conservation law and the Delgado-Candy trick as in [1], we can extend the local solution to global.

Corollary 3. *Let (s,r) satisfy (1.4) and $s \geq 0$. Then the Cauchy problem (1.1) is time globally well-posed in $H^s(\mathbb{R}) \times H^r(\mathbb{R})$.*

We may say that Theorem 1 (combined with the result of [1]) is sharp. Namely, the Cauchy problem is ill-posed with (s,r) in the remaining regions. We give the ill-posed result.

Theorem 4 ([9])**.** *The Cauchy problem (1.1) is ill-posed in $H^s(\mathbb{R}) \times H^r(\mathbb{R})$ if one of the followings holds:*

(a) $r > s$,

(b) $s > 0$ *and* $r < s - 1$,

(c) $s \neq 0$ *and* $r < -1/2$,

(d) $(s, r) = (1/2, -1/2)$,

(e) $(s, r) = (-1/2, -1/2)$,

(f) $s = 0$ *and* $r < -1$.

Note that (e) is the scale critical regularity. We thus have ill-posedness in $H^{-1/2}(\mathbb{R}) \times H^{-1/2}(\mathbb{R})$ although well-posedness holds in the scale critical Lebesgue space $L^1(\mathbb{R}) \times L^1(\mathbb{R})$ [8].

All the ill-posedness results in Theorem 4 come from the observing that the flow map is discontinuous. Furthermore, the discontinuity of the flow map is caused by the norm inflation expect for (f). Here, the norm inflation means that for any $\varepsilon > 0$ there exist a solution $(\psi, \mathbb{A})$ of (1.1) and $t > 0$ such that $(\psi_0, \mathbb{A}_0) \in \mathcal{S}(\mathbb{R}) \times \mathcal{S}(\mathbb{R})$,

$$\|\psi_0\|_{H^s} + \|\mathbb{A}_0\|_{H^r} < \varepsilon, \quad 0 < t < \varepsilon, \quad \|\psi(t, \cdot)\|_{H^s} + \|\mathbb{A}(t, \cdot)\|_{H^r} > \varepsilon^{-1}.$$

Because the charge conservation $\|\psi(t, \cdot)\|_{L^2} = \|\psi_0\|_{L^2}$ and discontinuity of the flow map is ascribed to the nonlinear part of the Dirac equation, we never expect the norm inflation in the case (f).

Here we also emphasize that we can deal with either massless case $m = 0$ and massive case $m > 0$ for the ill-posedness. As we see below, we need an extra technique to estimate the mass term for giving the norm inflation.

2 Reduction of the problem under some conditions

In the same way with the previous works [1], [6], we diagonalize (1.1) by setting $u_\pm = \psi_1 \pm \psi_2$, $A_\pm = A_0 \mp A_1$, $u_{\pm,0} = \psi_{0,1} \pm \psi_{0,2}$, and $a_\pm = A_{0,0} \mp A_{0,1}$ to have

$$\begin{cases} (\partial_t \pm \partial_x) u_\pm = -imu_\mp + iA_\mp u_\pm, \\ (\partial_t \pm \partial_x) A_\pm = \mp \Re(u_+ \overline{u_-}), \\ u_\pm(0, x) = u_{\pm,0}(x), \ A_\pm(0, x) = a_\pm(x). \end{cases} \tag{2.1}$$

We see in the left hand side that u_+ and u_- are the solutions for the transport equations which move to the positive and negative direction respectively when t increases. The functions A_+ and A_- are also in the same manner. In the right hand side, the two nonlinear terms $A_\mp u_\pm$ and $u_+ \bar{u}_-$ consist of the two functions which have different signs to each others. This is called null form. We will make use of this form very well to estimate the bilinear terms.

Here we introduce the reduction of the problem by taking the special conditions which we assumed in the previous paper [9]. We set $m=0$. Then, (2.1) is equivalent to the following system of integral equations:

$$\begin{aligned} u_\pm(t,x) &= u_{\pm,0}(x\mp t)\exp\left(i\int_0^t A_\mp(t',x\mp(t-t'))dt'\right), \\ A_\pm(t,x) &= a_\pm(x\mp t)\mp\int_0^t \Re(u_+\overline{u_-})(t',x\mp(t-t'))dt'. \end{aligned} \tag{2.2}$$

If we take $u_{-,0}=0$, which implies that $\Re(u_+\overline{u_-})=\Re(u_{+,0}\overline{u_{-,0}})=0$, (2.2) then becomes

$$\begin{aligned} &u_+(t,x)=u_{+,0}(x-t)\exp\left(\frac{i}{2}\int_{x-t}^{x+t}a_-(t')dt'\right), \\ &u_-(t,x)=0, \quad A_\pm(t,x)=a_\pm(x\mp t). \end{aligned}$$

Similar observations with this can be found in the papers by Chadam and Glassey [3] and Ozawa and Yamauchi [10] for the Dirac-Klein-Gordon system with Yukawa coupling.

Let K be a mapping from $H^s(\mathbb{R})\times\mathbb{H}^r(\mathbb{R})$ to $H^s(\mathbb{R})$ defined by

$$K(f,g):=fe^{ig}, \tag{2.3}$$

where $\mathbb{H}^r(\mathbb{R})$ denotes the space which consists of the real-valued functions in $H^r(\mathbb{R})$. Hence, under the special conditions, Theorem 1 is reduced to prove the mapping K is continuous if (s,r) satisfies (1.4). Theorem 4 (except for (a)) follows from observing that the mapping K is not continuous.

We remark, finally, we can remove these special conditions for the both Theorem 1 and Theorem 4.

3 Ill-posedness

For the proof of Theorem 4, we use the explicit representation of a solution of (1.1). Suppose $m=0$ which is the simplest case. We need some argument more for $m>0$.

We here recall the one dimensional Sobolev product estimate.

Proposition 5. *Let s_0, s_1, and s_2 be real numbers. Then,*

$$\|fg\|_{H^{-s_0}}\le C\|f\|_{H^{s_1}}\|g\|_{H^{s_1}}$$

holds if and only if

$$s_0+s_1+s_2\ge 1/2, \quad s_0+s_1+s_2\ge\max(s_0,s_1,s_2)$$

and that the two inequalities are not both equalities.

The ill-posedness in the cases (b), (c) with $s>0$, and (d) essentially follows from Proposition 5. The mapping K in (2.3) is not well-defined, of cause not continuous, in these cases. For the proof of ill-posedness for (e), we employ the duality argument and the ill-posedness in (d).

In the case (c) with $s < 0$, we expand the solution as follows:

$$u_{+,0}(x-t)\exp\left(\frac{i}{2}\int_{x-t}^{x+t} a_-(t')dt'\right) = u_{+,0}(x-t)\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{i}{2}\int_{x-t}^{x+t} a_-(t')dt'\right)^n,$$

which coincides with the series obtained by the successive approximation. Thanks to well-posedness result obtained in [1], this expansion makes sense if $-1/2 < r \le s \le r+1$. However, the convergence of the series is not assured if $s < 0$, $r < -1/2$. Using the modulation space $M^0_{2,1}(\mathbb{R})$ as in the paper [7] by Iwabuchi and Ogawa, we prove the ill-posedness. Since $M^0_{2,1}(\mathbb{R})$ is a Banach algebra, the well-posedness in $M^0_{2,1}(\mathbb{R})$ is easily obtained and the series converges in $C([-T,T];M^0_{2,1}(\mathbb{R}))$. We choose the initial data $u_{+,0}\in H^s(\mathbb{R})$ and $a_0 \in H^r(\mathbb{R})$ such that the second iterate $u^{(2)}$ is unbounded and the remaining iterates $u^{(n)}$ $(n \ne 2)$ are bounded. We have to estimate the each iterate $u^{(n)}, n = 1,2,3,\ldots$.

More precisely, we choose a sequence of the initial data $\{(u_{\pm,0,k}, a_{\pm,k})\}_{k\in\mathbb{N}} \subset \mathcal{S}(\mathbb{R}) \times \mathcal{S}(\mathbb{R})$ satisfying the following five conditions: (i) $u_{-,0,k} = 0$, (ii) the sequence of the initial data $\{(u_{+,0,k}, a_{\pm,k})\}_{k\in\mathbb{N}}$ goes to zero in $H^s(\mathbb{R}) \times H^r(\mathbb{R})$ and is unbounded in $M^0_{2,1}(\mathbb{R}) \times M^0_{2,1}(\mathbb{R})$, (iii) the existence time T_k which comes from the well-posedness in $M^0_{2,1}(\mathbb{R}) \times M^0_{2,1}(\mathbb{R})$ tends to zero as k goes to infinity, (iv) the sequence of the $C([-T_k,T_k];H^s(\mathbb{R}))$ norm of the second iterate $\{\|u^{(2)}_{\pm,k}\|_{L^\infty([-T_k,T_k];H^s)}\}_{k\in\mathbb{N}}$ is unbounded, (v) the sum of the sequences of the $C([-T_k,T_k];H^s(\mathbb{R}))$ norm of the remaining iterate $\{\|u^{(n)}_{\pm,k}\|_{L^\infty([-T_k,T_k];H^s)}\}_{k\in\mathbb{N}}$ $(n \ne 2)$ are bounded. In this case, the expansion

$$u_{+,k}(t,x) = \sum_{n=1}^{\infty} u^{(n)}_{+,k}, \quad u^{(n)}_{+,k} := u_{+,0,k}(x-t)\frac{1}{n!}\left(\frac{i}{2}\int_{x-t}^{x+t} a_{-,k}(t')dt'\right)^n$$

holds in $C([-T_k,T_k];M^0_{2,1}(\mathbb{R}))$. From the triangle inequality and conditions (iv) and (v), we obtain

$$\|u_{+,k}(T_k)\|_{H^s} \ge \|u^{(2)}_{+,k}(T_k)\|_{H^s} - \|u^{(1)}_{+,k}(T_k)\|_{H^s} - \sum_{n=3}^{\infty}\|u^{(n)}_{+,k}(T_k)\|_{H^s} \ge \frac{1}{2}\|u^{(2)}_{+,k}(T_k)\|_{H^s},$$

which implies the norm inflation. In the case (a), we apply a similar argument to consider the series obtained by the successive approximation for $A_\pm$.

4 Well-posedness

We rewrite the solution $(u_\pm, A_\pm) \to (u'_\pm, A'_\pm)$ from the idea based on the observation in the section 2. We put

$$u'_\pm(t,x) := u_\pm(t,x)\exp\left(-i\int_{x-t}^{x+t}\frac{a_\mp(t')}{2}dt'\right), \quad A'_\pm(t,x) := A_\pm(t,x) - a_\pm(x \mp t),$$
$$\Gamma(t,x) := \int_{x-t}^{x+t}\frac{a_+(t')-a_-(t')}{2}dt'.$$

Then, (2.1) is reduced to the following:

$$\begin{cases} (\partial_t \pm \partial_x)u'_\pm = -imu'_\mp e^{\pm i\Gamma} + iA'_\mp u'_\pm, \\ (\partial_t \pm \partial_x)A'_\pm = \mp\Re\big(u'_+\overline{u'_-}e^{-i\Gamma}\big), \\ u'_\pm(0,x) = u_{\pm,0}(x),\ A'_\pm(0,x) = 0. \end{cases} \tag{4.1}$$

Since the initial data for $A'_\pm$ are zero and Γ has regularity because of integration, we can prove the existence of a solution of the Cauchy problem for the modified Chern-Simons-Dirac system (4.1).

Lemma 6. *Let s and r satisfy $s-1 \leq r \leq s$ and $r > -s-1$. For $(u_{\pm,0}, a_\pm) \in H^s(\mathbb{R}) \times H^r(\mathbb{R})$, there exists $T > 0$ and a solution $(u'_\pm, A'_\pm) \in C([-T,T]; H^s(\mathbb{R}) \times H^{-1/2+\varepsilon}(\mathbb{R}))$ to (4.1) for some $\varepsilon > 0$. Moreover, the map from $(u_{\pm,0}, a_\pm)$ to $(u'_\pm, A'_\pm)$ is continuous from $H^s(\mathbb{R}) \times H^r(\mathbb{R})$ to $C([-T,T]; H^s(\mathbb{R}) \times H^{-1/2+\varepsilon}(\mathbb{R}))$.*

Using the Fourier restriction norm space (or the wave Sobolev space) and the contraction argument, we prove the existence part of this Lemma. Here, the norm of its space is defined as follows (see, for example, [1]):

$$\|u\|_{Z^{s,b}_\pm} := \|\langle \tau \mp \xi\rangle^s \langle \tau \pm \xi\rangle^b \widetilde{u}\|_{L^2_{\tau,\xi}}, \quad \|u\|_{Y^{s,b}_\pm} := \|\langle \xi\rangle^s \langle \tau \pm \xi\rangle^b \widetilde{u}\|_{L^2_\xi L^1_\tau}.$$

The weight $\tau \pm \xi$ corresponds to the linear part of (4.1). The $Z^{s,b}_\pm$ space enough to control the nonlinear parts in (4.1). However, it is not sufficient to obtain Lemma 6 because $Z^{s,b}_\pm$ is not contained inside $C(\mathbb{R}; H^s(\mathbb{R}))$ if s close to $-1/2$. Thus we use the auxiliary space $Y^{s,b}_\pm$ to control the $L^\infty H^s$ norm.

Thanks to the charge conservation law and the Delgado-Candy trick as in [1], we can extend the local solution to global. The Delgado-Candy trick is a technique to decompose $u_\pm$ into the "massless" part $u^L_\pm$ and the "massive" part $u^N_\pm$ ([4], [2]):

$$\begin{cases} (\partial_t \pm \partial_x)u^L_\pm = iA_\mp u^L_\pm, \\ (\partial_t \pm \partial_x)u^N_\pm = -imu_\mp + iA_\mp u^N_\pm, \\ u^L_\pm(0,x) = u_{\pm,0}(x),\ u^N_\pm(0,x) = 0. \end{cases} \tag{4.2}$$

The "massless" part has a representation like (2.2) and the "massive" part is essentially bounded:

$$\sup_{|t|\leq T} \|u^L_\pm(t)\|_{L^2} \leq \|u_{\pm,0}\|_{L^2}, \quad \sup_{|t|\leq T} \|u^N_\pm(t)\|_{L^2\cap L^\infty} \leq m(e^{mT} + T - 1)(\|u_{+,0}\|_{L^2} + \|u_{-,0}\|_{L^2}).$$

We hence obtain a priori bound for $A'_\pm$:

$$\sup_{|t|\leq T} \|A'(t)\|_{H^{-1/2+\varepsilon}} \leq C(e^{mT} + T)^2(\|u_{+,0}\|_{L^2} + \|u_{-,0}\|_{L^2})^2,$$

which yields the global existence.

Corollary 7. *Assume that $s \geq 0$ in Lemma 6. Then the local solution can be extended to a global solution $(u'_\pm, A'_\pm) \in C(\mathbb{R}; H^s(\mathbb{R}) \times H^{-1/2+\varepsilon}(\mathbb{R}))$.*

We remark Lemma 6 covers the almost all exponents in Theorem 1. But, the point $(s,r)=(0,-1)$ is excluded in the condition of Lemma 6. To treat this point, we employ the similar argument in [8].

Lemma 8. *Let $-1 \le r \le 0$. For $(u_{\pm,0}, a_\pm) \in L^2(\mathbb{R}) \times H^r(\mathbb{R})$, there exists a solution $(u'_\pm, A'_\pm) \in C(\mathbb{R}; L^2(\mathbb{R}) \times L^2(\mathbb{R}))$ to (4.1). Moreover, the map from $(u_{\pm,0}, a_\pm)$ to $(u'_\pm, A'_\pm)$ is continuous from $L^2(\mathbb{R}) \times H^r(\mathbb{R})$ to $C(\mathbb{R}; L^2(\mathbb{R}) \times L^2(\mathbb{R}))$.*

Machihara and Ogawa [8] used the contraction argument in the $L_t^\infty L_x^p \cap L_{t,x}^p$ space to obtain well-posedness. In the Lebesgue space, the function $e^{i\Gamma}$ is harmless because Γ is a real valued function. We can therefore apply the similar argument in [8] to (4.1).

Remark 2. Lemmas 6 and 8 show that the Duhamel part for $A_\pm$ is more regular than the initial data. More precisely, $A'(t)$ belongs to $H^{-1/2+\varepsilon}(\mathbb{R})$ even if the initial data $a_\pm \in H^r(\mathbb{R})$ with $r \le -1/2$.

From Lemmas 6 and 8, it suffices to prove that the mapping $u'_\pm \mapsto u_\pm$ is continuous in $C([-T,T]; H^s(\mathbb{R}))$.

Lemma 9. *Let ($|s| < 1/2$ and $r = -1/2$) or ($s = 0$ and $r \ge -1$). Then, $K : H^s(\mathbb{R}) \times \mathbb{H}^{r+1}(\mathbb{R}) \to H^s(\mathbb{R})$ is continuous.*

Here the range $|s| < 1/2$ is sharp. We can get the solution $(u'_\pm, A'_\pm)$ of the modified Chern-Simons-Dirac system in the wider range than (1.4). However, in order to obtain well-posedness, especially to have countinuous dependence of initail data, we have to restrict the range to (1.4) from the sharpness of Lemma 9.

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THE DYNAMICS OF VORTEX FILAMENTS WITH CORNERS

LUIS VEGA

1. Introduction

In these pages I shall sketch some recent work about the evolution of vortex filaments that follow the geometric law of the binormal: a point of the filament moves in the direction of the binormal with a speed that is proportional to the curvature. First I shall consider the case of a curve (filament) that is regular except at a point where it has a corner (joint work with V. Banica). Then, I shall look at the case of a regular polygon (joint work with F. de la Hoz).

Therefore at some point we will have to answer the simple question: **What is the velocity of the corner?**

The motivation of this geometric flow comes from fluid dynamics. In particular the well known smoke rings as those seen in figure 1. What we see in the picture are "vortex tubes" that propagate in a self-similar way. These vortex tubes have two distinguished parts. A first one which looks very much as a horseshoe, and a second one with a helicoidal shape. In figure 2 we see some vortices above an inclined triangular wing. Again self-similarity is evident together with the pair of symmetrical helices winding around two lines.

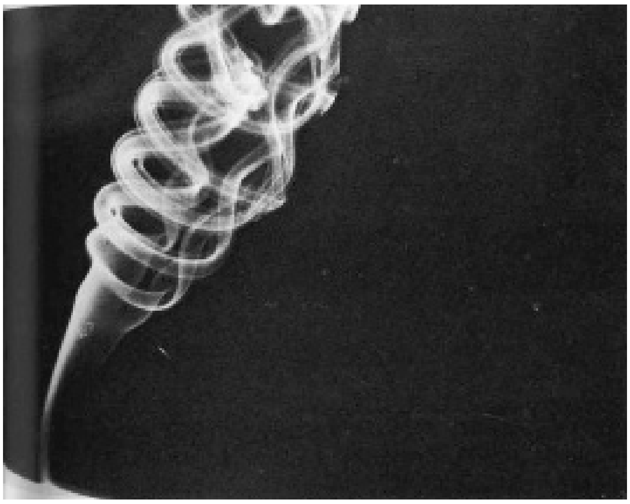

Figure 1.

Partially supported by the grants MTM2011-24054, UFI11/52 and IT-305-07.

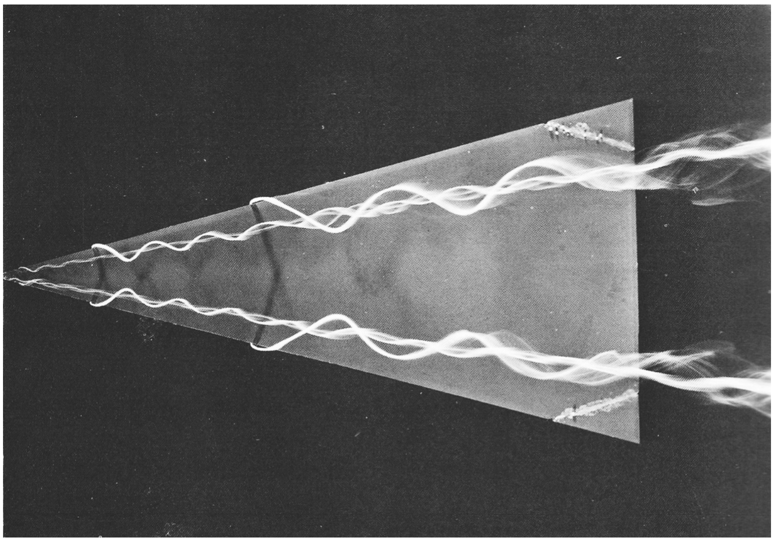

FIGURE 2.

A first simplification to describe mathematically what we see in the above two figures is to forget viscosity effects and consider that the fluids satisfy Euler equations. Therefore we have to give $\mathbf{u}$ the velocity field or as an alternative the vorticity

$$\omega = \operatorname{curl}\mathbf{u} = \nabla \wedge \mathbf{u}.$$

In our case the vorticity is a singular vector measure that has the support on a curve $\mathbf{X}$ in $\mathbb{R}^3$

$$\omega = \Gamma\mathbf{T}ds \qquad \mathbf{T} = \mathbf{X}_s.$$

Above Γ is the circulation that is a constant. Together with the inviscid condition

$$div\,\mathbf{u} = 0,$$

we can use the so-called Biot-Savart law

$$\mathbf{u}(P) = \frac{\Gamma}{4\pi}\int_{-\infty}^{\infty} \frac{\mathbf{X}(s) - \mathbf{P}}{|\mathbf{X}(s) - \mathbf{P}|^3} \wedge \mathbf{T}(s)ds, \tag{1}$$

to obtain the velocity at any point $\mathbf{P}$ that is not in the curve $\mathbf{X}(s,t)$. Particular examples are the straight lines and vortex rings. Straight lines do not move and are mathematical idealizations of bathtub vortices. Vortex rings do not change their shape and move perpendicularly to the plane where they are contained and in a direction that is determined by the sign of Γ. We encourage the reader to see the movie `http://news.uchicago.edu/multimedia/vortex-tied-knots` where the evolution of different vortex filaments can be seen and in particular the case of one ring. Finally particular solutions of Euler equations with $\mathbf{X}$ with a helicoidal shape are known since the work of Hardin [14].

In order to compute the velocity of the curve we should be able to compute (1) for a point of the curve, say $\mathbf{X}(s_0,t)$. A simple look at the Biot-Savart integral tells us that this is not an easy task unless some simplifications are done. The simplest one is to consider that just local effects are relevant and to make a Taylor expansion around $s = s_0$ to capture them. The first term is determined by the tangent at that point. If this term is the only one considered it is like changing the curve for the tangent line at that point and therefore the overall contribution to the velocity field is zero. As can be expected the next relevant term has a singularity that depends logarithmically with the distance of P to $\mathbf{X}(s_0, t)$. The usual procedure, that goes back to Da Rios in 1906 [7] (see also [16] for the details and the limitations of this approach) is to renormalize the time variable to avoid this singularity. After this renormalization and making all the relevant constants equal to unity we are lead to the equation

$$\mathbf{X}_t = \mathbf{X}_s \wedge \mathbf{X}_{ss}. \tag{2}$$

Calling

$$\begin{aligned} &c = c(s,t) && \text{curvature} \\ &\tau = \tau(s,t) && \text{torsion} \\ &\mathbf{n} = \mathbf{n}(s,t) \in \mathbb{R}^3 && \\ &\mathbf{b} = \mathbf{b}(s,t) && \text{binormal,} \end{aligned}$$

we get from Frenet equations

$$\begin{array}{rcrrr} \mathbf{T}_s & = & & c\mathbf{n} & \\ \mathbf{n}_s & = & -c\mathbf{T} & & +\tau\mathbf{b} \\ \mathbf{b}_s & = & & -\tau\mathbf{n} & \end{array} \tag{3}$$

that (2) can be rewritten as

$$\mathbf{X}_t = c\mathbf{b}. \tag{4}$$

Particular examples are the straight line, the circle, that are obvious, and the helix. The helix is easily obtained solving the equation that the tangent vector $\mathbf{T}$ satisfies. In fact, differentiating with respect to the spatial variable in (2) we get

$$\mathbf{T}_t = \mathbf{T} \wedge \mathbf{T}_{ss}. \tag{5}$$

If we look for traveling wave solutions $\mathbf{T}(s,t) = \mathbf{R}(s - bt)$ we get from (5)

$$-a\mathbf{R}' = \mathbf{R} \wedge \mathbf{R}''.$$

From (3) we obtain

$$-b\, c\mathbf{n} = \mathbf{R} \wedge \left(c'\mathbf{n} - c^2\mathbf{R} + c\tau\mathbf{b}\right).$$

As a consequence $c' = 0$. Hence $c = a$ and $\tau = b$, and therefore we have a helix.

Several remarks about (2) are in order

Remark 1.

- $|T|^2 =$ *constant. This follows immediately form* (5)*;*
- *Equation* (2) *is time reversible: If* $\mathbf{X}$ *is a solution, so is* $\widetilde{X}(s,t) = X(-s,-t)$*. In other words, a reorientation of the curves is equivalent to change the direction of time,*
- *The equation is rotation invariant.*

2. Self-similar Solutions

It seems very natural after looking at the figures 1 and 2 to look for self-similar solutions of (1). That is to say, solutions $\mathbf{X}$ that can be written as

$$\mathbf{X}(s,t) = \sqrt{t}\mathbf{G}\left(s/\sqrt{t}\right) \tag{6}$$

for some $\mathbf{G}$. If we call $\mathbf{G}_s = \mathbf{R}$ and take $\mathbf{T}(s,t) = \mathbf{R}\left(s/\sqrt{t}\right)$ we get from (3) that

$$-\frac{s}{2}\mathbf{R}' = \mathbf{R} \wedge \mathbf{R}_{ss}.$$

Using again Frenet equations (2) we obtain this time

$$-\frac{s}{2}c\mathbf{n} = \mathbf{R} \wedge \left(c'\mathbf{n} - c^2\mathbf{R} + c\tau\mathbf{b}\right).$$

Hence $c' = 0$ so that $c = a$ and $\tau = s/2$ (see [**?**]). In figure 3 we see a self-similar solution (6) for some choice of $\mathbf{G}$.

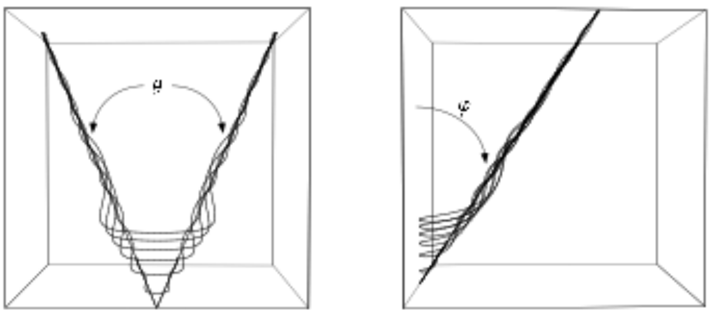

Figure 3.

The self-similar solutions and its characterizations were studied in [12]. In particular a characterization of them in terms of the parameter a is obtained. It is also proved that at time zero they have the shape of two half lines joined together at $s = 0$ with an angle θ. The curve $\mathbf{G}$ tends asymptotically to these two lines. Also precise expressions of θ and the angle φ in terms of a are given. Finally observe that the right hand side of figure 2 indicates that the corner moves in a very precise way: the speed is $a/\sqrt{t}$ and the direction is determined by the angle φ. In figure 4 we compare the dynamics of self-similar solutions with the vortices above an inclined triangular wing given in figure 2. The similitude at the qualitative level is quite appealing.

3. Schrödinger Equation

An important step in the understanding of (1) was given by Hasimoto in [13]. He introduces the transformation

$$\psi(s,t) = c(s,t)e^{i\int_0^s \tau(s',t)ds'}. \tag{7}$$

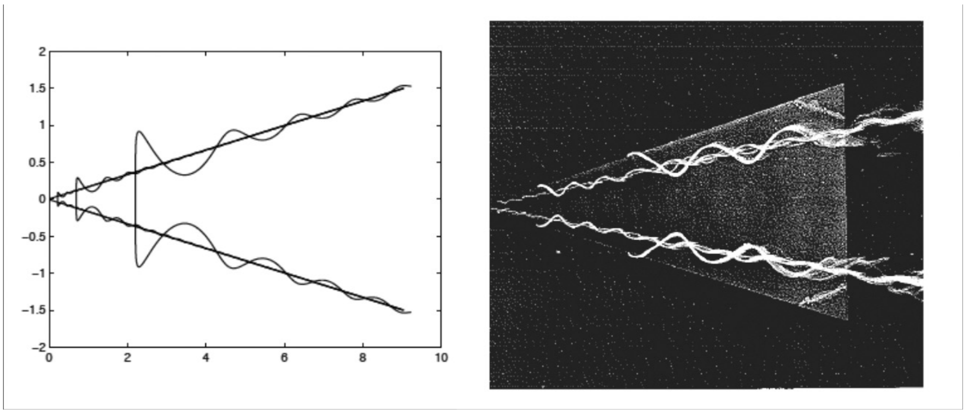

Figure 4.

After some calculations he proves that if $\mathbf{X}$ solves (1) then ψ solves

$$\partial_t\psi(s,t) = i\left(\partial_s^2\psi + \frac{1}{2}(|\psi|^2 + A(t))\psi\right) \tag{8}$$

for some $A(t) \in \mathbb{R}$.

Therefore it is straightforward that for regular solutions of (8)

$$\int_{-\infty}^{\infty} |\psi(s,t)|^2 ds = \int_{-\infty}^{\infty} |\psi(s,0)|^2 ds = \int_{-\infty}^{\infty} c^2(s,0) ds. \tag{9}$$

In fact to solve (8) under the condition that (9) is finite is nowadays quite standard. However in our case:

$$\psi(s,t) = \frac{a}{\sqrt{t}} e^{i\frac{s^2}{4t}} \qquad , \qquad \int_{-\infty}^{\infty} |\psi|^2 ds = +\infty.$$

Our next step is to extend the self-similar solutions for negative times. Formally this can be easily done using remark 1. As we said to change the direction of time is enough to change the orientation of the curve at time zero. In our case this can be done using a rotation ρ that interchanges the two lines that generate the initial curve. Then, it is enough to apply this rotation ρ to the curve $\sqrt{t}\mathbf{G}(s/\sqrt{t})$ as done in figure 5. We have the following result.

Theorem 3.1. *The self-similar solutions exhibited in figure 5 are stable. In particular, the creation/annihilation of a corner is a stable procedure.*

The proof of this result is a consequence of several papers done in collaboration with V. Banica (see [1], [2], [3], and [4]). We also refer the reader to [4] for a precise statement of the above theorem. The proof follows the following steps:

- A pseudo-conformal transformation of ψ is done. This implies to change the variable t into $1/t$ so that the initial value problem for (9) becomes a scattering problem. Therefore the existence of wave operator and its asymptotic completeness have to be settled.

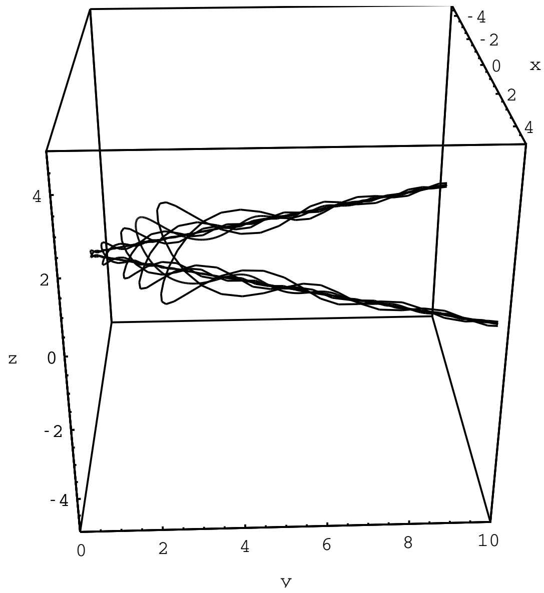

FIGURE 5.

- For dealing with the scattering problem the right space of functions has to be found.
- Extra difficulties come from the fact that the non-linear potential is a long range potential. This has an important consequence: cubic NLS equation (9) with the Dirac–delta as initial condition is ill–posed. Observe that the Dirac-delta is the initial condition of (9) with $A(t) = \frac{a^2}{t}$ associated to a self-similar solution.
- As a consequence, for describing the formation of a corner at time $t = 0$ one can not work just with (9). It is also necessary to work with (1) and (3). The recipe to go beyond $t = 0$ is to use a blow–up argument to "capture" the appropriate selfsimilar solution.
- For the last step the characterization of the self-similar solutions obtained in [12] plays a fundamental role.

4. A REGULAR POLYGON

Our next step is to consider solutions of (1) that at time $t = 0$ are given by a regular polygon. In terms of the Hasimoto function (7) a planar regular polygon with M sides is described as

$$\psi(s,0) = \frac{2\pi}{M} \sum_{k=-\infty}^{\infty} \delta\left(s - \frac{2\pi k}{M}\right). \tag{10}$$

Let us recall the so-called galilean transformations: if $\psi(s,t)$ is a solution of (9) then

$$\tilde{\psi}(s,t) \equiv e^{iks-ik^2t}\psi(s-2kt,t), \quad \forall\, k,t \in \mathbb{R},$$

is also solution of (9). Observe that for $\psi(s,0)$ as in (10) then

$$e^{2\pi ijMs}\psi(s,0)=\psi(s,0) \quad \forall\, j\in\mathbb{Z}.$$

Therefore if there were uniqueness for the initial value problem (9) with initial condition (10) then

$$\tilde{\psi}_k=\psi \quad \forall\, k\in\mathbb{Z}.$$

This has very strong consequences because if we define

$$\widehat{\psi}(j,t)=\frac{M}{2\pi}\int_0^{2\pi/M}e^{-iMjs}\psi(s,t)ds$$

then

$$\begin{aligned}
\widehat{\psi}(j,t) &= \frac{M}{2\pi}\int_0^{2\pi/M}e^{-iMjs}\psi(s,t)ds\\
&= \frac{M}{2\pi}\int_0^{2\pi/M}e^{-iMjs}\left[e^{iMks-i(Mk)^2t}\psi(s-2Mkt,t)\right]ds\\
&= \frac{Me^{-i(Mk)^2t}}{2\pi}\int_0^{2\pi/M}e^{-iM(j-k)s}\psi(s-2Mkt,t)ds\\
&= \frac{Me^{-i(Mk)^2t}}{2\pi}\int_0^{2\pi/M}e^{-iM(j-k)(s+2Mkt)}\psi(s,t)ds\\
&= \frac{Me^{-i(Mk)^2t-iM(j-k)(2Mkt)}}{2\pi}\int_0^{2\pi/M}e^{-iM(j-k)s}\psi(s,t)ds\\
&= e^{-i(Mk)^2t-iM(j-k)(2Mkt)}\widehat{\psi}(j-k,t).
\end{aligned}$$

Therefore

$$\widehat{\psi}(j,t)=e^{-i(Mj)^2t\widehat{\psi}}(0,t) \qquad \text{for all } j.$$

Now for "rational times"

$$t_{pq}=2\pi p/(M^2q)$$

we get

$$\begin{aligned}
\psi\left(s,t_{pq}\right) &= \widehat{\psi}\left(0,t_{pq}\right)\sum_{k=-\infty}^{\infty}e^{-i(Mk)^2 2\pi p/(M^2q)+iMks}\\
&= \widehat{\psi}\left(0,t_{pq}\right)\sum_{k=-\infty}^{\infty}e^{-2\pi i(p/q)k^2+iMks}\\
&= \widehat{\psi}\left(0,t_{pq}\right)\sum_{l=0}^{q-1}\sum_{k=-\infty}^{\infty}e^{-2\pi i(p/q)(qk+l)^2+iM(qk+l)s}\\
&= \widehat{\psi}\left(0,t_{pq}\right)\sum_{l=0}^{q-1}e^{-2\pi i(p/q)l^2+iMl)s}\sum_{k=-\infty}^{\infty}e^{iMqks}.
\end{aligned}$$

This is the so-called Talbot effect – see [5] and [8]. Observe that in our case this Talbot effect is understood geometrically by observing that at time t_{pq} a new polygon with Mq

sides is generated. This polygon is determined by the generalized quadratic Gauss sums that are defined by

$$\sum_{l=0}^{|c|-1} e^{2\pi i(al^2+bl)/c},$$

for given integers a, b, c, with $c \neq 0$.

It is well known that

$$G(-p,m,q) = \begin{cases} \sqrt{q}e^{i\theta m}, & \text{if } q \text{ is odd,} \\ \sqrt{2q}e^{i\theta m}, & \text{if } q \text{ is even and } q/2 \equiv m \mod 2, \\ 0, & \text{if } q \text{ is even and } q/2 \not\equiv m \mod 2, \end{cases} \tag{11}$$

for a certain angle θ_m that depends on m, p, and q.

Gathering all the information we get

$$\psi(s,t_{pq}) = \begin{cases} \sum_{m=0}^{q-1} (\alpha_m + i\beta_m)\,\delta\left(s - \frac{2\pi m}{Mq}\right), & \text{if } q \text{ is odd,} \\ \sum_{m=0}^{q/2-1} (\alpha_{2m+1} + i\beta_{2m+1})\,\delta\left(s - \frac{4\pi m + 2\pi}{Mq}\right), & \text{if } q/2 \text{ is odd,} \\ \sum_{m=0}^{q/2-1} (\alpha_{2m} + i\beta_{2m})\,\delta\left(s - \frac{4\pi m}{Mq}\right), & \text{if } q/2 \text{ is even,} \end{cases}$$

where

$$|\alpha_m + i\beta_m| = \begin{cases} \frac{2\pi}{M\sqrt{q}}\widehat{\psi}(0,t_{pq}), & \text{if } q \text{ is odd,} \\ \frac{2\pi}{M\sqrt{\frac{q}{2}}}\widehat{\psi}(0,t_{pq}), & \text{if } q \text{ is even and } q/2 \equiv m \mod 2, \\ 0, & \text{if } q \text{ is even and } q/2 \not\equiv m \mod 2, \end{cases}$$

so we conclude that the angle ρ between two adjacent sides is constant. Furthermore, writing

$$\alpha_m + i\beta_m = \rho e^{i\theta_m}$$

we see that the structure of the polygon is given by the angles θ_m.

Bearing in mind that ψ is given as a sum of δ-functions it is more appropriate to use the so-called parallel frame than the Frenet frame,

$$\begin{pmatrix} \mathbf{T} \\ \mathbf{e}_1 \\ \mathbf{e}_2 \end{pmatrix}_s = \begin{pmatrix} 0 & \alpha & \beta \\ -\alpha & 0 & 0 \\ -\beta & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} \mathbf{T} \\ \mathbf{e}_1 \\ \mathbf{e}_2 \end{pmatrix}.$$

Hence we have to solve systems of the type

$$\begin{pmatrix} \mathbf{T} \\ \mathbf{e}_1 \\ \mathbf{e}_2 \end{pmatrix}_s = \begin{pmatrix} 0 & a\delta & b\delta \\ -a\delta & 0 & 0 \\ -b\delta & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} \mathbf{T} \\ \mathbf{e}_1 \\ \mathbf{e}_2 \end{pmatrix}.$$

It is immediate to obtain

$$\begin{aligned} \begin{pmatrix} u_1(0^+) \\ u_2(0^+) \\ u_3(0^+) \end{pmatrix} &= \exp\left[\begin{pmatrix} 0 & a & b \\ -a & 0 & 0 \\ -b & 0 & 0 \end{pmatrix} \int_{0^-}^{0^+} \delta(s')ds'\right] \cdot \begin{pmatrix} u_1(0^-) \\ u_2(0^-) \\ u_3(0^-) \end{pmatrix} \\ &= \exp\left[\begin{pmatrix} 0 & a & b \\ -a & 0 & 0 \\ -b & 0 & 0 \end{pmatrix}\right] \cdot \begin{pmatrix} u_1(0^-) \\ u_2(0^-) \\ u_3(0^-) \end{pmatrix} \end{aligned}$$

We still have to determined $\widehat{\psi}(0,t)$. We will do it by imposing that the polygon has to be closed. Observe that to compute the k-side we have to write

$$\begin{pmatrix} \mathbf{T}\left(\frac{2\pi k}{Mq}^+\right)^T \\ \hline \mathbf{e}_1\left(\frac{2\pi k}{Mq}^+\right)^T \\ \hline \mathbf{e}_2\left(\frac{2\pi k}{Mq}^+\right)^T \end{pmatrix} = \mathbf{M}_k \cdot \mathbf{M}_{k-1} \cdot \ldots \cdot \mathbf{M}_1 \cdot \mathbf{M}_0 \cdot \begin{pmatrix} \mathbf{T}\ (0^-)^T \\ \hline \mathbf{e}_1\ (0^-)^T \\ \hline \mathbf{e}_2\ (0^-)^T \end{pmatrix}.$$

Hence in order the polygon to be closed it is necessary that

$$\mathbf{M}_{Mq-1} \cdot \mathbf{M}_{Mq-2} \cdot \ldots \cdot \mathbf{M}_1 \cdot \mathbf{M}_0 \equiv \mathrm{I}.$$

Let us define:

$$\mathbf{M} = \mathbf{M}_{q-1} \cdot \mathbf{M}_{q-2} \cdot \ldots \cdot \mathbf{M}_1 \cdot \mathbf{M}_0.$$

Hence $\mathbf{M}$ is an M–th root of the identity matrix and also a rotation matrix that induces a rotation of $2\pi/M$ degrees around a certain rotation axis. Hence

$$\begin{aligned} \mathrm{Tr}\,(\mathbf{M}) &= 1 + 2\cos\left(\frac{2\pi}{M}\right), \\ \sigma(\mathbf{M}) &= \left\{1, e^{2\pi i/M}, e^{-2\pi i/M}\right\}. \end{aligned}$$

From this fact and some strong numerical evidence we conjecture that

$$\cos(\rho) = \begin{cases} 2\cos^{2/q}(\pi/M) - 1, & \text{if } q \text{ is odd}, \\ 2\cos^{4/q}(\pi/M) - 1, & \text{if } q \text{ is even}. \end{cases}$$

Hence we have all the necessary information to compute $\mathbf{T}$ at $t = t_{pq}$. Except the trajectory of one point. This is obtained using the symmetries of the $M-$polygon as it is explained in [15].

In figure 6 we compare the numerical solution with the theoretical solution, the construction of which we have just sketched. We see that there is a remarkable agreement between the two.

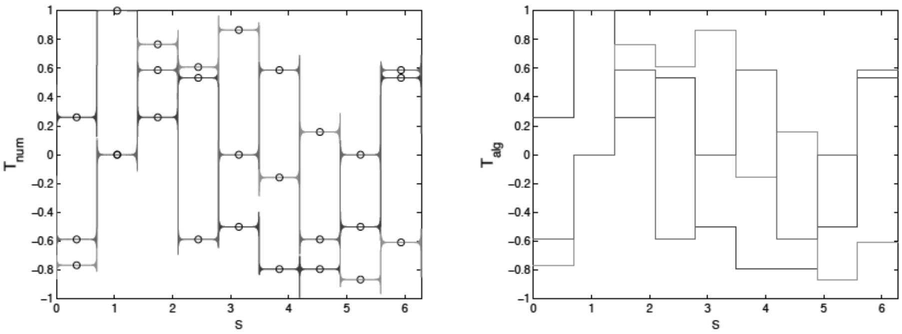

FIGURE 6. $\mathbf{T}_{num}$ versus $\mathbf{T}_{alg}$, for $M = 3$, at $T_{1,3} = \dfrac{2\pi}{27}$. T_1 appears in blue, T_2 in green, T_3 in red. In $\mathbf{T}_{num}$, the Gibbs phenomenon is clearly visible. The black circles denote the points chosen for the comparisons.

It is interesting to observe the trajectory of one of the corners as for example $\mathbf{X}(0, t)$. Using again the symmetries of the M-polygon is easy to see that this trajectory falls in a plane. In figures 7 and 8 we can see the particular cases of the equilateral triangle and the square.

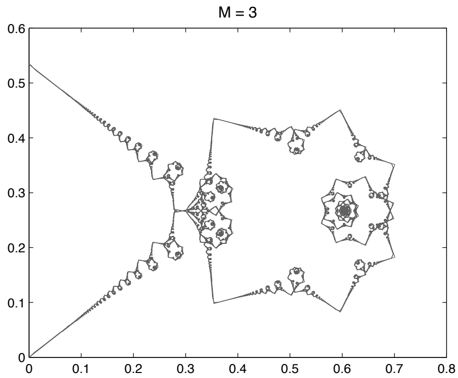

FIGURE 7.

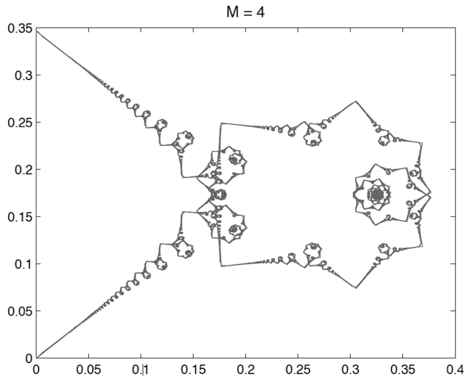

FIGURE 8.

These pictures are quite reminiscent of the so-called Riemann's non-differentiable function

$$\phi(t) = \sum_{k \neq 0} \frac{e^{-2\pi i t k^2}}{k^2}. \tag{12}$$

Observe that $\phi(1) = \phi(0)$. Renormalizing $\mathbf{X}(0,t)$ accordingly we see in figures 9, 10 and 11 that at the qualitative level there is a big similitude between them. This similitude becomes stronger the bigger the M. This is numerically proved in [15]. In [9] it is proved that the graph given by ϕ in (12) is a fractal that satisfies the so-called Frisch-Parisi conjecture –see [10] and [11].

I want to finish with a few remarks about the behavior of real fluids. The results in [15], some of which have been sketched in this note, suggest that at half of a period the starting regular polygon with M sides becomes the same polygon but with the axis switched with an angle equal to π/M. Observe that at those times, and due to the fact that $q = 2$, the Gauss sum is zero for half of the cases in (11), so that the polygon has M sides instead of $2M$. Then, the the axis switching phenomena follows from the analysis of which are the precise values of m in (11) that make the Gauss sum trivial. It turns out that this phenomena has been observed in real fluids, in particular for non-circular jets, and it is well documented. We refer the reader to [15] for the appropriate references. In figure 12 the reader will find some pictures of a domestic experiment. A smoke cannon made with a cardboard box that has a hole with the shape of an equilateral triangle is used. After introducing some smoke through the hole the box is suddenly hit at the back. A camera in front of the box is located. The first three pictures clearly illustrate the flip-flop of the triangle as a consequence of the switching of the axis by a $\pi/6$-angle. Or course the corners are smoothed out due to the effects of the viscosity. Our analysis also implies that skew polygons with six sides should appear as a consequence of the Talbot effect mentioned above. The last picture in figure 12 is not conclusive and further evidence in this direction is needed.

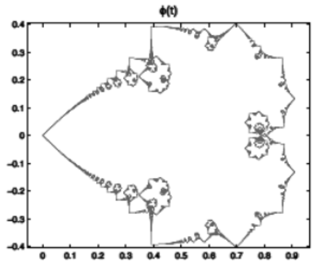

FIGURE 9.

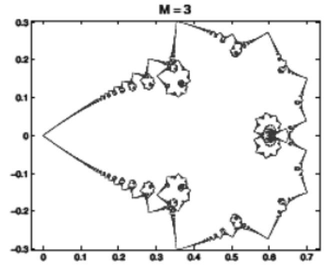

FIGURE 10.

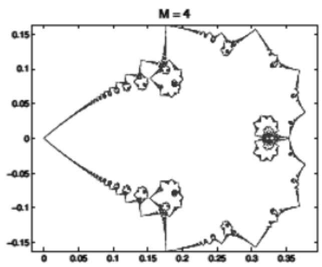

FIGURE 11.

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FIGURE 12.

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ON A FRACTIONAL YAMABE PROBLEM

YANNICK SIRE

Abstract. We investigate the prescription of constant curvature on manifolds, where the curvature is related to conformally covariant operators of fractional order.

1. Introduction

Let $(M, \bar{g})$ be a compact smooth connected n-dimensional Riemannian manifold, $n \geq 3$. The 'standard' regular Yamabe problem concerns the existence and geometric properties of conformal metrics of the form $g = u^{\frac{4}{n-2}}\bar{g}$ with constant scalar curvature. This corresponds to solving the partial differential equation

$$\Delta_{\bar{g}} u + \frac{n-2}{4(n-1)} R^{\bar{g}} u = \frac{n-2}{4(n-1)} R^{g}\, u^{\frac{n+2}{n-2}}, \qquad u > 0, \tag{1.1}$$

where the scalar curvature R^g is constant. In the previous equation, the operator

$$L_{\bar{g}} = \Delta_{\bar{g}} u + \frac{n-2}{4(n-1)} R^{\bar{g}}$$

is the so-called conformal laplacian on M. It satisfies the conformality property: if f is any (smooth) function and $g = u^{\frac{4}{n-2}}\,\bar{g}$ for some $u > 0$, then

$$L_{\bar{g}}(uf) = u^{\frac{n+2}{n-2}} L_g(f). \tag{1.2}$$

On the other hand, the "standard" singular Yamabe problem concerns the existence and geometric properties of complete metrics of the form $g = u^{\frac{4}{n-2}}\bar{g}$ with constant scalar curvature on $M \backslash \Lambda$, where $\Lambda \subset M$ is a closed set. It boils down to solving the equation (1.1) with a 'boundary condition' that $u \to \infty$ sufficiently quickly at Λ so that g is complete.

The purpose of this note is to introduce a notion of curvature on compact manifolds interpolating between already known curvatures and their associated regular and singular Yamabe metrics.

Setting $f \equiv 1$ in (1.2) yields the familiar relationship (1.1) between the scalar curvatures $R^{\bar{g}}$ and R^g. $L_{\bar{g}}$ is the first in a sequence of conformally covariant elliptic operators, $P_k^{\bar{g}}$, which exist for all $k \in \mathbb{N}$ if

n is odd, but only for $k \in \{1, \dots, n/2\}$ if k is even. The first construction of these operators, by Graham-Jenne-Mason-Sparling [8] (for which reason they are known as the GJMS operators), proceeded by trying to find lower order geometric correction terms to Δ^k in order to obtain nice transformation properties under conformal changes of metric. Beyond the case $k = 1$ which we have already discussed, the operator

$$P_2^{\bar{g}} = \Delta_{\bar{g}}^2 + \delta\Big(a_n R\overline{g} + b_n Ric\Big)d + \tfrac{n-4}{2}Q^n,$$

called the Paneitz operator, had also been discovered much earlier than the operators $P_k^{\bar{g}}$ with $k > 2$. The natural question is then to consider conformally covariant operators $P_\gamma^{\bar{g}}$ of fractional order γ and their curvature given by

$$Q_\gamma^g = P_\gamma^g(1).$$

We focus here only on the operators $P_\gamma^{\bar{g}}$ when $\gamma \in \mathbb{R}$, $|\gamma| \leq n/2$. These have the following properties: first, $P_0^{\bar{g}} = \mathrm{Id}$, and more generally, $P_k^{\bar{g}}$ is the k^{th} GJMS operator, $k = 1, \dots, n/2$; next, $P_\gamma^{\bar{g}}$ is elliptic of order 2γ with principal symbol $\sigma_{2\gamma}(P_\gamma^{\bar{g}}) = |\xi|_{\bar{g}}^{2\gamma}$, hence (since M is compact), $P_\gamma^{\bar{g}}$ is Fredholm on L^2 when $\gamma > 0$; if $P_\gamma^{\bar{g}}$ is invertible, then $P_{-\gamma}^{\bar{g}} = (P_\gamma^{\bar{g}})^{-1}$; finally,

$$\text{if } g = u^{\frac{4}{n-2\gamma}}\bar{g}, \qquad \text{then } P_\gamma^{\bar{g}}(uf) = u^{\frac{n+2\gamma}{n-2\gamma}} P_\gamma^g(f) \tag{1.3}$$

for any smooth function f.

2. Fractional laplacians and fractional curvature

We now provide a more careful description of the construction of the family of conformally covariant operators $P_\gamma^{\bar{g}}$.

Graham and Zworski [9] discovered a beautiful connection between the scattering theory of the Laplacian on an asymptotically hyperbolic Einstein manifold and the GJMS operators on its conformal infinity. Let (M, g) be a compact n-dimensional Riemannian manifold. Suppose that X is a smooth compact manifold with boundary, with $\partial X = M$, and denote by x a defining function for the boundary, i.e. $x \geq 0$ on X, $x = 0$ precisely on ∂X and $dx \neq 0$ there. A metric G on the interior of X is called conformally compact if $x^2 G = \overline{G}$ extends as a smooth nondegenerate metric on the closed manifold with boundary. It is not hard to check that G is complete and, provided that $|dx|_{\overline{G}} = 1$ at ∂X, the sectional curvatures of G all tend to -1 at 'infinity'. The metric G is called Poincaré-Einstein if it is conformally compact and also satisfies the Einstein equation $Ric^G = -nG$. It is only necessary to consider asymptotically Poincaré-Einstein metrics; by definition, these are conformally compact metrics which satisfy $Ric^G = -nG + O(x^N)$ for some suitably large N (typically, $N > n$ is sufficient).

The conformal infinity of G is the conformal class of $\overline{G}\big|_{T\partial X}$; only the conformal class is well defined since the defining function x is defined up to a positive smooth multiple. If g is any representative of this conformal class, then there is a unique defining function x for M such that $G = x^{-2}(dx^2 + g(x))$ where $g(x)$ is a family of metrics on M (or rather, the level sets of x), with $g(0)$ the given initial metric.

We now define the scattering operator $S(s)$ for (X, G). Fix any $f_0 \in C^\infty(M)$; then for all but a discrete set of values $s \in \mathbb{C}$, there exists a unique generalized eigenfunction u of the Laplace operator on X with eigenvalue $s(n-s)$. In other words, u satisfies

$$\begin{cases} (\Delta_G - s(n-s))u = 0 \\ u = fx^{n-s} + \tilde{f}x^s, \quad \text{for some } f, \tilde{f} \in C^\infty(\overline{X}) \quad \text{with } f|_{x=0} = f_0. \end{cases} \tag{2.1}$$

By definition, $S(s)f_0 = \tilde{f}|_{x=0}$. This is an elliptic pseudodifferential operator of order $2s - n$ which depends meromorphically on s; it is known to always have simple poles at the values $s = n/2, n/2+1, n/2+2, \ldots$. These locations are independent of (X, G), hence are called the trivial poles of the scattering operator. Letting $s = n/2 + \gamma$, we now define

$$P_\gamma^g = 2^{2\gamma}\frac{\Gamma(\gamma)}{\Gamma(-\gamma)}S\Big(\frac{n}{2} + \gamma\Big); \tag{2.2}$$

because of these prefactors, one has

$$\sigma_{2\gamma}(P_\gamma^g) = |\eta|_g^{2\gamma}. \tag{2.3}$$

The scattering operator satisfies a functional equation, $S(s)S(n-s) = \mathrm{Id}$, which implies that

$$P_\gamma^g \circ P_{-\gamma}^g = \mathrm{Id}. \tag{2.4}$$

Finally, it is proved in [9] that the operators P_γ^g satisfy the conformal covariance equation (1.3).

This definition of the operators P_γ^g depends crucially on the choice of the Poincaré-Einstein filling (X, G). Graham and Zworski point out that it is only necessary that the metric G satisfy the Einstein equation to sufficiently high order as $x \to 0$ in order that the properties of the P_γ^g listed above be true (for γ in a finite range which depends on the order to which G satisfies the Einstein equation). As we have discussed in the introduction, it is always possible to find such metrics, and we suppose that one has been fixed.

Remark 2.1. *In the flat metric, the operator P^g_γ is the fractional laplacian $(-\Delta)^\gamma$, a Fourier multiplier of symbol $|\xi|^{2\gamma}$.*

3. Results on the singular fractional Yamabe problem

In view of the previous discussion, one can formulate the following fractional Yamabe problem: given a metric $\bar g$ on a compact manifold M, find $u > 0$ so that if $g = u^{4/(n-2\gamma)}\bar g$, then Q^g_γ is constant. This amounts to solving

$$P^{\bar g}_\gamma u = Q^g_\gamma u^{\frac{n+2\gamma}{n-2\gamma}}, \quad u > 0, \tag{3.1}$$

for Q^g_γ = const. Following the seminal works [1], [14], [6], Qing and Gonzalez were able to solve the problem in most of the cases [12]. The only remaining part is the Schoen result using the Positive mass [14, 15], which is not known for these operators.

The singular fractional Yamabe problem is much less understood. In the local case $\gamma = 1$, it is known that solutions with $R^g < 0$ exist quite generally if Λ is large in a capacitary sense [10], whereas for $R^g > 0$ existence is only known when Λ is a smooth submanifold (possibly with boundary) of dimension $k < (n-2)/2$, see [11]. On the geometric side is a well-known theorem by Schoen and Yau [13] stating that if (M, h) is a compact manifold with a locally conformally flat metric h of positive scalar curvature, then the developing map D from the universal cover $\widetilde{M}$ to S^n, which by definition is conformal, is injective, and moreover, $\Lambda := S^n \backslash D(\widetilde{M})$ has Hausdorff dimension less than or equal to $(n-2)/2$. Regarding the lifted metric $\tilde h$ on $\widetilde{M}$ as a metric on Ω, this provides an interesting class of solutions of the singular Yamabe problem which are periodic with respect to a Kleinian group, and for which the singular set Λ is typically nonrectifiable. More generally, that paper also shows that if $\bar g$ is the standard round metric on the sphere and if $g = u^{\frac{4}{n-2}}\bar g$ is a complete metric with positive scalar curvature and bounded Ricci curvature on a domain $\Omega = S^n \setminus \Lambda$, then $\dim \Lambda \leq (n-2)/2$.

A first approach to generalize the Schoen-Yau theory is the following result (see [7])

Theorem 3.1. *Suppose that $(M^n, \bar g)$ is compact and $g = u^{\frac{4}{n-2\gamma}}\bar g$ is a complete metric on $\Omega = M \setminus \Lambda$, where Λ is a smooth k-dimensional submanifold. Assume furthermore that u is polyhomogeneous along Λ with leading exponent $-n/2+\gamma$. If $0 < \gamma \leq \frac{n}{2}$, and if $Q^g_\gamma > 0$ everywhere for any choice of asymptotically Poincaré-Einstein extension (X, G) which defines $P^{\bar g}_\gamma$ and hence Q^g_γ, then n, k and γ are restricted by the* *inequality*

$$\Gamma\Big(\frac{n}{4}-\frac{k}{2}+\frac{\gamma}{2}\Big)\Big/\Gamma\Big(\frac{n}{4}-\frac{k}{2}-\frac{\gamma}{2}\Big)>0, \tag{3.2}$$

where Γ *is the ordinary Gamma function. This inequality holds in particular when* $k<(n-2\gamma)/2$, *and in this case there is a unique extension of* u *to a distribution on all of* M *which solves the same equation, or in other words,* u *extends uniquely to a weak solution of* (3.1) *on all of* M.

Remark 3.2. *Recall that* u *is said to be polyhomogeneous along* Λ *if in terms of any cylindrical coordinate system* (r,θ,y) *in a tubular neighbourhood of* Λ, *where* r *and* θ *are polar coordinates in disks in the normal bundle and* y *is a local coordinate along* Λ, u *admits an asymptotic expansion*

$$u\sim\sum a_{jk}(y,\theta)r^{\mu_j}(\log r)^k$$

where μ_j *is a sequence of complex numbers with real part tending to infinity, for each* j, a_{jk} *is nonzero for only finitely many nonnegative integers* k, *and such that every coefficient* $a_{jk}\in C^\infty$. *The number* μ_0 *is called the leading exponent* $\Re(\mu_j)>\Re(\mu_0)$ *for all* $j\neq 0$.

Inequality (3.2) is satisfied whenever $k<(n-2\gamma)/2$, and in fact is equivalent to this simpler inequality when $\gamma=1$. When $\gamma=2$, i.e. for the standard $Q-$curvature, this result is already known: it is shown in [5] that complete metrics with $Q>0$ and positive scalar curvature must have singular set with dimension less than $(n-4)/2$, which again agrees with (3.2). The previous estimate provides an insight on the solvability of the singular fractional Yamabe problem.

The simplest case of a singular set one can imagine is an isolated singularity. Consider the equation

$$(-\Delta)^\gamma u=u^{\frac{n+2\gamma}{n-2\gamma}}\quad\text{in } B_1\setminus\{0\}\subset\mathbb{R}^n \tag{3.3}$$

By means of the Caffarelli-Silvestre extension [3], one can reformulate equation (3.3) in terms of the following boundary problem

$$\begin{cases}\operatorname{div}(t^{1-2\gamma}\nabla U)=0 & \text{in } B_1^+\subset\mathbb{R}^{n+1}_+,\\ \dfrac{\partial U}{\partial\nu^\gamma}(x,0)=U^{\frac{n+2\gamma}{n-2\gamma}}(x,0) & \text{on } \partial B_1^+\setminus\{0\},\end{cases} \tag{3.4}$$

In the case of conformally flat manifolds, one then has [2]

Theorem 3.3. *Suppose that* U *is a nonnegative solution of* (3.4). *Then either* u *can be extended as a continuous function near* 0, *or* *there exist two positive constants c_1 and c_2 such that*

$$c_1|x|^{-\frac{n-2\gamma}{2}} \leq u(x) \leq c_2|x|^{-\frac{n-2\gamma}{2}}. \tag{3.5}$$

and

Theorem 3.4. *If U is a nonnegative solution of* (3.4)*, then*

$$u(x) = \bar{u}(|x|)(1 + O(|x|)) \quad \text{as } x \to 0,$$

where $\bar{u}(|x|) = \frac{1}{|\mathbb{S}^n|}\int_{\mathbb{S}^n} u(|x|\theta)\,d\theta$ is the spherical average of u.

The previous theorems are a nonlocal version of famous results by Caffarelli, Gidas and Spruck [4].

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INVERSE PROBLEMS ON COMPACT SETS FOR PDE

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Abstract

In the paper we will consider elements of the theory of inverse and ill-posed problems including the quasisolutions method and the method of extending compacts. We will show how these methods can be applied for solving inverse problems for PDE.

1. Well-posed and ill-posed problems

Let us consider an operator equation:

$$Az = u,$$

where A is a linear operator acting from a normed space Z into a normed space U. It is required to find a solution of the operator equation z corresponding to a given inhomogeneity (or right-hand side) u.

This equation is a typical mathematical model for many physical so called inverse problems if it is supposed that unknown physical characteristics z cannot be measured directly. As results of experiments, it is possible to obtain only data u connected with z with help of an operator A.

French mathematician J. Hadamard formulated the following conditions of well-posedness of mathematical problems. Let us consider these conditions for the operator equation above. The problem of solving the operator equation is called to be well-posed (according to Hadamard) if the following three conditions are fulfilled:

1) the solution exists $\forall u \in U$;
2) the solution is unique;
3) if $u_n \to u$, $Az_n = u_n$, $Az = u$, then $z_n \to z$.

The condition 2) can be realized then and only then the operator A is one-to-one (injective). The conditions 1) and 2) imply that an inverse operator A^{-1} exists, and its domain $D(A^{-1})$ (or the range of the operator A $R(A)$) coincides with U. It is equivalent to that the operator A is bijective. The condition 3) means that the inverse operator A^{-1} is continuous, i.e., to "small" perturbations of the right-hand side u "small" changes of the solution z correspond. Moreover, J. Hadamard believed that well-posed problems only can be considered while solving practical problems. However, there are well known a lot of examples of ill-posed problems that should be numerically solved when practical problems are investigated. It should be noted that stability or instability of solutions depends on definition of the space of solutions *Z*. Usually, a choice of the space of solutions (including a choice of the norm) is determined by requirements of an applied problem. A mathematical problem can be ill-posed or well-posed depending on a choice of a norm in a functional space.

Numerous inverse (including ill-posed) problems can be found in different branches of physics. E.g., an astrophysicist has no possibility to influent actively on processes in remote stars and galaxies. He is induced to make conclusions about physical characteristics of very remote objects using their indirect manifestations measured on the Earth surface or near the Earth on space stations. Excellent examples of ill-posed problems are in medicine. Firstly, let us point out computerized tomography. A lot of applications of ill-posed problems are in geophysics. Indeed, it is easier and cheaper to judge about what is going under the Earth surface solving inverse problems than drilling deep boreholes. Other examples are in radio astronomy, spectroscopy, nuclear physics, plasma diagnostics, etc., etc.

A completely continuous operator acting in infinite dimensional Banach spaces has an inverse operator that is not continuous (not bounded). Moreover, a range of a completely continuous operator acting between infinite dimensional Banach spaces is not closed. Therefore, in any neighborhood of the right-hand side $u(x)$ such that the equation has a solution there exists infinite number of right-hand sides such that the equation is not solvable.

A mathematical problem can be ill-posed in connection also with errors in an operator. The simplest example gives the problem to find a normal pseudosolution of a system of linear algebraic equations. Instability of this problem is determined by errors in a matrix.

2. **Definition of the regularizing algorithm**

Let us given an operator equation:

$$Az = u,$$

where A is an operator acting between normed spaces *Z* and *U*. In 1963 A.N. Tikhonov formulated a famous definition of the regularizing algorithm (RA) that is a basic conception in the modern theory of ill-posed problems.

Definition. Regularizing algorithm (regularizing operator) $R(\delta, u_\delta) \equiv R_\delta(u_\delta)$ is called an operator possessing two properties**:**

1) $R_\delta(u_\delta)$ is defined for any $\delta > 0$, $u_\delta \in U$, and is mapping $(0, +\infty) \times U$ into *Z*;
2) For any $z \in Z$ and for any $u_\delta \in U$ such that $Az = u$, $\|u - u_\delta\| \le \delta, \delta > 0$, $z_\delta = R_\delta(u_\delta) \underset{\delta \to 0}{\to} z$.

A problem of solving an operator equation is called to be regularizable if there exists at least one regularizing algorithm. Directly from the definition it follows that if there exists one regularizing algorithm then number of them is infinite.

At the present time, all mathematical problems can be divided into following classes:

1) well-posed problems;

2) ill-posed regularizable problems;

3) ill-posed nonregularizable problems.

All well-posed problems are regularizable as it can be taken $R_\delta(u_\delta) = A^{-1}$. Let us note that knowledge of $\delta > 0$ is not obligatory in this case.

Not all ill-posed problems are regularizable, and it depends on a choice of spaces *Z*, *U.* Russian mathematician L.D. Menikhes constructed an example of an integral operator with a continuous closed kernel acting from $C[0,1]$ into $L_2[0,1]$ such that an inverse problem (that is, solving a Fredholm integral equation of the 1st kind) is nonregularizable. It depends on properties of the space $C[0,1]$. If Z is the Hilbert space, and an operator A is bounded and injective, then the problem of solving of the operator equation is regularizable. This result is valid for some Banach spaces, not for all (for reflexive Banach spaces only). In particular, the space $C[0,1]$ does not belong to such spaces.

An equivalent definition of the regularizing algorithm is following. Let be given an operator (mapping) $R_\delta(u_\delta)$ defined for any $\delta > 0$, $u_\delta \in U$, and reflecting $(0,+\infty)\times U$ into *Z.* An accuracy of solving an operator equation in a point $z \in Z$ using an operator $R_\delta(u_\delta)$ under condition that the right-hand side defined with an error $\delta > 0$ is defined as $\Delta(R_\delta, \delta, z) = \sup\limits_{u_\delta \in U: \|u_\delta - u\| \le \delta,\ Az=u} \| R_\delta u_\delta - z \|$. An operator $R_\delta(u_\delta)$ is called a regularizing algorithm (operator) if for any $z \in Z$ $\Delta(R_\delta, \delta, z) \underset{\delta\to 0}{\to} 0$. This definition is equivalent to the definition above.

Similarly, a definition of the regularizing algorithm can be formulated for a problem of calculating values of an operator (see the end of the previous section), that is for a problem of calculating values of mapping $G:\ D(G) \to Y$, $D(G) \subseteq X$ under condition that an argument of G is specified with an error (*X*, *Y* are metric or normed spaces). Of course, if A is an injective operator then a problem of solving an operator equation can be considered as a problem of calculating values of A^{-1}.

It is very important to get an answer to the following question: is it possible to solve an ill-posed problem (i.e., to construct a regularizing algorithm) without knowledge of an error level δ?

Evidently, if a problem is well-posed then a stable method of its solution can be constructed without knowledge of an error δ. E.g., if an operator equation is under consideration then it can be taken $z_\delta = A^{-1}u_\delta \to z = A^{-1}u$ as $\delta \to 0$. It is impossible if a problem is ill-posed. A.B. Bakushinsky proved the following theorem for a problem of calculating values of an operator. An analogous theorem is valid for a problem of solving operator equations.

Theorem. If there exists a regularizing algorithm for calculating values of an operator G on a set $D(G) \subseteq X$, and the regularizing algorithm does not depend on δ (explicitly), then an extension of G from $D(G) \subseteq X$ to X exists, and this extension is continuous on $D(G) \subseteq X$.

So, construction of regularizing algorithms not depending on errors explicitly is feasible only for well-posed on its domains problems.

The next very important property of ill-posed problems is impossibility of error estimation for a solution even if an error of a right-hand side of an operator equation or an error of an argument in a problem of calculating values of an operator is known. This basic result was also obtained by A.B. Bakushinsky for solving operator equations.

Theorem. Let $\Delta(R_\delta,\delta,z)=\sup\limits_{u_\delta\in U:\|u_\delta-u\|\le\delta,\ Az=u}\|R_\delta u_\delta - z\|\le\varepsilon(\delta)\underset{\delta\to 0}{\longrightarrow}0$ for any $z\in D\subseteq Z$. Then a contraction of the inverse operator on the set AD: $A^{-1}\big|_{AD\subseteq U}$ is continuous on AD.

So, a uniform on z error estimation of an operator equation on a set $D\subseteq Z$ exists then and only then if the inverse operator is continuous on AD. The theorem is valid also for nonlinear operator equations, in metric spaces at that.

From the definition of the regularizing algorithm it follows immediately if one exists then there is infinite number of them. While solving ill-posed problems, it is impossible to choose a regularizing algorithm that finds an approximate solution with the minimal error. It is impossible also to compare different regularizing algorithms according to errors of approximate solutions. Only including a priori information in a statement of the problem can give such a possibility, but in this case a reformulated problem is well-posed in fact. We will consider examples below.

Regularizing algorithms for operator equations in infinite dimensional Banach spaces could not be compared also according to convergence rates of approximate solutions to an exact solution as errors of input data tend to zero. The author of this principal result is V.A. Vinokurov.

In conclusion of the section let formulate a definition of the regularizing algorithm in the case when an operator can also contain an error, i.e., instead of an operator A it is given a bounded linear operator A_h: $A_h:Z\to U$, such that $\|A_h-A\|\le h,\quad h\ge 0$. Briefly, let us note a pair of errors δ, h as η=(δ, h).

Definition. Regularizing algorithm (regularizing operator) $R(\eta,u_\delta,A_h)\equiv R_\eta(u_\delta,A_h)$ is called an operator possessing two properties**:**

1) $R_\eta(u_\delta,A_h)$ is defined for any $\delta>0$, $h\ge 0$, $u_\delta\in U$, $A_h\in L(Z,U)$, and is mapping $(0,+\infty)\times[0,+\infty)\times U\times L(Z,U)$ into Z;

2) for any $z\in Z$, for any $u_\delta\in U$ such that $Az=u$, $\|u-u_\delta\|\le\delta,\delta>0$ and for any $A_h\in L(Z,U)$ such that $\|A_h-A\|\le h,\quad h\ge 0$, $z_\eta=R_\eta(u_\delta,A_h)\underset{\eta\to 0}{\longrightarrow}z$.

Here $L(Z,U)$ is a space of linear bounded operators acting from Z into U with the usual operator norm.

Similarly, it possible to define what is it a regularizing algorithm if an operator equation is considered on a set $D\subseteq Z$, i.e., a priori information that an exact solution $z\in D\subseteq Z$ is available.

For ill-posed SLAE A.N. Tikhonov was the first who proved impossibility to construct a regularizing algorithm that does not depend explicitly on *h*.

3. Ill-posed problems on compact sets

Let us consider an operator equation:

$$Az = u,$$

A is a linear injective operator acting between normed spaces *Z* and *U*. Let $\bar{z}$ be an exact solution of an operator equation, $A\bar{z} = \bar{u}$, $\bar{u}$ is an exact right-hand side, and it is given an approximate right-hand side such that $\|\bar{u} - u_\delta\| \le \delta, \delta > 0$.

A set $Z_\delta = \{z_\delta : \|Az_\delta - u_\delta\| \le \delta\}$ is a set of approximate solutions of the operator equation. For linear ill-posed problems $diam\, Z_\delta = \sup\{\| z_1 - z_2 \|: z_1, z_2 \in Z_\delta\} = \infty$ for any $\delta > 0$ since the inverse operator A^{-1} is not bounded.

The question is that: is it possible to use a priori information in order to restrict a set of approximate solutions or (it is better) to reformulate a problem to be well-posed. A.N. Tikhonov proposed a following idea: if it is known the set of solutions is a compact then a problem of solving an operator equation is well-posed under condition that an approximate right-hand side belongs to the image of the compact. A.N. Tikhonov proved this assertion using as basis the following theorem.

Theorm. Let an injective continuous operator A be mapping: $D \in Z \to AD \in U$, where Z, U are normed spaces, D is a compact. Then the inverse operator A^{-1} is continuous on AD.

The theorem is true for nonlinear operators also. So, a problem of solving an operator equation is well-posed under condition that an approximate right-hand side belongs to AD. This idea made possible to M.M. Lavrentiev to introduce a conception of a well-posed according to A.N. Tikhonov mathematical problem (it is supposed that a set of well-posedness exists), and to V.K. Ivanov to define a quasisolution of an ill-posed problem.

The theorem above is not valid if $u_\delta \notin R(A)$. So, it should be generalized.

Definition. An element $z_\delta \in D$ such that $z_\delta = \arg\min_{z \in D} \|Az - u_\delta\|$ is called a quasisolution of an operator equation on a compact D ($z_\delta = \arg\min_{z \in D} \|Az - u_\delta\|$ means that $\|A\, z_\delta - u_\delta\| = \min\{\|A\, z - u_\delta\| : z \in D\}$).

A quasisolution exists but maybe is nonunique. Though, any quasisolution tends to an exact solution: $z_\delta \to \bar{z}$ as $\delta \to 0$. In this case, knowledge of an error δ is not obligatory. If δ is known then:

1) any element $z_\delta \in D$ satisfying an inequality : $\|Az_\delta - u_\delta\| \le \delta$, can be chosen as an approximate solution with the same property of convergence to an exact solution (δ-quasisolution);

2) it is possible to find an error of an approximate solution solving an extreme problem: find $max\|z-z_\delta\|$ maximizing on all $z \in D$ satisfying an inequality: $\|Az - u_\delta\| \le \delta$ (it is obviously that an exact solution satisfying the inequality).

Thus, the problem of quasisolving an operator equation does not differ strongly from a well-posed problem. A condition of uniqueness only maybe does not satisfy.

If an operator A is specified with an error then the definition of a quasisolution can be modified changing an operator A to an operator A_h.

Definition. An element $z_\eta \in D$ such that $z_\eta = \arg\min_{z\in D} \|A_h z - u_\delta\|$ is called a quasisolution of an operator equation on a compact D.

Any element $z_\eta \in D$ satisfying an inequality: $\|Az_\eta - u_\delta\| \le \delta + h \| z_\eta \|$ can be chosen as an approximate solution (η-quasisolution).

If Z and U are Hilbert spaces then many numerical methods of finding quasisolutions of linear operator equations are based on convexity and differentiability of the discrepancy functional $\|Az - u_\delta\|^2$. If D is a convex compact then finding a quasisolution is a problem of convex programming. The inequalities written above and defining approximate solutions can be used as stopping rules for minimizing the discrepancy procedures. The problem of calculating errors of an approximate solution is a nonstandard problem of convex programming because it is necessary to maximize (not to minimize) a convex functional.

Some sets of correctness are very well known in applied sciences. First of all, if an exact solution belongs to a family of functions depending on finite number of bounded parameters then the problem of finding parameters can be well-posed. The same problem without such a priori information can be ill-posed.

If an unknown function $z(s)$, $s \in [a, b]$, is monotonic and bounded then it is sufficient to define a compact set in the space $L_2[a, b]$. After finite-dimensional approximation the problem of finding a quasisolution is a quadratic programming problem. For numerical solving, known methods such a method of projections of conjugate gradients or a method of conditional gradient can be applied. Similar approach can be used also when the solution is monotonic and bounded, or monotonic and convex, or has given number of maxima and minima. In these cases, an error of an approximate solution can be calculated.

4. Ill-posed problems with sourcewise represented solutions

Let an operator A be linear injective continuous and mapping $Z \to U$; Z, U are normed spaces. Let the following a priori information be valid: it is known that an exact solution $\bar{z}$ for an equation $\bar{u} = A\bar{z}$ is represented in the form $B\bar{v} = \bar{z}$, $\bar{v} \in V$; $B : V \to Z$; B is an injective completely continuous operator; V is a Hilbert space. Let suppose that an approximate right-hand side u_δ such that $\left\|\bar{u} - u_\delta\right\| \le \delta$, and its error $\delta > 0$ is known. Such a priori information is typical for many physical problems.

V.K. Ivanov and I.N. Dombrovskaya proposed an idea of a method of extending compacts. Let describe a version of this method below investigated by A.G. Yagola and K.Yu. Dorofeev.

Let preset an iteration number n=1, and define a closed ball in the space V: $\overline{S_n}(0) = \{v : \|v\| \le n\}$. Its image $Z_n = B\overline{S_n}(0)$ is a compact since B is a completely continuous operator and V is a Hilbert space. After that let us find $\min\limits_{z \in B(\overline{S_n}(0))} \|Az - u_\delta\|$, where u_δ is given approximate right-hand side $\|\bar{u} - u_\delta\| \le \delta, \delta > 0$. Existence of the minimum is guaranteed by compactness of Z_n and continuity of A. If $\min\limits_{z \in B(\overline{S_n}(0))} \|Az - u_\delta\| \le \delta$, then the iteration process should be stopped, and the number $n(\delta) = n$ defined. An approximate solution of the operator equation can be chosen as any element $z_{n(\delta)} : z_{n(\delta)} \in B(\overline{S_{n(\delta)}}(0))$ satisfying $\| Az_{n(\delta)} - u_\delta \| \le \delta$. If $\min\limits_{z \in B(\overline{S_n}(0))} \|Az - u_\delta\| > \delta$ then the compact should be extended. For this purpose n changes to n +*1*, and the process repeats.

<u>Theorem.</u> The process described above converges: $n(\delta) < +\infty$. There exists $\delta_0 > 0$ (generally speaking, depending on $\bar{z}$) such that $n(\delta) = n(\delta_0) \quad \forall \delta \in (0, \delta_0]$. Approximate solutions $z_{n(\delta)}$ strongly converge to the exact solution $\bar{z}$ as $\delta \to 0$.

It is clear why the method is referred to as "an extending compacts method". It appears that using this method so called an a posteriori error estimate can be defined. It means that there exists a function $\chi(u_\delta, \delta)$ such that $\chi(u_\delta, \delta) \to 0$ as $\delta \to 0$, and $\chi(u_\delta, \delta) \ge \| z_{n(\delta)} - \bar{z} \|$ at least for sufficiently small $\delta > 0$. As an a posteriori error estimate $\chi(u_\delta, \delta) = \max\{\| z_{n(\delta)} - z \| : z \in Z_{n(\delta)}, \| Az - u_\delta \| \le \delta\}$ can be taken.

An a posteriori error estimate is not an error estimate in a general sense, error estimates cannot be constructed for ill-posed problems. However, for sufficiently small $\delta > 0$ (notably $\forall \delta \in (0, \delta_0]$) an a posteriori error estimate is an error estimate for a solution of an ill-posed problem if an a priori information about sourcewise representability is available.

This approach was generalized to cases when both operators A and B are specified with errors, also to nonlinear ill-posed problems under condition of sourcewise representation of an exact solution.

Numerical methods for solving linear ill-posed problems under condition of sourcewise representation were constructed, including methods for an a posteriori error estimation. To use a sequence of natural numbers as radii of balls in the space V is not obligatory. Any unbounded monotonically increasing sequence of positive numbers can be taken.

5. Applications to PDE

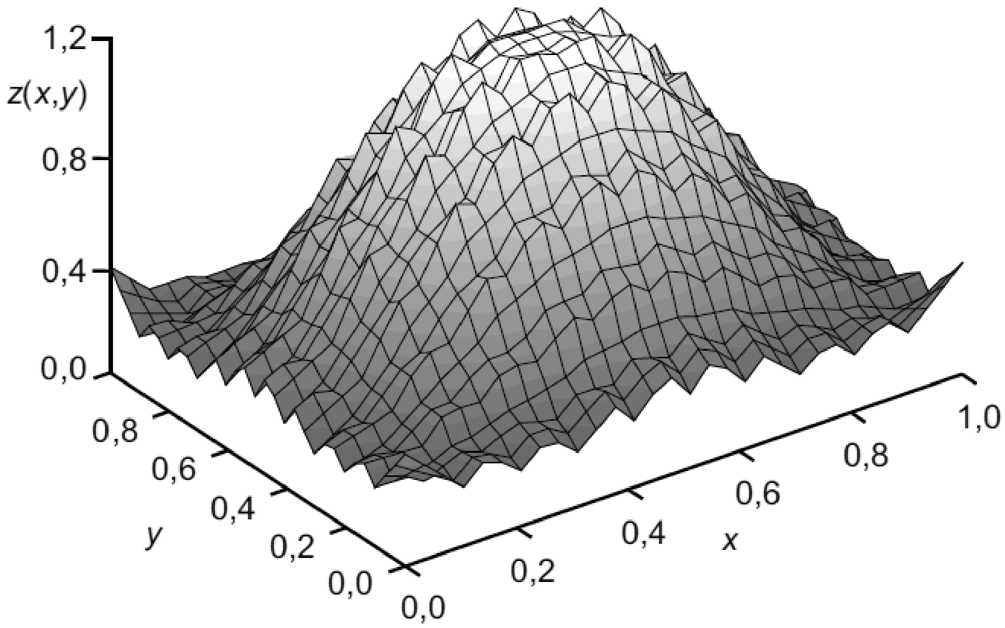
Many inverse and ill-posed problems for PDE can be formulated as operator equations. In this case, regularizing algorithms for solving ill-posed problems can be applied. If it is available a priori information that an unknown solution is an element of a given compact then Ivanov's quasisolution method can be used for constructing an approximate solution [1]. Moreover, an error of this solution can be calculated [2]. Sometimes, there exists a priori information that an unknown solution is sourcewise represented with a completely continuous operator. Then the method of extended compacts can be applied, and so called a posteriori error estimate can be calculated [2-3].

As examples of applications we considered:

1) Inverse problems of heat conductivity [4].
2) Cauchy problems for Laplace equation [5].
3) The Black-Scholes option pricing model [6].

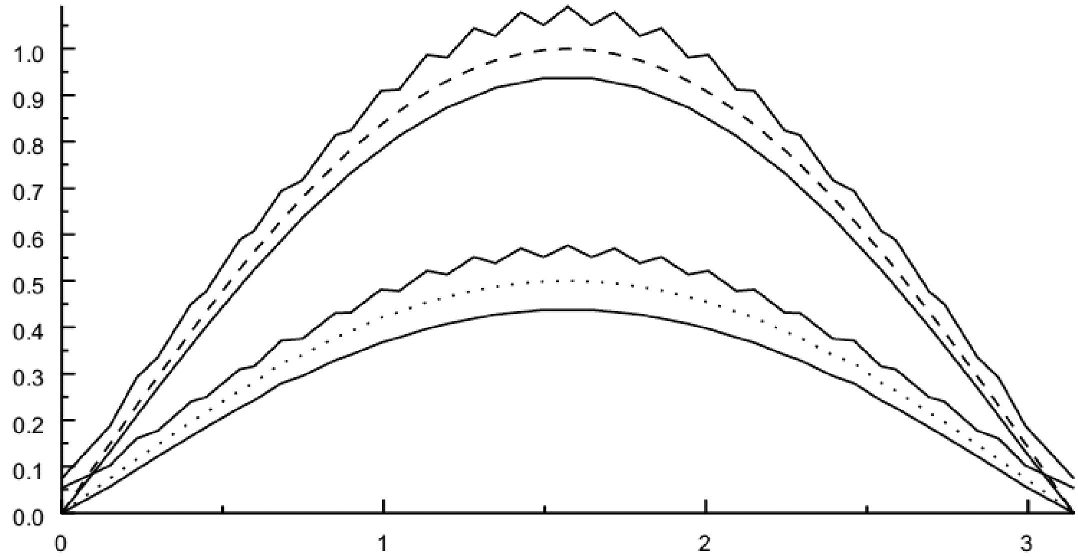
Some results of calculations are below.

On the picture it is upper error estimation for a concave nonnegative initial temperature. It is a result of numerical solution of an inverse problem for 2D heat conductivity equation on a compact set of nonnegative functions concave along coordinate lines.



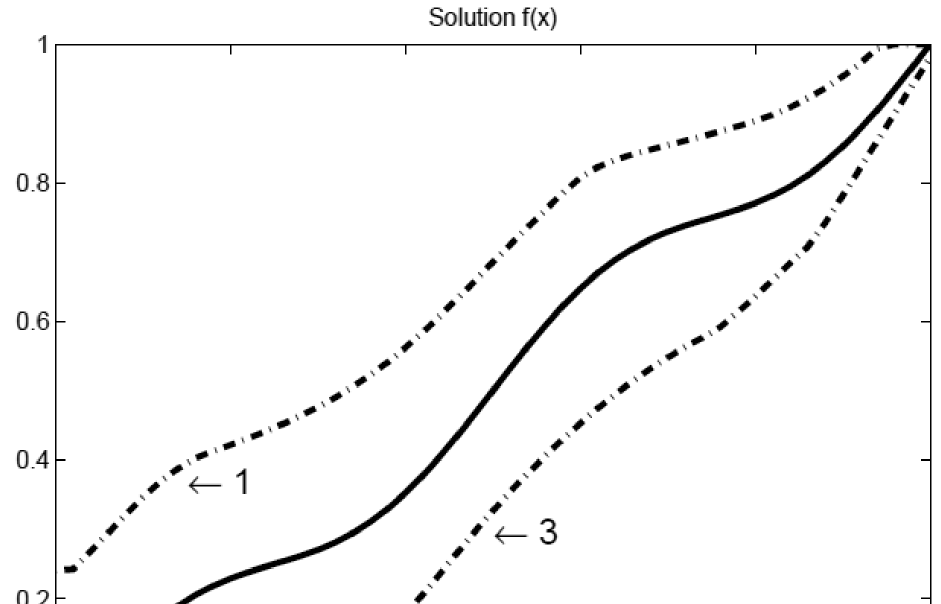
In the paper [4] it was shown how to apply the method of extending compacts for restoration and error estimation for the temperature at t>0 if the temperature is measured at T (if 0<t<T).

Results of solution and error estimation for Cauchy problem in the cartesian frame under condition that the exact solution is a concave nonnegative function are below. On the picture, there are the unknown function on the upper boundary of the Cartesian frame, its upper and lower error estimation so as its normal derivative with errors.



In our joint paper [6] we considered an inverse problem of parameter identification for a parabolic equation. The underlying practical example is the reconstruction of the unknown drift in the extended Black–Scholes option pricing model. Using a priori information about the unknown solution (i.e. its Lipschitz constant), we provide a solution to this nonlinear ill-posed problem, as well as an error estimate. Other types of a priori information may be used (for example, monotonicity and/or convexity of the unknown solution).

The following figure is from the paper [6]. It is an approximate solution and its upper and lower error estimations for given error of input data.



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Generalized eigenvalue problems for (p, q)-Laplacian *

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1 Introduction

In this talk, we consider the existence or non-existence of λ for which the following equation has a non-trivial (mainly, *positive*) solution:

$$(GEV;\lambda,\mu) \quad \begin{cases} -\Delta_p u - \mu\Delta_q u = \lambda(m_p(x)|u|^{p-2}u + \mu m_q(x)|u|^{q-2}u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with C^2 boundary $\partial\Omega$, $\lambda \in \mathbb{R}$, $1 < q < p < \infty$, $\mu \geq 0$, and the weights m_p, $m_q \in L^\infty(\Omega)$ are such that the Lebesgue measure of $\{x \in \Omega\,;\,, m_r(x) > 0\}$, with $r = p$, q, is positive. Here Δ_r stands for the usual r-Laplacian, i.e., $\Delta_r u = \operatorname{div}(|\nabla u|^{r-2}\nabla u)$ with $r \in (1, +\infty)$.

We say that $u \in W_0^{1,p}(\Omega)$ is a solution of $(GEV;\lambda,\mu)$ if it holds

$$\begin{aligned} &\textstyle\int_\Omega |\nabla u|^{p-2}\nabla u\nabla\varphi\,dx + \mu\int_\Omega |\nabla u|^{q-2}\nabla u\nabla\varphi\,dx \\ &= \lambda\textstyle\int_\Omega (m_p|u|^{p-2}u + \mu m_q|u|^{q-2}u)\varphi\,dx \end{aligned}$$

for all $\varphi \in W_0^{1,p}(\Omega)$.

Letting $\mu \to +0$, our equation $(GEV;\lambda,\mu)$ turns into the following $(p-1)$-homogeneous equation known as the usual weighted eigenvalue problem for the p-Laplacian:

$$(EV;p,\lambda) \qquad \begin{cases} -\Delta_p u = \lambda m_p(x)|u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Moreover, after multiplying our equation by $1/\mu$ and then letting $\mu \to +\infty$, $(GEV;\lambda,\mu)$ becomes the equation:

$$(EV;q,\lambda) \qquad \begin{cases} -\Delta_q u = \lambda m_q(x)|u|^{q-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Thus, we can understand that there is a close relationship between our equation $(GEV;\lambda,\mu)$ and the usual weighted eigenvalue problems for the p-Laplacian and q-Laplacian, i.e., problems $(EV;p,\lambda)$ and $(EV;q,\lambda)$.

*This talk is based on the results in [9], [12] and [13]

We briefly discuss the motivation for the given formulation of the eigenvalue problem $(GEV;\lambda,\mu)$. Let us observe that setting $A_r(t) := |t|^{r-2}$, with any $1 < r < +\infty$, the basic eigenvalue problem $(EV;r,\lambda)$ can be written as

$$-\mathrm{div}\,(A_r(|\nabla u|)\nabla u) = \lambda m_r(x)A_r(u)u \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega.$$

Now in line with this, if $m := m_p \equiv m_q$ and $\mu = 1$, then setting $A_{p,q}(t) := A_p(t) + A_q(t)$ there corresponds the equation

$$-\mathrm{div}\,(A_{p,q}(|\nabla u|)\nabla u) = \lambda m(x)A_{p,q}(u)u \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega.$$

We have thus exactly arrived at the statement of problem $(GEV;\lambda,\mu)$ encompassing the natural formulation for the generalization of the eigenvalue problem for the (p,q)-Laplace operator.

Nonlinear eigenvalue problems for elliptic equations have been thoroughly studied (see [10] for a comprehensive survey of different developments). Recently, many authors have studied (p,q)-Laplace equations (see [4], [7], [11], [14], [15]). Also, see Section 3 for the special cases.

2 Homogeneous eigenvalue problems

For later use, we review a few facts related to the limiting cases when $\mu = 0$ or $\mu = +\infty$ in $(GEV;\lambda,\mu)$. We recall that λ is an eigenvalue of $-\Delta_r$ ($r \in (1,+\infty)$) with weight function m_r if problem $(EV;r,\lambda)$ has a non-trivial solution. We denote the set of all eigenvalues of $-\Delta_r$ with weight function m_r by $\sigma(-\Delta_r, m_r)$. In particular, in the case of $m_r \equiv 1$, we write $\sigma(-\Delta_r)$ instead of $\sigma(-\Delta_r, 1)$. It is well known that the first positive eigenvalue $\lambda_1(r,m_r)$ of $-\Delta_r$ with weight function m_r is obtained by minimizing the Rayleigh quotient:

$$\lambda_1(r,m_r) := \inf\left\{ \frac{\int_\Omega |\nabla u|^r\,dx}{\int_\Omega m_r|u|^r\,dx}\,;\, u \in W_0^{1,r}(\Omega),\ \int_\Omega m_r|u|^r\,dx > 0 \right\}.$$

Since there exist *no non-negative* eigenvalues provided $m_r \le 0$, we set

$$\lambda_1(r,-m_r) = +\infty \quad \text{if} \quad m_r \ge 0. \tag{1}$$

It is also worth mentioning that $\lambda_1(r,m_r)$ has positive eigenfunctions $\varphi_1(r,m_r) \in C_0^{1,\alpha_r}(\overline{\Omega})$ with some $\alpha_r \in (0,1)$. Furthermore, the first positive eigenvalue is simple and *isolated*. Moreover, $(EV;r,\lambda)$ has no constant sign solutions (other than the trivial solution) provided $\pm\lambda \ne \lambda_1(r,\pm m_r)$.

Finally, we recall the second (positive) eigenvalue $\lambda_2(r,m_r)$ of $-\Delta_r$ with weight function m_r. It is defined by

$$\lambda_2(r,m_r) = \min\{\lambda > \lambda_1(r,m_r)\,;\, \lambda \in \sigma(-\Delta_r,m_r)\,\}.$$

We note that by the definition of $\lambda_2(r,m_r)$, if $\lambda_1(r,\pm m_r) < \pm\lambda < \lambda_2(r,\pm m_r)$, then $(EV;r,\lambda)$ has *no non-trivial* solutions.

3 Special cases of $m_p \equiv 0$ or $m_q \equiv 0$

Benouhiba and Belyacine ([1], [2]) showed the existence of the principal eigenvalue and of a *continuous* family of eigenvalues for the equation

$$-\Delta_p u - \Delta_q u = \lambda g(x)|u|^{p-2}u \quad \text{in } \mathbb{R}^N$$

In [3, Theorem 4.2], Cingolani and Degiovanni proved the existence of a *non-trivial* solution for the equation

$$-\Delta_p u - \mu\Delta u = \lambda|u|^{p-2}u + g(u) \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega$$

in the case where $p > 2(= q)$, $g \in C^1$ and $\lambda \notin \sigma(-\Delta_p)$. Under the *Neumann boundary condition*, Mihăilescu [8] determined the set of eigenvalues for the equation $-\Delta_p u - \Delta u = \lambda u$ in Ω, where $p > 2(= q)$.

Note that all results above are the special case where m_p or m_q disappears from our equation $(GEV; \lambda, \mu)$. In this section, we consider the following equation as the special case of our equation $(GEV; \lambda, \mu)$:

$$(GEV; r, \lambda, \mu) \qquad \begin{cases} -\Delta_r u - \mu\Delta_{r^*} u = \lambda m_r(x)|u|^{r-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $1 < r \neq r^* < \infty$. Recall that letting $\mu \to +0$, equation $(GEV; r, \lambda, \mu)$ above turns into the homogeneous equation $(EV; r, \lambda)$.

Throughout this section, we put $p = \max\{r, r^*\}$. We define the functional $\Phi_{(r,r^*,\mu)}$ on $W_0^{1,p}(\Omega)$ as follows:

$$\Phi_{(r,r^*,\mu)}(u) := \|\nabla u\|_r^r + \frac{\mu r}{r^*}\|\nabla u\|_{r^*}^{r^*} \tag{2}$$

for $u \in W_0^{1,p}(\Omega)$. The following proposition is the result on Rayleigh quotient to solve our problems $(GEV; r, \lambda, \mu)$.

Proposition 1 *([12, Proposition 4]) For $\mu > 0$, we set*

$$\underline{\lambda}(r, r^*, \mu, m_r) := \inf\left\{\frac{\Phi_{(r,r^*,\mu)}(u)}{\int_\Omega m_r|u|^r\,dx}\,;\, u \in W_0^{1,p}(\Omega),\ \int_\Omega m_r|u|^r\,dx > 0\right\}, \tag{3}$$

where $\Phi_{(r,r^*,\mu)}$ is the functional defined in (2)*. Then,*

$$\underline{\lambda}(r, r^*, \mu, m_r) = \lambda_1(r, m_r)$$

holds for every $\mu > 0$. In addition, for every $\mu > 0$, the infimum in (3) *is not attained.*

According to the standard argument using Rayleigh quotient, it is proved that if $-\lambda_1(r, -m_r) < \lambda < \lambda_1(r, m_r)$ holds, then $(EV; r, \lambda)$ has no non-trivial solutions. Similarly, by using Proposition 1, we obtain the following theorem.

Theorem 2 *([12, Theorem 1]) If $-\lambda_1(r, -m_r) \leq \lambda \leq \lambda_1(r, m_r)$ holds, then for any $\mu > 0$, $(GEV; r, \lambda, \mu)$ has no non-trivial solutions.*

It is also known that the homogeneous equation $(EV; r, \lambda)$ has no *positive* (or negative) solutions provided $\pm\lambda \neq \lambda_1(r, \pm m_r)$ (refer to [5, Section 6.2]). On the contrary, our equation $(GEV; r, \lambda, \mu)$ has a positive solution in the other cases other than that treated in Theorem 2.

Theorem 3 *([12, Theorem 2] and [13, Theorem 1]) If $\lambda > \lambda_1(r, m_r)$ or $\lambda < -\lambda_1(r, -m_r)$ holds, then for any $\mu > 0$, $(GEV; r, \lambda, \mu)$ has at least one positive solution. In particular, in the case of $r < r^*$, for any $\mu > 0$, $(GEV; r, \lambda, \mu)$ has a unique positive solution u_μ, that is, $u_\mu = \mu^{1/(r-r^*)}u_1$.*

Due to Theorem 2 and Theorem 3, we know that $(GEV; r, \lambda, \mu)$ has at least one positive solution if and only if

$$\lambda \in \begin{cases} (\lambda_1(r, m_r), +\infty) & \text{if } m_r \geq 0, \\ (-\infty, -\lambda_1(r, -m_r)) \cup (\lambda_1(r, m_r), +\infty) & \text{otherwise.} \end{cases}$$

By the definition of the second (positive) eigenvalue $\lambda_2(r, m_r)$, if $\lambda_1(r, \pm m_r) < \pm\lambda < \lambda_2(r, \pm m_r)$, then $(EV; r, \lambda)$ has no non-trivial solutions. This assertion is generalized to a non-existence of a *sign-changing* solution for our non-homogeneous equation as follows.

Theorem 4 *([12, Theorem 3]) If $\lambda_1(r, m_r) < \lambda \leq \lambda_2(r, m_r)$ holds, then for any $\mu > 0$, $(GEV; r, \lambda, \mu)$ has no sign-changing solutions.*

Theorem 5 *([13, Theorem 2]) Assume $r < r^*$. If $\pm\lambda > \lambda_2(r, \pm m_r)$ holds respectively, then for any $\mu > 0$, $(GEV; r, \lambda, \mu)$ has at least one sing-changing solution.*

4 Positive solution for $(GEV; \lambda, \mu)$

An essential part in our approach is that problem $(GEV; \lambda, \mu)$ is equivalent to another eigenvalue problem $(GEV; \lambda)$ where we have only one parameter in the case of $\mu = 1$, that is,

$$(GEV; \lambda) \qquad \begin{cases} -\Delta_p u - \Delta_q u = \lambda(m_p(x)|u|^{p-2}u + m_q(x)|u|^{q-2}u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

In fact, if u is a solution of $(GEV; \lambda, \mu)$, multiplying $(GEV; \lambda, \mu)$ by s^{p-1} $(s > 0)$, then $v = su$ is a solution of

$$-\Delta_p v - s^{p-q}\mu\Delta_q v = \lambda(m_p(x)|v|^{p-2}v + s^{p-q}\mu m_q(x)|v|^{q-2}v) \text{ in } \Omega, \quad v = 0 \text{ on } \partial\Omega.$$

So, choosing s_0 such that $s_0^{p-q}\mu = 1$, we see that $s_0 u$ is a solution of $(GEV; \lambda)$. Conversely, if u is a solution of $(GEV; \lambda)$, by the same argument above, then we easily see that $v = su$ with $s = \mu^{1/(p-q)}$ is a solution of $(GEV; \lambda, \mu)$.

As stated in (1), we recall that we defined

$$\lambda_1(r, 0) = \infty.$$

Then, we see the results developed from Theorem 2 and Theorem 3.

Theorem 6 *([9, Theorem 1]) If it holds*

$$-\min\{\lambda_1(p,-m_p),\lambda_1(q,-m_q)\}<\lambda<\min\{\lambda_1(p,m_p),\lambda_1(q,m_q)\},$$

then $(GEV;\lambda)$ has no non-trivial solutions.

Moreover, if the following $(H;\pm m_p,\pm m_q)$ holds, then problem $(GEV;\lambda)$ with $\lambda=\pm\min\{\lambda_1(p,\pm m_p),\lambda_1(q,\pm m_q)\}$ (respectively), has no non-trivial solutions:

$(H;\pm m_p,\pm m_q)$: *the following (i) or (ii) holds:*

(i) $\lambda_1(p,\pm m_p)\neq\lambda_1(q,\pm m_q)$;

(ii) $\varphi_1(p,\pm m_p)\neq t\varphi_1(q,\pm m_q)$ *for all* $t>0$,

where $\varphi_1(p,\pm m_p)$ and $\varphi_1(q,\pm m_q)$ are positive eigenfunctions corresponding to $\lambda_1(p,\pm m_p)$ and $\lambda_1(q,\pm m_q)$, respectively (see Section 2).

Remark 7 *If $(H;m_p,m_q)$ does not hold, that is, $\lambda_1(p,m_p)=\lambda_1(q,m_q)$ and $\varphi_1(p,m_p)=t\varphi_1(q,m_q)$ for some $t>0$, then $\varphi_1(q,m_q)$ and $\varphi_1(p,m_p)$ are positive solutions of $(GEV;\lambda)$ with $\lambda=\lambda_1(p,m_p)=\lambda_1(q,m_q)$. Indeed, since $\varphi_1(p,m_p)$ is a positive solution of $(EV;p,m_p)$, which is a $(p-1)$-homogeneous equation, then $\varphi_q=\varphi_1(q,m_q)$ solves the equation $-\Delta_p\varphi_q=\lambda m_p\varphi_q^{p-1}$ in Ω. On the other hand, because φ_q is a positive solution of $(EV;q,m_q)$, φ_q satisfies also the equation $-\Delta_q\varphi_q=\lambda m_q\varphi_q^{q-1}$ in Ω. Therefore, φ_q is a positive solution of $(GEV;\lambda)$ for $\lambda=\lambda_1(p,m_p)=\lambda_1(q,m_q)$.*

Theorem 6 can be proved by using the following result on the Rayleigh quotient:

Proposition 8 *([9, Proposition 7 and 8]) Let*

$$\underline{\lambda}:=\inf\left\{\frac{\Phi(u)}{\Psi(u)}\,;\,u\in W_0^{1,p}(\Omega),\ \Psi(u)>0\right\},\tag{4}$$

where

$$\Phi(u):=\frac{1}{p}\|\nabla u\|_p^p+\frac{1}{q}\|\nabla u\|_q^q,$$
$$\Psi(u):=\frac{1}{p}\int_\Omega m_p|u|^p\,dx+\frac{1}{q}\int_\Omega m_q|u|^q\,dx$$

for all $u\in W_0^{1,p}(\Omega)$. Then it holds $\underline{\lambda}=\min\{\lambda_1(p,m_p),\lambda_1(q,m_q)\}$. Moreover, $(H;m_p,m_q)$ as in Theorem 6 holds, Then the infimum in (4) is not attained.

The following theorem is our main existence result on $(GEV;\lambda)$ (or $(GEV;\lambda,\mu)$) in the non-resonant case.

Theorem 9 *([9, Theorem 3]) Assume that there holds $\lambda_1(p,m_p)\neq\lambda_1(q,m_q)$ (resp. $\lambda_1(p,-m_p)\neq\lambda_1(q,-m_q)$). If*

$$\min\{\lambda_1(p,m_p),\lambda_1(q,m_q)\}<\lambda<\max\{\lambda_1(p,m_p),\lambda_1(q,m_q)\}$$
$$(resp.\quad -\max\{\lambda_1(p,-m_p),\lambda_1(q,-m_q)\}<\lambda<-\min\{\lambda_1(p,-m_p),\lambda_1(q,-m_q)\}),$$

then $(GEV;\lambda)$ has at least one positive solution.

Finally, we state our existence results for problem $(GEV;\lambda)$ in the (second) resonant case $\lambda = \pm\max\{\lambda_1(p,\pm m_p), \lambda_1(q,\pm m_q)\}$.

Theorem 10 *([9, Theorem 4 and 5]) Let $r = p$ or q, and set $p^* = q$ and $q^* = p$. Assume that*

$$\pm\lambda = \lambda_1(r,\pm m_r) > \lambda_1(r^*,\pm m_{r^*}) \tag{5}$$

and

$$\int_\Omega |\nabla\varphi_1(r,\pm m_r)|^{r^*}\,dx - \lambda\int_\Omega m_{r^*}\varphi_1(r,\pm m_r)^{r^*}\,dx > 0, \tag{6}$$

respectively. Then $(GEV;\lambda)$ has at least one positive solution.

Remark 11 *We can easily produce examples where the hypotheses of the formulated results are fulfilled. For instance, let us take: $N = 1$, $\Omega = (0,\pi)$, $1 < q = 2 < p$, $m_q \equiv 1$ and $m_{p,n}(x) = 1 - h(x)/n$ with $0 \le h \in L^\infty(\Omega)$ and $h \not\equiv 0$. Then it is clear that $\lambda_1(2,m_2) = 1$ and $\varphi_1 = \varphi_1(2,m_2) = \sin x$. By easy computation, we have*

$$\int_0^\pi |\varphi_1'|^p\,dx - \int_0^\pi m_{p,n}\varphi_1^p\,dx = \frac{1}{n}\int_0^\pi h(x)\sin^p x\,dx > 0$$

for every $n \in \mathbb{N}$, so (6) (with $r = q = 2$, $r^* = p$, $m_r \equiv 1$) holds true. On the other hand, we see that

$$\lambda_1(p,1) < \frac{\int_0^\pi |\varphi_1'|^p\,dx}{\int_0^\pi \varphi_1^p\,dx} = 1$$

by the definition of $\lambda_1(p,1)$ and the facts that $\varphi_1(2,1) \ne t\varphi_1(p,1)$ for any $t > 0$ and $\lambda_1(p,1)$ is simple. By the continuity of $\lambda_1(p,m_p)$ with respect to m_p, it follows that for a sufficiently large n we have $\lambda_1(p,m_{p,n}) < 1$, so (5) is valid, too.

Comment 1: In the case $m_p \ge 0$ and $m_q \ge 0$ (non-negative weights), it is possible to prove that there exists λ^* satisfying

$$\max\{\lambda_1(p,m_p), \lambda_1(q,m_q)\} \le \lambda^* \le \infty$$

such that $(GEV;\lambda)$ has at least one or no positive solutions provided

$$\min\{\lambda_1(p,m_p), \lambda_1(q,m_q)\} < \lambda < \lambda^* \quad \text{or} \quad \lambda^* < \lambda,$$

respectively. Moreover, if $\infty > \lambda^* > \max\{\lambda_1(p,m_p), \lambda_1(q,m_q)\}$, then $(GEV;\lambda)$ has at least one positive solution in the case of $\lambda = \lambda^*$ also. However, as far as the author knows, there are no information about λ^* except one dimension case with no weight functions (see the next comment).

Comment 2: In the remaining case, that is, $\pm\lambda > \max\{\lambda_1(p,m_p), \lambda_1(q,m_q)\}$, we do not know that our equation has a positive solution or not. Concerning this question, we can refer to [6] for the special case $N = 1$, $m_p \equiv 1$ and $m_q \equiv 1$. According to the results in [6], $\lambda^* < \infty$ holds and both cases $\lambda^* > \max\{\lambda_1(p,1), \lambda_1(q,1)\}$ and $\lambda^* = \max\{\lambda_1(p,1), \lambda_1(q,1)\}$ occur (it depends on the relation between p, q and the length of the interval). The results in [6] are shown by using the bifurcation method and time map.

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An Application of Microlocal Analysis to an Inverse Problem Arising from Financial Markets

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1 Introduction

This is a joint work with Shinichi Doi in Osaka University. In this talk we consider our new model based on the Black-Scholes Model and formulated a new mathematical approach for an inverse problem in financial markets. Financial derivatives are contracts wherein payment is derived from an underlying asset such as a stock, bond, commodity, interest, or exchange rate. An underlying asset S_t at time t is modeled by the following stochastic differential equation:

$$dS_t = \mu(t, S_t)S_t dt + \sigma(t, S_t)S_t dW_t, \tag{1.1}$$

where the process W_t is the Brownian motion. The parameters $\mu(t, S)$ and $\sigma(t, S)$ are called the real drift and the local volatility of the underlying asset, respectively.

Black and Sholes first found how to construct a dynamic portfolio Π_t of the derivative security and the underlying asset [1]. Their approach is developed in probability theory, and the hedging and pricing theory of the derivative security is established as mathematical finance. By Ito's lemma, the stochastic behavior of the derivative security $u(t, S)$ is governed by the following stochastic differential equation:

$$du = \left(\frac{\partial u}{\partial t} + \mu(t, S)S\frac{\partial u}{\partial S} + \frac{1}{2}\sigma(t, S)^2\frac{\partial^2 u}{\partial S^2}\right)dt + \sigma(t, S)S\frac{\partial u}{\partial S}dW. \tag{1.2}$$

In the absence of arbitrage opportunities, the instantaneous return of this portfolio must be equal to the interest rate r, the return on a riskless asset, such as a bank deposit. Therefore, this equality takes the form of the following partial differential equation:

$$\frac{\partial u}{\partial t} + \frac{1}{2}\sigma(t, S)^2 S^2\frac{\partial^2 u}{\partial S^2} + (r - \delta)S\frac{\partial u}{\partial S} - ru = 0, \tag{1.3}$$

where r and the divided rate δ are the known constants.

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Their approach provides us a useful, simple method of pricing with financial derivatives, risk premium, and default probability estimation under the assumption that the risky asset is log-normally distributed. However, theoretical prices of options with different strike prices calculated by the Black-Scholes model differ from real market prices. Specifically, when we apply the Black-Scholes model to default probability estimation, we must be careful of the deviation between the expected and observed values. Merton has formulated a default probability estimation using a model based on [1] by considering the value of the firm instead of the stock, the firm's debt instead of strike price, and its equity instead of option price [8]. However, as shown in deriving the Black-Scholes model (see [1]), under the no arbitrage property of the financial market, the real drift μ does not enter equation (1.3). In [9], taking this into account, we have derived the following new model, by using A_t instead of S_t:

$$\frac{\partial u}{\partial t} + \frac{1}{2}\sigma(t,A)^2 A^2 \frac{\partial^2 u}{\partial A^2} + \mu(t,A)A\frac{\partial u}{\partial A} - ru = 0. \tag{1.4}$$

Moreover, in [9] we have established an inverse problem to the reconstruction of the real drift from the observable data, but only an binary option case.

In this talk, we prove a uniqueness for the solution of an inverse problem with respect to the real drift by using an application for microlocal analysis. To give a brief description of our problem, we see the method in [4]. In [4], they used the standard linearization method to an option pricing inverse problem and derived a partial differential equation for a principal part V. Since, after a change of variables, this equation is reduced to the heat equation with the right-hand side linear with respect to f, they wrote the well known integral representation for the solution V as follows:

$$V(\tau,x) = \int_{\mathbf{R}}\int_0^{\tau^*} \frac{1}{\sqrt{2\pi(\tau^*-\theta)\sigma_0^2}} e^{-\frac{|x-y|^2}{2\sigma_0^2(\tau^*-\theta)}} w(\theta,y)f(y)d\theta dy, \tag{1.5}$$

where $w(\tau,y)$ is represented by

$$w(\tau,y) = \frac{s^*}{\sqrt{2\pi\tau\sigma_0^2}} e^{-\frac{|y|^2}{2\sigma_0^2\tau}}. \tag{1.6}$$

For the above equation they applied the Laplace transform to exactly evaluate an integral with respect to the time. As a result, they derived the integral equation for f of the following form

$$V(\tau,x) = \int_{\mathbf{R}} B(x,y;\tau)f(y)dy \tag{1.7}$$

with the kernel

$$B(x,y;\tau) = \frac{s^*}{\sigma_0^2\sqrt{\pi}} \int_{\frac{|x-y|+|y|}{\sigma_0\sqrt{2\tau}}}^{\infty} e^{-\theta^2} d\theta \tag{1.8}$$

given by the error function and proved the uniqueness for the linearized inverse problem. In our case, since the linearized solution $\tilde{V}$ is the following form:

$$\tilde{V}(\tau,x)=\int_{\mathbf{R}}\int_0^{\tau}\frac{1}{\sqrt{4\pi(\tau-s)\sigma_0^2}}e^{-\frac{|y-x|^2}{4(\tau-\theta)\sigma_0^2}-\frac{\mu_0+\sigma_0^2}{2\sigma_0}y}\tilde{w}(\theta,y)f(y)d\theta dy, \tag{1.9}$$

where $\tilde{w}(\theta,y)$ is the following form

$$\tilde{w}(\tau,y)=\int_0^{\infty}\frac{1}{\sqrt{4\pi\tau\sigma_0^2}}e^{-\frac{|x-y|^2}{4\tau\sigma_0^2}-\frac{\mu_0+\sigma_0^2}{2\sigma_0}y}dx. \tag{1.10}$$

Therefore we can't derive an integral equation by Laplace transform as (1.7) that is, in our case $\tilde{w}(\tau,y)$ is not Gauss function as $w(\tau,y)$ but Error function. In the present paper, taking this into account, we shall prove a uniqueness in the inverse problem of the real trend by applying FBI transform to (1.9).

2 Inverse problem of the real drift

As we have seen in Section1, we have derived the new arbitrage model and formulated an inverse option pricing problem for a reconstruction of a real trend in the binary option case. In this section we explain how we can formulate an inverse problem of our new arbitrage model and reconstruct the real drift.

Here, we consider the following problem that the local volatility function $\sigma(t,A)$ is a positive constant $\sigma_0>0$ and the real drift function $\mu(t,A)$ is a space-dependentin our new equation (1.4) with a suitable condition,

$$u(t,A)|_{t=T}=\max\{A-D,0\} \tag{2.1}$$

where D is a price of the firm's debt at the maturity date T.

By the following changes of variables and substitutions

$$\begin{aligned} y&=\log\frac{A}{D}, & \tau&=T-t,\\ \mu(y)&=\mu(De^y), & U(\tau,y)&=u(T-\tau,De^y)/D, \end{aligned} \tag{2.2}$$

one can transform the equation (1.4) and the initial data into the following inverse problem of the real drift :

$$\begin{cases} \dfrac{\partial U}{\partial\tau}=\dfrac{1}{2}\sigma_0^2\dfrac{\partial^2 U}{\partial y^2}-\Big(\dfrac{1}{2}\sigma_0^2-\mu(y)\Big)\dfrac{\partial U}{\partial y}-rU & (y,\tau)\in\mathbf{R}\times(0,\tau^*),\\ U(\tau,y)|_{\tau=0}=\max\{e^y-1,0\} & y\in\mathbf{R}, \end{cases} \tag{2.3}$$

$$U(\tau^*, y) = U^*(y) \qquad\qquad y \in \omega \subseteq \mathbf{R}, \tag{2.4}$$

where $\tau^* = T - t^* > 0$, t^* is the current time and ω is an interval of **R**.

Here, we define that the inverse problem of the real drift (2.3) and (2.4) seeks $\mu(y)$ from given $U^*(y)$. However, since this inverse problem is a nonlinear, there are difficulties with uniqueness and existence of the solution to one. Therefore, we will formulate the inverse problem of the real drift by means of the method of a linearization in [3] and [4].

To linearize around the constant coefficient μ_0, we assume that

$$\mu(y) = \mu_0 + f(y),$$

where $f(y)$ denotes a small perturbation. Thus, we observe

$$U = U_0 + V + \nu,$$

where U_0 solves the Cauchy problem (2.3) with $\mu(y) \equiv \mu_0$, ν is quadratically small with respect to f, and V is the principal part of the perturbed solution U. Substituting this into the expression for u and neglecting terms of high order with respect to f, we reach the linearized inverse problem of the real drift.

Linearized Inverse Problem of the real Drift (LIPD). *The parameters τ^*, μ_0, σ_0, and r are given. From the option price $V^*(y) = \{U^*(y) - U_0(\tau^*, y)\}$, identify the perturbation $f(y)$ satisfying*

$$\begin{cases} \dfrac{\partial V}{\partial \tau} - \dfrac{1}{2}\sigma_0^2 \dfrac{\partial^2 V}{\partial y^2} + \left(\dfrac{1}{2}\sigma_0^2 - \mu_0\right)\dfrac{\partial V}{\partial y} + rV = \dfrac{\partial u_0}{\partial y} f(y), \\[2ex] \qquad\qquad V(\tau, y)|_{\tau=0} = 0, \end{cases} \tag{2.5}$$

$$V(\tau^*, y) = V^*(y). \tag{2.6}$$

3 Main results

In this section we shall prove the uniqueness of the solution to LIPD by using the method of microlocal analysis. Before describing the main theorem, we shall transform the equation of (2.4) into simple form and derive an integral equation of Fredholm type.

We set

$$a_0 = \frac{\sigma_0^2 - 2\mu_0}{2\sigma_0^2}, \qquad b_0 = r + \frac{1}{2}\sigma_0^2 a_0^2$$

$$H_a = -\left(\frac{\partial}{\partial y} - a\right)^2 \qquad (a = a_0 - 1)$$

then (2.5) can be rewritten as

$$\begin{cases} \left(\dfrac{\partial}{\partial\tau} + \dfrac{1}{2}\sigma_0^2 H_a\right) v(\tau, y) = f(y)\left(e^{-y+b_0\tau}\dfrac{\partial U_0}{\partial y}\right) & (y,\tau) \in \mathbf{R}\times(0,\tau^*), \\ v(\tau,y)|_{\tau=0} = 0 & y \in \mathbf{R}, \end{cases} \tag{3.1}$$

where, $v(\tau, y) = e^{-y+b_0\tau}V(\tau, y)$. From the well-known representation of the solution to the Cauchy problem (3.1), we have the following an integral equation of Fredholm type:

$$v(\tau^*, x) = \int_0^{\tau^*} U_a(\tau^* - s)[w(s,\cdot)f(\cdot)](y)ds. \tag{3.2}$$

Here

$$(U_a(\tau)\varphi)(y) = \int_R K_a(\tau, y-x)\varphi(x)dx,$$

where

$$K_a(\tau, y) = \frac{1}{\sqrt{4\pi\tau}}e^{-\frac{|y|^2}{4\tau}+ay}$$

and $w(\tau, x)$ is represented the following form:

$$\begin{aligned} w(\tau,x) :=& (U_a(\tau)H_+)(x) \\ =& \int_0^\infty \frac{1}{\sqrt{4\pi\tau}}e^{-\frac{|x-y|^2}{4\tau}+a(x-y)}dx \\ =& \frac{1}{\sqrt{\pi}}e^{\tau a^2}\int_{-\infty}^{\frac{x-2\tau a}{\sqrt{4\tau}}} e^{-\theta^2}d\theta, \end{aligned} \tag{3.3}$$

where $H_+(x) = 1_{[0,\infty]}(x)$.

We will describe results about LIPD in the following theorem.

Theorem 3.1. *Let $\tau^* > 0$ and $f(y) \in L^2(\boldsymbol{R})$. Assume that supp$f \subset [-L, \infty)$ with some $L \geq 0$. Then a solution f(y) to the integral equation (3.2) and hence, to the inverse problem of the real drift (2.5) and (2.6) is unique.*

Proof. To prove the claim of Theorem 3.1 it suffices to prove $f = 0$ assuming that the left-hand side of (3.2) is zero.

We assume that $v(\tau^*, y)$ is zero, that is,

$$\int_0^{\tau^*} U_a(\tau^* - s)[w(s,\cdot)f(\cdot)](y)ds = 0. \tag{3.4}$$

To prove that f is zero, we apply the Fourier-Bros-Iagolnitzer (for short, FBI) transform of (3.4) and we prove that the FBI transform of f is exponentially small(see Definition 4.1.1 in [7]).

We first write the integral equation (3.4) as the some of two parts as follows:

$$\begin{aligned}
&\int_0^{\tau^*} U_a(\tau^*-s)[w(s,\cdot)f(\cdot)](y)ds \\
&= \int_0^{\tau_0^*} U_a(\tau^*-s)[w(s,\cdot)f(\cdot)](y)ds + \int_{\tau_0^*}^{\tau^*} U_a(\tau^*-s)[w(s,\cdot)f(\cdot)](y)ds. \\
&\equiv I_1(y) + I_2(y), \qquad (3.5)
\end{aligned}$$

where τ_0^* is a positive constant such that $0 < \tau_0^* < \tau^*$. In the remaining part of this proof, to derive exponentially small of Tf, we shall consider the L^2 estimate of (3.5) with H_a, and we assume that $L_0 = L + 1$. Then we can show that $TH_aI_1(x,\xi;h)$ is exponentially small by several lemmas.

Next, to consider the L^2 estimate of $TH_aI_2(x,\xi;h)$, we regard H_a as a pseudodifferential operator acting on f(see [7] for details), that is,

$$\begin{aligned}
H_aI_2(y) &= \int_{\tau_0^*}^{\tau^*} H_aU_a(\tau^*-s)[w(s,\cdot)f(\cdot)](y)ds \\
&= \mathrm{Op}_\mathrm{h}^1(p)f(y),
\end{aligned}$$

where the symbol of the above pseudodifferential operator is the following form

$$\begin{aligned}
p(y,\xi) &= (\xi+ia)^2\int_{\tau_0^*}^{\tau^*} e^{-(\tau^*-s)(\xi+ia)^2}w(s,y)ds \\
&= \int_{\tau_0^*}^{\tau^*} \frac{\partial}{\partial s}(e^{-(\tau^*-s)(\xi+ia)^2})w(s,y)ds \\
&= w(\tau^*,y) - e^{-(\tau^*-\tau_0^*)(\xi+ia)^2}w(\tau_0^*,y) - \int_{\tau_0^*}^{\tau^*} e^{-(\tau^*-s)(\xi+ia)^2}\frac{\partial w}{\partial s}(s,y)ds. \qquad (3.6)
\end{aligned}$$

Here, let $\chi_1(\xi) \in C_0^\infty(\mathbf{R})$ be such that $\chi_1 = 0$ if $\xi < \frac{1}{4}, \chi_1 = 1$ if $\xi > \frac{1}{2}$ and we set

$$p_j(x,\xi;h) = p(x,\frac{\xi}{h})\chi_j(\xi) \ \ (j=1,2), \qquad (3.7)$$

where $\chi_2 = 1 - \chi_1(\xi)$. Moreover, let the real-valued function $\psi \in C_0^\infty(\mathbf{R})$ be such that $\psi = 0$ if $\xi < 1, \psi = 1$ if $\xi > 2$ and we set

$$T^\varepsilon u(\,\cdot\,,\,\cdot\,;h) = e^{\varepsilon\psi(\xi)/h}Tu(\,\cdot\,,\,\cdot\,;h). \qquad (3.8)$$

Now we apply Corollary 3.5.5 (in [7]) with $T = T^\varepsilon$, $f = 1$ and $\psi = \varepsilon\psi$, where $\varepsilon > 0$ will be taken small enough later. we obtain

$$\begin{aligned}&||T^\varepsilon \mathrm{Op}_h^1(p_1)f||_{L^2}^2\\ &\qquad \geq ||p_1(y - i\varepsilon\partial_\xi\psi(\xi), \xi - \varepsilon\partial_\xi\psi(\xi); h)T^\varepsilon f||_{L^2}^2 - Ch||T^\varepsilon f||_{L^2}^2.\end{aligned} \tag{3.9}$$

By using Taylor's formula, we can estimate the right hand side of (3.9) as follows:

$$\geq ||p_1(y, \xi; h)T^\varepsilon f||_{L^2}^2 - C_1(\varepsilon + h)||T^\varepsilon f||_{L^2}^2$$

$$\geq ||w(\tau^*, y)\chi_1(\xi)T^\varepsilon f||_{L^2}^2 - C_2(\varepsilon + h)||T^\varepsilon f||_{L^2}^2$$

Here, by $\chi_1(\xi) = 1 - \chi_2(\xi)$, (3.8) ans several lemmas we have

$$\begin{aligned}&\geq \left\{\frac{C_0^2}{2} - C_2(\varepsilon + h)\right\}||T^\varepsilon f||_{L^2([-L_0,\infty)\times\mathbf{R})}^2\\ &\qquad - C^2||Tf||_{L^2}^2 - C_2(\varepsilon + h)||T^\varepsilon f||_{L^2((-\infty,-L_0)\times\mathbf{R})}^2\end{aligned}$$

where we used that $\psi(\xi) = 0$ if $\xi \leq 1$.

On the other hand, since we can rewrite the left-hand side of (3.9),

$$\begin{aligned}T^\varepsilon[\mathrm{Op}_h^1(p_1)f] &= T^\varepsilon[-H_a I_1(y) - \mathrm{Op}_h^1(p_2)f]\\ &= -T^\varepsilon H_a I_1(y) - T^\varepsilon[\mathrm{Op}_h^1(p_2)f],\end{aligned}$$

by using the follows

$$||T^\varepsilon I_1||_{L^2(\mathbf{R}\times\{|\xi|\leq 1\})}^2 = ||TI_1||_{L^2(\mathbf{R}\times\{|\xi|\leq 1\})}^2 \leq C_3||Tf||_{L^2}^2$$

and

$$||T^\varepsilon \mathrm{Op}_h^1(p_2)f||_{L^2(\mathbf{R}\times\{|\xi|\leq 1\})} = \quad ||T\mathrm{Op}_h^1(p_2)f||_{L^2(\mathbf{R}\times\{|\xi|\leq 1\})} \leq C_4||Tf||_{L^2}^2,$$

we have the following estimates applying several lemmas,

$$\begin{aligned}&||T^\varepsilon \mathrm{Op}_h^1(p_1)f||_{L^2}^2\\ &\quad \leq ||T^\varepsilon H_a I_1||_{L^2(\mathbf{R}\times\{|\xi|\leq 1\})}^2 + ||T^\varepsilon H_a I_1||_{L^2(\mathbf{R}\times\{|\xi|>1\})}^2\\ &\qquad + ||T^\varepsilon \mathrm{Op}_h^1(p_2)f]||_{L^2(\mathbf{R}\times\{|\xi|\leq 1\})}^2 + ||T^\varepsilon \mathrm{Op}_h^1(p_2)f]||_{L^2(\mathbf{R}\times\{|\xi|>1\})}^2\\ &\quad \leq 2C_5||Tf||_{L^2}^2\\ &\qquad + 2\left(||T^\varepsilon H_a I_1||_{L^2(\mathbf{R}\times\{|\xi|>1\})}^2 + ||T^\varepsilon \mathrm{Op}_h^1(p_2)f]||_{L^2(\mathbf{R}\times\{|\xi|>1\})}^2\right).\end{aligned} \tag{3.10}$$

Then, the estimates gives

$$\left\{\frac{C_0^2}{2} - C_2(\varepsilon + h)\right\}||T^\varepsilon f||^2_{L^2([-L_0,\infty)\times\mathbf{R})}$$
$$-C^2||Tf||^2_{L^2} - C_2(\varepsilon+h)||T^\varepsilon f||^2_{L^2((-\infty,-L_0)\times\mathbf{R})}$$
$$\leq C_5||Tf||^2_{L^2} + 2\left(||T^\varepsilon H_a I_1||^2_{L^2(\mathbf{R}\times\{|\xi|>1\})} + ||T^\varepsilon \mathrm{Op}_h^1(p_2)f]||^2_{L^2(\mathbf{R}\times\{|\xi|>1\})}\right).$$

Since we can get the following

$$||T^\varepsilon f||^2_{L^2((-\infty,-L_0)\times\mathbf{R})} = ||e^{\varepsilon\frac{\psi(\xi)}{h}} Tf||^2_{L^2((-\infty,-L_0)\times\mathbf{R})} = O(e^{\frac{2\varepsilon-\delta}{h}})||Tf||^2_{L^2},$$

$$||T^\varepsilon H_a I_1||^2_{L^2(\mathbf{R}\times|\xi|>1)} = ||e^{\varepsilon\frac{\psi(\xi)}{h}} H_a I_1||^2_{L^2(\mathbf{R}\times\{|\xi|>1\})} = O(e^{\frac{2\varepsilon-\delta}{h}})||Tf||^2_{L^2}$$

and

$$||T^\varepsilon \mathrm{Op}_h^1(p_2)f]||^2_{\mathbf{R}\times L^2(|\xi|>1)} = ||e^{\varepsilon\frac{\psi(\xi)}{h}} \mathrm{Op}_h^1(p_2)f]||^2_{L^2(\mathbf{R}\times\{|\xi|>1\})} = O(e^{\frac{2\varepsilon-\delta}{h}})||Tf||^2_{L^2},$$

for $\delta > 0$ if ε is chosen small enough, we have

$$||e^{\varepsilon\frac{\psi(\xi)}{h}} Tf||^2_{L^2([-L_0,\infty)\times\mathbf{R})} = O(1)||Tf||^2_{L^2}.$$

In paticular, since $\psi(\xi) = 0$ if $|\xi| \geq 2$, therefore we obtain

$$||Tf||^2_{L^2([-L_0,\infty)\times\{|\xi|\geq 2\})} = O(e^{-\frac{\delta}{h}}). \tag{3.11}$$

In particular, we deduce form (3.11) that

$$[-L_0,\infty)\times\{|\xi|\geq 2\}\cap \mathrm{WF}_a(f) = \emptyset,$$

where WF_a is called analytic wave front set of f (see [7] for details), hence we obtain that f is real analytic in $(-L_0,\infty)$.

Since $f = 0$ in $(-L_0,-L)$ by the assumption, then we conclude that f is identically zero on **R**.

The proof is complete. □

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Type II blow-up mechanisms in semilinear parabolic equations

Yukihiro Seki *

1 Introduction

In this talk we discuss blow-up mechanisms for semilinear parabolic equations whose typical form is:

$$u_t = \Delta u + |u|^{p-1}u, \quad x \in \mathbf{R}^N,\ t > 0, \tag{1.1a}$$

$$u(x,0) = u_0, \quad x \in \mathbf{R}^N, \tag{1.1b}$$

where Δ denotes the Laplace operator in the Euclidean space $\mathbf{R}^N$ with $N \geq 1$, $p > 1$ is a constant and u_0 is a bounded function in $\mathbf{R}^N$. Local-in-time existence of a unique classical solution of (1.1a)-(1.1b) is well known. As usual, we say that the solution u of (1.1a)-(1.1b) blows up in a finite time T if the solution stays bounded for $0 < t < T$ and

$$\limsup_{t \to T} \|u(\cdot,t)\|_{L^\infty(\mathbf{R}^N)} = +\infty. \tag{1.2}$$

Various criteria on given data for blow-up in finite time are known. For example, if $u_0 \in H^1 \cap L^{p+1}(\mathbf{R}^N)$ and

$$\frac{1}{2}\int_{\mathbf{R}^N} |\nabla u_0|^2 dx - \frac{1}{p+1}\int_{\mathbf{R}^N} |u_0|^{p+1} dx < 0, \tag{1.3}$$

then the solution of (1.1a)-(1.1b) blows up in finite time (cf. [8, 15]).

The main focus of this talk is to describe singularity mechanisms for blow-up solutions. More precisely, we are interested in the blow-up rate of $\|u(\cdot,t)\|_{L^\infty(\mathbf{R}^N)}$ as t approaches the blow-up time.

1.1 Known results on blow-up rates

The following definition is due to [10].

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Definition 1.1. Let u be a solution of (1.1a)-(1.1b) that blows up in a finite time T. The blow-up is called of **type I** if there exists a positive constant K such that

$$\|u(\cdot,t)\|_{L^\infty(\mathbf{R}^N)} \leq K(T-t)^{-1/(p-1)}; \tag{1.4}$$

whereas the blow-up is called of **type II** otherwise, i.e.,

$$\limsup_{t\nearrow T}(T-t)^{1/(p-1)}\|u(\cdot,t)\|_{L^\infty(\mathbf{R}^N)} = +\infty. \tag{1.5}$$

When a solution blows up in finite time and the blow-up is of type II, we call the solution type II blow-up solution.

We just review some known results on blow-up rates.

1. Sobolev subcritical case: $N = 1, 2$ or $p < (N+2)/(N-2) =: p_{\mathrm{S}}$.

   Giga, Matsui, and Sasayama [3] proved that blow-ups of all the solutions of (1.1a) are of type I for all subcritical range of p, thus improving considerably the result of an earlier work by Giga and Kohn [2].

2. Sobolev critical case: $N \geq 3$ and $p = p_{\mathrm{S}}$.

   Nonexistence of type II blow-up was proven for positive radial solutions by Matano and Merle [10], whereas sign-changing type II blow-up solutions exist when $3 \leq N \leq 6$ according to a formal matched asymptotic method in Filippas, Herrero, and Velázquez [1]. It has recently proven by Schweyer [16] that type II blow-up does occur for $N = 4$ in the radial case.

3. Sobolev supercritical case: $N \geq 3$ and $p_{\mathrm{S}} < p$.

   In this case another exponent p_{JL} defined by

   $$p_{\mathrm{JL}} := \begin{cases} +\infty, & N \leq 10, \\ \dfrac{N - 2\sqrt{N-1}}{N - 4 - 2\sqrt{N-1}}, & N \geq 11. \end{cases}$$

   plays an essential role. The importance of this exponent was first shown in [6].

   (a) Joseph-Lundgren subcritical case: $N \geq 3$ and $p_{\mathrm{S}} < p < p_{\mathrm{JL}}$.
   Matano and Merle [10, 11] and Mizoguchi [13] proved that type II blow-up cannot occur for radial solutions under some mild assumptions on initial data.

   (b) Joseph-Lundgren supercritical case: $N \geq 11$ and $p_{\mathrm{JL}} < p$.
   Type II blow-up may actually occur as was shown in Herrero, and Velázquez [4, 5]. A matched asymptotic method plays a crucial role in constructing type II blow-up solutions in these articles. The result is described in detail in §3. Based upon these specific solutions constructed in [4, 5], further progress has been established in Matano [9] and Mizoguchi [14].

As for positive radial solutions, we may understand that Joseph-Lundgren exponent divides the range of p into two parts in terms of existence/nonexistence of type II blow-up. It should be noticed that this fact has already been conjectured in [4].

2 Main result

A natural question that arises from these results is whether or not type II blow-up would occur in the **Joseph-Lundgren critical case**: $p = p_{\mathrm{JL}}$. As far as the speaker knows, no conjecture has circulated for this open question. The aim of this talk is to give a formal result, based on a matched asymptotic method, that suggests the existence of type II blow-up solutions. The main result may be formally stated as follows:

Main result. *Let $N \geq 11$ and $p = p_{\mathrm{JL}}$. Then there exist radial solutions that blow up in finite time and the blow-ups are of type II.*

The blow-up mechanisms of these solutions are different from those of any type II blow-up solutions having been found for $p > p_{\mathrm{JL}}$. Further details will be presented in the talk.

3 Herrero–Velázquez' solutions

We shall recall the result of [4, 5] in detail. Throughout this talk we use the following notation:

$$\beta = \frac{1}{p-1}; \tag{3.1a}$$

$$\gamma = \frac{N-2-\sqrt{16\beta^2-4(N-2)\beta+(N-2)(N-10)}}{2}. \tag{3.1b}$$

Notice that $\gamma > 0$ is a real root of the quadratic equation:

$$\gamma^2-(N-2)\gamma+2(N-2\beta-2)(\beta+1)=0 \tag{3.2}$$

if and only if $N \geq 11$ and $p \geq p_{\mathrm{JL}}$. Quadratic equation (3.2) is related to the asymptotic expansions as $|x| \to \infty$ of stationary solutions $U_\eta(|x|)$ to be given in §§4.2 below.

Proposition 3.1. *(Herrero and Velázquez* [5, Theorem 1]*). Assume that $N \geq 11$ and $p > p_{\mathrm{JL}}$ and let $T > 0$ be any constant. Then for every positive integer ℓ such that $\lambda_\ell := \ell - \gamma/2 + 1/(p-1) > 0$, there exists a radial solution u_ℓ of* (1.1a)-(1.1b) *which blows up at $t = T$, $x = 0$, and satisfies* (1.5).

Moreover, the solution satisfies $\|u(\cdot,t)\|_\infty = u(0,t)$ and:

$$C_1\,(T-t)^{-\beta-2\beta\omega_\ell} \leq u_\ell(0,t) \leq C_2\,(T-t)^{-\beta-2\beta\omega_\ell} \tag{3.3a}$$

$$\text{with} \quad \omega_\ell := \frac{\lambda_\ell}{\gamma-2\beta} > 0 \tag{3.3b}$$

for some positive constants C_1 and C_2 depending only on p, N and ℓ.

We shall call the solutions Herrero–Velázquez' solutions or **HV solutions** for short.

4 Preliminary results

Let us considr the radial stationary version of equation (1.1a):

$$\frac{d^2U}{dr^2} + \frac{N-1}{r}\frac{dU}{dr} + U^p = 0 \qquad \text{for } r > 0. \tag{4.1}$$

Structures of solutions of (4.1) play important roles in the study of existence/nonexistence of type II blow-ups. Up to now, many important properties on those solutions are available. We just review some of them.

4.1 Singular stationary solutions

Proposition 4.1. *Assume that $N \geq 3$ and $p > N/(N-2)$. Then there exists a singular stationary solution U_∞ of (4.1) given by*

$$U_\infty(r) = c_{p,N} r^{-2\beta}, \quad c_{p,N}^{p-1} = 2\beta\left(N-2-2\beta\right). \tag{4.2}$$

Moreover, function $x \mapsto U_\infty(|x|)$ belongs to $H^1_{loc}\left(\boldsymbol{R}^N\right)$ when $p > p_S$.

4.2 Regular stationary solutions

We just recall some properties on regular solutions of (4.1). Given a constant $\alpha > 0$, we investigate regular solutions U_α of (4.1) satisfying

$$U(0) = \alpha, \quad U'(0) = 0. \tag{4.3}$$

Proposition 4.2. *(Infinitely many intersection / ordered structure) Assume that $p > p_S$. Then for every $\alpha > 0$ there exists a unique solution U_α of (4.1) satisfying (4.3). The solutions $U_\alpha(r)$ are monotone decreasing in r and*

$$U_\alpha(r) \to U_\infty(r) \tag{4.4}$$

as $r \to \infty$ and also as $\alpha \to \infty$. Moreover,

1. *If $p_S < p < p_{JL}$, then the graphs of $U_\alpha(r)$ and $U_\infty(r)$ intersect infinitely many times:*

$$\mathcal{Z}_{(0,\infty)}\left(U_\alpha - U_\infty\right) = +\infty, \tag{4.5}$$

where $\mathcal{Z}_{(0,\infty)}(F)$ denotes the number of zeros of function F in the interval $(0,\infty)$.

2. *If $N \geq 11$ and $p \geq p_{JL}$, the solutions are ordered according to their values at $r = 0$. Namely, if $0 < \alpha_1 < \alpha_2$, it follows that*

$$U_{\alpha_1}(r) < U_{\alpha_2}(r) < U_\infty(r) \qquad \text{for all } r > 0. \tag{4.6}$$

The fact of (4.5) is a key property to prove nonexistence of type II blow-up for $p_{\mathrm{S}} < p < p_{\mathrm{JL}}$ [10, 11, 13]. On the other hand, for $N \geq 11$ and $p > p_{\mathrm{JL}}$, the ordered structure (4.6) and the asymptotic formula (4.7a) below were essentially used to construct type II blow-up solutions in [5]. As for the asymptotic expansions, a logarithmic factor appears in the first corrective term (cf. (4.7b) below) when $p = p_{\mathrm{JL}}$, which violates the argument of [5]. This fact gives an essential difficulty in the critical case.

Proposition 4.3. *(Asymptotic expansions) For every $\alpha > 0$, the following asymptotic expansions as $r \to \infty$ hold:*

$$p > p_{JL} \implies U_\alpha(r) = U_\infty(r) - h(\alpha) r^{-\gamma} + o(r^{-\gamma}); \tag{4.7a}$$

$$p = p_{JL} \implies U_\alpha(r) = U_\infty(r) - h_1(\alpha) r^{-\gamma} \log r + h_2(\alpha) r^{-\gamma} + o(r^{-\gamma}) \tag{4.7b}$$

where $\gamma (> 2\beta)$ is the real number given in (3.1b), *and where $h(\alpha), h_1(\alpha)$ and $h(\alpha)$ are positive constants depending only on p, N and α.*

Proposition 4.3 was proven in [7, Lemma 4.3–4.4], where more precise formulas were obtained.

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α-GAUSS CURVATURE FLOWS WITH FLAT SIDES

LAMI KIM *

We consider α-Gauss curvature flow with flat sides, which is given by the flow

$$\begin{aligned} \frac{\partial X}{\partial t}(x,t) &= -K^{\alpha}(x,t)\,\nu(x,t) \\ X(x,0) &= X_0(x) \end{aligned} \tag{1}$$

where ν denotes the unit outward normal vector and $1/2 < \alpha \leq 1$. This flow is related to the deformation of 2-dimensional compact convex surfaces in $\mathbb{R}^3$ moving with collision from any random angle.

Let Σ_0 be a compact convex initial surface and $\alpha > 0$. Then there exists viscosity solution Σ_t which has uniform Lipschitz bound for $0 < t < T_0$, [2]. For $\frac{1}{2} < \alpha \leq 1$, Σ_t has a uniform $C^{1,1}$-estimate for $0 < t < T_0$, [2, 12]. The C^{∞}_{δ}-regularity of the strictly convex part of the surface and the smoothness of the interface between the strictly convex part and flat side have been studied for $\alpha = 1$ in [10]. For n-dimensional compact convex hypersurfaces and $\alpha \leq \frac{1}{n}$, it becomes more singular and the solution gets regular instantaneously. On the other hand, if $\alpha > \frac{1}{n}$, it becomes degenerate and the flat side of the hypersurface persists for a moment, [2, 4].

We assume that the initial surface Σ_0 has only one flat side, namely that at t we have $\Sigma_t = \Sigma_t^1 \cup \Sigma_t^2$ where Σ_t^1 is the flat side and Σ_t^2 is strictly convex part of Σ_t. The intersection between two regions is the free boundary $\Gamma_t = \Sigma_t^1 \cap \Sigma_t^2$. Then the lower part of the surface Σ_0 can be written as a graph $z = f(x)$ and we can also write the lower part of Σ_t as $z = f(x,t)$ for $x \in \Omega \subset \mathbb{R}^2$ where Ω is an open subset of $\mathbb{R}^2$.

In this talk, we prove that there exists smooth solution if Σ_0 is smooth and strictly convex and that there is $C^{1,1}$-viscosity solution before the collapsing time if Σ_0 is only convex. Furthermore, we show that Σ_t^1 will stay for a while. We also discuss Γ_t remains smooth on $0 < t < T_0$ under the following conditions for the function f, where T_0 is the vanishing time of Σ_t^1.

Condition 1. *Set* $\Lambda(f) = \{f = 0\}$ *and* $\Gamma(f) = \partial\Lambda(f)$.

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This talk is based on joint work with Ki-ahm Lee and Eunjai Rhee.

(I) *(Nondegeneracy Condition)*
The function f vanishes of the order dist$(X, \Lambda(f))^{\frac{3\alpha-1}{2\alpha-1}}$ and the interface $\Gamma(f)$ is strictly convex so that $\Gamma(f)$ moves with finite nondegenerate speed. Setting $g = (\beta f)^{\frac{1}{\beta}}$, we assume that at time $t = 0$ the function g satisfies the following nondegeneracy condition: at $t = 0$,

$$0 < \lambda < |Dg(X)| < \frac{1}{\lambda} \qquad \text{and} \quad 0 < \lambda^2 < D^2_{\tau\tau} g(X) < \frac{1}{\lambda^2} \tag{2}$$

for all $X \in \Gamma_0$ and some positive number $\lambda > 0$, where $D^2_{\tau\tau}$ denotes the second order tangential derivative at Γ. Then the initial speed of free boundary has the speed, at $t = 0$,

$$0 < \lambda^{4\alpha-1} < |\gamma_t| < \frac{1}{\lambda^{4\alpha-1}}. \tag{3}$$

(II) *(Before Focusing of Flat Side)*
Let T be any number on $0 < T < T_0$, so that Σ^1_t is non-zero. Since the area is non-zero, Σ^1_t contains a disc D_{ρ_0} for some $\rho_0 > 0$. We assume that

$$D_{\rho_0} = \left\{X \in \mathbb{R}^2 \,:\, |X| \leq \rho_0\right\} \subset \Sigma^1_t \quad \text{for } 0 \leq t \leq T_0. \tag{4}$$

(III) *(Graph on a Neighborhood of the Flat Side)*
Without loss of generality, we assume that

$$\max_{x\in\Omega(t)} f(\cdot, t) \geq 2, \quad 0 \leq t \leq T_0 \tag{5}$$

where $\Omega(t) = \{X = (x, y) \in \mathbb{R}^2 \,:\, |Df|(X, t) < \infty\}$. Set

$$\Omega_P(t) = \left\{x \in \mathbb{R}^2 \,:\, f(x, y, t) \leq f(P)\right\}. \tag{6}$$

The following is the first our main result. Let us assume $\frac{1}{2} < \alpha \leq 1$.

Theorem 2. *For a compact convex initial surface Σ_0, any viscosity solution Σ_t of* (1) *is $C^{1,1}$ for $0 < t < T_0$. Furthermore, Σ^2_t is smooth for $0 < t < T_0$.*

In [8], authors proved the following short time existence of C^∞_s-solution with a flat side. From the conditions (2), our linearized equation is in the same class of operators considered in [8]. Hence their Schauder theory can apply to our linearized equation and then we get the short time existence by the application of implicit function theorem as in [8].

Theorem 3. *[Short Time Regularity]* [8] *Let us assume that* $g = (\beta f)^{\frac{1}{\beta}}$ *is of class* $C^{2+\gamma}$ *up to the interface* $z = 0$ *at time* $t = 0$, *for some* $0 < \gamma < 1$, *and satisfies Conditions 1 for* f. *Then there exists a time* $T > 0$ *such that the equation* (1) *admits a solution* $\Sigma(t)$ *on* $0 \le t \le T$. *Moreover, the function* $g = (\beta f)^{\frac{1}{\beta}}$ *is smooth up to the interface* $z = 0$ *on* $0 < t \le T$. *In particular, the interface* $\Gamma(t)$ *will be a smooth curve for all* t *in* $0 < t \le T$.

Then we have the long time regularity of the solution.

Theorem 4. *[Long Time Regularity] Under the assumptions of Theorem 3,* $g = (\beta f)^{\frac{1}{\beta}}$ *remains smooth up to the interface* $z = 0$ *on* $0 < t < T$ *for all* $T < T_0$. *In addition, the interface* Γ_t *will be smooth curve for all* t *in* $0 < t < T_0$.

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Singular perturbation problems
for
nonlinear elliptic equations in degenerate settings

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0. Introduction

Singular perturbation problems for nonlinear elliptic equations has been studied by many mathematicians. Especially the following singularly perturbed nonlinear Schrödinger equations is well studied since the pioneering work of Floer-Weinstein [**FW**]:

$$\begin{cases} -\varepsilon^2\Delta u + V(x)u = g(u) & \text{in } \mathbf{R}^N, \\ u > 0 & \text{in } \mathbf{R}^N, \\ u \in H^1(\mathbf{R}^N), \end{cases} \qquad (0.1)_\varepsilon$$

Here $N \geq 1$, $g(s) \in C(\mathbf{R}, \mathbf{R})$ is a function with a subcritical growth, $V(x) \in C(\mathbf{R}^N, \mathbf{R})$ is a positive function and $0 < \varepsilon \ll 1$. Among solutions of $(0.1)_\varepsilon$, we are interested in concentrating families (u_ε) of solutions, which have the following behavior:

(i) $u_\varepsilon(x)$ has a local maximum at $x_\varepsilon \in \mathbf{R}^N$ and x_ε converges to some $x_0 \in \mathbf{R}^N$ as $\varepsilon \to 0$.

(ii) rescaled function $v_\varepsilon(y) = u_\varepsilon(\varepsilon y + x_\varepsilon)$ converges as $\varepsilon \to 0$ to a solution $\omega(y) \in H^1(\mathbf{R}^N)$ of the limit equation:

$$-\Delta\omega + V(x_0)\omega = g(\omega),\ \omega > 0 \quad \text{in } \mathbf{R}^N, \quad \omega \in H^1(\mathbf{R}^N). \qquad (0.2)$$

The limit equation (0.2) plays important roles in the study of $(0.1)_\varepsilon$. When solutions of (0.2) are unique up to translation and non-degenerate, we can use Lyapunov-Schmidt reduction method and we can reduce $(0.1)_\varepsilon$ to a finite dimensional problem and interesting family of solutions with multiple concentrating points are constructed. See [**ABC**, **DKW**, **KW**, **O1**, **O2**, **W**] and references therein. However, uniqueness and non-degeneray of solutions of (0.2) is verified only restricted classes of nonlinearities including $g(u) = u^p$ $(1 < p < \frac{N+2}{N-2})$.

There are a lot of efforts to relax the non-degenerate condition using variational methods, especially for general nonlinearities. See [**BJ**, **BT2**, **BT3**, **DPR**, **DF1**, **DF2**, **DF3**, **G**, **JT**] and references therein.

In this talk, I would like to talk about another class of a singular perturbation problem, in which a domain depends on singular perturbation parameter $\varepsilon \in (0,1)$:

$$\begin{cases} -\Delta u = u^p, \ u > 0 & \text{in } \Omega_\varepsilon, \\ u = 0 & \text{on } \partial\Omega_\varepsilon. \end{cases} \tag{$*$}_\varepsilon$$

Here $1 < p < \frac{N+2}{N-2}$ $(N \geq 3)$, $1 < p < \infty$ $(N = 2)$ and $\Omega_\varepsilon \subset \mathbf{R}^k \times \mathbf{R}^\ell$ $(N = k + \ell)$ is given by

$$\Omega_\varepsilon = \{(x,y) \in \mathbf{R}^k \times \mathbf{R}^\ell;\ (\varepsilon x, y) \in \Omega_1\} \left(= \bigcup_{x \in \mathbf{R}^k} (\{x\} \times D_{\varepsilon x}) \right).$$

Here

$$\Omega_1 = \bigcup_{z \in \mathbf{R}^k} (\{z\} \times D_z)$$

and $D_z \subset \mathbf{R}^\ell$ be a family of bounded smooth domains which depends on $z \in \mathbf{R}^k$ smoothly.

Such a problem naturally appeared when we studied a problem in an expanding tubular type domain in [**BT4**] (c.f. [**DY**, **ACP**]). We would like to give a partial answer to the following questions:

(i) Where the peaks appears?

(ii) What happens if the section depends on the location $z \in M$.

This talk is based on my joint works with Jaeyoung Byeon, KAIST, Korea.

1. Setting of our problem

1.1. Domain Ω_ε

First we give a precise definition of the domain Ω_ε. We assume that Ω_1 satisfies the following conditions.

(Ω1) $D \subset \mathbf{R}^\ell$ is a bounded domain with a smooth boundary ∂D.

(Ω2) $\varphi(z,y) :\ \mathbf{R}^k \times \overline{D} \to \mathbf{R}^\ell$ is a smooth map such that

(i) For $z \in \mathbf{R}^k$, set $D_z = \varphi(z, D)$. Then

$$\varphi(z,\cdot) :\ \overline{D} \to \overline{D}_z \quad \text{is a diffeomorphism for each } z \in \mathbf{R}^k.$$

(ii) All derivatives of $\varphi(z,y)$ is bounded in $\mathbf{R}^k \times \overline{D}$ and there exists $C_0 > 0$ such that $\det \left[\frac{\partial\varphi}{\partial y}(z,y) \right] \geq C_0$ on $\mathbf{R}^k \times \overline{D}$.

(Ω3) We set

$$\Omega_1 = \bigcup_{z\in\mathbf{R}^k} (\{z\}\times D_z),$$
$$\Omega_\varepsilon = \bigcup_{x\in\mathbf{R}^k} (\{x\}\times D_{\varepsilon x}) = \{(x,\varphi(\varepsilon x,y));\ (x,y)\in\Omega_1\} \quad \text{for } \varepsilon\in(0,1].$$

1.2. Variational formulation

We consider

$$\begin{cases} -\Delta u = u^p,\ u>0 & \text{in } \Omega_\varepsilon, \\ u=0 & \text{on } \partial\Omega_\varepsilon. \end{cases} \tag{$*$}_\varepsilon$$

This problem is reduced to a problem finding a critical point of

$$u\mapsto \int_{\Omega_\varepsilon} \frac{1}{2}|\nabla u|^2 - \frac{1}{p+1}u_+^{p+1}\,dxdy;\ H_0^1(\Omega_\varepsilon)\to\mathbf{R}.$$

Using a transformation appeared in (Ω1)–(Ω3), it can be written as a functional:

$$\begin{aligned} I_\varepsilon(u) &= \int_{\mathbf{R}^k\times D} F_\varepsilon(\varepsilon x,y,\nabla u,u)\,dxdy \in C^1(H_0^1(\mathbf{R}^k\times D),\mathbf{R}), \\ F_\varepsilon(z,y,\nabla u,u) &= \Big(\frac{1}{2}|\nabla_x u + \varepsilon B(z,y)\nabla_y u|^2 + \frac{1}{2}|A(z,y)\nabla_y u|^2 \\ &\qquad - \frac{1}{p+1}u_+^{p+1}\Big)\det\left[\frac{\partial\varphi}{\partial y}(z,y)\right], \end{aligned}$$

where $A(z,y)$, $B(z,y)$ are matrices defined using $\varphi(z,y)$. We also set the limit functional at $z\in\mathbf{R}^k$ by

$$L(z,u) = \int_{\mathbf{R}^k\times D} F_0(z,y,\nabla u,u)\,dxdy \in C^1(H_0^1(\mathbf{R}^k\times D),\mathbf{R}).$$

Here $F_0(z,y,\nabla,u)$ is defined by setting $\varepsilon=0$ in the definition of $F_\varepsilon(z,y,\nabla u,u)$.

We note that $L(z,v)$ plays a role of the limit functional of $I_\varepsilon(u)$. In fact, for $u(x,y)\in C_0^\infty(\mathbf{R}^k\times D)$ and $z\in\mathbf{R}^k$, we have

$$I_\varepsilon(u(x-\frac{z}{\varepsilon},y))\to L(z,u) \quad \text{as } \varepsilon\to 0.$$

We also note that $L(z,u)$ is corresponding to the following limit problem:

$$\begin{cases} -\Delta u = u^p,\ u>0 & \text{in } \mathbf{R}^k\times D_z, \\ u=0 & \text{on } \mathbf{R}^k\times\partial D_z. \end{cases} \tag{$**$}_z$$

1.3. Properties of the limit problem

It is known that the limit problem has the following properties:

1° Solutions of $(**)$ has symmetry $u(x,y) = u(|x|,y)$ after a suitable shift in x and set of solutions

$$S_z = \{\omega(|x|,y);\ \omega \neq 0,\ D_u L(z,\omega) = 0\}$$

is compact in $H^1_{0,s}(\mathbf{R}^k \times D)$.

2° For each z, $(**)_z$ has a least energy solution; we denote the least energy level by $m(z)$:

$$m(z) = \inf\{L(z,\omega);\ \omega \in S_z\}.$$

Moreover $m(z) : \mathbf{R}^k \to \mathbf{R}$ is continuous.

In general, least energy solutions are not unique.

3° $m(z)$ has a property:

$$D_z \subset D_{z'} \implies m(z') \leq m(z).$$

We refer to Gidas-Ni-Nirenberg [**GNN**] and Byeon [**B**] for the symmetry of solutions. We also refer to Esteban [**E**] and Byeon-Tanaka [**BT1**] for the existence of least energy solutions. We also note that the natural space to deal with $L(z,u)$ is the following space

$$H^1_{0,s}(\mathbf{R}^k \times D) = \{u(x,y) \in H^1_0(\mathbf{R}^k \times D);\ u(x,y) = u(|x|,y)\}$$

by the property 1°. In what follows, we regard $L(z,u)$ is a functional defined on $\mathbf{R}^k \times H^1_{0,s}(\mathbf{R}^k \times D)$.

2. Our results

First we recall a non-existence result due to Esteban-Lions [**EL**]. In the following theorem we denote the outward normal vector of Ω_ε at $(x,y) \in \partial\Omega_\varepsilon$ by $N(x,y) \in \mathbf{R}^N$.

Theorem 1 (Esteban-Lions [EL]). *If Ω_1 is monotone in one direction, that is, there is a vector $T \in \mathbf{R}^N$ satisfying*

$$(N(x,y),T) > 0 \quad \text{for all } (x,y) \in \partial\Omega_\varepsilon,$$

then $(*)_\varepsilon$ does not have non-trivial solutions.

From this result, we cannot expect the existence of concentrating solution in monotone parts of Ω_1.

2.1. Concentration at a thick part

First we deal with thick parts of Ω_1. By the property 3°, we have the property (2.1) in the following theorem at a thick part O of Ω_1.

Theorem 2. *Suppose that a bounded open set $O \subset \mathbf{R}^k$ satisfies*

$$\inf_{z\in O} m(z) < \inf_{z\in\partial O} m(z). \tag{2.1}$$

Then for $\varepsilon > 0$ small, $(*)_\varepsilon$ has a positive solution $u_\varepsilon(x,y)$ concentrating in O. More precisely, any sequence (ε_n) with $\varepsilon_n \to 0$ has a subsequence (ε_{n_j}), $(x_{n_j}) \subset \mathbf{R}^k$ and $(z_0,\omega_0) \in O \times H^1_{0,s}(\mathbf{R}^k \times D)$ such that

$$u_{\varepsilon_{n_j}}(x - x_{n_j}, y) \to \omega_0(x,y) \text{ in } H^1_0(\mathbf{R}^k \times D), \quad \varepsilon_{n_j} x_{n_j} \to z_0 \in O.$$

Here (z_0,ω_0) is a critical point of $L(z,u)$, i.e., $D_{z,u}L(z_0,\omega_0) = 0$, satisfying

$$L(z_0,\omega_0) = m(z_0) = \inf_{z\in O} m(z).$$

.

2.2. Concentration at a thin part

Next we consider thin parts of Ω_1. Thin parts correspond to high energy solutions and we need more assumptions.

Condition (E). For a bounded open set $O \subset \mathbf{R}^k$, we say that O satisfies (E) if and only if

$$\left(\frac{\partial\varphi}{\partial z}(z,y)n(z), \frac{\partial\varphi}{\partial y}(z,y)\nu(y)\right) > 0 \quad \text{for all } z \in \partial O \text{ and } y \in \partial D.$$

Here $n(z) \in \mathbf{R}^k$ ($\nu(y) \in \mathbf{R}^\ell$ resp.) is a unit outward normal vector of O (D resp.) at $z \in \partial O$ ($y \in \partial D$ resp.).

Remark. If $O \subset \mathbf{R}^k$ satisfies (E), then we have for some $\delta_0 > 0$

$$D_z \subset D_{z+tn(z)} \text{ for } z \in \partial O \text{ and } t \in [0,\delta_0].$$

Under (E) we have the following existence result.

Theorem 3. *Assume that $O \subset \mathbf{R}^k$ satisfies (E). Then for $\varepsilon > 0$ small, $(*)_\varepsilon$ has a positive solution $u_\varepsilon(x,y)$ concentrating in O. More precisely, any sequence (ε_n) with $\varepsilon_n \to 0$ has a subsequence (ε_{n_j}), $(x_{n_j}) \subset \mathbf{R}^k$ and $(z_0,\omega_0) \in O \times H^1_{0,s}(\mathbf{R}^k \times D)$ such that*

$$u_{\varepsilon_{n_j}}(x - x_{n_j}, y) \to \omega_0(x,y) \text{ in } H^1_0(\mathbf{R}^k \times D), \quad \varepsilon_{n_j} x_{n_j} \to z_0 \in O.$$

Here (z_0,ω_0) is a critical point of $L(z,u)$.

Remark. In Theorem 3, we only have

$$L(z_0, \omega_0) \geq \max_{z \in \overline{O}} m(z).$$

We conjecture that the equality does not hold in general and ω_0 is not a least energy solution of the limit problem. In contrast, for a singular perturbation problem for NLS, we have

$$L(z_0, \omega_0) = \max_{z \in \overline{O}} m(z).$$

3. Our approach

To show our Theorems 2–3, we take the following approach:

Step 1: Analysis of the limit problem.

We introduce a minimax method to the limit functional $L(z,u) \in C^1(H^1_{0,s}(\mathbf{R}^k \times D), \mathbf{R})$:

$$b = \inf_{\gamma \in \Gamma} \max_{(s,z) \in [0,1] \times \overline{O}} L(\gamma(s,z)).$$

We show

$$K_b = \{(z,\omega) \in O \times H^1_{0,s}(\mathbf{R}^k \times D);\ L(z,\omega) = b,\ DL(z,\omega) = 0\}$$

is non-empty and compact in $O \times H^1_{0,s}(\mathbf{R}^k \times D)$.
Here we use a deformation argument in a manifold with a boundary, which is due to Majer [**M**]. We note that the condition (E) gives us a useful property of $L(z,u)$:

$$DL(z,u) \neq -\lambda(n(z), 0) \quad \text{for } (z,u) \in \partial O \times H^1_{0,s}(\mathbf{R}^k \times D),\ \lambda \geq 0.$$

Step 2: Construction of a critical point $u_\varepsilon \in H^1_0(\mathbf{R}^k \times D)$ of $I_\varepsilon(u)$ related to K_k.

We try to find a family (u_ε) of critical points of $I_\varepsilon(u)$ such that for some $(z_0, \omega_0) \in K_b$

$$\varepsilon X(u_\varepsilon) \to z_0,$$
$$u_\varepsilon(x - X(u_\varepsilon), y) \to \omega_0,$$

Here $X(u) : H^1_0(\mathbf{R}^k \times D) \to \mathbf{R}^k$ is a center of mass of $u \in H^1_0(\mathbf{R}^k \times D)$. To find such a family, we develop a new local deformation argument in a neighborhood of

$$A_b^{(\varepsilon)} = \{\omega(x - \frac{z}{\varepsilon}, y);\ (z,\omega) \in K_b\},$$

which is an extension of the arguments in [**BT2**, **BT3**].

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Heat equation with a nonlinear boundary condition and uniformly local L^r spaces

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This is joint work with my student Ryuichi Sato (Tohoku University) and it is concerned with the heat equation with a nonlinear boundary condition,

$$\begin{cases} \partial_t u = \Delta u, & x \in \Omega,\ t > 0, \\ \nabla u \cdot \nu(x) = |u|^{p-1}u, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = \varphi(x), & x \in \Omega, \end{cases} \tag{1}$$

where $N \geq 1$, $p > 1$, Ω is a smooth domain in $\mathbf{R}^N$, $\partial_t = \partial/\partial t$ and $\nu = \nu(x)$ is the outer unit normal vector to $\partial\Omega$. For any $\varphi \in BUC(\Omega)$, problem (1) has a unique solution

$$u \in C^{2,1}(\Omega \times (0,T]) \cap C^{1,0}(\overline{\Omega} \times (0,T]) \cap BUC(\overline{\Omega} \times [0,T])$$

for some $T > 0$ and the maximal existence time $T(\varphi)$ of the solution can be defined. If $T(\varphi) < \infty$, then

$$\limsup_{t \to T(\varphi)} \|u(t)\|_{L^\infty(\Omega)} = \infty$$

and we call $T(\varphi)$ the blow-up time of the solution u.

Problem (1) has been studied in many papers from various points of view (see e.g. [1]–[5], [7]–[11], [12]–[17], [18], [19], [20] and references therein) while there are few results related to the dependence of the blow-up time on the initial function even in the case $\Omega = \mathbf{R}^N_+$. We remark that the blow-up time for problem (1) cannot be chosen uniform for all initial functions lying in a bounded set of $L^r(\mathbf{R}^N_+)$ with $1 \leq r \leq N(p-1)$.

For $1 \leq r < \infty$ and $\rho > 0$, let $L^r_{uloc,\rho}(\Omega)$ be the uniformly local L^r space in Ω equipped with the norm

$$||f||_{r,\rho} := \sup_{x \in \overline{\Omega}} \left(\int_{\Omega \cap B(x,\rho)} |f(y)|^r dy \right)^{1/r}.$$

We denote by $\mathcal{L}^r_{uloc,\rho}(\Omega)$ the completion of bounded uniformly continuous functions in Ω with respect to the norm $\|\cdot\|_{r,\rho}$, that is,

$$\mathcal{L}^r_{uloc,\rho}(\Omega) := \overline{BUC(\Omega)}^{\|\cdot\|_{r,\rho}}.$$

We set $L^\infty_{uloc,\rho}(\Omega) = L^\infty(\Omega)$ and $\mathcal{L}^\infty_{uloc,\rho}(\Omega) = BUC(\Omega)$.

In this talk we prove the local existence and the uniqueness of the solutions of problem (1) with initial functions in $\mathcal{L}^r_{uloc,\rho}(\Omega)$, and study the dependence of the blow-up time on the initial functions. As an application of the main results of this paper, we study the asymptotic behavior of the blow-up time $T(\varphi)$ with $\varphi = \lambda\psi$ as $\lambda \to 0$ or $\lambda \to \infty$ and show the validity of our arguments. Furthermore, we obtain a lower estimate of the blow-up rate of the solutions.

Throughout this talk we assume that $\Omega \subset \mathbf{R}^N$ is a uniformly regular domain of class C^1. For any $x \in \mathbf{R}^N$ and $\rho > 0$, define

$$B(x,\rho) := \{y \in \mathbf{R}^N : |x-y| < \rho\}, \;\; \Omega(x,\rho) := \Omega \cap B(x,\rho), \;\; \partial\Omega(x,\rho) := \partial\Omega \cap B(x,\rho).$$

By the trace inequality for $W^{1,1}(\Omega)$-functions and the Gagliardo-Nirenberg inequality we can find $\rho_* \in (0,\infty]$ with the following properties.

- There exists a positive constant c_1 such that

$$\int_{\partial\Omega(x,\rho)} |v|\, d\sigma \le c_1 \int_{\Omega(x,\rho)} |\nabla v|\, dy \tag{2}$$

  for all $v \in C_0^1(B(x,\rho))$, $x \in \overline{\Omega}$ and $0 < \rho < \rho_*$.

- Let $1 \le \alpha,\, \beta \le \infty$ and $\sigma \in [0,1]$ be such that

$$\frac{1}{\alpha} = \sigma\left(\frac{1}{2} - \frac{1}{N}\right) + (1-\sigma)\frac{1}{\beta}. \tag{3}$$

  Assume, if $N \ge 2$, that $\alpha \ne \infty$ or $N \ne 2$. Then there exists a constant c_2 such that

$$\|v\|_{L^\alpha(\Omega(x,\rho))} \le c_2 \|v\|^{1-\sigma}_{L^\beta(\Omega(x,\rho))} \|\nabla v\|^{\sigma}_{L^2(\Omega(x,\rho))} \tag{4}$$

  for all $v \in C_0^1(B(x,\rho))$, $x \in \overline{\Omega}$ and $0 < \rho < \rho_*$.

We remark that, in the case

$$\Omega = \{(x', x_N) \in \mathbf{R}^N \,:\, x_N > \Phi(x')\},$$

where $N \ge 2$ and $\Phi \in C^1(\mathbf{R}^{N-1})$ with $\|\nabla\Phi\|_{L^\infty(\mathbf{R}^{N-1})} < \infty$, (2) and (4) hold with $\rho_* = \infty$. Inequalities (2) and (4) are used to treat the nonlinear boundary condition.

Next we state the definition of the solution of (1).

Definition 1 *Let $0 < T \le \infty$ and $1 \le r < \infty$. Let u be a continuous function in $\overline{\Omega} \times (0,T]$. We say that u is a $L^r_{uloc}(\Omega)$-solution of* (1) *in $\Omega \times [0,T]$ if*

- *$u \in L^\infty(\tau,T : L^\infty(\Omega)) \cap L^2(\tau, T : W^{1,2}(\Omega \cap B(0,R)))$ for any $\tau \in (0,T)$ and $R > 0$,*
- *$u \in C([0,T) : L^r_{uloc,\rho}(\Omega))$ with $\lim_{t\to 0} \|u(t) - \varphi\|_{r,\rho} = 0$ for some $\rho > 0$,*
- *u satisfies*

$$\int_0^T \int_\Omega \{-u\partial_t\phi + \nabla u \cdot \nabla\phi\}\, dyds = \int_0^T \int_{\partial\Omega} |u|^{p-1}u\phi\, d\sigma ds \tag{5}$$

  for all $\phi \in C_0^\infty(\mathbf{R}^N \times (0,T))$.

Here $d\sigma$ is the surface measure on $\partial\Omega$. Furthermore, for any continuous function u in $\overline{\Omega} \times (0,T)$, we say that u is a $L^r_{uloc}(\Omega)$-solution of (1) *in $\Omega \times [0,T)$ if u is a $L^r_{uloc}(\Omega)$-solution of* (1) *in $\Omega \times [0,\eta]$ for any $\eta \in (0,T)$.*

Now we are ready to state the main results of this talk. Let $p_* = 1 + 1/N$.

Theorem 1 *Let $N \geq 1$ and $\Omega \subset \mathbf{R}^N$ be a uniformly regular domain of class C^1. Let ρ_* satisfy (2) and (4). Then, for any $1 \leq r < \infty$ with*

$$\begin{cases} r \geq N(p-1) & \text{if} \quad p > p_*, \\ r > 1 & \text{if} \quad p = p_*, \\ r \geq 1 & \text{if} \quad 1 < p < p_*, \end{cases} \tag{6}$$

there exists a positive constant γ_1 such that, for any $\varphi \in \mathcal{L}^r_{uloc,\rho}(\Omega)$ with

$$\rho^{\frac{1}{p-1}-\frac{N}{r}} \|\varphi\|_{r,\rho} \leq \gamma_1 \tag{7}$$

for some $\rho \in (0, \rho_*/2)$, problem (1) possesses a $L^r_{uloc}(\Omega)$-solution u of (1) in $\Omega \times [0, \mu\rho^2]$ satisfying

$$\sup_{0<t<\mu\rho^2} \|u(t)\|_{r,\rho} \leq C\|\varphi\|_{r,\rho}, \tag{8}$$

$$\sup_{0<t<\mu\rho^2} t^{\frac{N}{2r}} \|u(t)\|_{L^\infty(\Omega)} \leq C\|\varphi\|_{r,\rho}. \tag{9}$$

Here C and μ are constants depending only on N, Ω, p and r.

Theorem 2 *Assume the same conditions as in Theorem 1. Let v and w be $L^r_{uloc}(\Omega)$-solutions in $\Omega \times [0, T)$ such that $v(x,0) \leq w(x,0)$ for almost all $x \in \Omega$, where $T > 0$ and r is as in (6). Assume, if $r = 1$, that*

$$\limsup_{t \to +0} t^{\frac{1}{2(p-1)}} \left[\|v(t)\|_{L^\infty(\Omega)} + \|w(t)\|_{L^\infty(\Omega)} \right] < \infty. \tag{10}$$

Then there exists a positive constant γ_2 such that, if

$$\rho^{\frac{1}{p-1}-\frac{N}{r}} [\|v(0)\|_{r,\rho} + \|w(0)\|_{r,\rho}] \leq \gamma_2 \tag{11}$$

for some $\rho \in (0, \rho_*/2)$, then

$$v(x,t) \leq w(x,t) \quad \text{in} \quad \Omega \times (0, T).$$

We give some comments related to Theorems 1 and 2.

(i) Let u be a $L^r_{uloc}(\Omega)$-solution of (1) in $\Omega \times [0, T)$. It follows from Definition 1 that $u \in L^\infty(\tau, \sigma : L^\infty(\Omega))$ for any $0 < \tau < \sigma < T$. This together with Theorem 6.2 of [5] implies that $u(t) \in BUC(\Omega)$ for any $t \in (0, T)$. This means that $u(0) \in \mathcal{L}^r_{uloc,\rho}(\Omega)$ for any $\rho > 0$.

(iii) Let $1 \leq r < \infty$. If, either

(a) $f \in L^r_{uloc,1}(\Omega)$, $r > N(p-1)$ or (b) $f \in L^r(\Omega)$, $r \geq N(p-1)$,

then, for any $\gamma > 0$, we can find a constant $\rho > 0$ such that $\rho^{\frac{1}{p-1}-\frac{N}{r}} \|f\|_{r,\rho} \leq \gamma$.

As a corollary of Theorem 1, we have:

Corollary 1 *Assume the same conditions as in Theorem* 1 *and* $p > p_*$.

(i) *For any* $\varphi \in L^{N(p-1)}(\Omega)$, *problem* (1) *has a unique* $L^{N(p-1)}_{uloc}(\Omega)$*-solution in* $\Omega \times [0, T]$ *for some* $T > 0$.

(ii) *Assume* $\rho_* = \infty$. *Then there exists a constant* γ *such that, if*

$$\|\varphi\|_{L^{N(p-1)}(\Omega)} \leq \gamma, \tag{12}$$

then problem (1) *has a unique* $L^{N(p-1)}_{uloc}(\Omega)$*-solution* u *such that*

$$\sup_{0<t<\infty} \|u(t)\|_{L^{N(p-1)}(\Omega)} + \sup_{0<t<\infty} t^{\frac{1}{2(p-1)}} \|u(t)\|_{L^\infty(\Omega)} < \infty.$$

Furthermore, as an application of our theorems, we give a lower blow-up estimate of the solution u of (1).

Corollary 2 *Let* $N \geq 1$ *and* $\Omega \subset \mathbf{R}^N$ *be a uniformly regular domain of class* C^1. *Let* u *be a solution of* (1) *blowing up at* $t = T < \infty$. *Then*

$$\liminf_{t \to T} (T - t)^{\frac{1}{2(p-1)} - \frac{N}{2r}} \|u(t)\|_{L^r(\Omega)} > 0, \tag{13}$$

where

$$\begin{cases} N(p-1) \leq r \leq \infty & \text{if} \quad p > 1 + 1/N, \\ 1 < r \leq \infty & \text{if} \quad p = 1 + 1/N, \\ 1 \leq r \leq \infty & \text{if} \quad 1 < p < 1 + 1/N. \end{cases} \tag{14}$$

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FLOW MONOTONICITY AND STRICHARTZ INEQUALITIES

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This talk is largely based on joint work [5] with Jonathan Bennett and Marina Iliopoulou.

Background

A number of important inequalities from geometric analysis may be understood via the monotonicity of an appropriate *functional*, often referred to as a Lyapunov functional, as the input evolves under a well-chosen *flow*. For instance, consider the sharp Young convolution inequality on euclidean space, due to Beckner [1] and Brascamp–Lieb [8], which states that whenever $d \geq 1$ and $p_1, p_2, p \geq 1$ satisfy the scaling hypothesis $1 + \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, then

$$\|f_1 * f_2\|_{L^p(\mathbb{R}^d)} \leq \mathbf{C}(p_1, p_2)\|f_1\|_{L^{p_1}(\mathbb{R}^d)}\|f_2\|_{L^{p_2}(\mathbb{R}^d)} \tag{1}$$

for each $f_1 \in L^{p_1}(\mathbb{R}^d)$ and $f_2 \in L^{p_2}(\mathbb{R}^d)$. Here, the constant $\mathbf{C}(p_1, p_2)$ is given by

$$\mathbf{C}(p_1, p_2) = \|H_{\sigma_1}^{1/p_1} * H_{\sigma_2}^{1/p_2}\|_{L^p(\mathbb{R}^d)}$$

where the parameters $(\sigma_1, \sigma_2) \in (0, \infty)^2$ satisfy $p_1 p_1' \sigma_1 = p_2 p_2' \sigma_2$, and H_t is the heat kernel on $\mathbb{R}^d$ given by

$$H_t(x) = \frac{1}{(4\pi t)^{d/2}} e^{-\frac{|x|^2}{4t}}.$$

Written in this way, the constant $\mathbf{C}(p_1, p_2)$ is easily seen to be best possible by taking

$$(f_1, f_2) = (H_{\sigma_1}^{1/p_1}, H_{\sigma_2}^{1/p_2})$$

and using the fact that the heat kernel has unit mass for each time. The left-hand side of (1) with

$$(f_1, f_2) \to ((e^{\sigma_1 t\Delta} f_1^{p_1})^{1/p_1}, (e^{\sigma_2 t\Delta} f_2^{p_2})^{1/p_2})$$

gives rise to the quantity

$$Q(t) = \|(e^{\sigma_1 t\Delta} f_1^{p_1})^{1/p_1} * (e^{\sigma_2 t\Delta} f_2^{p_2})^{1/p_2}\|_{L^p(\mathbb{R}^d)}$$

which generates the sharp inequality (1). In particular, for (sufficiently nice) non-negative f_1 and f_2, we have

$$\lim_{t\to 0+} Q(t) = \|f_1 * f_2\|_{L^p(\mathbb{R}^d)}, \qquad \lim_{t\to\infty} Q(t) = \mathbf{C}(p_1, p_2)\|f_1\|_{L^{p_1}(\mathbb{R}^d)}\|f_2\|_{L^{p_2}(\mathbb{R}^d)},$$

and it was shown in [2] that Q is nondecreasing.

This is an example of an inequality with certain gaussian input functions as extremisers. In fact, it can be shown that all extremisers must be gaussian, and thus the above example indicates that when the objective is to establish an inequality in sharp form, the choice of flow is constrained by the class of extremisers.

We also note that under the above assumptions on $(p_1, p_2, \sigma_1, \sigma_2)$, except now $0 < p_1, p_2 \leq 1$, the Lyapunov functional Q is nonincreasing for (sufficiently nice) $f_1 \in L^{p_1}(\mathbb{R}^d)$ and $f_2 \in L^{p_2}(\mathbb{R}^d)$ (see [2]). This monotonicity yields the sharp reverse form of the Young convolution inequality

$$\|f_1 * f_2\|_{L^p(\mathbb{R}^d)} \geq \mathbf{C}(p_1, p_2)\|f_1\|_{L^{p_1}(\mathbb{R}^d)}\|f_2\|_{L^{p_2}(\mathbb{R}^d)}$$

for each *nonnegative* $f_1 \in L^{p_1}(\mathbb{R}^d)$ and $f_2 \in L^{p_2}(\mathbb{R}^d)$. With the best constant (in this case, the largest), this inequality was first established by Brascamp–Lieb [8] who also observed a fundamental link with geometry by showing that the Prékopa–Leindler inequality follows in the limiting case as p tends to zero. The Prékopa–Leindler inequality is a functional form of the Brunn–Minkowski inequality, a core inequality in geometric analysis which, for example, quickly implies the classical isoperimetric inequality.

Lieb [18] also observed that the sharp (forward) Young convolution inequality implies the Shannon entropy power inequality from information theory, which states that

$$e^{2H(X+Y)} \geq e^{2H(X)} + e^{2H(Y)}$$

for independent random variables X and Y in $\mathbb{R}$, with equality when X and Y are gaussian random variables. Here, for a random variable X in $\mathbb{R}$ with appropriate probability density function f, $H(X)$ is the entropy of X and is given by

$$H(X) = -\int_{\mathbb{R}} f \log f.$$

An early proof of the entropy power inequality was given by Stam [21] (and Blachman [7], including higher dimensions) using heat-flow monotonicity. Crucially, the time derivative of the entropy functional H along heat-flow coincides with the Fisher information I along heat-flow (de Bruijn's identity), the latter being of quadratic nature and thus more accessible. Interestingly, the Blachman–Stam inequality, a certain subadditivity of the Fisher information, is a crucial ingredient in the heat-flow monotonicity approach to the entropy power inequality, and a stronger version of this inequality is key in establishing the monotonicity of the above functional Q for the Young convolution inequalities (see Toscani [22] for this observation).

Switching focus, we turn our attention to the main purpose of the talk which is to discuss monotonicity phenomena in the context of Strichartz inequalities for various evolution equations.

The Schrödinger equation

It is natural to begin with the free Schrödinger propagator whose classical Strichartz estimates

$$\|e^{is\Delta}f\|_{L^{2\frac{d+2}{d}}_{s,x}(\mathbb{R}\times\mathbb{R}^d)} \lesssim \|f\|_{L^2(\mathbb{R}^d)} \tag{2}$$

are conjectured to have gaussian extremisers for each $d \geq 1$; see Foschi [15] and also Hundertmark–Zharnitsky [16], where the conjecture is stated and established for $d = 1, 2$.

It was proved in [3] that a monotonicity phenomenon exists in this context, where the initial data evolves under the quadratic heat-flow

$$f \to (e^{t\Delta}|f|^2)^{1/2}.$$

Theorem 1. ([3]) *Let $d = 1, 2$ and $f \in L^2(\mathbb{R}^d)$. Then*

$$Q(t) = \|e^{is\Delta}(e^{t\Delta}|f|^2)^{1/2}\|_{L^{2\frac{d+2}{d}}_{s,x}(\mathbb{R}\times\mathbb{R}^d)}$$

is nondecreasing for each $t > 0$.

Underpinning this is the monotonicity of the Cauchy–Schwarz functional

$$\int_{\mathbb{R}^d} f_1 f_2$$

under such quadratic heat-flow; see work of Bennett–Carbery–Christ–Tao [6] and Carlen–Lieb–Loss [10].

A limitation of Theorem 1 is that it appears to be rather rigid and does not seem to extend in several desirable directions, including higher dimensions, and to a broader class of Sobolev–Strichartz inequalities for other dispersive or wave-like propagators, including initial data measured in the scale of classical Sobolev spaces. Here, we shall see that we can achieve this, to some extent, by considering *linear* flows rather than quadratic. Additionally, in Theorems 2, 3, 5 and 6, stated in terms of a Lyapunov functional $Q(t)$, the monotonicity may be strengthened to *complete monotonicity* in the sense that the kth derivative of Q has sign $(-1)^k$ for every $k \in \mathbb{N}_0$.

To begin to describe these results, we state a linear heat-flow counterpart to Theorem 1.

Theorem 2. ([5]) *Suppose $f \in L^2(\mathbb{R}^d)$. Then, for $d = 1$,*

$$Q(t) = \frac{1}{2\sqrt{3}}\|e^{t\Delta}f\|^6_{L^2(\mathbb{R})} - \|e^{is\Delta}e^{t\Delta}f\|^6_{L^6_{s,x}(\mathbb{R}\times\mathbb{R})}$$

is nonincreasing for $t > 0$, and for $d = 2$,

$$Q(t) = \frac{1}{4}\|e^{t\Delta}f\|^4_{L^2(\mathbb{R}^2)} - \|e^{is\Delta}e^{t\Delta}f\|^4_{L^4_{s,x}(\mathbb{R}\times\mathbb{R}^2)}$$

is nonincreasing for $t > 0$.

Theorems 1 and 2 both recover the sharp Strichartz inequalities in [15] and [16] by comparing $Q(t)$ as $t \to 0+$ and $t \to \infty$. In the former case, this is analogous to the earlier recovery of the Young convolution inequalities, with $\lim_{t\to 0+} Q(t)$ giving the left-hand side of (2) and $\lim_{t\to\infty} Q(t)$ giving the sharp form of the right-hand side of (2). In the latter case, the quantities $Q(t)$ in Theorem 2 tend to zero as $t \to \infty$ and the sharp form of (2) is equivalent to the nonnegativity of $\lim_{t\to 0+} Q(t)$. The idea of flowing the difference of two sides of an inequality (first raised to an appropriate power) is not new, and can be seen, for example, in work of Carlen–Carrillo–Loss [9] in the context of the sharp Hardy–Littlewood–Sobolev inequality.

Theorem 2 is, in fact, a special case of a result which holds in all spatial dimensions. If we write

$$\mathfrak{I}(f) = \int_{(\mathbb{R}^d)^m} |\widehat{f} \otimes \cdots \otimes \widehat{f}(\xi)|^2 \Big(\sum_{1 \leq i < j \leq m} |\xi_i - \xi_j|^2 \Big)^{\frac{1}{2}(d(m-1)-2)} \mathrm{d}\xi$$

then whenever $d \geq 1$, $m \geq 3$, or $d \geq 2$, $m \geq 2$, and whenever $\mathfrak{I}(f) < \infty$, the quantity

$$Q(t) = \frac{|\mathbb{S}^{(m-1)d-1}|}{2m^{\frac{dm-2}{2}}(2\pi)^{(2m-1)d-1}} \mathfrak{I}(e^{t\Delta} f) - \|e^{is\Delta} e^{t\Delta} f\|^{2m}_{L^{2m}_{s,x}(\mathbb{R} \times \mathbb{R}^d)}$$

is nonincreasing for each $t > 0$. To help ground this, by looking at the case $m = 2$, we see that the implied inequality, by comparing initial and eternal times t, is simply

$$\|e^{is\Delta} f\|^4_{L^4_{s,x}(\mathbb{R} \times \mathbb{R}^d)} \leq \frac{|\mathbb{S}^{d-1}|}{2^d (2\pi)^{3d-1}} \int_{(\mathbb{R}^d)^2} |\widehat{f}(\xi_1)|^2 |\widehat{f}(\xi_2)|^2 |\xi_1 - \xi_2|^{d-2} \,\mathrm{d}\xi.$$

This sharp estimate is due to Carneiro [11], holds for each $d \geq 2$, and has gaussian extremisers. One may view this sharp inequality as a relative of

$$\|e^{is\Delta} f\|_{L^4_{s,x}(\mathbb{R} \times \mathbb{R}^d)} \lesssim \|f\|_{\dot{H}^{\frac{d-2}{4}}(\mathbb{R}^d)}, \tag{3}$$

which is a classical Sobolev–Strichartz estimate.

Ozawa–Tsutsumi [19] also proved a sharp relative of (3) where no additional regularity on the initial data beyond square-integrability is assumed, and this is compensated for by measuring the modulus square of the solution in a classical homogeneous Sobolev space with nonpositive order; specifically

$$\|(-\Delta)^{\frac{2-d}{4}} |e^{is\Delta} f|^2\|^2_{L^2_{s,x}(\mathbb{R} \times \mathbb{R}^d)} \leq \frac{|\mathbb{S}^{d-1}|}{4(2\pi)^{d-1}} \|f\|^4_{L^2(\mathbb{R}^d)} \tag{4}$$

for each $f \in L^2(\mathbb{R}^d)$. This inequality is valid for each $d \geq 2$ and the given constant is sharp with gaussian extremisers (when $d = 1$, (4) is in fact an identity). Furthermore, it may be seen as a consequence of the following monotonicity.

Theorem 3. ([5]) *Let $d \geq 2$ and $f \in L^2(\mathbb{R}^d)$. Then*

$$Q(t) = \frac{|\mathbb{S}^{d-1}|}{4(2\pi)^{d-1}} \|e^{t\Delta} f\|^4_{L^2(\mathbb{R}^d)} - \|(-\Delta)^{\frac{2-d}{4}} |e^{is\Delta} e^{t\Delta} f|^2\|^2_{L^2_{s,x}(\mathbb{R} \times \mathbb{R}^d)}$$

is nonincreasing for each $t > 0$.

Our proofs of Theorems 2 and 3 proceed via Fourier analysis. The linear nature of the flows in these theorems allows such a approach, in stark contrast to the quadratic flow in Theorem 1. A significant advantage of using Fourier analysis is that Strichartz inequalities for other dispersive and wave-like equations may be handled in a similar manner; see the forthcoming section.

Up to now, we have stated various monotonicity phenomena, each of which generates the sharp constant in the underlying inequality. These results are, however, restricted to inequalities where the Lebesgue space exponent on the solution is an even integer. Our next result demonstrates that if one is willing to sacrifice the sharpness of the constant, then the monotonicity phenomenon in Theorem 2 may be extended to the full range of admissible exponents, including the mixed space-time norm regime.

Theorem 4. ([5]) *Suppose $(p,q,d) \neq (2,\infty,2)$ is such that $2 \leq p, q \leq \infty$ and*

$$\frac{2}{p} + \frac{d}{q} = \frac{d}{2}.$$

Then there exists a constant $C_{p,q}$ such that

$$Q(t) = C_{p,q} \|e^{t\Delta} f\|^p_{L^2(\mathbb{R}^d)} - \|e^{is\Delta} e^{t\Delta} f\|^p_{L^p_s L^q_x(\mathbb{R}\times\mathbb{R}^d)}$$

is nonincreasing for each $t > 0$.

In the above generality, a Fourier analytic approach appears to be difficult to implement, and we prove Theorem 4 using PDE methods.

The wave and Klein–Gordon equations

Using the Fourier analytic approach, we may prove the following analogous results to Theorem 2 for the one-sided wave and Klein–Gordon propagators.

Theorem 5. ([5]) *Suppose $f \in \dot{H}^{\frac{1}{2}}(\mathbb{R}^d)$. Then for $d = 2$,*

$$Q(t) = \frac{1}{2\pi} \|e^{-t\sqrt{-\Delta}} f\|^6_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^2)} - \|e^{is\sqrt{-\Delta}} e^{-t\sqrt{-\Delta}} f\|^6_{L^6_{s,x}(\mathbb{R}\times\mathbb{R}^2)}$$

is nonincreasing for each $t > 0$, and for $d = 3$,

$$Q(t) = \frac{1}{2\pi} \|e^{-t\sqrt{-\Delta}} f\|^4_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} - \|e^{is\sqrt{-\Delta}} e^{-t\sqrt{-\Delta}} f\|^4_{L^4_{s,x}(\mathbb{R}\times\mathbb{R}^3)}$$

is nonincreasing for each $t > 0$.

Regarding notation, $\dot{H}^{\frac{1}{2}}(\mathbb{R}^d)$ denotes the homogeneous Sobolev space on $\mathbb{R}^d$ of order $\frac{1}{2}$. The extremisers for the inequalities generated by the monotone quantities in Theorem 5 were found by Foschi [15] and include initial data f such that

$$\widehat{f}(\xi) = \frac{e^{-|\xi|}}{|\xi|},$$

from which we see the relevance of the flow $e^{-t\sqrt{-\Delta}}$.

For the Klein–Gordon equation, we obtain the following monotonicity phenomena under the flow $e^{-t\sqrt{1-\Delta}} f$.

Theorem 6. ([5]) *Suppose $f \in H^{\frac{1}{2}}(\mathbb{R}^d)$. Then for $d = 2$,*

$$Q(t) = \frac{1}{2} \|e^{-t\sqrt{1-\Delta}} f\|^4_{H^{\frac{1}{2}}(\mathbb{R}^2)} - \|e^{is\sqrt{1-\Delta}} e^{-t\sqrt{1-\Delta}} f\|^4_{L^4_{s,x}(\mathbb{R}\times\mathbb{R}^2)}$$

is decreasing for each $t > 0$, and for $d = 3$,

$$Q(t) = \frac{1}{2\pi}\|e^{-t\sqrt{1-\Delta}}f\|^4_{H^{\frac{1}{2}}(\mathbb{R}^3)} - \|e^{is\sqrt{1-\Delta}}e^{-t\sqrt{1-\Delta}}f\|^4_{L^4_{s,x}(\mathbb{R}\times\mathbb{R}^3)}$$

is decreasing for each $t > 0$.

Here, $H^{\frac{1}{2}}(\mathbb{R}^d)$ denotes the inhomogeneous Sobolev space of order $\frac{1}{2}$. The Strichartz inequalities generated by the strict monotonicity of Q in Theorem 6 *do not* have extremisers. However, if we write

$$\widehat{f}_a(\xi) = \frac{e^{-a\sqrt{1+|\xi|^2}}}{\sqrt{1+|\xi|^2}}$$

then (f_a) is an extremising sequence as $a \to \infty$ for $d = 2$, and $a \to 0+$ for $d = 3$ (Quilodrán [20] first established the underlying sharp inequalities, the lack of extremisers and identified such extremising sequences). Despite the lack of extremisers, we see the relevance of the flow $e^{-t\sqrt{1-\Delta}}f$.

The kinetic transport equation

We conclude with some observations regarding flow monotonicity in the context of the kinetic transport equation

$$\partial_s u(s,x,v) + v\cdot\nabla_x u(s,x,v) = 0, \qquad u(0,x,v) = f(x,v)$$

for $(s,x,v) \in \mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d$. The Strichartz estimates for the solution of this equation are of the form

$$\|\rho(f)\|_{L^q_s L^p_x(\mathbb{R}\times\mathbb{R}^d)} \lesssim \|f\|_{L^a_{x,v}(\mathbb{R}^d\times\mathbb{R}^d)}, \tag{5}$$

where the velocity averaging operator ρ, often called the macroscopic density, is given by

$$\rho(f)(s,x) = \int_{\mathbb{R}^d} f(x-vs,v)\,\mathrm{d}v.$$

Based on results in [12], [17] and [4], it is now known that (5) holds if and only if (a,p,q) satisfies

$$q > a, \qquad p \geq a, \qquad \frac{2}{q} = d\left(1-\frac{1}{p}\right), \qquad \frac{1}{a} = \frac{1}{2}\left(1+\frac{1}{p}\right).$$

Given Theorem 1 for the Schrödinger equation, it is natural to look for monotonicity in the context of (5) in the particular case

$$\|\rho(f)\|_{L^{\frac{d+2}{d}}_{s,x}(\mathbb{R}\times\mathbb{R}^d)} \lesssim \|f\|_{L^{\frac{d+2}{d+1}}_{x,v}(\mathbb{R}^d\times\mathbb{R}^d)} \tag{6}$$

corresponding to $(a,p,q) = (\frac{d+2}{d+1}, \frac{d+2}{d}, \frac{d+2}{d})$. The dual estimate to (6) is

$$\|\rho^*(g)\|_{L^{d+2}_{x,v}(\mathbb{R}^d\times\mathbb{R}^d)} \lesssim \|g\|_{L^{\frac{d+2}{2}}_{s,x}(\mathbb{R}\times\mathbb{R}^d)} \tag{7}$$

where the operator ρ^* is given by

$$\rho^*(g)(x,v) = \int_{\mathbb{R}} g(s,x+vs)\,\mathrm{d}s.$$

The sharp constant in the space-time X-ray transform estimate (7) was identified by Drouot [13], who observed that the function

$$g(s,x) = \frac{1}{1+s^2+|x|^2}$$

is amongst the class of extremisers. Thanks to work of Flock [14], it is known that, modulo symmetries of the inequality, all extremisers are of this type and thus rules out a heat-flow monotonicity phenomena in this context.

We conclude by presenting some evidence in the form of the following result (for the 2-plane transform rather than the X-ray type transform above) to suggest that it is reasonable to expect monotonicity under certain fast diffusion flows.

Theorem 7. ([5]) *Suppose $d \geq 2$, $m = \frac{d+1}{d+3}$ and let $g \in L^{2\frac{d+1}{d+3}}(\mathbb{R}^{d+1})$ be nonnegative and of compact support. If $u : [0,\infty) \times \mathbb{R}^{d+1} \to [0,\infty)$ satisfies*

$$\partial_t u = \Delta(u^m); \quad u(0,\cdot) = g, \tag{8}$$

then

$$Q(t) = C_d^2 \|u(t,\cdot)\|^2_{L^{2\frac{d+1}{d+3}}(\mathbb{R}^{d+1})} - \|T_{2,d+1}(u(t,\cdot))\|^2_{L^2(\mathcal{M}_{2,d+1})}$$

is nonincreasing for each $t > 0$.

Here, $T_{2,d+1}$ denotes the classical 2-plane transform on $\mathbb{R}^{d+1}$ and $\mathcal{M}_{2,d+1}$ the Grassmann manifold of all affine 2-planes in $\mathbb{R}^{d+1}$. The constant C_d is the sharp constant in the inequality

$$\|T_{2,d+1}(g)\|_{L^2(\mathcal{M}_{2,d+1})} \leq C_d \|g\|_{L^{2\frac{d+1}{d+3}}(\mathbb{R}^{d+1})}$$

whose extremisers include the function

$$g(s,x) = \frac{1}{(1+s^2+|x|^2)^{\frac{d+3}{2}}}.$$

This explains the appearance of the fast diffusion flow in (8) whose asymptotic profiles (Barenblatt profiles) take this shape for the given value of m. Theorem 7 is simply proved by combining an identity of Drury which connects $\|T_{2,d+1}(g)\|_{L^2(\mathcal{M}_{2,d+1})}$ with the diagonal Hardy–Littlewood–Sobolev functional, and the recent observation of Carlen–Carrillo–Loss [9] that certain cases of the sharp Hardy–Littlewood–Sobolev inequality may be established via fast diffusion monotonicity. It is reasonable to hope that other versions of Drury's identity may lead to monotonicity phenomena under fast diffusion in the context of the kinetic transport equation for the Strichartz inequalities (6) and (7).

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Poster Session

1 Tomoro Asai (The University of Tokyo)
On self-similar solutions to the surface diffusion flow equations with contact angle boundary conditions

2 Takashi Kagaya (Hokkaido University)
Motion by curvature flow with constant driving force term for free boundary problem

3 Yoichi Miyazaki (Nihon University)
Hölder regularity theorem for elliptic equations in a non-smooth domain

4 Fumio Nakajima (Iwate University)
Is a smooth part of the surface of Mt. Fuji harmonic ?

5 Atsushi Nakayasu (The University of Tokyo)
On metric viscosity solutions for Hamilton-Jacobi equations of evolution type

6 Tokinaga Namba (The University of Tokyo)
On cell problems for Hamilton-Jacobi equations with non-coercive Hamiltonians and its application to homogenization problems

7 Takahiro Okabe (Hirosaki University)
Space-time asymptotics of the 2D Navier-Stokes flow in the whole plane

8 Takuya Suzuki (The University of Tokyo)
Analyticity of semigroups generated by higher order elliptic operators in spaces of bounded functions on C^1 domains

9 Igor Trushin (Tohoku University)
Inverse Scattering on Graphs

10 Kota Uriya (Tohoku University)
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11 Erika Ushikoshi (Tamagawa University)
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12 Kyouhei Wakasa (Hokkaido University)
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13 Masakazu Yamamoto (Hirosaki University)
Large-time behavior of solutions to the drift-diffusion equation with critical dissipation