

ON THE POINTWISE DECAY ESTIMATE FOR THE WAVE EQUATION WITH COMPACTLY SUPPORTED FORCING TERM

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Dedicated to Professor Gustavo Ponce on the occasion of his 60th birthday

ABSTRACT. In this paper we derive a new type of pointwise decay estimates for solutions to the Cauchy problem for the wave equation in 2D, in the sense that one can diminish the weight in the time variable for the forcing term if it is compactly supported in the spatial variables. As an application of the estimate, we also establish an improved decay estimate for the solution to the exterior problem in 2D.

Keywords. Wave equation, Pointwise decay estimate, Exterior problem

1. INTRODUCTION

In this paper we derive a new type of pointwise decay estimates for solutions to the following Cauchy problem for the wave equation:

$$(1.1) \quad (\partial_t^2 - \Delta)v = g(t, x), \quad (t, x) \in (0, T) \times \mathbf{R}^2,$$

$$(1.2) \quad v(0, x) = v_0(x), \quad (\partial_t v)(0, x) = v_1(x), \quad x \in \mathbf{R}^2,$$

where $u = u(t, x)$ is an unknown function, $\Delta = \partial_1^2 + \partial_2^2$, $\partial_t = \partial_0 = \partial/\partial t$, and $\partial_j = \partial/\partial x_j$ ($j = 1, 2$). In addition, we assume that v_0 and v_1 are smooth functions on \mathbf{R}^2 , and that g is a smooth function on $[0, T) \times \mathbf{R}^2$. As is well known, different kind of pointwise estimates has been used for treating nonlinear wave equations. This approach goes back to the seminal work of John [5].

On the other hand, the point of the present study is to understand how we can feel influence coming from such an assumption on the forcing term $g(t)$ that for each fixed $t \in [0, T)$, its support is contained in a ball with a fixed radius centered at the origin. This question is closely related to the exterior problem (see the section 4 below for details). But our pointwise estimate (1.3) itself is of interest, because it tells us that if the forcing term is compactly supported in the spatial variables, then one can diminish the weight in the time variable for the forcing term, compared with the standard pointwise decay estimate (2.4) below (see also Remark 1 in the below of Proposition 3.2).

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Theorem 1.1. *Let $a > 0$ and $1 < \nu < 3/2$. If $\text{supp } g(t, \cdot) \subset \{x \in \mathbf{R}^2 \mid |x| \leq a\}$ for any $t \in [0, T)$, then we have*

$$(1.3) \quad \begin{aligned} & \langle t + |x| \rangle^{1/2} (\min\{\langle x \rangle, \langle t - |x| \rangle\})^{\nu-1} |v(t, x)| \\ & \leq C \sup_{y \in \mathbf{R}^2} \langle y \rangle^{\nu+(1/2)} \left(\sum_{|\alpha| \leq 1} |\partial_x^\alpha v_0(y)| + |v_1(y)| \right) + C \sup_{(s,y) \in [0,t) \times \mathbf{R}^2} \langle s \rangle^{\nu-(1/2)} |g(s, y)|, \end{aligned}$$

where C is a constant, and $\langle x \rangle = \sqrt{1 + |x|^2}$.

This paper is organized as follows. In the next section we collect basic notations and recall known pointwise estimates for the problem (1.1)-(1.2). In the section 3 we give a proof of Theorem 1.1 in a slightly generalized form. The section 4 is devoted to establish an improved decay estimate for the exterior problem as an application of Theorem 1.1.

2. PRELIMINARIES

2.1. Notation. Let us start with some standard notation.

- We put $\langle y \rangle := \sqrt{1 + |y|^2}$ for $y \in \mathbf{R}^d$ with $d \in \mathbf{N}$.
- Let $A = A(y)$ and $B = B(y)$ be two positive functions of some variable y , such as $y = (t, x)$ or $y = x$, on suitable domains. We write $A \lesssim B$ if there exists a positive constant C such that $A(y) \leq CB(y)$ for all y in the intersection of the domains of A and B .
- The norm $\|\cdot\|_{L^\infty}$ without any other index stands for $\|\cdot\|_{L^\infty(\mathbf{R}^2)}$.
- For a time-space depending function u satisfying $u(t, \cdot) \in X$ for $0 \leq t < T$ with a Banach space X , we put $\|u\|_{L_T^\infty X} := \sup_{0 \leq t < T} \|u(t, \cdot)\|_X$. For the brevity of the description, we sometimes use the expression $\|h(s, y)\|_{L_t^\infty L^\infty}$ with dummy variables (s, y) for a function h on $[0, t) \times \mathbf{R}^2$, which means $\sup_{0 \leq s < t} \|h(s, \cdot)\|_{L^\infty}$.
- B_r stands for an open ball with radius r centered at the origin of \mathbf{R}^2 .

Following Klainerman [10], we introduce the vector fields $\Gamma = (\Gamma_0, \dots, \Gamma_3)$:

$$\Gamma_0 = \partial_0 = \partial_t, \quad \Gamma_j = \partial_j \quad (j = 1, 2), \quad \Gamma_3 = O_{12} = x_1 \partial_2 - x_2 \partial_1.$$

We put $\partial = (\partial_0, \partial_1, \partial_2)$, and $\tilde{\Gamma} = (\nabla_x, O_{12})$. The standard multi-index notation will be used for these sets of vector fields, such as $\partial^\alpha = \partial_0^{\alpha_0} \partial_1^{\alpha_1} \partial_2^{\alpha_2}$ with $\alpha = (\alpha_0, \alpha_1, \alpha_2)$ and $\Gamma^\gamma = \Gamma_0^{\gamma_0} \dots \Gamma_3^{\gamma_3}$ with $\gamma = (\gamma_0, \dots, \gamma_3)$.

For simplicity, we set

$$(2.1) \quad |\varphi(t, x)|_m = \sum_{|\alpha| \leq m} |\Gamma^\alpha \varphi(t, x)|$$

for a real-valued smooth function $\varphi(t, x)$ and a non-negative integer m .

Denoting $[A, B] := AB - BA$, we have

$$(2.2) \quad [\Gamma_i, \partial_t^2 - \Delta] = 0, \quad i = 0, 1, 2, 3,$$

and also for $i, j = 0, 1, 2, 3$, we have $[\Gamma_i, \Gamma_j] = \sum_{k=0}^3 c_{ij}^k \Gamma_k$ with suitable constants c_{ij}^k .

2.2. Decay estimates for the Cauchy problem. In this subsection we recall known decay estimates for the solution of the Cauchy problem (1.1)-(1.2). We denote by $S_0[\vec{v}_0, g](t, x)$ the solution to this problem with $\vec{v}_0 = (v_0, v_1) \in (C^\infty(\mathbf{R}^2))^2$ decaying rapidly at spatial infinity. Let $K_0[\vec{v}_0](t, x)$ and $L_0[g](t, x)$ be the solutions with $g = 0$ and $\vec{v}_0 = (0, 0)$, respectively.

The following decay estimate for solutions of the homogeneous wave equation follows from Proposition 2.1 with $\kappa = \nu - (1/2) > 0$ in [15]. Indeed, one can reduce the estimate to the case $m = 0$, thanks to (2.2).

Lemma 2.1. *Let $1 < \nu < 3/2$ and m be a non-negative integer. For $\vec{v}_0 = (v_0, v_1) \in (C^\infty(\mathbf{R}^2))^2$ we have*

$$(2.3) \quad \begin{aligned} & \langle t + |x| \rangle^{1/2} \langle t - |x| \rangle^{\nu-1} |K_0[\vec{v}_0](t, x)|_m \\ & \lesssim \mathcal{B}_{\nu+(1/2), m}[\vec{v}_0] := \sum_{|\gamma| \leq k+1} \|\langle \cdot \rangle^{\nu+(1/2)} \tilde{\Gamma}^\gamma v_0\|_{L^\infty} + \sum_{|\gamma| \leq k} \|\langle \cdot \rangle^{\nu+(1/2)} \tilde{\Gamma}^\gamma v_1\|_{L^\infty} \end{aligned}$$

for $(t, x) \in [0, T] \times \mathbf{R}^2$.

The following lemma is concerned with the inhomogeneous wave equation. For the proof, see [13, Lemma 3.4], also Di Flaviano [1].

Lemma 2.2. *Let $1 < \nu < 3/2$, $\kappa > 1$, and m is a non-negative integer. If g is a smooth function on $[0, T] \times \mathbf{R}^2$, then we have*

$$(2.4) \quad \begin{aligned} & \langle t + |x| \rangle^{1/2} \langle t - |x| \rangle^{\nu-1} |L_0[g](t, x)|_m \\ & \lesssim \sum_{|\beta| \leq m} \|\langle y \rangle^{1/2} \langle s + |y| \rangle^\nu (\min\{\langle |y| \rangle, \langle s - |y| \rangle\})^\kappa \Gamma^\beta g(s, y)\|_{L_t^\infty L^\infty} \end{aligned}$$

for $(t, x) \in [0, T] \times \mathbf{R}^2$.

3. PROOF OF THEOREM 1.1

In this section we shall prove the following pointwise decay estimate which is involved with the generalized derivatives Γ 's. This kind of generalization of Theorem 1.1 would be useful, when we wish to study the nonlinear wave equations.

Theorem 3.1. *Let $a > 0$, $1 < \nu < 3/2$ and m be a non-negative integer. If $\text{supp } g(t, \cdot) \subset \overline{B_a}$ for any $t \in [0, T]$, then we have*

$$(3.1) \quad \begin{aligned} & \langle t + |x| \rangle^{1/2} (\min\{\langle |x| \rangle, \langle t - |x| \rangle\})^{\nu-1} |S_0[\vec{v}_0, g](t, x)|_m \\ & \lesssim \mathcal{B}_{\nu+(1/2), m}[\vec{v}_0] + \sum_{|\alpha| \leq m} \|\langle s \rangle^{\nu-(1/2)} \partial^\alpha g(s, y)\|_{L_t^\infty L^\infty}. \end{aligned}$$

In order to prove (3.1), it suffices to consider the solution of the inhomogeneous wave equation, in view of (2.3).

Proposition 3.2. *Under the same assumptions of Theorem 3.1, we have*

$$(3.2) \quad \begin{aligned} & \langle t + |x| \rangle^{1/2} (\min\{\langle |x| \rangle, \langle t - |x| \rangle\})^{\nu-1} |L_0[g](t, x)|_m \\ & \lesssim \sum_{|\alpha| \leq m} \|\langle s \rangle^{\nu-(1/2)} \partial^\beta g(s, y)\|_{L_t^\infty L^\infty}. \end{aligned}$$

Remark 1. Let us compare (3.2) with (2.4). It is clear that (2.4) implies

$$(3.3) \quad \begin{aligned} & \langle t + |x| \rangle^{1/2} (\min\{\langle |x| \rangle, \langle t - |x| \rangle\})^{\nu-1} |L_0[g](t, x)|_m \\ & \lesssim \sum_{|\alpha| \leq m} \|\langle y \rangle^{1/2+\kappa} \langle s \rangle^\nu \Gamma^\beta g(s, y)\|_{L_t^\infty L^\infty}. \end{aligned}$$

In particular, if $\text{supp } g(s, \cdot) \subset \overline{B_a}$, then the right hand side is evaluated by

$$\sum_{|\alpha| \leq m} \|\langle s \rangle^\nu \partial^\beta g(s, y)\|_{L_t^\infty L^\infty}.$$

Therefore, (3.2) is actually different from (2.4) in the sense that one can diminish the weight on the forcing term with respect to the time variable as long as the forcing term is compactly supported in the spatial variable.

Proof of Proposition 3.2. It follows from (2.2) and the uniqueness for the classical solution that

$$(3.4) \quad Z^\alpha L_0[g] = L_0[Z^\alpha g] + K_0[(\phi_\alpha, \psi_\alpha)], \quad |\alpha| \leq m,$$

where we put $\phi_\alpha(x) = (Z^\alpha L_0[g])(0, x)$, $\psi_\alpha(x) = (\partial_t Z^\alpha L_0[g])(0, x)$. From the equation (1.1) we get

$$\phi_\alpha(x) = \sum_{|\beta| \leq |\alpha|-2} C_\beta (Z^\beta g)(0, x), \quad \psi_\alpha(x) = \sum_{|\beta| \leq |\alpha|-1} C'_\beta (Z^\beta g)(0, x)$$

with suitable constants C_β, C'_β . Therefore, by (2.3) with $1 < \nu < 3/2$, we get

$$\begin{aligned} \langle t + |x| \rangle^{1/2} \langle t - |x| \rangle^{\nu-1} |K_0[(\phi_\alpha, \psi_\alpha)](t, x)| & \lesssim \sum_{|\alpha| \leq m-1} \|\langle \cdot \rangle^{\nu+(1/2)} (Z^\alpha g)(0)\|_{L^\infty} \\ & \lesssim \sum_{|\alpha| \leq m} \|\langle s \rangle^{\nu-(1/2)} \partial^\beta g(s, y)\|_{L_t^\infty L^\infty}, \end{aligned}$$

because of the assumption on g . Hence, from (3.4) we see that it is enough to show

$$(3.5) \quad \langle t + |x| \rangle^{1/2} (\min\{\langle r \rangle, \langle t - r \rangle\})^{\nu-1} |L_0[g](t, x)| \lesssim \|\langle s \rangle^{\nu-(1/2)} g(s, y)\|_{L_t^\infty L^\infty}.$$

Let χ_A be the characteristic function for a set A . Then, following the opening of the section 4 in [11], we get

$$(3.6) \quad \begin{aligned} |L_0[g](t, x)| &\lesssim \|\langle s \rangle^{\nu-(1/2)} g(s, y)\|_{L_t^\infty L^\infty} \\ &\times \left(\int_0^t ds \int_{|t-s-r|}^{t-s+r} \chi_{[0,a]}(\lambda) \langle s \rangle^{-(\nu-(1/2))} K_1(\lambda, r, t-s) d\lambda \right. \\ &\left. + \int_0^{(t-r)_+} ds \int_0^{t-s-r} \chi_{[0,a]}(\lambda) \langle s \rangle^{-(\nu-(1/2))} K_2(\lambda, r, t-s) d\lambda \right) \end{aligned}$$

with $r = |x|$. Here we denoted

$$\begin{aligned} K_1(\lambda, r, t) &= \frac{2}{\pi} \sqrt{\lambda} \int_{|\lambda-r|}^t \frac{\rho h(\lambda, \rho, r)}{\sqrt{t^2 - \rho^2}} d\rho, \quad |t-r| < \lambda < t+r, \\ K_2(\lambda, r, t) &= \frac{2}{\pi} \sqrt{\lambda} \int_{|\lambda-r|}^{\lambda+r} \frac{\rho h(\lambda, \rho, r)}{\sqrt{t^2 - \rho^2}} d\rho, \quad 0 < \lambda < t-r \end{aligned}$$

with $h(\lambda, \rho, r) = (\rho^2 - (\lambda-r)^2)^{-1/2} ((\lambda+r)^2 - \rho^2)^{-1/2}$, and they are estimated as follows:

$$K_1(\lambda, r, t-s) \leq \frac{\sqrt{\lambda}}{\sqrt{\lambda-s+r+t} \sqrt{\lambda+s+r-t}}$$

for $0 \leq s \leq t$, $|t-s-r| < \lambda < t-s+r$, and

$$K_2(\lambda, r, t-s) \leq \frac{\sqrt{\lambda}}{\sqrt{t-r-\lambda-s} \sqrt{t+r+\lambda-s}}$$

for $0 \leq s \leq t$, $0 < \lambda < t-s-r$. Indeed, these estimates follow from the proofs of (4.18) and (4.22) in [11], because $\sqrt{\lambda}$ is simply evaluated by $\sqrt{\alpha}$ when $\alpha \geq \beta$, which is equivalent to $s \geq 0$, in their proofs (notice that $K_1(\lambda, s, r, t)$ and $K_2(\lambda, s, r, t)$ in [11] are actually equal to $K_1(\lambda, r, t-s)$ and $K_2(\lambda, r, t-s)$). Thus we have

$$|L_0[g](t, x)| \lesssim (I_1(t, r) + I_2(t, r)) \|\langle s \rangle^{\nu-(1/2)} g(s, y)\|_{L_t^\infty L^\infty},$$

where we put

$$(3.7) \quad I_1(t, r) = \int_0^t ds \int_{|t-s-r|}^{t-s+r} \frac{\chi_{[0,a]}(\lambda) \langle s \rangle^{-(\nu-(1/2))}}{\sqrt{\lambda-s+r+t} \sqrt{\lambda+s+r-t}} d\lambda,$$

$$(3.8) \quad I_2(t, r) = \int_0^{(t-r)_+} ds \int_0^{t-s-r} \frac{\chi_{[0,a]}(\lambda) \langle s \rangle^{-(\nu-(1/2))}}{\sqrt{t-r-\lambda-s} \sqrt{t+r+\lambda-s}} d\lambda.$$

Therefore, in order to prove (3.5), we have only to show

$$(3.9) \quad I_1(t, r) \lesssim \langle t+r \rangle^{-1/2} \langle t-r \rangle^{-(\nu-1)},$$

$$(3.10) \quad I_2(t, r) \lesssim \langle t+r \rangle^{-1/2} (\min\{\langle r \rangle, \langle t-r \rangle\})^{-(\nu-1)}.$$

First we estimate $I_1(r, t)$. Since $I_1 \equiv 0$ if $r - t \geq a$, we may assume $t - r \geq -a$. Notice that if $\lambda \geq |(t - r) - s|$ and $\lambda \leq a$, then $\langle s \rangle$ is equivalent to $\langle t - r \rangle$ and $\lambda - s + r + t \geq 2r$. Therefore from (3.7) we have

$$(3.11) \quad I_1(t, r) \lesssim \frac{\langle t - r \rangle^{-(\nu-(1/2))}}{\sqrt{2r}} \int_0^t ds \int_{|t-s-r|}^{t-s+r} \frac{\chi_{[0,a]}(\lambda)}{\sqrt{\lambda + s + r - t}} d\lambda.$$

To proceed further, we divide the argument into four cases.

1. $|t - r| \leq a$ and $0 < r \leq a$: It follows that

$$I_1(t, r) \lesssim \frac{1}{\sqrt{r}} \int_0^t ds \int_{|t-s-r|}^{t-s+r} \frac{1}{\sqrt{\lambda + s + r - t}} d\lambda \lesssim t.$$

Since $t \leq |t - r| + r \leq 2a$ in this case, the above estimate yields (3.9).

2. $|t - r| \leq a$ and $r \geq a$: We change the order of the integration to obtain

$$\begin{aligned} \sqrt{r} I_1(t, r) &\lesssim \int_0^{(t-r)_+} d\lambda \int_{-\lambda+t-r}^{\lambda+t-r} \frac{1}{\sqrt{\lambda + s + r - t}} ds \\ &\quad + \int_{(t-r)_+}^a d\lambda \int_0^{\lambda+t-r} \frac{1}{\sqrt{\lambda + s + r - t}} ds \lesssim 1. \end{aligned}$$

Since $4r \geq 2r + a + (t - r) = r + t + a$ in this case, this estimate implies (3.9).

3. $t - r \geq a$ and $0 < r \leq a$: It follows from (3.11) that

$$\begin{aligned} \langle t - r \rangle^{\nu-(1/2)} I_1(t, r) &\lesssim \frac{1}{\sqrt{r}} \int_0^r d\lambda \int_{-\lambda+t-r}^{\lambda+t-r} \frac{1}{\sqrt{\lambda + s + r - t}} ds \\ &\quad + \frac{1}{\sqrt{r}} \int_r^a d\lambda \int_{-\lambda+t-r}^{-\lambda+t+r} \frac{1}{\sqrt{\lambda + s + r - t}} ds \lesssim 1. \end{aligned}$$

Since $t \geq r + a \geq 2r$ in this case, $\langle t - r \rangle$ is equivalent to $\langle t + r \rangle$, so that the above estimate yields (3.9).

4. $t - r \geq a$ and $r \geq a$: From (3.11) we have

$$\langle t - r \rangle^{\nu-(1/2)} \langle r \rangle^{1/2} I_1(t, r) \lesssim \int_0^a d\lambda \int_{-\lambda+t-r}^{\lambda+t-r} \frac{1}{\sqrt{\lambda + s + r - t}} ds \lesssim 1,$$

which yields (3.9). Indeed, if $t \geq 2r$, then $\langle t - r \rangle$ is equivalent to $\langle t + r \rangle$. On the other hand, if $t \leq 2r$, then $\langle r \rangle$ is equivalent to $\langle t + r \rangle$. Therefore, the above estimate is enough to conclude that the desired one holds.

Next we estimate $I_2(r, t)$. We may assume $t - r > 0$, and we divide the argument into three cases.

1. $0 < t - r \leq 4a$ and $0 < r \leq a$: It follows (3.8) that

$$I_2(t, r) \lesssim \int_0^{t-r} ds \int_0^{t-s-r} \frac{1}{\sqrt{t-r-\lambda-s}\sqrt{\lambda}} d\lambda \lesssim a\pi.$$

Thus we get (3.10).

2. $0 < t - r \leq 4a$ and $r \geq a$: Changing the order of the integration, we get

$$I_2(t, r) \lesssim \int_0^{t-r} d\lambda \int_0^{-\lambda+t-r} \frac{1}{\sqrt{t-r-\lambda-s}\sqrt{t+r+\lambda-s}} ds.$$

Since $t+r+\lambda-s \geq 2r+2\lambda \geq 2r$ for $\lambda \geq 0$ and $s \leq -\lambda+t-r$, we obtain

$$I_2(t, r) \lesssim \frac{(t-r)^{3/2}}{\sqrt{2r}} \lesssim \frac{1}{\sqrt{r}},$$

which implies (3.10).

3. $t - r \geq 4a$: It follows from (3.8) that

$$(3.12) \quad \begin{aligned} I_2(t, r) &\lesssim \int_0^a d\lambda \int_{-\lambda+2^{-1}(t-r)}^{-\lambda+t-r} \frac{\langle s \rangle^{-(\nu-(1/2))}}{\sqrt{t-r-\lambda-s}\sqrt{t+r+\lambda-s}} ds \\ &\quad + \int_0^a d\lambda \int_0^{-\lambda+2^{-1}(t-r)} \frac{\langle s \rangle^{-(\nu-(1/2))}}{\sqrt{t-r-\lambda-s}\sqrt{t+r+\lambda-s}} ds \\ &\equiv I_{2,1}(t, r) + I_{2,2}(t, r). \end{aligned}$$

First we shall show

$$(3.13) \quad I_{2,1}(t, r) \lesssim \langle t+r \rangle^{-1/2} (\min\{\langle r \rangle, \langle t-r \rangle\})^{-(\nu-1)}.$$

Since $-\lambda+2^{-1}(t-r) \geq 4^{-1}(t-r)$ for $\lambda \leq a$ and $t-r \geq 4a$, we have

$$\langle t-r \rangle^{\nu-(1/2)} I_{2,1}(t, r) \lesssim \int_0^a d\lambda \int_{-\lambda+2^{-1}(t-r)}^{-\lambda+t-r} \frac{1}{\sqrt{t-r-\lambda-s}\sqrt{t+r+\lambda-s}} ds.$$

When $t \leq 2r$ and $r \geq a$, we have only show

$$(3.14) \quad I_{2,1}(t, r) \lesssim r^{-1/2} \langle t-r \rangle^{-(\nu-1)}.$$

Since $t+r+\lambda-s \geq 2r+2\lambda \geq 2r$ for $s \leq -\lambda+t-r$ and $\lambda \geq 0$, we have

$$\begin{aligned} \langle t-r \rangle^{\nu-(1/2)} I_{2,1}(t, r) &\lesssim \frac{1}{\sqrt{2r}} \int_0^a d\lambda \int_{-\lambda+2^{-1}(t-r)}^{-\lambda+t-r} \frac{1}{\sqrt{t-r-\lambda-s}} ds \\ &\lesssim r^{-1/2} (t-r)^{1/2}, \end{aligned}$$

which implies (3.14). When $t \geq 2r$ and $r \geq a$, we have only show

$$(3.15) \quad I_{2,1}(t, r) \lesssim r^{-(\nu-1)} \langle t-r \rangle^{-1/2}.$$

For simplicity, we put $\theta = \nu - 1$. Since $1 < \nu < 3/2$, we have $0 < \theta < 1/2$. It follows that

$$\langle t-r \rangle^{\nu-(1/2)} I_{2,1}(t, r) \lesssim \frac{1}{(2r)^\theta} \int_0^a J(t, r, \lambda) d\lambda,$$

where we set

$$J(t, r, \lambda) = \int_{-\lambda+2^{-1}(t-r)}^{-\lambda+t-r} \frac{1}{\sqrt{t-r-\lambda-s} (t+r+\lambda-s)^{(1/2)-\theta}} ds.$$

For $\lambda \geq 0$ we have

$$(3.16) \quad J(t, r, \lambda) \lesssim \int_{-\lambda+2^{-1}(t-r)}^{-\lambda+t-r} (t-r-\lambda-s)^{-1+\theta} ds \lesssim (t-r)^\theta.$$

Thus we get (3.15). When $0 < r \leq a$, we have only show

$$(3.17) \quad I_{2,1}(t, r) \lesssim \langle t-r \rangle^{-1/2}.$$

By (3.16) we have

$$\langle t-r \rangle^{\nu-(1/2)} I_{2,1}(t, r) \lesssim \int_0^a \frac{1}{(2\lambda)^\theta} J(t, r, \lambda) d\lambda \lesssim (t-r)^\theta,$$

which yields (3.17). From (3.14), (3.15), and (3.17), we obtain (3.13).

Next we consider $I_{2,2}$. Since $t-r-\lambda-s \geq (t-r)/2$ and $t+r+\lambda-s \geq (t+r)/2$ for $s \leq -\lambda+2^{-1}(t-r)$ and $\lambda \geq 0$, we get

$$\begin{aligned} I_{2,2}(t, r) &\lesssim \frac{1}{\sqrt{t+r}\sqrt{t-r}} \int_0^a d\lambda \int_0^{2^{-1}(t-r)} \langle s \rangle^{-(\nu-(1/2))} ds \\ &\lesssim \frac{\langle t-r \rangle^{-\nu+(3/2)}}{\sqrt{t+r}\sqrt{t-r}} \lesssim \langle t+r \rangle^{-1/2} \langle t-r \rangle^{-(\nu-1)}. \end{aligned}$$

Combining this estimate with (3.13), we find (3.10).

Thus the proof of (3.2) is complete. \square

4. ON THE EXTERIOR PROBLEM FOR THE WAVE EQUATION

As an application of Theorem 3.1, we examine the pointwise decay estimate for the exterior problem in this section.

Let Ω be an unbounded domain in \mathbf{R}^2 with compact and smooth boundary $\partial\Omega$. We put $\mathcal{O} := \mathbf{R}^2 \setminus \Omega$, which is called an obstacle and is assumed to be non-empty. Throughout this section we shall assume $0 \in \mathcal{O}$ so that we have $|x| \geq c_0$ for $x \in \Omega$ with some positive constant c_0 . We shall also assume that $\overline{\mathcal{O}} \subset B_1$. Thus a function $v = v(x)$ on Ω vanishing for $|x| \leq 1$ can be naturally regarded as a function on \mathbf{R}^2 .

Given $T > 0$, we consider the mixed problem for the wave equation :

$$(4.1) \quad (\partial_t^2 - \Delta_x)u = f(t, x), \quad (t, x) \in (0, T) \times \Omega,$$

$$(4.2) \quad u(t, x) = 0, \quad (t, x) \in (0, T) \times \partial\Omega,$$

$$(4.3) \quad u(0, x) = u_0(x), \quad (\partial_t u)(0, x) = u_1(x), \quad x \in \Omega.$$

We assume $u_0, u_1 \in C_0^\infty(\overline{\Omega})$, namely, they are smooth functions on $\overline{\Omega}$ vanishing outside some ball. On the other hand, $f \in C^\infty([0, T] \times \overline{\Omega})$ is assumed to satisfy $f(t, \cdot) \in C_0^\infty(\overline{\Omega})$ for any fixed $t \in [0, T]$. In order to obtain a smooth solution to (4.1)-(4.3), we need the compatibility condition to infinite order, i.e., $u_j(x)$ vanishes on $\partial\Omega$ for any non-negative integer j , where $u_j(x)$ is determined by

$$(4.4) \quad u_j(x) = \Delta u_{j-2}(x) + (\partial_t^{j-2} f)(0, x), \quad j \geq 2.$$

We denote by $X(T)$ the set of all such data $\Xi = (u_0, u_1, f) = (\vec{u}_0, f)$. For Ξ we denote by $S[\Xi](t, x)$ the solution of the mixed problem (4.1)-(4.3). Besides, we set $K[\vec{u}_0] = S[(\vec{u}_0, 0)]$ and $L[f] = S[(0, 0, f)]$.

Then we have the following decay estimates.

Theorem 4.1. *Let $1 < \nu < 3/2$, $\kappa > 1$, and k be a non-negative integer. Assume that \mathcal{O} is star-shaped, and $\Xi = (\vec{u}_0, f) \in X(T)$. Then we have*

$$(4.5) \quad \begin{aligned} & \langle t + |x| \rangle^{1/2} (\min\{\langle x \rangle, \langle t - |x| \rangle\})^{\nu-1} |S[\Xi](t, x)|_k \\ & \lesssim \mathcal{A}_{\nu+(1/2), k+3}[\vec{u}_0] + \sum_{|\delta| \leq k+3} \| |y|^{1/2} W_{\nu, \kappa}(s, y) \Gamma^\delta f(s, y) \|_{L_t^\infty L_\Omega^\infty} \end{aligned}$$

for $(t, x) \in [0, T) \times \bar{\Omega}$. Here we put

$$(4.6) \quad \begin{aligned} W_{\nu, \kappa}(s, y) &= \langle s + |y| \rangle^\nu (\min\{\langle |y| \rangle, \langle s - |y| \rangle\})^\kappa, \\ \mathcal{A}_{\rho, k}[u_0, u_1] &= \sum_{|\gamma| \leq k+1} \| \langle \cdot \rangle^\rho \tilde{\Gamma}^\gamma u_0 \|_{L_\Omega^\infty} + \sum_{|\gamma| \leq k} \| \langle \cdot \rangle^\rho \tilde{\Gamma}^\gamma u_1 \|_{L_\Omega^\infty}. \end{aligned}$$

Our proof of the theorem is based on the cut-off method used in Shibata and Tsutsumi [19], where the nonlinear problem was handled when the space dimension is greater than 3 (see also [2, 3], [4], [6], [8, 9], [12], [16], [17], [18], and the references cited therein). Because the decaying rate of the solution to the wave equation is weaker and weaker as the space dimension is lower and lower, in our previous works [7, 13, 14] which treat the two-dimensional case, we could not obtain such a decay estimate of the solution itself as above. But, thanks to (3.2), we succeed to derive the same decay estimate as in the boundaryless case. The estimate (4.5) will be used in the forthcoming papers concerning the nonlinear wave equation in the exterior domain.

4.1. Decomposition of solutions. For $a > 0$, we denote by ψ_a a smooth radially symmetric function on \mathbf{R}^2 satisfying

$$(4.7) \quad \begin{cases} \psi_a(x) = 0, & |x| \leq a, \\ \psi_a(x) = 1, & |x| \geq a + 1. \end{cases}$$

Besides, for $a \geq 1$ we set

$$\Omega_a = \Omega \cap B_a.$$

Since $\bar{\mathcal{O}} \subset B_1$, we see that $\Omega_a \neq \emptyset$ for any $a \geq 1$. Then we have the following decomposition (for the proof, we refer to [19], [12, Lemma 3.1]).

Lemma 4.2. *Fix $a \geq 1$. Let $\Xi = (u_0, u_1, f) \in X(T)$. Assume that*

$$\text{supp } f(t, \cdot) \subset \overline{\Omega_{t+a}} \quad \text{and} \quad \text{supp } u_0 \subset \overline{\Omega_a}, \quad \text{supp } u_1 \subset \overline{\Omega_a}$$

holds for any $t \in (0, T)$. Then we have

$$(4.8) \quad S[\Xi](t, x) = \psi_a(x) S_0[\psi_{2a}\Xi](t, x) + \sum_{i=1}^4 S_i[\Xi](t, x),$$

where

$$(4.9) \quad S_1[\Xi](t, x) = (1 - \psi_{2a}(x))L[\psi_a, -\Delta]S_0[\psi_{2a}\Xi](t, x),$$

$$(4.10) \quad S_2[\Xi](t, x) = -L_0[\psi_{2a}, -\Delta]L[\psi_a, -\Delta]S_0[\psi_{2a}\Xi](t, x),$$

$$(4.11) \quad S_3[\Xi](t, x) = (1 - \psi_{3a}(x))S[(1 - \psi_{2a})\Xi](t, x),$$

$$(4.12) \quad S_4[\Xi](t, x) = -L_0[\psi_{3a}, -\Delta]S[(1 - \psi_{2a})\Xi](t, x).$$

4.2. Proof of Theorem 4.1. For $a > 1$, we denote by $X_a(T)$ the set of all $(u_0, u_1, f) \in X(T)$ satisfying

$$u_0(x) = u_1(x) = f(t, x) \equiv 0 \text{ for } |x| \geq a \text{ and } t \in [0, T].$$

It is useful to prepare the following lemma for the proof of Theorem 4.1.

Lemma 4.3. *Let \mathcal{O} be star-shaped. Let $a, b > 1$, $0 < \rho \leq 1$, $1 < \nu < 3/2$, $\kappa > 1$, and m be a non-negative integer.*

(i) *Let χ be a smooth function on \mathbf{R}^2 satisfying $\text{supp } \chi \subset B_b$. If $\Xi = (\vec{u}_0, f) \in X_a(T)$, then we have*

$$(4.13) \quad \begin{aligned} & \langle t \rangle^\rho |\chi S[\Xi](t, x)|_m \\ & \lesssim \mathcal{A}_{0, m+1}[\vec{u}_0] + \sum_{|\beta| \leq m+1} \|\langle s \rangle^\rho \partial^\beta f(s, y)\|_{L_t^\infty L^\infty(\Omega_a)} \end{aligned}$$

for $(t, x) \in [0, T] \times \overline{\Omega}$.

(ii) *Let g be a smooth function on $[0, T] \times \mathbf{R}^2$ and satisfy $\text{supp } g(t, \cdot) \subset \overline{B_a} \setminus B_1$ for any $t \in [0, T]$. Then we have*

$$(4.14) \quad \begin{aligned} & \langle t + |x| \rangle^{1/2} (\min\{\langle |x| \rangle, \langle t - |x| \rangle\})^{\nu-1} |L_0[g](t, x)|_m \\ & \lesssim \sum_{|\beta| \leq m} \|\langle s \rangle^{\nu-(1/2)} \partial^\beta g(s, y)\|_{L_t^\infty L^\infty(\Omega_a)} \end{aligned}$$

for $(t, x) \in [0, T] \times \overline{\Omega}$.

(iii) *Let \vec{v}_0 and g be smooth functions on \mathbf{R}^2 and on $[0, T] \times \mathbf{R}^2$, respectively. If $\vec{v}_0(x) = g(t, x) = 0$ for any $(t, x) \in [0, T] \times B_1$, then*

$$(4.15) \quad \begin{aligned} & \langle t \rangle^{\nu-(1/2)} |S_0[(\vec{v}_0, g)](t, x)|_m \\ & \lesssim \mathcal{A}_{\nu+(1/2), m}[\vec{v}_0] + \sum_{|\beta| \leq m} \|\langle y \rangle^{1/2} W_{\nu, \kappa}(s, y) \Gamma^\beta g(s, y)\|_{L_t^\infty L_\infty^\infty} \end{aligned}$$

for $(t, x) \in [0, T] \times \overline{\Omega_b}$.

Proof. One can find the proof of (4.13) in [13, Lemma 4.1]. Since $\text{supp } g(t, \cdot) \subset \overline{B_a} \setminus B_1 \subset \overline{\Omega_a}$, we get (4.14) from (3.2).

Finally we prove (4.15) by using (2.3) and (2.4) with (4.6). It follows that

$$\begin{aligned} & \langle t + |x| \rangle^{1/2} \langle t - |x| \rangle^{\nu-1} |S_0[\vec{v}_0, g](t, x)|_m \\ & \lesssim \mathcal{B}_{\nu+(1/2), m}[\vec{v}_0] + \sum_{|\beta| \leq m} \|\langle y \rangle^{1/2} W_{\nu, \kappa}(s, y) \Gamma^\beta g(s, y)\|_{L_t^\infty L^\infty} \end{aligned}$$

for $(t, x) \in [0, T] \times \mathbf{R}^2$. Observe that $\langle t - |x| \rangle$ is equivalent to $\langle t \rangle$ when $x \in \overline{\Omega}_b$. Therefore we get (4.15). This completes the proof. \square

Now we are in a position to prove Theorem 4.1.

Proof of Theorem 4.1. According to Lemma 4.2 with $a = 1$, we can write

$$(4.16) \quad S[\Xi](t, x) = \psi_1(x) S_0[\psi_2 \Xi](t, x) + \sum_{i=1}^4 S_i[\Xi](t, x)$$

for $(t, x) \in [0, T] \times \overline{\Omega}$, where ψ_a is defined by (4.7) and $S_i[\Xi]$ for $1 \leq i \leq 4$ are defined by (4.9)–(4.12) with $a = 1$. It is easy to check that

$$(4.17) \quad [\psi_a, -\Delta]h(t, x) = h(t, x)\Delta\psi_a(x) + 2\nabla_x h(t, x) \cdot \nabla_x \psi_a(x)$$

for $(t, x) \in [0, T] \times \overline{\Omega}$, $a \geq 1$ and any smooth function h . Note that this identity implies

$$(4.18) \quad (0, 0, [\psi_a, -\Delta]h) \in X_{a+1}(T)$$

because $\text{supp } \nabla_x \psi_a \cup \text{supp } \Delta \psi_a \subset \overline{B_{a+1}} \setminus B_a$.

Let $1/2 < \nu < 3/2$ and $\kappa > 1$ in the following. Applying (2.3) and (2.4), we have

$$\begin{aligned} & \langle t + |x| \rangle^{1/2} \langle t - |x| \rangle^{\nu-1} |S_0[\psi_2 \Xi](t, x)|_k \\ & \lesssim \mathcal{B}_{\nu+(1/2), k}[\psi_2 \vec{u}_0] + \sum_{|\delta| \leq k} \|\langle y \rangle^{1/2} W_{\nu, \kappa}(s, y) \Gamma^\delta(\psi_2 f)(s, y)\|_{L_t^\infty L^\infty} \\ & \lesssim \mathcal{A}_{\nu+(1/2), k}[\vec{u}_0] + \sum_{|\delta| \leq k} \|\langle y \rangle^{1/2} W_{\nu, \kappa}(s, y) \Gamma^\delta f(s, y)\|_{L_t^\infty L_\Omega^\infty}, \end{aligned}$$

so that

$$(4.19) \quad \begin{aligned} & \langle t + |x| \rangle^{1/2} \langle t - |x| \rangle^{\nu-1} |\psi_1(x) S_0[\psi_2 \Xi](t, x)|_k \\ & \lesssim \mathcal{A}_{\nu+(1/2), k}[\vec{u}_0] + \sum_{|\delta| \leq k} \|\langle y \rangle^{1/2} W_{\nu, \kappa}(s, y) \Gamma^\delta f(s, y)\|_{L_t^\infty L_\Omega^\infty}. \end{aligned}$$

Next we estimate $S_1[\Xi]$ and $S_3[\Xi]$. Using (4.13) and then (4.17), we get

$$\begin{aligned} & \langle t \rangle^{\nu-(1/2)} |S_1[\Xi](t, x)|_k \\ & \lesssim \sum_{|\beta| \leq k+1} \|\langle s \rangle^{\nu-(1/2)} \partial^\beta ([\psi_1, -\Delta] S_0[\psi_2 \Xi])(s, y)\|_{L_t^\infty L^\infty(\Omega_2)} \\ & \lesssim \sum_{|\beta| \leq k+2} \|\langle s \rangle^{\nu-(1/2)} \partial^\beta S_0[\psi_2 \Xi](s, y)\|_{L_t^\infty L^\infty(\Omega_2)}. \end{aligned}$$

Since $(\psi_2 \Xi)(t, x) = 0$ if $|x| \leq 2$, we can use (4.15) to obtain

$$(4.20) \quad \begin{aligned} & \langle t \rangle^{\nu-(1/2)} |S_1[\Xi](t, x)|_k \\ & \lesssim \mathcal{A}_{\nu+(1/2), k+2}[\psi_2 \vec{u}_0] + \sum_{|\beta| \leq k+2} \| |y|^{1/2} W_{\nu, \kappa}(s, y) \Gamma^\beta f(s, y) \|_{L_t^\infty L_\Omega^\infty} \end{aligned}$$

for $(t, x) \in [0, T) \times \overline{\Omega}$.

Because $(1 - \psi_2)\Xi \in X_3(T)$ for $\Xi \in X(T)$, it follows from (4.11) and (4.13) that

$$(4.21) \quad \begin{aligned} & \langle t \rangle^{\nu-(1/2)} |S_3[\Xi](t, x)|_k \\ & \lesssim \mathcal{A}_{0, k+1}[\vec{u}_0] + \sum_{|\beta| \leq k+1} \| \langle s \rangle^{\nu-(1/2)} \partial^\beta f(s, y) \|_{L_t^\infty L^\infty(\Omega_3)} \end{aligned}$$

for $(t, x) \in [0, T) \times \overline{\Omega}$.

By using the trivial inequality $\langle s \rangle^{\nu-(1/2)} \lesssim |y|^{1/2} W_{\nu, \kappa}(s, y)$ in $[0, T) \times \Omega_3$, from (4.20) and (4.21) we see that

$$(4.22) \quad \begin{aligned} & \langle t \rangle^{\nu-(1/2)} (|S_1[\Xi](t, x)|_k + |S_3[\Xi](t, x)|_k) \\ & \lesssim \mathcal{A}_{\nu+(1/2), k+2}[\vec{u}_0] + \sum_{|\beta| \leq k+2} \| |y|^{1/2} W_{\nu, \kappa}(s, y) \Gamma^\beta f(s, y) \|_{L_t^\infty L_\Omega^\infty}. \end{aligned}$$

Moreover, noting that $S_1[\Xi]$ and $S_3[\Xi]$ are supported on $\overline{\Omega_4}$, we can replace the weight on the left hand side as follows:

$$(4.23) \quad \begin{aligned} & \langle t + |x| \rangle^{1/2} \langle t - |x| \rangle^{\nu-1} (|S_1[\Xi](t, x)|_k + |S_3[\Xi](t, x)|_k) \\ & \lesssim \mathcal{A}_{\nu+(1/2), k+2}[\vec{u}_0] + \sum_{|\beta| \leq k+2} \| |y|^{1/2} W_{\nu, \kappa}(s, y) \Gamma^\beta f(s, y) \|_{L_t^\infty L_\Omega^\infty}. \end{aligned}$$

Finally we consider $S_2[\Xi]$ and $S_4[\Xi]$. Setting $g_j[\Xi] = (\partial_t^2 - \Delta)S_j[\Xi]$ for $j = 2, 4$, and recalling (4.10), (4.12), and the definition of L_0 , we have

$$\begin{aligned} g_2[\Xi] &= -[\psi_2, -\Delta]L[\psi_1, -\Delta]S_0[\psi_2 \Xi], \\ g_4[\Xi] &= -[\psi_3, -\Delta]S[(1 - \psi_2)\Xi]. \end{aligned}$$

Having in mind (4.17) we can say that g_2 and g_4 have the same structures as S_1 and S_3 , but they contain one more derivative. Therefore, arguing similarly to the derivation of (4.22), we arrive at

$$(4.24) \quad \begin{aligned} & \langle t \rangle^{\nu-(1/2)} (|g_2[\Xi](t, x)|_k + |g_4[\Xi](t, x)|_k) \\ & \lesssim \mathcal{A}_{\nu+(1/2), k+3}[\vec{u}_0] + \sum_{|\beta| \leq k+3} \| |y|^{1/2} W_{\nu, \kappa}(s, y) \Gamma^\beta f(s, y) \|_{L_t^\infty L_\Omega^\infty}. \end{aligned}$$

Since $S_j[\Xi] = L_0[g_j]$ for $j = 2, 4$, and g_2, g_4 are supported on $\overline{B_4} \setminus \overline{B_2}$, we are in a position to apply (4.14) and we get

$$(4.25) \quad \begin{aligned} & \langle t + |x| \rangle^{1/2} (\min\{\langle |x| \rangle, \langle t - |x| \rangle\})^{\nu-1} (|S_2[\Xi](t, x)| + |S_4[\Xi](t, x)|) \\ & \lesssim \mathcal{A}_{\nu+(1/2), k+3}[\vec{u}_0] + \sum_{|\beta| \leq k+3} \| |y|^{1/2} W_{\nu, \kappa}(s, y) \Gamma^\beta f(s, y) \|_{L_t^\infty L_\Omega^\infty}. \end{aligned}$$

Now (4.5) follows from (4.19), (4.23) and (4.25). This completes the proof. \square

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