DISCONTINUOUS STATIONARY SOLUTION TO GENERALIZED EIKONAL-CURVATURE EQUATION AND ITS STABILITY

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Dedicated to Professor Fumihiro Sato on the occasion of his 65th birthday

1. INTRODUCTION

In this paper we consider stability of stationary solutions to an eikonal-curvature flow equation

$$(1.1) \qquad V = C + \mathcal{K} \quad \text{on } \Gamma_t$$

for evolving interface Γ_t in a domain $\Omega \subset \mathbb{R}^N$ with a level set method, where $V, \mathcal{K} = \sum_{j=1}^{N} \kappa_j$ and κ_j is the normal velocity, mean and principal curvature of Γ_t defined with the outer continuous unit normal vector field $\mathbf{n} \in S^{N-1}$ of Γ_t (we call $\mathbf{n}$ the orientation of the evolution), respectively. Note that "interface" means the boundary of an open set called "interior" so that $\mathcal{K}$ is not positive for smooth boundary of a convex open set. When Ω has smooth boundary $\partial\Omega$, we add the right angle condition

$$(1.2) \qquad \Gamma_t \perp \partial\Omega$$

between Γ_t and $\partial\Omega$ to (1.1).

Level set method, which is introduced by Osher and Sethian [28] in the numerical analysis on evolving interfaces, describes the evolving interfaces Γ_t by

$$(1.3) \qquad \Gamma_t = \{x \in \overline{\Omega}; \ u(t,x) = c\}$$

with an auxiliary function $u\colon [0,\infty) \times \overline{\Omega} \to \mathbb{R}$ and a constant $c \in \mathbb{R}$. The orientation of the evolution is given as

$$\mathbf{n} = -\frac{\nabla u}{|\nabla u|}$$

by setting an interior of Γ_t as the superlevel set $\{x \in \Omega; \ u(t,x) > c\}$, where $\nabla = (\partial/\partial x_1, \ldots, \partial/\partial x_N)$. Then we obtain the level set equation of (1.1) as

$$(1.4) \qquad u_t - |\nabla u| \left\{ \operatorname{div} \frac{\nabla u}{|\nabla u|} + C \right\} = 0 \quad \text{in } (0,T) \times \Omega,$$

where

$$\operatorname{div} P(x) = \sum_{j=1}^{N} \frac{\partial p_j}{\partial x_j}(x)$$

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for $P(x) = (p_1(x), \ldots, p_N(x)) \in \mathbb{R}^N$. The right angle condition (1.2) is represented as the Neumann boundary condition of u;

$$(1.5) \qquad \langle \vec{\nu}, \nabla u \rangle = 0 \quad \text{in } (0, T) \times \partial\Omega,$$

where $\vec{\nu} \in S^{N-1}$ is the outer unit normal vector field of $\partial\Omega$. The equation (1.4)–(1.5) includes an important extension of (1.1)–(1.2); (1.4)–(1.5) means that all the level set of u evolve with (1.1)–(1.2) although (1.1)–(1.2) holds only on Γ_t. See [16] for details.

Level set method is powerful to study the evolution of Γ_t including singularities, i.e., vanishing of Γ_t in finite time, or collision with each other. It is well known that the evolving planer closed simple curve by (1.1) with $C = 0$ becomes convex in finite time and converges to a single point; see [13, 19]. Although the first behavior does not hold for closed compact hypersurface in $\mathbb{R}^3$ (see [20]), the vanishing property is easily extended to the evolving closed compact hypersurface in $\mathbb{R}^N$ by a level set method. The author, Y.-H. R. Tsai and Y. Giga [27] extend the idea of the level set method to spiral curves on the plane. One can find in [27], moreover, their formulation works well even if the topological change of the curves occurs during the evolution of spirals.

However, this method also has difficulties caused by the implicit representation of interfaces, which is the main topic of this paper. In this paper we consider the existence and stability of stationary solutions to (1.1) with the level set formulation. One can easily find some stationary interfaces of (1.1) as the hypersurface with the constant mean curvature. We first prove that there exist discontinuous (so that weak) stationary solutions to (1.4) describing an interface with a constant mean curvature which is a boundary of N-dimensional submanifold. We next prove that there are no continuous stationary solutions describing just a sphere in $\mathbb{R}^N$ provided $C \neq 0$, or a hyperplane contacting to $\partial\Omega$ of a sandglass-type domain $\Omega \subset \mathbb{R}^N$ at its neck with the right angle provided that $C = 0$. By the proof of the above we also derive that the first stationary solution is unstable, and the second one is asymptotically stable with set-theoretic approach.

The stationary solution to (1.1) provided that $C \neq 0$ means an interface with a constant mean curvature. In this paper we treat only sphere or cylindrical surface, however there exist various hypersurfaces with constant mean curvature; see [31, 24, 25]. A compact hypersurface with a constant mean curvature is characterized as a stable solution to the variational problem minimizing surface area provided that the measure of the domain enclosed by the surface is a constant. The result of this paper means that a boundary of a domain with a constant mean curvature is unstable from a view point of eikonal-curvature flow (1.1) even if the domain and boundary are compact.

From a view point of stability of stationary solution to an evolution equation, Ei and Yanagida [10, 11] or Ei, Sato and Yanagida [9] investigate the stability or instability of stationary solutions to (1.1) with contact angle condition. Their method is extended to a surface diffusion flow by [14]. As in these papers we often consider a linearization of problems around a stationary solution: see [2] or [26] for details of linearization of nonlinear ordinary or partial differential equations, respectively. In particular, the existence of solutions to (1.1) in [9] is guaranteed with aid of level set method. Our result

means that the stationary solution as in [9] which is a center of linearization
is given by unstationary continuous or discontinuous stationary viscosity so-
lution. Moreover, our result points out that there may be no suitable center
of linearization around a stationary solution if we consider the stability or
instability of stationary interface only with a level set method. We also
refer [17] for the result of convergence to stationary solution in cylindrical
domain by strong maximum principle of a stationary problem to (1.4)–(1.5)
with $C = 0$.

We also mention on a discontinuous solution from a positive view point.
The discontinuous stationary solutions we find is based on characteristic
function of an open set, and then they are examples of set-theoretic solution
which is introduced in [16]. There is a pioneering work on set-theoretic
approach to (1.1) including anisotropic evolution by Soner [30], which is
based on the signed distance function. The definition by [16] corresponds to
some approximation algorithms to (1.1): see [5, 1, 4, 6, 12]. One can find
a lot of characterization of set-theoretic solution in [16]. We calculate weak
derivatives of characteristic function in viscosity solution sense in this paper.
Moreover, we observe that discontinuous solutions to (1.4) plays important
role to describe stationary interfaces to (1.1).

This paper is organized as follows. We first summarize the definition and
some properties of viscosity solutions in §2. We also give some properties of
the characteristic functions for an open set from the view point of viscosity
solutions. We next consider the stationary ball solution to (1.1) with $C \neq 0$
in $\Omega = \mathbb{R}^N$ in §3. We finally consider the stationary hyperplane to (1.1)
with $C = 0$ in the axis-symmetric domain in §4.

2. Viscosity solutions

In this section we summarize the theory of viscosity solutions to the de-
generate parabolic equation, and show some fundamental properties which
is used to the following sections.

2.1. Definitions. Let $\Omega \subset \mathbb{R}^N$ be a domain and $T > 0$. We consider a
geometric degenerate parabolic equation of the form

$$(2.1) \qquad u_t + F(\nabla u, \nabla^2 u) = 0 \quad \text{in } (0, T) \times \Omega$$

including (1.1). We also consider the Neumann boundary condition

$$(2.2) \qquad \langle \vec{\nu}, \nabla u \rangle = 0 \quad \text{in } (0, T) \times \partial\Omega$$

if $\partial\Omega \neq \emptyset$. The equation (3.1) is represented with

$$(2.3) \qquad F(p, X) = F_0(p, X) - C|p|$$

with

$$(2.4) \qquad F_0(p, X) = -\text{trace} \left\{ \left(I_N - \frac{p \otimes p}{|p|^2} \right) X \right\}$$

for $p \in \mathbb{R}^N \setminus \{0\}$ and $X \in \mathbb{S}^N$, where $\mathbb{S}^N$ is the space of real $N \times N$ symmetric
matrices, and $p \otimes q = (p_i q_j)_{1 \leq i,j \leq N}$ for $p = (p_1, \ldots, p_N)$, $q = (q_1, \ldots, q_N) \in \mathbb{R}^N$.

Remark that the equation (2.1) for (1.1) is not defined where $\nabla u = 0$
though such a situation may occur. In the theory of viscosity solutions

we often extend equations or solutions with upper or lower semicontinuous envelope to overcome such a difficulty. For $f\colon \mathbb{R}^d \supset D \to \mathbb{R}$ define $f^*\colon \overline{D} \to \mathbb{R} \cup \{\infty\}$ or $f_*\colon \overline{D} \to \mathbb{R} \cup \{-\infty\}$ as

$$f^*(x) = \lim_{r \to 0} \sup\{f(y);\ |y - x| < r\}, \quad f_*(x) = \lim_{r \to 0} \inf\{f(y);\ |y - x| < r\},$$

respectively. We call f^* or f_* upper or lower semicontinuous envelope, respectively. Note that $f_* \leq f \leq f^*$ in D, and f is upper (resp. lower) semicontinuous if and only if $f^* = f$ (resp. $f_* = f$).

We now list the assumptions for F.

(F1) $F\colon \mathcal{J} := (\mathbb{R}^N \setminus \{0\}) \times \mathbb{S}^N \to \mathbb{R}$ is continuous.

(F2) $-\infty < F_*(0, O) = F^*(0, O) < +\infty$, where O is the zero matrix.

(F3) F is degenerate elliptic, i.e.,

$$F(p, X + Y) \leq F(p, X)$$

for $(p, X) \in \mathcal{J}$, $Y \in \mathbb{S}^N$ provided that $Y \geq 0$,

(F4) F is geometric, i.e.,

$$F(\lambda p, \lambda X + \mu p \otimes p) = \lambda F(p, X) \quad \text{for } (p, X) \in \mathcal{J},\ \lambda > 0,\ \mu \in \mathbb{R},$$

(F5) There exist positive constants K_1, K_2, K_3, K_4 such that, if $X, Y \in \mathbb{S}^N$ and nonnegative constants $\gamma_1, \gamma_2, \gamma_3$ satisfying

$$\langle X\xi, \xi \rangle + \langle Y\eta, \eta \rangle$$
$$\leq \gamma_1 |\xi - \eta|^2 + \gamma_2(|\xi|^2 + |\eta|^2) + \gamma_3 |\xi - \eta|(|\xi| + |\eta|)$$

for $\xi, \eta \in \mathbb{R}^N$, then

$$F(p, X) - F(q, -Y)$$
$$\geq -K_1 \gamma_1 |\bar{p} - \bar{q}|^2 - K_2 \gamma_2 - K_3 \gamma_3 |\bar{p} - \bar{q}| - K_4 |p - q|$$

for $p, q \in \mathbb{R}^N \setminus \{0\}$, where $\bar{p} = p/|p|$,

Note that (2.4) and then (2.3) satisfy (F1)–(F5). Moreover, note that F_*, F^* also satisfies (F3) or (F4) if F is so, respectively.

We now recall the definition of viscosity solutions, which is a weak solution to a degenerate parabolic or elliptic equation based on the maximum principle of C^2 function.

Definition 2.1. We say $u\colon (0, T) \times \Omega \to \mathbb{R}$ is viscosity sub- (resp. super-) solution to (2.1) if the followings hold;

(S1) $u^* < \infty$ (resp. $u_* > -\infty$) in $[0, T] \times \overline{\Omega}$.

(S2) for each $(\hat{t}, \hat{x}) \in (0, T) \times \Omega$ and $\varphi \in C^2((0, T) \times \Omega)$ satisfying

$$u^*(t, x) - \varphi(t, x) \leq u^*(\hat{t}, \hat{x}) - \varphi(\hat{t}, \hat{x})$$
$$(\text{resp. } u_*(t, x) - \varphi(t, x) \geq u_*(\hat{t}, \hat{x}) - \varphi(\hat{t}, \hat{x}))$$

for $(t, x) \in (0, T) \times \Omega$,

(2.5) $$\varphi_t(\hat{t}, \hat{x}) + F_*(\nabla \varphi(\hat{t}, \hat{x}), \nabla^2 \varphi(\hat{t}, \hat{x})) \leq 0$$

(2.6) $$(\text{resp. } \varphi_t(\hat{t}, \hat{x}) + F^*(\nabla \varphi(\hat{t}, \hat{x}), \nabla^2 \varphi(\hat{t}, \hat{x})) \geq 0)$$

holds.

We say u is viscosity sub- (resp. super-) solution to (2.1)–(2.2) if u satisfies (S1) and the following (S2)' instead of (S2) hold;

(S2)' for each $(\hat{t}, \hat{x}) \in (0, T) \times \overline{\Omega}$ and $\varphi \in C^2((0, T) \times \overline{\Omega})$ satisfying

$$u^*(t, x) - \varphi(t, x) \le u^*(\hat{t}, \hat{x}) - \varphi(\hat{t}, \hat{x})$$

$$(\text{resp. } u_*(t, x) - \varphi(t, x) \ge u_*(\hat{t}, \hat{x}) - \varphi(\hat{t}, \hat{x}))$$

for $(t, x) \in (0, T) \times \overline{\Omega}$,
(a) (2.5) (resp. (2.6)) holds if $\hat{x} \in \Omega$,
(b) either (2.5) (resp. (2.6)) or

$$\langle \vec{\nu}, \nabla\varphi(\hat{t}, \hat{x}) \rangle \le 0 \quad (\text{resp. } \langle \vec{\nu}, \nabla\varphi(\hat{t}, \hat{x}) \rangle \ge 0)$$

holds if $\hat{x} \in \partial\Omega$.

We say u is a viscosity solution if u is viscosity sub- and supersolution.

See [8] for details of viscosity solutions, or [7] for theory of viscosity solutions to degenerate parabolic equations. Note that C^2 classical sub- or supersolution to (2.1) is viscosity sub- or supersolution if (F3) holds. These theory is extended to the Neumann boundary value problem in a bounded domain of geometric and degenerate parabolic equation; see [18] or [29]. In this paper we consider all solutions in viscosity solution sense so that we omit the word of "viscosity" here and hereafter.

It is convenient to introduce an equivalent definition to the conditions of (S2) by [23].

Lemma 2.2 ([3])**.** *Assume that (F3) hold. Then, the condition (S2) is equivalent to Then, it is equivalent to the following conditions.*

(i) *If* $|\nabla\varphi(\hat{t}, \hat{x})| \ne 0$, *then*

$$\varphi_t + F(\nabla\varphi, \nabla^2\varphi) \le 0 \ (\text{resp. } \varphi_t + F(\nabla\varphi, \nabla^2\varphi) \ge 0) \quad at \ (\hat{t}, \hat{x}).$$

(ii) *If* $|\nabla\varphi(\hat{t}, \hat{x})| = 0$, *then*

$$\varphi_t + F_*(0, O) \le 0 \ (\text{resp. } \varphi_t + F^*(0, O) \ge 0) \quad at \ (\hat{t}, \hat{x})$$

provided that $\nabla\varphi(\hat{t}, \hat{x}) = 0$ *and* $\nabla^2\varphi(\hat{t}, \hat{x}) = O$.

Note that [3] derive the above for a geometric evolution equation and then the condition (ii) is

$$\varphi_t(\hat{t}, \hat{x}) \le 0 \quad (\text{resp. } \varphi_t(\hat{t}, \hat{x}) \ge 0)$$

provided that $\nabla\varphi(\hat{t}, \hat{x}) = 0$ and $\nabla^2\varphi(\hat{t}, \hat{x}) = O$; in fact $F^*(0, O) = F_*(0, O) = 0$. However, we can prove this lemma without the assumption (F4) by revising the conclusion of the case (ii) as the above. Since the proof is same as that in [3] we omit the proof.

2.2.   Remark on the existence and uniqueness. The existence of solution to (2.1) is established by Perron's method due to H. Ishii [21]. This method is based on the following two propositions.

Proposition 2.3. *Let $\mathcal{S}$ be a non-empty set of subsolutions (resp. supersolutions) to (2.1) in $(0,T) \times \Omega$. Assume that functions in $\mathcal{S}$ is locally uniformly bounded in $(0,T) \times \Omega$. Then,*

$$u(t,x) = \sup\{v(t,x);\ v \in \mathcal{S}\} \quad (resp.\ u(t,x) = \inf\{v(t,x);\ v \in \mathcal{S}\})$$

is still a subsolution (resp. supersolution) to (2.1) in $(0,T) \times \Omega$.

Proposition 2.4 (Perron's method)**.** *Assume that (F3) holds.*
Let $f,g\colon (0,T) \times \Omega \to \mathbb{R}$ be a locally bounded sub- and supersolution to (2.1) in $(0,T) \times \Omega$ satisfying $f \leq g$ in $(0,T) \times \Omega$. Then,

$$u(t,x) = \sup\left\{ v(t,x);\ \begin{array}{l} v \text{ is a subsolution to } (2.1) \text{ in } (0,T) \times \Omega, \\ f \leq v \leq g \quad in\ [0,T] \times \overline{\Omega} \end{array} \right\}$$

is a solution to (2.1) in $(0,T) \times \Omega$.

This method is established by [21] for Hamilton-Jacobi equations, and is extended not only to second order degenerate elliptic equation in [22]. This method is also extended to the degenerate parabolic equation in [7] with an interpretation of (2.1) to a second order degenerate elliptic equation

$$E(Du, D^2u) = u_t + F(\nabla u, \nabla^2 u)$$

with a differential operator

$$D = \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_N} \right).$$

It is also extended to the Neumann boundary value problem in [29]. One also can find in [7] or [29] the detailed construction of a solution with initial data $u_0 \in BUC(\overline{\Omega})$ to (2.1), where $BUC(\overline{\Omega})$ denotes the space of bounded and uniformly continuous functions. By combining the comparison principle, which is explained later, one observe that the solution is continuous.

The uniqueness of solutions to (2.1) with respect to the initial data $u|_{t=0} = u_0$ is derived from the following comparison principle.

Comparison principle(CP): Let u and v be a sub- and supersolution to (2.1) in $(0,T) \times \Omega$. Under a suitable assumptions from boundary condition, if

$$(2.7) \qquad\qquad u^*(0,x) \leq v_*(0,x) \quad \text{for } x \in \overline{\Omega},$$

then

$$u^*(t,x) \leq v_*(t,x) \quad \text{for } (t,x) \in [0,T] \times \overline{\Omega}.$$

We now summarize the known results on the comparison principle. In the following sections we consider two type of domains;

- $\Omega = \mathbb{R}^N$ (in §3),

- axis-symmetric nonconvex domain (in §4):

(2.8)
$$\Omega = \{(x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R}; \ |x'| < r(x_N)\}$$

with a smooth function $r \colon \mathbb{R} \to (0, \infty)$ satisfying
(Ω1) $r' < 0$ in $(-\infty, 0)$, $r' > 0$ in $(0, \infty)$,
(Ω2) there exists $\delta_0 > 0$ such that

$$\overline{B_{\delta_0}(x + \delta_0 \vec{\nu}(x))} \cap \overline{\Omega} = \overline{B_{\delta_0}(x - \delta_0 \vec{\nu}(x))} \cap \Omega^c = \{x\}$$

for $x \in \partial\Omega$, where $B_\delta(x_0) = \{x \in \mathbb{R}^N; \ |x - x_0| < \delta\}$.

If Ω is bounded, (CP) is established by [7] at least (2.1) with (F1)–(F3) and additional assumption from Dirichlet boundary condition as

$$u^*(t, x) \le v_*(t, x) \quad \text{in } (0, T) \times \partial\Omega.$$

Their proof is easily extend to the case $\Omega = \mathbb{R}^N$ with (F1)–(F3) and

(2.9)
$$\begin{cases} \text{there exist } \alpha, \beta \in \mathbb{R} \text{ and } R > 0 \text{ such that} \\ u(t, x) = \alpha, \quad v(t, x) = \beta \quad \text{if } |x| > R \text{ and } t \in [0, T). \end{cases}$$

The above condition is essentially same as the Dirichlet boundary condition. Note that $\alpha \le \beta$ by (2.7). In general (CP) for unbounded domain by [15] with additional assumptions of asymptotic behavior for sub- and supersolutions as $|x| \to \infty$ and boundedness assumptions for F; see [15] for details. For the Neumann boundary value problem on bounded nonconvex domain (CP) is established by [18] for (2.1) satisfying (F1)–(F3) and (F5). When we consider the axis-symmetric domain, we additionally assume that

(2.10)
$$\begin{cases} \text{there exist } \alpha_j, \beta_j \in \mathbb{R} \text{ for } j = 1, 2 \text{ and } R > 0 \text{ such that} \\ u(t, x) = \alpha_1, \quad v(t, x) = \alpha_2 \quad \text{if } x_N < -R, \\ u(t, x) = \beta_1, \quad v(t, x) = \beta_2 \quad \text{if } x_N > R \\ \text{for } t \in [0, T) \text{ and } x = (x', x_N) \in \overline{\Omega}. \end{cases}$$

Then (CP) is established by applying the proof in [18]. By (CP) we obtain the solution constructed from $u_0 \in BUC(\overline{\Omega})$ which satisfies (2.9) or (2.10) is not only unique but also continuous in $[0, \infty) \times \overline{\Omega}$.

The level set method establishes the evolution of interfaces by extracting implicitly described Γ_t as in (1.3) with a solution to the level set equation of the evolution equation. However, we remark that the initial data u_0 is not unique for given Γ_0 and implicit description (1.3); for example u_0^3 still describes Γ_0 if u_0 describes it with $c = 0$ level set. Thus, even if solution to the level set equation is unique with respect to initial data u_0, one can obtain several level sets started from Γ_0. It is very important property if Γ_t is determined uniquely with respect to Γ_0.

We shall conclude this section with mentioning the uniqueness of level sets. It is obtained from the comparison of interior or exterior sets. In the level set method we regard a set $\{x \in \overline{\Omega}; u(t, x) > c\}$ or $\{x \in \overline{\Omega}; u(t, x) < c\}$ is interior or exterior of Γ_t given by (1.3), respectively. We now deduce a comparison principle of interior or exterior set from (CP). We first recall the stability result of solutions.

Proposition 2.5. *Let $u_k \colon (0,T) \times \Omega \to \mathbb{R}$ be a sub- (resp. super-) solution to (2.1) in $(0,T) \times \Omega$ for $k \in \mathbb{N}$. Assume that u_k converges to $u \colon (0,T) \times \Omega \to \mathbb{R}$ locally uniformly in $(0,T) \times \Omega$ as $k \to \infty$. If $u^* < \infty$ (resp. $u_* > -\infty$) in $(0,T) \times \Omega$, then u is a sub- (resp. super-) solution to (2.1) in $(0,T) \times \Omega$.*

Proposition 2.5 is established for more general equations in [7] and extended to the Neumann boundary problem in [29]. We now show the following rescaling invariance of dependent variable for geometric evolution equations.

Lemma 2.6. *Assume that (F1)–(F4) hold. Let u be a sub- (resp. super-) solution to (2.1) in $(0,T) \times \Omega$ and $G \colon \mathbb{R} \to \mathbb{R}$ be uniformly continuous and nondecreasing function. Then, $G(u^*(t,x))$ (resp. $G(u_*(t,x))$) is still a sub- (resp. super-) solution to (2.1) in $(0,T) \times \Omega$.*

Proof. In this proof we demonstrate only that $w(t,x) = G(u^*(t,x))$ is a subsolution if u is so. Note that w is upper semicontinuous and thus $w^* = w$ in this case.

We first demonstrate that $G \in C^2(\mathbb{R})$ and $G' > 0$ in $\mathbb{R}$. Let $\varphi \in C^2((0,\infty) \times \Omega)$ and assume that

$$w(t,x) - \varphi(t,x) \le w(\hat{t}, \hat{x}) - \varphi(\hat{t}, \hat{x}) \quad \text{for } (t,x) \in (0,\infty) \times \Omega$$

We may assume that $w(\hat{t}, \hat{x}) - \varphi(\hat{t}, \hat{x}) = 0$ by considering $\varphi(t,x) - (w(\hat{t}, \hat{x}) - \varphi(\hat{t}, \hat{x}))$ instead of φ.

Since $G' > 0$ there exists $H = G^{-1} \in C^2(\mathbb{R})$ and $H' > 0$ in $\mathbb{R}$. We now define $\psi(t,x) = H(\varphi(t,x))$. Then,

$$u^*(t,x) - \psi(t,x) \le u^*(\hat{t}, \hat{x}) - \psi(\hat{t}, \hat{x}) = 0 \quad \text{for } (t,x) \in (0,\infty) \times \Omega.$$

In fact, since $H' > 0$ and $w(t,x) = G(u^*(t,x)) \le \varphi(t,x)$ we obtain

$$u^*(t,x) \le H(\varphi(t,x)) = \psi(t,x).$$

Moreover, we have

$$\psi(\hat{t}, \hat{x}) = H(\varphi(\hat{t}, \hat{x})) = H(G(u^*(\hat{t}, \hat{x}))) = u^*(\hat{t}, \hat{x}),$$

which implies

$$u^*(t,x) - \psi(t,x) \le 0 = u^*(\hat{t}, \hat{x}) - \psi(\hat{t}, \hat{x})$$

for $(t,x) \in (0,\infty) \times \Omega$.

By straightforward calculation we obtain

$$\psi_t = H'(\varphi)\varphi_t, \quad \nabla\psi = H'(\varphi)\nabla\varphi,$$
$$\nabla^2\psi = H'(\varphi)\nabla^2\varphi + H''(\varphi)\nabla\varphi \otimes \nabla\varphi.$$

Since u^* is a subsolution to (2.1) then we have

$$\psi_t + F_*(\hat{t}, \hat{x}, \nabla\psi, \nabla^2\psi) \le 0 \quad \text{at } (\hat{t}, \hat{x}),$$

which implies

$$H'(\varphi)(\varphi_t + F_*(\hat{t}, \hat{x}, \nabla\varphi, \nabla^2\varphi)) \le 0 \quad \text{at } (\hat{t}, \hat{x})$$

by (A4). Since $H' > 0$ we obtain

$$\varphi_t + F_*(\hat{t}, \hat{x}, \nabla\varphi, \nabla^2\varphi)) \le 0 \quad \text{at } (\hat{t}, \hat{x}).$$

Let G is uniformly continuous and nondecreasing. We now approximate G with smooth and strictly increasing G_ε. Let $G_\varepsilon = (G * \rho_\varepsilon)(s) + \varepsilon \tanh s$, where $G * \rho_\varepsilon$ is a convolution between G and ρ_ε, and $\rho_\varepsilon \in C^\infty(\mathbb{R})$ is a mollifier, i.e., $\rho_\varepsilon \geq 0$, $\int_\mathbb{R} \rho_\varepsilon = 1$ and $\mathrm{supp}\rho_\varepsilon \subset [-\varepsilon, \varepsilon]$. Then, $G_\varepsilon \in C^\infty(\mathbb{R})$, $G' > 0$ and $\lim_{\varepsilon \to 0} G_\varepsilon = G$ uniformly in $\mathbb{R}$.

We now define $w_\varepsilon(t, x) = G_\varepsilon(u^*(t, x))$. Then, $\lim_{\varepsilon \to 0} w_\varepsilon(t, x) = w(t, x)$ uniformly on $[0, \infty) \times \Omega$ and thus w is a subsolution to (2.1) by Proposition 2.5. $\qquad \square$

We are now in the position to state the comparison principle of interior and exterior sets.

Theorem 2.7. *Either following (i) or (ii) holds.*

(I) *For the case $\Omega = \mathbb{R}^N$, assume (F1)–(F4) hold. Let u and v be sub- and supersolutions to (2.1) in $(0, T) \times \Omega$ satisfying (2.9).*

(II) *For the case Ω given by (2.8) satisfying ($\Omega 1$)–($\Omega 2$), assume that (F1)–(F5) hold. Let let u and v be sub- and supersolutions to (2.1)–(2.2) in $(0, T) \times \overline{\Omega}$ satisfying (2.10).*

If

$$(2.11) \qquad \{x \in \overline{\Omega};\ u^*(0, x) > c_1\} \subset \{x \in \overline{\Omega};\ v_*(0, x) > c_2\}$$

$$(2.12) \qquad (resp.\ \{x \in \overline{\Omega};\ u^*(0, x) < c_1\} \supset \{x \in \overline{\Omega};\ v_*(0, x) < c_2\})$$

for some $c_1, c_2 \in \mathbb{R}$. Then,

$$(2.13) \qquad \{x \in \overline{\Omega};\ u^*(t, x) > c_1\} \subset \{x \in \overline{\Omega};\ v_*(t, x) > c_2\}$$

$$(2.14) \qquad (resp.\ \{x \in \overline{\Omega};\ u^*(t, x) < c_1\} \supset \{x \in \overline{\Omega};\ v_*(t, x) < c_2\})$$

for $t \in [0, T]$.

Note that (CP) is available for the situations Theorem 2.7 considered. However, Theorem 2.7 requires no relation between u^* and v_* except (2.11) or (2.12) though (CP) is crucial to prove that. This comparison principle is the generalized result of [7, Theorem 7.1]. Moreover no regularity assumptions for initial data $u(0, \cdot)$ and $v(0, \cdot)$ are required in Theorem 2.7, which is the advantage to the result in [7, §7]. The proof is similar to that in [7], but we verify it here because we relax the assumptions.

Proof. We may assume that $c_1 = c_2 = 0$ without loss of generality by considering $u - c_1$ and $v - c_2$ instead of u and v, respectively. We here mention only the case (I) since the argument for the case (II) is parallel.

We divide the proof into three steps.

Step 1. We construct a rescaling function to apply the comparison principle. We now define

$$G(s) = \sup\{(u^*(0, y))_+;\ v_*(0, y) \leq s\},$$

where $(a)_+ = \max\{0, a\}$. Then, the followings hold.

(i) G is monotone nondecreasing, $G \geq 0$,

(ii) $G(s) = 0$ if $s \leq 0$,

(iii) $u^*(0, x) \leq G(v_*(0, x))$ for $x \in \overline{\Omega}$,

(iv) G is upper semicontinuous in $\mathbb{R}$, and continuous on $(-\infty, 0]$.

We now demonstrate these properties. The property (i) is derived directly from the definition of G. The property (ii) is derived from (2.11). In fact, from (2.11) we have $u^*(0, x) \leq 0$ if $v_*(0, x) \leq 0$, which implies

$$(u^*(0, x))_+ = 0 \quad \text{for } x \in \{y \in \overline{\Omega}; \ v_*(0, y) \leq 0\}$$

and thus $G(s) = 0$ for $s \leq 0$. The property (iii) follows from definition of G. In fact, for fixed $x \in \overline{\Omega}$ we have $x \in \{y \in \overline{\Omega}; \ v_*(0, y) \leq v_*(0, x)\}$, which implies

$$u^*(0, x) \leq (u^*(0, x))_+ \leq G(v_*(0, x)).$$

Finally we demonstrate that (iv). Let

$$s^* := \sup\{\bar{s}; \ G(s) = 0 \text{ for } s \in (-\infty, \bar{s})\},$$

then it is clear that G is continuous in $(-\infty, s^*)$ since $G = 0$ in $(-\infty, s^*)$. Let $\hat{s} \geq s^*$. Then, for each $k \in \mathbb{N}$ there exists $y_k \in \overline{\Omega}$ such that

$$(2.15) \qquad u^*(0, y_k) > G(\hat{s} + k^{-1}) - k^{-1}, \quad v_*(0, y_k) \leq \hat{s} + k^{-1}$$

by the definition of G and s^* since $G(\hat{s} + k^{-1}) > 0$. We now divide the case into two cases.

Case 1. Assume that there exists k_0 such that $|y_k| > R$ for $k > k_0$, where $R > 0$ is such that $u^*(0, y) = \alpha$ and $v_*(0, y) = \beta$ if $|y| > R$. Then, $v_*(0, y_k) = \beta \leq \hat{s} + k^{-1}$ provided $k > k_0$, which implies $v_*(0, y_k) \leq \hat{s}$ for $k > k_0$ so that we have

$$u^*(0, y_k) \leq G(\hat{s})$$

provided that $k > k_0$. Then, for $\varepsilon > 0$ we choose $k \in (k_0, \infty)$ satisfying $k^{-1} < \varepsilon$. If $r \in (0, k^{-1})$ then

$$(2.16) \qquad G(\hat{s} + r) \leq G(\hat{s} + k^{-1}) = u^*(0, y_k) + k^{-1} < G(\hat{s}) + \varepsilon.$$

Case 2. Assume that there exists a subsequence of y_k, which we denoted also by y_k, satisfying $|y_k| \leq R$. Then, we may assume that $\lim_{k \to \infty} y_k = y_0 \in \mathbb{R}^N$ by taking a subsequence of y_k if necessary. Then the second inequality of (2.15) and lower semicontinuity of v_* imply that

$$v_*(0, y_0) \leq \hat{s}$$

and thus $u^*(0, y_0) \leq G(\hat{s})$. For $\varepsilon > 0$ we choose $k \in (0, \infty)$ such that $k^{-1} < \varepsilon/2$ and $u^*(0, y_k) < u^*(0, y_0) + \varepsilon/2$ by the upper semicontinuity of u^*. If $r \in (0, k^{-1})$ then

$$(2.17) \qquad G(\hat{s} + r) \leq G(\hat{s} + k^{-1}) \leq u^*(0, y_k) + k^{-1} < u^*(0, y_0) + \varepsilon$$
$$\leq G(\hat{s}) + \varepsilon.$$

The inequalities (2.16) and (2.17) imply $\overline{\lim}_{s \to \hat{s}+0} G(s) = G(\hat{s})$: here we have used $G(\hat{s} + r) \geq G(\hat{s})$. On the other hand $\overline{\lim}_{s \to \hat{s}-0} G(s) \leq G(\hat{s})$ by (i). Hence we upper semicontinuity of G.

It remains the continuity of G at $s = 0$. Note that $s^* \geq 0$ by (ii), thus it suffices to prove the continuity of G at $s = 0$ provided that $s^* = 0$. In this case there exists sequence $y_k \in \mathbb{R}^N$ satisfying (2.15) with $\hat{s} = 0$, since

$G(k^{-1}) > 0$. Then, by the parallel argument of the above Case 1 or Case 2 yields that either the following Case 3 or 4 holds;

Case 3. There exists $k_0 > 0$ such that $v_*(0, y_k) \leq 0$ provided that $k > k_0$,

Case 4. There exists $y_0 = \lim_{k \to \infty} y_k$ satisfying $v_*(0, y_0) \leq 0$ by taking subsequence of y_k.

If Case 3 holds then $u^*(0, y_k) \leq 0$ provided that $k > k_0$, and if Case 4 holds then $u^*(0, y_0) \leq 0$ by (2.11). This implies that $\overline{\lim}_{s \to +0} G(s) \leq 0$ by the parallel argument of the above Case 1 and 2. Since $G \geq 0$ in $\mathbb{R}$ and $G = 0$ on $(-\infty, 0]$ we obtain $\lim_{s \to 0} G(s) = 0 = G(0)$.

Step 2. We now mollify G to uniformly continuous, monotone nondecreasing function.

Since $u(0, \cdot) \in D_\alpha$ and u^* is upper semicontinuous, there exists $K > 0$ such that $u^*(0, x) \leq K$ for $x \in \mathbb{R}^N$. We first define

$$\bar{G}(2^k) := G(2^{k+1}) \quad \text{if } k \in \mathbb{Z} \cap (-\infty, 0],$$
$$\bar{G}(2) := K.$$

We next define $\bar{G}$ in $[2^k, 2^{k+1}]$ for $k \in \mathbb{Z} \cap (-\infty, 0]$ with linear interpolation, i.e.,

$$\bar{G}(s) := \bar{G}(2^k)\frac{2^{k+1} - s}{2^{k+1} - 2^k} + \bar{G}(2^{k+1})\frac{s - 2^k}{2^{k+1} - 2^k} \quad \text{for } s \in [2^k, 2^{k+1}]$$

for $k \in \mathbb{Z} \cap (-\infty, 0]$. Finally, we define

$$\bar{G}(s) := \begin{cases} \bar{G}(2) & \text{if } s \in [2, \infty), \\ 0 & \text{if } s \in (-\infty, 0]. \end{cases}$$

Then, $\bar{G}$ satisfies

    (v) $\bar{G}$ is monotone nondecreasing, $\bar{G} \geq 0$ in $\mathbb{R}$,

    (vi) $\bar{G}(s) = 0$ if $s \leq 0$,

    (vii) $u^*(0, x) \leq \bar{G}(v_*(0, x))$ for $x \in \mathbb{R}^N$,

    (viii) $\bar{G}$ is uniform continuous on $\mathbb{R}$.

We now demonstrate the above properties. The properties (v), (vi) and (viii) follows directly from the definition of $\bar{G}$.

It remains to prove (vii). Note that $G \leq \bar{G}$ on $\mathbb{R}$ so that (vii) follows from (iii). In fact, if $s \in (0, 1]$ then there exists $k \in \mathbb{Z} \cap (-\infty, 0]$ such that $s \in [s_{k-1}, s_k]$, which implies

$$\bar{G}(s) \geq \bar{G}(s_{k-1}) = G(s_k) \geq G(s).$$

If $s \leq 0$, then $\bar{G}(s) = 0 = G(s)$. If $s \geq 2$, then $\bar{G}(s) = K \geq G(s)$ by definition of G. If $s \in [1, 2]$, then

$$\bar{G}(s) = (K - G(2))(s - 1) + G(2) \geq G(2) \geq G(s).$$

Step 3. We are in the position to prove (2.13).

Let $w(t, x) = \bar{G}(v_*(t, x))$. Then, w is a supersolution to (2.1) by Lemma 2.6, and lower semicontinuous i.e., $w_* = w$. Moreover, we observe that

$$u^*(0, x) \leq w(0, x) \quad \text{for } x \in \mathbb{R}^N$$

by (vii). Then, we obtain

$$u^*(t,x) \leq w(t,x) \quad \text{for } (t,x) \in (0.T) \times \mathbb{R}^N$$

by (CP). The above implies

$$\{x \in \mathbb{R}^N; \ u^*(t,x) > 0\} \subset \{x \in \mathbb{R}^N; \ w(t,x) > 0\}$$

for $t \in [0,T)$. By (vi) we have

$$\{x \in \mathbb{R}^N; \ w(t,x) > 0\} \subset \{x \in \mathbb{R}^N; \ v_*(t,x) > 0\}$$

for $t \in [0,T)$ so that we obtain (2.13).

To derive (2.14) from (2.12) we construct a rescaled subsolution $\bar{H}(u^*(t,x))$ with

$$H(s) = \inf\{(v_*(0,y))_-; \ u^*(0,y) \geq s\},$$

where $(a)_- = \min\{0,a\}$, and its relaxation $\bar{H}$ by the parallel argument of the above Step 1, 2, and 3. Then, $\bar{H}$ satisfies

- $\bar{H}$ is monotone nondecreasing, $\bar{H} \leq 0$ in $\mathbb{R}$,
- $\bar{H}(s) = 0$ if $s \geq 0$,
- $\bar{H}(u^*(0,x)) \leq v_*(0,x)$ for $x \in \mathbb{R}^N$,
- $\bar{H}$ is uniform continuous on $\mathbb{R}$.

Hence we obtain (2.14). $\qquad\square$

2.3. Set theoretic solution. Set-theoretic solution for a geometric evolution equation is introduced in [16]; we say D_t is set-theoretic sub- (resp super-) solution to (1.1) if

$$w(t,x) := \chi_{D_t}(x) = \begin{cases} 1 & \text{if } x \in D_t \\ 0 & \text{otherwise} \end{cases}$$

is sub- (resp. super-) solution to the level set equation of (1.1), i.e., (2.1) with (2.3).

The characterizations of the set-theoretic solution are also introduced in [16]; see for details. In this section we derive a direct calculation of derivatives of a characteristic function in viscosity solution sense.

Theorem 2.8. *Let $D \subset \mathbb{R}^N$ be a N-dimensional submanifold with smooth boundary $\Gamma = \partial D$. If $\varphi(x) \in C^2(\mathbb{R}^N)$ and $\hat{x} \in \Gamma$ satisfies*

$$(\chi_D)^*(x) - \varphi(x) \leq (\chi_D)^*(\hat{x}) - \varphi(\hat{x})$$

$$(\text{resp. } (\chi_D)_*(x) - \varphi(x) \geq (\chi_D)_*(\hat{x}) - \varphi(\hat{x})) \quad \text{for } x \in \mathbb{R}^N$$

and $|\nabla\varphi(\hat{x})| \neq 0$, then

$$-\operatorname{div}\frac{\nabla\varphi}{|\nabla\varphi|} + \mathcal{K}_\Gamma \leq 0 \ \left(\text{resp. } -\operatorname{div}\frac{\nabla\varphi}{|\nabla\varphi|} + \mathcal{K}_\Gamma \geq 0\right) \quad \text{at } \hat{x},$$

where $\mathcal{K}_\Gamma = \mathcal{K}_\Gamma(x)$ is the mean curvature of Γ at $x \in \Gamma$ in the direction of the outer unit normal vector field $\mathbf{n}$ of ∂D.

Proof. We here demonstrate only the subsolution case since the proof on supersolution case is parallel.

By the definition of N-dimensional submanifold with boundary Γ is smooth $N-1$-dimensional submanifold, i.e., smooth hypersurface in $\mathbb{R}^N$. We here choose the orthonormal basis τ_j $(j = 1, \ldots, N-1)$ of tangential space $T_{\hat{x}}\Gamma$ at $\hat{x}$. Then the outer unit normal vector field $\mathbf{n}$ of Γ is defined, and there exists $\delta_0 > 0$ such that

$$\hat{x} + \delta \hat{\mathbf{n}} \in \mathbb{R}^N \setminus \overline{D}, \quad \hat{x} - \delta \hat{\mathbf{n}} \in \mathrm{Int} D,$$

for $\delta \in (0, \delta_0)$, where $\hat{\mathbf{n}} = \mathbf{n}(\hat{x})$. We next choose a smooth curve ζ_j on Γ, which is defined in a neighborhood of $0 \in \mathbb{R}$, satisfying

$$\zeta_j(0) = \hat{x}, \quad \zeta_j'(0) = \tau_j$$

for $j = 1, \ldots, N-1$.

We now define functions $\Phi_j(\sigma)$, $\Psi(\sigma)$, which are defined in a neighborhood of $0 \in \mathbb{R}$, given as

$$\Phi_j(\sigma) = (\chi_D)^*(\zeta_j(\sigma)) - \varphi(\zeta_j(\sigma)),$$
$$\Psi(\sigma) = (\chi_D)^*(\hat{x} + \sigma \hat{\mathbf{n}}) - \varphi(\hat{x} + \sigma \hat{\mathbf{n}}).$$

Then, both Φ_j and Ψ take their maximum at $\sigma = 0$. Moreover, since $(\chi_D)^* = \chi_{\overline{D}}$, Φ_j is smooth in a neighborhood of 0 for $j = 1, \ldots, N-1$, and Ψ is continuous on $[-\delta_0, 0]$ and smooth in $(-\delta_0, 0)$. Thus we first obtain

$$\Phi_j'(0) = -\langle \nabla \varphi(\hat{x}), \tau_j \rangle = 0$$

for $j = 1, \ldots, N-1$, which and $|\nabla \varphi(\hat{x})| \neq 0$ yield that

$$\frac{\nabla \varphi(\hat{x})}{|\nabla \varphi(\hat{x})|} = \hat{\mathbf{n}} \text{ or } -\hat{\mathbf{n}}.$$

We next obtain

$$0 \geq \Psi(-\sigma) - \Psi(0) = 1 - \varphi(\hat{x} - \sigma \hat{\mathbf{n}}) - (1 - \varphi(\hat{x}))$$
$$= -\varphi(\hat{x} - \sigma \hat{\mathbf{n}}) + \varphi(\hat{x}) = \sigma \langle \nabla \varphi(\hat{x}), \hat{\mathbf{n}} \rangle + O(\sigma^2),$$

which implies $\langle \nabla \varphi(\hat{x}), \hat{\mathbf{n}} \rangle \leq 0$ and thus

$$\frac{\nabla \varphi(\hat{x})}{|\nabla \varphi(\hat{x})|} = -\hat{\mathbf{n}}.$$

Moreover, we obtain $\Phi_j''(0) \leq 0$ for $j = 1, \ldots, N-1$ since $\Phi_j(\sigma)$ attains its maximum at $\sigma = 0$, which implies

$$0 \geq \sum_{j=1}^{N-1} (-\langle \nabla^2 \varphi(\hat{x}) \tau_j, \tau_j \rangle - \langle \varphi(\hat{x}), \zeta_j''(0) \rangle)$$
$$= \sum_{j=1}^{N-1} \left[-\mathrm{trace} \left\{ (\tau_j \otimes \tau_j) \nabla^2 \varphi(\hat{x}) \right\} + |\nabla \varphi(\hat{x})| \langle \hat{\mathbf{n}}, \zeta_j''(0) \rangle \right].$$

Note that

$$I_N - \sum_{j=1}^{N-1} \tau_j \otimes \tau_j = \hat{\mathbf{n}} \otimes \hat{\mathbf{n}} = \frac{\nabla \varphi(\hat{x})}{|\nabla \varphi(\hat{x})|} \otimes \frac{\nabla \varphi(\hat{x})}{|\nabla \varphi(\hat{x})|},$$

and

$$\sum_{j=1}^{N-1} \langle \hat{\mathbf{n}}, \zeta_j''(0) \rangle = - \sum_{j=1}^{N-1} \langle D_{\tau_j} \hat{\mathbf{n}}, \tau_j \rangle = \mathcal{K}_\Gamma(\hat{x}).$$

Then the above yields

$$-\text{trace}\left\{ \left(I_N - \frac{\nabla\varphi \otimes \nabla\varphi}{|\nabla\varphi|^2} \right) \nabla^2\varphi \right\} + \mathcal{K}_\Gamma |\nabla\varphi| \leq 0 \quad \text{at } \hat{x}.$$

Since

$$\text{trace}\left\{ \left(I_N - \frac{\nabla\varphi \otimes \nabla\varphi}{|\nabla\varphi|^2} \right) \nabla^2\varphi \right\} = |\nabla\varphi| \text{div} \frac{\nabla\varphi}{|\nabla\varphi|}$$

we obtain the conclusion of Theorem 2.8. $\qquad\qquad\qquad\qquad\qquad\square$

The following corollary states clear result however we need Theorem 2.8 and Lemma 2.2 to prove.

Corollary 2.9. *A domain with suitable constant mean curvature boundary is a stationary set theoretic solution to* (1.1)*. For example,*

  (i) *Ball:* $\{x \in \mathbb{R}^N; \ |x - x_0| < (N-1)/C\}$ *for* $x_0 \in \mathbb{R}^N$ *provided that* $C \neq 0$,
 (ii) *Generalized cylindrical surface:*
     $\{(x', x_{k+1}, \ldots, x_N) \in \mathbb{R}^k \times \mathbb{R}^{N-k}; \ |x'| < (N-1-k)/C\}$ *for* $k \in [2, N-1] \cap \mathbb{Z}$ *and its rotation provided that* $C \neq 0$,
(iii) *For* $e \in S^{N-1}$ *and* $x_0 \in \mathbb{R}^N$, *the set under a hyperplane* $\{x \in \mathbb{R}^N; \ \langle e, x - x_0 \rangle \leq 0\}$ *provided that* $C = 0$.

Remark 2.10. We remark that the mean curvature of $\partial\{x \in \mathbb{R}^N; \ |x - x_0| < (N-1)/C\}$ is $-C$ since the curvature is defined with the outer unit normal vector field $\mathbf{n}$ and we do not take the average of principal curvature as the mean curvature.

3. EIKONAL-CURVATURE FLOW

In this section we consider a stationary ball for (1.1) in $\mathbb{R}^N$ with a level set formulation. The level set equation of (1.1) is of the form

$$(3.1) \qquad\qquad u_t + F(\nabla u, \nabla^2 u) = 0 \quad \text{in } (0, \infty) \times \mathbb{R}^N,$$

with (2.3) and (2.4). To consider stability of a stationary ball we consider evolution of compact interface with a level set formulation so that we consider spatially profile of initial data or solutions to (3.1) in

$$\mathfrak{X}_\alpha := \{f \colon \mathbb{R}^N \to \mathbb{R}; \ \text{supp}(f - \alpha) \text{ is compact}\}.$$

3.1.   Barrier solutions for the uniqueness. The comparison principle is established by [7] or [15]. Note that we now consider $\Omega = \mathbb{R}^N$ thus Ω is unbounded. However, if we assume (2.9) for a sub- and supersolution u, v to (3.1), then we obtain (CP) in this problem with the argument as in [7]. Note that if we assume (2.7) then we observe $\alpha \leq \beta$ for α, β in (2.9). Theorem 2.7 is also available in this problem.

The existence of viscosity solution with initial data $u_0 \in BUC(\mathbb{R}^N)$, where $BUC(\Omega)$ denotes the space of bounded uniformly continuous functions on Ω, is also established in [7] with Perron's method. For the uniqueness of solutions we have to verify that

$$u(t,x) = \alpha \quad \text{if } |x| > R(t)$$

with at least locally bounded $R(t)$. For this purpose we now construct barrier sub- and supersolutions. We here construct only the supersolution since the construction of subsolution is established parallel.

We now assume that there exists $R_0 > 0$ such that

$$u_0(x) = u(0,x) = \alpha \quad \text{if } |x| \geq R_0.$$

We now set

$$b(t,x) = Bt + A(-|x| + R_0 + 1)$$

for constants $A, B > 0$ determined later. Then, for $x \neq 0$ we have

$$\nabla b = -\frac{Ax}{|x|}, \quad \nabla^2 b = -\frac{A}{|x|}I + \frac{Ax \otimes x}{|x|^3}.$$

Thus we obtain

$$F(\nabla b, \nabla^2 b) = F\left(-\frac{A}{|x|}x, -\frac{A}{|x|}I\right) = A\left(\frac{(N-1)}{|x|} - C\right)$$

for $x \neq 0$ by (F4). We now set $B = AC$. Then

$$b_t + F(\nabla b, \nabla^2 b) = \frac{A(N-1)}{|x|} > 0$$

for $x \neq 0$.

We now set $A = \|u_0\|_\infty - \alpha$ and define

$$b^+(t,x) = \min\{\theta_\alpha^+(b(t,x) + \alpha), \|u_0\|_\infty\}$$

with a cut-off function

$$(3.2) \qquad \theta_\alpha^+(\sigma) = \begin{cases} \alpha & \text{if } \sigma < \alpha, \\ \sigma & \text{otherwise,} \end{cases}$$

where $\|u_0\|_\infty = \sup_{\mathbb{R}^N} |u_0|$. Note that a constant function is a solution to (3.1) and then b^+ is a supersolution. Then, we observe that b^+ is a supersolution to (3.1) in $(0, \infty) \times \mathbb{R}^N$ from Lemma 2.6 and Proposition 2.3, and satisfies

(3.3) $b^+(0,x) = \|u_0\|_\infty$ if $|x| \leq R_0$,

(3.4) $b^+ \geq \alpha$ in $[0, T] \times \mathbb{R}^N$, and $b^+(t,x) = \alpha$ if $|x| \geq R_0 + 1 + Bt/A$.

The property (3.3) and (3.4) implies that $b^+(0,x) \geq u_0(x)$. Then, the solution u constructed by Perron's method implies $u(t,x) \leq b^+(t,x)$ by taking infimum of the constructed supersolutions and b^+.

By the parallel argument of the above with $A = \|u_0\|_\infty + \alpha$ we obtain a barrier subsolution $b^-(t,x) = \max\{\theta_\alpha^-(-b(t,x) + \alpha), -\|u_0\|_\infty\}$ to (3.1) in $(0,\infty) \times \mathbb{R}^N$ satisfying

$$b^-(0,x) = -\|u_0\|_\infty \text{ if } |x| \leq R_0,$$

$$b^- \leq \alpha \text{ in } [0,T] \times \mathbb{R}^N, \text{ and } b^-(t,x) = \alpha \text{ if } |x| \geq R_0 + 1 + Bt/A,$$

where

$$(3.5) \qquad\qquad \theta_\alpha^-(\sigma) = \begin{cases} \alpha & \text{if } \sigma > \alpha, \\ \sigma & \text{otherwise.} \end{cases}$$

The above implies that the solution $u(t,x)$ to (3.1) with $u(0,x) = u_0(x)$ satisfies

$$u(t,x) = \alpha \quad \text{if } |x| \geq R(t) := R_0 + 1 + Ct,$$

and then the solution u is unique with respect to $u_0 \in UC(\mathbb{R}^N) \cap \mathfrak{X}_\alpha$ in $[0,T) \times \mathbb{R}^N$ for arbitrary $T > 0$.

3.2. Instability of stationary ball. By Corollary 2.9

$$\Xi = \left\{ x \in \mathbb{R}^N; \ |x - x_0| < \frac{N-1}{C} \right\}.$$

is a set-theoretic solution to (1.1) with a constant $C \neq 0$. It is also easy to find that the above stationary solution is unstable by considering a ball with the different radius from $(N-1)/C$; if the radius is less than $(N-1)/C$ then the ball will vanish, and if the radius is larger than that then the ball will spread whole domain. The following result expresses the above phenomena from a view point with a level set formulation.

Theorem 3.1. *Let $u \in C([0,\infty) \times \mathbb{R}^N)$ be a solution to (3.1) with initial data $u(0,\cdot) = u_0 \in UC(\mathbb{R}^N) \cap \mathfrak{X}_\alpha$ satisfying*

$$\Xi = \{x \in \mathbb{R}^N; \ u(0,x) > c\}, \quad \Gamma = \partial\Xi = \{x \in \mathbb{R}^N; u(0,x) = c\}$$

for fixed $c \in \mathbb{R}$. Then

$$\lim_{t \to \infty} u(t,x) = c.$$

Proof. We may assume that $c = 0$ and $x_0 = 0$ without loss of generality by considering $\tilde{u}(t,x) = u(t, x + x_0) - c$ instead of u.

We first note that, for arbitrary $T > 0$ we have

$$(3.6) \qquad u \geq 0 \quad \text{on } [0,T) \times \overline{\Xi}, \quad u \leq 0 \quad \text{on } [0,T) \times \Xi^c$$

by (CP), which implies that

$$(3.7) \qquad\qquad u \equiv 0 \quad \text{on } [0,\infty) \times \Gamma.$$

In fact, $\chi_-(x) = -\|u_0\|_\infty \chi_{\Xi^c}(x)$ is a solution to (3.1) by Corollary 2.9 and satisfies

$$\chi_-^*(x) \leq u(0,x) \quad \text{for } x \in \mathbb{R}^N.$$

Note that $u^* = u_* = u$ since $u \in C([0,\infty) \times \mathbb{R}^N)$. This implies

$$\chi_-^*(x) \leq u(t,x) \quad \text{for } (t,x) \in (0,T) \times \mathbb{R}^N$$

with arbitrary $T > 0$ by (CP). Note that $\chi_-^*(x) = \|u_0\|_\infty(\chi_{\overline{\Xi}}(x) - 1)$ and then we obtain the first inequality of (3.6). The second inequality of (3.6) is obtained the parallel argument of the above with $\chi^+(x) = \|u_0\|_\infty\chi_\Xi(x)$.

Let $\varepsilon > 0$. Then, there exists $\delta > 0$ such that

$$\{x \in \mathbb{R}^N; \ u(0, x) > \varepsilon\} \subset \Xi_\delta := \left\{x \in \mathbb{R}^N; \ |x| < \frac{N-1}{C} - \delta\right\}$$

since u is uniformly continuous in $\mathbb{R}^N$. We may assume that $\delta \to 0$ as $\varepsilon \to 0$ by taking $\delta' = \min\{\delta, \varepsilon\}$ instead of δ.

We now define

$$v_\delta^+(t, x) = -A_1 t - |x|^2 + \left(\frac{N-1}{C} - \delta\right)^2,$$

where $A_1 > 0$ is a constant determined later. Then,

$$\Xi_\delta = \{x \in \mathbb{R}^N; \ v_\delta^+(0, x) > 0\}$$

which implies

(3.8) $$\{x \in \mathbb{R}^N; \ u(0, x) > \varepsilon\} \subset \{x \in \mathbb{R}^N; \ v_\delta^+(0, x) > 0\}.$$

By straightforward calculation we obtain

$$(v_\delta^+)_t = -A_1, \quad \nabla v_\delta^+ = -2x, \quad \nabla^2 v_\delta^+ = -2I$$

and then

$$(v_\delta^+)_t + F^*(\nabla v_\delta^+, \nabla^2 v_\delta^+) = -A_1 + 2(N-1) - 2C|x|.$$

We here have used the fact that

$$F_0^*(0, X) = -\sum_{j=1}^{N-1} \lambda_j,$$

where λ_j is an eigenvalue of X satisfying $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_N$. Then, if $x \in \Xi_{\delta/4}$, then

$$(v_\delta^+)_t + F^*(\nabla v_\delta^+, \nabla^2 v_\delta^+)$$

$$\geq -A_1 + 2(N-1) - 2(N-1) + \frac{C\delta}{2} = -A_1 + \frac{C\delta}{2}.$$

Hence we obtain v_δ^+ is a supersolution to (3.1) in $(0, \infty) \times \Xi_{\delta/4}$ provided that $A_1 \in (0, C\delta/2)$.

We extend v_δ^+ to the whole domain. Fix $A_1 \in (0, C\delta/2)$ and let $G^+: \mathbb{R} \to \mathbb{R}$ be a function such that

$$G^+(s) = \begin{cases} -1 & \text{if } s < -M_1, \\ \dfrac{s}{M_1} & \text{if } -M_1 \leq s < 0, \\ s & \text{if } s \geq 0, \end{cases}$$

where

$$M_1 = \left(\frac{N-1}{C} - \frac{\delta}{2}\right)^2 - \left(\frac{N-1}{C} - \delta\right)^2.$$

Then, G^+ is continuous and monotone nondecreasing function and thus

$$\tilde{v}_\delta^+(t, x) = G^+(v_\delta^+(t, x))$$

is still a supersolution to (3.1) in $(0, T) \times \Xi_{\delta/4}$. Moreover, if $|x| \geq C^{-1}(N - 1) - \delta/2$, then $\tilde{v}_\delta^+(t, x) = -1$ for $t > 0$ since

$$v_\delta^+(t, x) \leq - \left(\frac{N - 1}{C} - \frac{\delta}{2} \right)^2 + \left(\frac{N - 1}{C} - \delta \right)^2 = -M_1.$$

We now define

$$w_\delta^+(t, x) = \begin{cases} \tilde{v}_\delta^+(t, x) & \text{if } x \in \left\{ y \in \mathbb{R}^N; \ |x| < \dfrac{N - 1}{C} - \dfrac{\delta}{4} \right\}, \\ -1 & \text{otherwise.} \end{cases}$$

Then, w_δ^+ is a supersolution to (3.1) in $(0, \infty) \times \mathbb{R}^N$. In fact, $w_\delta^+ \equiv -1$ in $(0, \infty) \times \{x \in \mathbb{R}^N; \ |x| > (N - 1)/C - \delta/2\}$, which implies

$$(w_\delta^+)_t + F^*(\nabla w_\delta^+, \nabla^2 w_\delta^+) = 0 + F^*(0, O) = 0$$

in $(0, \infty) \times \{x \in \mathbb{R}^N; \ |x| > (N - 1)/C - \delta/2\}$.

Theorem 2.7 and (3.8) yields

$$\{x \in \mathbb{R}^N; \ u(t, x) > \varepsilon\} \subset \{x \in \mathbb{R}^N; \ w_\delta^+(t, x) > 0\}$$

for $t \in (0, T)$ with arbitrary fixed $T > 0$. Note that

$$\{x \in \mathbb{R}^N; \ w^+(t, x) > 0\} = \emptyset$$

provided that $t > T_\delta := ((N - 1)/C - \delta)^2/A_1$. Hence we obtain

$$\{x \in \mathbb{R}^N; \ u(t, x) > \varepsilon\} = \emptyset \quad \text{provided that } t > T_\delta,$$

which implies

$$\varlimsup_{t \to \infty} u(t, x) \leq \varepsilon \quad \text{for } x \in \Xi_\delta.$$

By tending $\varepsilon \to 0$ yields that

$$\varlimsup_{t \to \infty} u(t, x) \leq 0 \quad \text{for } x \in \Xi,$$

which and the second inequality of (3.6) implies

$$(3.9) \qquad \varlimsup_{t \to \infty} u(t, x) \leq 0 \quad \text{for } x \in \mathbb{R}^N.$$

The lower estimate of u, i.e.,

$$\varliminf_{t \to \infty} u(t, x) \geq 0 \quad \text{for } x \in \mathbb{R}^N$$

is derived with similar way. For $\varepsilon > 0$ we first choose $\delta > 0$ such that $\lim_{\varepsilon \to 0} \delta = 0$ and

$$(3.10) \quad \{x \in \mathbb{R}^N; \ u(0, x) < -\varepsilon\} \subset \Xi^\delta := \left\{ x \in \mathbb{R}^N; \ |x| > \frac{N - 1}{C} + \delta \right\}.$$

We introduce the function v_δ^- of the form

$$v_\delta^-(t, x) = A_2 t - |x|^2 + \left(\frac{N - 1}{C} + \delta \right)^2.$$

Then we can find suitable A_2 such that is a subsolution to (3.1) in $(0, \infty) \times \Xi^{\delta/4}$ by the parallel way of the case of v_δ^+. Then, we extend v_δ^- into $(0, T) \times \mathbb{R}^N$ by similar way with

$$G^-(s) = \begin{cases} -1 & \text{if } s \le -1, \\ s & \text{if } -1 < s \le 0, \\ \dfrac{s}{M_2} & \text{if } 0 < s \le M_2, \\ 1 & \text{otherwise,} \end{cases}$$

where

$$M_2 = \left(\frac{N-1}{C} + \delta \right)^2 - \left(\frac{N-1}{C} + \frac{\delta}{2} \right)^2,$$

i.e.,

$$w_\delta^-(t, x) = \begin{cases} G^-(v_\delta^-(t, x)) & \text{if } x \in \Xi^{\delta/4}, \\ 1 & \text{otherwise.} \end{cases}$$

Then we obtain

$$\{x \in \mathbb{R}^N; \ u(t, x) < -\varepsilon\} \subset \{x \in \mathbb{R}^N; \ w_\delta^-(t, x) < 0\}$$

for $t \in [0, T)$ with arbitrary fixed $T > 0$ from (3.10) and $\Xi^\delta = \{x \in \mathbb{R}^N; \ w_\delta^-(0, x) < 0\}$. The above implies

$$u(t, x) \ge -\varepsilon \quad \text{if } |x| \le \sqrt{A_2 t + \left(\frac{N-1}{C} + \delta \right)^2}$$

and then $\varliminf_{t \to \infty} u(t, x) \ge -\varepsilon$. By tending $\varepsilon \to 0$ we obtain

(3.11) $$\lim_{t \to \infty} u(t, x) \ge 0 \quad \text{for } x \in \mathbb{R}^N.$$

We obtain the conclusion in Theorem 3.1 by (3.9), (3.11) and (3.7). $\qquad \square$

Note that the estimate (3.11) is not uniform with respect to $x \in \mathbb{R}^N$ and thus $\lim_{t \to \infty} \|u(t, \cdot) - c\|_\infty \ne 0$.

Theorem 3.1 means that every continuous stationary solution to (3.1) at least has no strict local maximum and minimum. This is generalized as follows.

Theorem 3.2. *There are no continuous stationary solutions $u \in \mathfrak{X}_\alpha$ for $\alpha \in \mathbb{R}$ to (3.1) such that u has a connected component $U \ne \emptyset$ of super- or sublevel set whose closure is included in Ξ, i.e., $\Xi \supset \overline{\{x \in \Xi; \ u(x) > c\}} \ne \emptyset$ or $\Xi \supset \overline{\{x \in \Xi; \ u(x) < c\}} \ne \emptyset$ for a constant $c \in \mathbb{R}$ and a center $x_0 \in \mathbb{R}^N$.*

Proof. We may assume that $c = 0$ and $x_0 = 0$ without loss of generality. We also assume that $\overline{\{x \in \Xi; \ u(x) > 0\}} \subset \Xi$ since the proof of the other case is parallel.

Fix $\varepsilon \in (0, \max_{\overline{\Xi}} u(x))$. Since $\max_{\partial \Xi} u(x) \le 0$, the function $\tilde{u}$ defined as

$$\tilde{u} = \begin{cases} \max\{\varepsilon/2, u(x)\} & \text{if } x \in \Xi, \\ \varepsilon/2 & \text{otherwise} \end{cases}$$

is still a subsolution to (3.1) by Proposition 2.3 and $\tilde{u} \cap \mathfrak{X}_{\varepsilon/2}$. However, the superlevel set $\{x \in \Xi;\ u(x) > \varepsilon\}$ must vanish by the similar argument in the proof of Theorem 3.1, which is the contradiction. $\square$

Remark 3.3.   (i) The proof of Theorem 3.1 is naturally extended to the solution $u(t,x) \in C([0,\infty) \times \mathbb{R}^k \times (\mathbb{R}/\mathbb{Z})^{N-k})$ describing the generalized cylindrical surface

$$\Gamma = \left\{ (x', x_{k+1}, \ldots, x_N) \in \mathbb{R}^k \times \mathbb{R}^{N-k};\ |x'| = \frac{N-1-k}{C} \right\}$$

for $k \in [2, N-1] \cap \mathbb{Z}$ with the comparison principle for unbounded domain as in [15]. Note that the assumption (2.9) for the above problems should be revised as follows.

(2.9)' there exists $R > 0$ such that $u(t,x) = \alpha$ and $v(t,x) = \beta$ if $|x'| > R$ for $t \in [0, T)$ and $x = (x', x_{k+1}, \ldots, x_N) \in \mathbb{R}^k \times \mathbb{R}^{N-k}$.

(ii) We also obtain the nonexistence result like as Theorem 3.2 corresponding to the stationary cylindrical surface in (i).

4.   Curvature flow equation on axis-symmetric domain

Let $\Omega \subset \mathbb{R}^N$ be an axis-symmetric domain

$$\Omega = \{(x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R};\ |x'| < r(x_N)\}$$

with some smooth positive function r satisfying (Ω1)–(Ω2) (see §2.2). In this section we consider evolving hypersurface $\Gamma_t \subset \overline{\Omega}$ by (1.1) with $C = 0$ and the right angle condition, i.e.,

$$(4.1) \qquad\qquad\qquad V = \mathcal{K} \quad \text{on } \Gamma_t,$$

$$(4.2) \qquad\qquad\qquad \Gamma_t \perp \partial\Omega.$$

The level set equation is of the form

$$(4.3) \qquad\qquad u_t + F_0(\nabla u, \nabla^2 u) = 0 \quad \text{in } (0, T) \times \Omega,$$

$$(4.4) \qquad\qquad \langle \nabla u, \vec{\nu} \rangle = 0 \quad \text{on } (0.T) \times \partial\Omega$$

with (2.4), where $\vec{\nu}$ is the outer unit normal vector field of $\partial\Omega$.

Note that Ω is non-convex. In this section we consider the situation such that (x', x_N) is in interior or exterior if $x_N \to -\infty$ or $x_N \to \infty$, respectively. To describe this situation with level set method we consider a spatially profile of initial data or solutions in

$$\mathfrak{Y}_{\alpha,\beta} = \left\{ f : \overline{\Omega} \to \mathbb{R};\ \begin{array}{l} \text{there exists } R > 0 \text{ such that} \\ f(x', x_N) = \alpha \text{ if } x_N < -R, \\ f(x', x_N) = \beta \text{ if } x_N > R \end{array} \right\}.$$

4.1.   Barrier solutions for the uniqueness. We here remark on the uniqueness and existence of solutions to (4.3)–(4.4). The comparison principle for (4.3)–(4.4) for a nonconvex and bounded Ω is established by [18]. Although Ω in our problem is unbounded, the comparison principle for our problem is also derived by applying the proof of [18] for sub- and supersolution u and v satisfying (2.10).

The existence of solution u to (4.3)–(4.4) with initial data $u_0 \in UC(\overline{\Omega}) \cap \mathfrak{Y}_{\alpha,\beta}$ is also derived by the Perron's method; see [29]. To show the uniqueness of solution we have to see $u(t,\cdot) \in \mathfrak{Y}_{\alpha,\beta}$ with at least locally bounded $R = R(t) > 0$. For this purpose we make barrier sub- and supersolutions as well as in §3 and demonstrate that the above R is independent of time $t \geq 0$.

We now assume there exists $R_0 > 0$ such that

$$u_0(x) = \alpha \text{ if } x_N < -R_0, \quad u_0(x) = \beta \text{ if } x_N > R_0$$

for $x = (x', x_N)$ and $u_0 = u(0,\cdot)$. Then,

$$b(x) = \begin{cases} -A(x_N + R_0)^4 + \|u_0\|_\infty & \text{if } x_N < -R_0, \\ \|u_0\|_\infty & \text{if } |x_N| \leq R_0, \\ -A(x_N - R_0)^4 + \|u_0\|_\infty & \text{otherwise} \end{cases}$$

is a C^2 supersolution to (4.3)–(4.4) in $(0,\infty) \times \mathbb{R}^N$. In fact, by straightforward calculation we have $\nabla^2 b = c(x) \nabla b \otimes \nabla b$ for $x \in \overline{\Omega}$, which implies

$$b_t + F_0^*(\nabla b, \nabla^2 b) = 0 + F_0^*(\nabla b, O) = 0 \quad \text{in } (0,T) \times \Omega$$

by (F4). For (4.4) we have

$$\langle \nabla b, \vec{\nu} \rangle = \begin{cases} \dfrac{4A(x_N + R_0)^3 r'(x_N)}{\sqrt{1 + r'(x_N)^2}} & \text{if } x_N < -R_0, \\ 0 & \text{if } |x_N| \leq R_0, \\ \dfrac{4A(x_N - R_0)^3 r'(x_N)}{\sqrt{1 + r'(x_N)^2}} & \text{otherwise,} \end{cases}$$

which implies $\langle \nabla b, \vec{\nu} \rangle \geq 0$ on $(0,T) \times \partial\Omega$ from $(\Omega 1)$.

We now define

$$b^+(x) := \begin{cases} \theta_\alpha^+(b(x)) & \text{if } x_N < 0, \\ \theta_\beta^+(b(x)) & \text{if } x_N \geq 0, \end{cases}$$

where θ_α^+ or θ_β^+ is defined in (3.2). Then, $b^+(x)$ is a still supersolution to (4.3)–(4.4) in $(0,\infty) \times \overline{\Omega}$ satisfying $b^+ \geq u_0$. Then, we observe that the solution by Perron's method satisfies $b^+(x) \geq u(t,x)$ for $t > 0$ and $x \in \overline{\Omega}$. By the parallel argument of the above with $-b$ and cut-off functions $\theta_\alpha^-, \theta_\beta^-$ as in (3.5) we also obtain the subsolution

$$b^-(x) := \begin{cases} \theta_\alpha^-(-b(x)) & \text{if } x_N < 0, \\ \theta_\beta^-(-b(x)) & \text{if } x_N \geq 0 \end{cases}$$

satisfying $b_-(x) \leq u(t,x)$ for $(t,x) \in (0,\infty) \times \overline{\Omega}$. By the definitions of $b^\pm$ there exists $R = R_0 + O(1/A) > R_0$ such that

$$b^-(x) = b^+(x) = \alpha \quad \text{if } x_N < -R,$$
$$b^-(x) = b^+(x) = \beta \quad \text{if } x_N > R$$

and thus

$$u(t,x) = \begin{cases} \alpha & \text{if } x_N < -R, \\ \beta & \text{if } x_N > R, \end{cases}$$

for $t \geq 0$: note that R is independent of t. Hence we obtain the solution $u \in C([0,\infty) \times \overline{\Omega})$ for $u_0 \in UC(\mathbb{R}^N) \cap \mathfrak{Y}_{\alpha,\beta}$ is unique.

Moreover, the idea of barrier solutions $b^{\pm}$ yields the following fundamental property.

Lemma 4.1. *Let u be a solution to (4.3)–(4.4) in $(0,\infty) \times \overline{\Omega}$. If*

$$\{x \in \overline{\Omega}; \ u(t_0,x) = c\} \subset \overline{\Omega}_I := \overline{\Omega} \cap (\mathbb{R}^{N-1} \times I)$$

for a connected interval $I \subset \mathbb{R}$ and $t_0 \in [0,\infty)$, then,

$$\{x \in \overline{\Omega}; \ u(t,x) = c\} \subset \overline{\Omega}_{\bar{I}}$$

for $t > t_0$.

Proof. Let $p = \inf I$, $q = \sup I$, and

$$\overline{\Omega}_I^{c-} = \{(x',x_N) \in \overline{\Omega}; \ x_N < p\}, \quad \overline{\Omega}_I^{c+} = \{(x',x_N) \in \overline{\Omega}; \ x_N > q\}.$$

Note that $\overline{\Omega}_I^{c-}$ or $\overline{\Omega}_I^{c+}$ is empty if $p = -\infty$ or $q = \infty$, respectively. By assumption we have that each $\overline{\Omega}_I^{c\pm}$ is included in $\{x \in \overline{\Omega}; \ u(t_0,x) > c\}$ or $\{x \in \overline{\Omega}; \ u(t_0,x) < c\}$.

We now demonstrate that

(4.5) $\{x \in \overline{\Omega}; \ u(t,x) = c\} \subset \overline{\Omega}_{[p,\infty)}$ for $t \geq t_0$ provided that $p > -\infty$.

We now assume $\overline{\Omega}_I^{c-} \subset \{x \in \overline{\Omega}; \ u(t,x) > c\}$. Then, the subsolution

$$\bar{b}(x) = \begin{cases} \theta_1^-(|x - (p+1)|^4 - 1) & \text{if } x_N < p+1, \\ -1 & \text{otherwise} \end{cases}$$

satisfies $\bar{b} \in \mathfrak{Y}_{1,-1}$ and

$$\{x \in \overline{\Omega}; \ \bar{b}(x) > 0\} = (-\infty, p) \subset \{x \in \overline{\Omega}; \ u(t_0,x) > c\}.$$

Then, by Theorem 2.7 we obtain

$$(-\infty, p) \subset \{x \in \overline{\Omega}; \ u(t,x) > c\}$$

for $t > t_0$, which implies (4.5). If $\overline{\Omega}_I^{c-} \subset \{x \in \overline{\Omega}; \ u(t,x) < c\}$, then we deduce (4.5) with similar argument as the above with the supersolution

$$\tilde{b}(x) = \begin{cases} \theta_{-1}^+(-|x - (p+1)|^4 + 1) & \text{if } x < p+1, \\ 1 & \text{otherwise.} \end{cases}$$

We also deduce

$$\{x \in \overline{\Omega}; \ u(t,x) = c\} \subset \overline{\Omega}_{(-\infty,q]}$$ for $t \geq t_0$ provided that $q < \infty$

with the parallel argument.   □

4.2.   Stability of plane at the neck. By Corollary 2.9 we obtain that the plane at $x_N = 0$, i.e.,

$$(4.6) \quad \Xi = \{(x', x_N) \in \overline{\Omega}; \; x_N < 0\}, \quad \Gamma = \partial\Xi = \{(x', 0) \in \overline{\Omega}; \; x' \in \mathbb{R}^{N-1}\}$$

is a set theoretic solution to (4.1)–(4.2). The goal of this section is to show the discontinuous solution is stable in the following sense.

Theorem 4.2. *Let $u \in C([0, \infty) \times \overline{\Omega})$ be a solution to (4.3)–(4.4) in $(0, \infty) \times \overline{\Omega}$ with initial data $u_0 \in \mathfrak{Y}_{\alpha,\beta}$. Then,*

$$\lim_{t \to \infty} u(t, x) = \begin{cases} \alpha & \text{if } x_N < 0, \\ \beta & \text{if } x_N > 0. \end{cases}$$

In other words, for every $c \in \mathbb{R}$ between α and β the level set $\{x \in \overline{\Omega}; \; u(t, x) = c\}$ converges to Γ as $t \to \infty$.

Proof. We may assume that $\alpha > \beta$ without loss of generality. Let $R_0 > 0$ be such that

$$u_0(x) = \begin{cases} \alpha & \text{if } x_N < -R_0, \\ \beta & \text{if } x_N > R_0 \end{cases}$$

for $x = (x', x_N) \in \overline{\Omega}$. Then, for $\mu > 0$

$$\{x \in \overline{\Omega}; \; u_0(x) > \alpha + \mu \text{ or } u_0(x) < \beta - \mu\} \subset \overline{\Omega}_{(-R_0, R_0)},$$
$$\{x \in \overline{\Omega}; \; u_0(x) < \alpha - \mu\} \subset \overline{\Omega}_{(-R_0, +\infty)},$$
$$\{x \in \overline{\Omega}; \; u_0(x) > \beta + \mu\} \subset \overline{\Omega}_{(-\infty, R_0)}.$$

We now prove that, for $\varepsilon \in (0, \min\{1, R_0\})$, there exists $T_\varepsilon > 0$ such that

$$\{x \in \overline{\Omega}; \; u(t, x) > \alpha + \mu \text{ or } u(t, x) < \beta - \mu\} \subset \overline{\Omega}_{(-\varepsilon, \varepsilon)},$$
$$\{x \in \overline{\Omega}; \; u(t, x) < \alpha - \mu\} \subset \overline{\Omega}_{(-\varepsilon, +\infty)},$$
$$(4.7) \qquad \{x \in \overline{\Omega}; \; u(t, x) > \beta + \mu\} \subset \overline{\Omega}_{(-\infty, \varepsilon)}$$

for $t > T_\varepsilon$. We here demonstrate only (4.7) since the proof of the others are parallel.

Let

$$v(x) = -|x - y|^2 + z = -|x'|^2 - |x_N + y_N|^2 + z$$

for $y = (0', -y_N)$, $y_N > 0$ and $z > 0$. We choose y_N and z as

$$(4.8) \qquad v(x', R_0) = 0 \quad \text{if } (x', R_0) \in \partial\Omega,$$
$$(4.9) \qquad v(0', R_0 + 1) \leq -1,$$
$$(4.10) \qquad v(x', \varepsilon/2) \geq c_\varepsilon + 1 \quad \text{if } (x', \varepsilon/2) \in \partial\Omega, \text{ where } c_\varepsilon = v(0', \varepsilon),$$
$$(4.11) \qquad \langle \nabla v, \vec{\nu} \rangle \geq 0 \quad \text{for } (x', x_N) \in \partial\Omega \text{ if } x_N \in I_\varepsilon = [\varepsilon/2, R_0 + 1].$$

From (4.8) we set

$$z = r(R_0)^2 + (R_0 + y_N)^2.$$

For (4.9), (4.10) and (4.11) it suffices to choose y_N satisfying

$$y_N \geq \max\left\{ \frac{r(R_0)^2}{2} - R_0, \frac{r(R_0)^2}{\varepsilon} + \frac{1}{\varepsilon} - \frac{3}{4}\varepsilon, \frac{r(R_0 + 1)}{m_\varepsilon} \right\},$$

where $m_\varepsilon = \inf_{I_\varepsilon} r' > 0$, since

$$\langle \nabla v, \vec{\nu} \rangle = -\frac{2(|x'| - r'(x_N)(x_N + y_N))}{\sqrt{1 + r'(x_N)^2}} \geq -\frac{2(r(R_0 + 1) - m_\varepsilon y_N)}{\sqrt{1 + r'(x_N)^2}}$$

for $(x', x_N) \in \partial\Omega$ by $(\Omega1)$, and $\vec{\nu} = (1 + r'(x_N)^2)^{-1/2}(x'/|x'|, -r'(x_N))$.

We now introduce

$$w(t, x) = -Bt + Av(x)$$

for $A, B > 0$ chosen later. Then we have

$$\nabla w = -2A(x', x_N + y_N), \quad \nabla^2 w = -2AI,$$

which implies

$$F_0^*(\nabla w, \nabla^2 w) = 2A(N - 1).$$

Thus, we set $B = 2A(N - 1) > 0$ to get

$$w_t + F_0^*(\nabla w, \nabla^2 w) = 0 \quad \text{in } (0, \infty) \times \Omega.$$

Moreover we have

$$\langle \nabla w, \vec{\nu} \rangle \geq 0 \quad \text{on } (0, \infty) \times (\partial\Omega \cap \overline{\Omega}_{I_\varepsilon}).$$

Hence we obtain $w(t, x)$ is a supersolution to (4.3)–(4.4) on $(0, \infty) \times \overline{\Omega}_{I_\varepsilon}$.

Let $T_\varepsilon > 0$ be such that

$$w(T_\varepsilon, (0', \varepsilon)) = -BT_\varepsilon + Ac_\varepsilon = 0, \quad \text{i.e., } T_\varepsilon = \frac{Ac_\varepsilon}{B} = \frac{c_\varepsilon}{2(N - 1)}.$$

Then, we have

$$(4.12) \qquad w(t, (x', R_0 + 1)) \leq -A \quad \text{for } (t, (x', R_0 + 1)) \in [0, T_\varepsilon] \times \overline{\Omega}$$

$$(4.13) \qquad w(t, (x', \varepsilon/2)) \geq A \quad \text{for } (t, (x', \varepsilon/2)) \in [0, T_\varepsilon] \times \overline{\Omega}$$

from (4.9) and (4.10), respectively. Fix $A > 1$. Then (4.12) and (4.13) implies that there exists $\varepsilon_A > 0$ such that

$$w(t, (x', R_0 + 1)) < -1 \quad \text{for } (t, (x', R_0 + 1)) \in [0, T_\varepsilon + \varepsilon_A] \times \overline{\Omega},$$
$$w(t, (x', \varepsilon/2)) > 1 \quad \text{for } (t, (x', \varepsilon/2)) \in [0, T_\varepsilon + \varepsilon_A] \times \overline{\Omega}.$$

We now define

$$\bar{w}(t, x) = \min\{1, \max\{-1, w(t, x)\}\} = \bar{\theta}(w(t, x)),$$

where

$$\bar{\theta}(\sigma) = \begin{cases} -1 & \text{if } \sigma < -1, \\ \sigma & \text{if } |\sigma| \leq 1, \\ 1 & \text{otherwise.} \end{cases}$$

Then, $\bar{w}$ is still a supersolution to (4.3)–(4.4) in $(0, T_\varepsilon + \varepsilon_A) \times \overline{\Omega}$. Moreover, (4.8) implies

$$\{x \in \overline{\Omega}; \ \bar{w}(0, x) > 0\} \supset \overline{\Omega}_{(-\infty, R_0)} \supset \{x \in \overline{\Omega}; \ u_0(x) > \beta + \mu\}.$$

Thus we obtain

$$\{x \in \overline{\Omega}; \ \bar{w}(T_\varepsilon, x) > 0\} \supset \{x \in \overline{\Omega}; \ u(T_\varepsilon, x) > \beta + \mu\}$$

by Theorem 2.7. The definition of T_ε implies

$$\bar{w}(T_\varepsilon, (x', x_N)) \leq \bar{w}(T_\varepsilon, (x', \varepsilon)) \leq \bar{w}(T_\varepsilon, (0', \varepsilon)) = 0$$

if $x_N \geq \varepsilon$, which and Lemma 4.1 yield

$$\overline{\Omega}_{(-\infty,\varepsilon)} \supset \{x \in \overline{\Omega};\ u(t,x) > \beta + \mu\} \quad \text{for } t \geq T_\varepsilon.$$

Hence we obtain

$$\varlimsup_{t\to\infty} u(t,x) \leq \beta + \mu \quad \text{if } x_N \geq \varepsilon.$$

Tending $\varepsilon \to 0$, and next $\mu \to 0$ yields that

$$\varlimsup_{t\to\infty} u(t,x) \leq \beta \quad \text{if } x_N > 0.$$

The parallel arguments making a subsolution with $Bt - Av(x)$ we obtain that

$$\varliminf_{t\to\infty} u(t,x) \geq \beta \quad \text{if } x_N > 0.$$

Hence we obtain $\lim_{t\to\infty} u(t,x) = \beta$ if $x_N > 0$.

We also deduce $\lim_{t\to\infty} u(t,x) = \alpha$ if $x_N < 0$ by the parallel argument constructing a sub- and supersolution by v with $y = (0, y_N)$, $y_N > 0$.   □

One can easily deduced the asymptotic stability of Γ from the proof of Theorem 4.2.

Corollary 4.3. *There is no continuous stationary solution $u \in \mathfrak{Y}_{\alpha,\beta}$ to (4.3)–(4.4) satisfying $\Gamma = \{x \in \overline{W};\ u(x) = c\}$ in (4.6) for $c \in \mathbb{R}$ between α and β.*

5.   Concluding remarks

The following problems are still open:

  (i) Are there continuous stationary solutions to (3.1) describing a constant mean curvature interface?
  (ii) Are there nonconstant stationary solutions to (3.1)?

However, at least every continuous stationary solution to (3.1) has neither strict local maximum nor minimum by Theorem 3.2. It is very strong restriction to continuous stationary solutions except constant functions.

Although we prove the nonexistence of continuous stationary solutions describing stationary interface Γ considered in §3 or §4, but there exist continuous and exactly unstationary solution describing Γ by [7] or [29]. It means that we have to choose the unstationary solution as the center of linearization if we consider the stability of stationary interface to (1.1) with a level set method from a view point of eigenvalue problem.

We conclude this section to mention anisotropic curvature equation, which we do not treat in this paper. The exact solution to the level set equation of an anisotropic mean curvature equation is presented in [16, §1.7.2]. Thus, one can easily find a vanishing or spreading self similar solution to an anisotropic curvature flow with constant driving force by the parallel argument of the proof of Theorem 3.1.

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