L^1 MAXIMAL REGULARITY AND LOCAL EXISTENCE OF A SOLUTION TO THE COMPRESSIBLE NAVIER-STOKES-POISSON SYSTEM IN A CRITICAL BESOV SPACE

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1. Introduction

We consider the Cauchy problem of the compressible Navier-Stokes-Poisson system in $\mathbb{R}^n$ with $n \geq 2$.

$$
\begin{cases}
\partial_t \rho + \mathrm{div}\ (\rho u) = 0, & (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n, \\
\partial_t(\rho u) + \mathrm{div}\ (\rho u \otimes u) + \nabla P(\rho) & \\
\qquad = \mu \Delta u + (\mu + \lambda) \nabla \mathrm{div}\ u + \kappa \rho \nabla \psi, & (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n, \\
-\Delta \psi = \rho - \bar{\rho}, & (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n, \\
\qquad u(0,x) = u_0(x),\ \rho(0,x) = \rho_0(x), & x \in \mathbb{R}^n,
\end{cases}
\tag{1.1}
$$

where $\rho = \rho(t,x)$, $u = u(t,x)$ and $\psi = \psi(t,x)$ are the unknown fluid density, the velocity vector and the potential force, respectively. $P = P(\rho)$ is the pressure given by ρ, and $u \otimes u$ denotes the tensor product of velocity vector u. μ, λ are Lamé constants satisfying $\mu + 2\lambda > 0$, $\kappa = \pm$ is a coupling constant and $\bar{\rho}$ is a given background density. Without losing generality, we assume that $\bar{\rho} = 1$. The system is strongly relevant to the simplified system of degenerate drift-diffusion equations and the Smoluchowski-Poisson system appeared in a semiconductor divise models (cf. [16]).

Introducing the perturbed density by $a(t,x) \equiv \rho(t,x) - 1$ with $a_0(x) \equiv \rho_0(x) - 1$, the problem (1.1) is reduced into the following problem of (a,u):

$$
\begin{cases}
\partial_t a + u \cdot \nabla a = -(1+a) \mathrm{div}\ u, \\
\partial_t u - \mathcal{L} u + \nabla(-\Delta)^{-1} a = -\dfrac{a}{1+a} \mathcal{L} u - u \cdot \nabla u - \nabla(Q(a)), \\
u(0,x) = u_0,\ a(0,x) = a_0.
\end{cases}
\tag{1.2}
$$

Here, we denote the elliptic operator $\mathcal{L}$ by $\mathcal{L} = \mu\Delta + (\lambda + \mu)\nabla \mathrm{div}$ and Q is a smooth function determined by P by

$$
Q(a) := -\int_0^t \frac{P'((1+z)^{-1})}{(1+z)^2} dz.
$$

Nash [20] considered the local well-posedness of the compressible Navier-Stokes system for smooth data away from a vacuum. Itaya [15] also obtained the existence and uniqueness of the system assuming sufficient smoothness to the data. Matsumura-Nishida [19] proved the existence of global classical solution provided the initial data with high regularity is close to the equilibrium state.

We recall that the compressible Navier-Stokes system (1.1) has a scaling invariance: For $\nu > 0$,

$$\begin{cases} \rho_\nu(t,x) = \rho(\nu^2 t, \nu x), \\ u_\nu(t,x) = \nu u(\nu^2 t, \nu x) \end{cases} \tag{1.3}$$

provided the pressure term has been changed accordingly. Extending the classical idea initiated by Fujita-Kato [11] applied to the incompressible Navier-Stokes system, Danchin [5], [8] considered the local existence and the uniqueness of the solution for the problem in the *"scaling-critical"* homogeneous Besov space. Haspot [14] improved Danchin's result [8] to the general Besov space and a larger space for the density by introducing an effective velocity. In order to consider the critical solvability, we necessarily introduce the homogeneous Besov spaces. Since the system (1.1) involves the hyperbolic equation for the density equation, it is required to consider the equation in the suitable space, where the supremum of the density has to be controlled. To this end, Danchin introduced the homogeneous Besov space that embedded into $L^\infty(\mathbb{R}^n)$.

Let $\{\phi_j\}_{j\in\mathbb{Z}}$ be the Littlewood-Paley dyadic decomposition of unity satisfying that

$$\sum_{j\in\mathbb{Z}} \hat{\phi}_j(\xi) = 1$$

for all $\xi \neq 0$ and supp $\hat{\phi}_j \subset \{\xi \in \mathbb{R}^n | \ 2^{j-1} < |\xi| < 2^{j+1}\}$. For $s \in \mathbb{R}$ and $1 \le p, \sigma \le \infty$, we define the homogeneous Besov space $\dot{B}^s_{p,\sigma}(\mathbb{R}^n)$ by

$$\dot{B}^s_{p,\sigma}(\mathbb{R}^n) = \{f \in \mathcal{S}^*/\mathcal{P}; \|f\|_{\dot{B}^s_{p,\sigma}} < \infty\}$$

with the norm

$$\|f\|_{\dot{B}^s_{p,\sigma}} \equiv \begin{cases} \Big(\sum_{j\in\mathbb{Z}} 2^{js\sigma}\|\phi_j * f\|_p^\sigma\Big)^{1/\sigma}, 1 \le \sigma < \infty, \\ \sup_{j\in\mathbb{Z}} 2^{js}\|\phi_j * f\|_p, \ \sigma = \infty, \end{cases} \tag{1.4}$$

where $\mathcal{P}$ denotes polynomials (see Triebel [23] for details).

Since our system involves the Poisson term and it brings our problem disturbing the low frequency in the Fourier spaces. Namely, the inverse operator of the Laplacian gives a stronger restriction for the low frequency part of the solution (cf. Yukawa potential case [2]). To handle with low frequency part, we also introduce another homogeneous Besov space of hybrid type; $\dot{B}^{s_*}_{p,\sigma} \oplus \dot{B}^{s^*}_{p,\sigma}$ by

$$\|f\|_{\dot{B}^{s_*}_{p,\sigma}\oplus\dot{B}^{s^*}_{p,\sigma}} \equiv \left(\sum_{j\le 0} 2^{\sigma s_* j}\|\phi_j * f\|_p^\sigma + \sum_{j>0} 2^{\sigma s^* j}\|\phi_j * f\|_p^\sigma\right)^{1/\sigma}$$

for all $1 \le p, \sigma \le \infty$ and s_*, $s^* \in \mathbb{R}$. We note that if $s_* < s^*$, then it holds that

$$\dot{B}^{s_*}_{p,\sigma} \oplus \dot{B}^{s^*}_{p,\sigma} = \dot{B}^{s_*}_{p,\sigma} \cap \dot{B}^{s^*}_{p,\sigma}$$

and hereafter we only use this setting. We define the *critical* inhomogeneous space as follows:

$$a \in L^\infty(0,T; \dot{B}^{\frac{N}{p}}_{p,1}), \ u \in L^\infty(0,T; \dot{B}^{\frac{N}{p}-1}_{p,1}), \ f \in L^1_{loc}(\mathbb{R}^+; \dot{B}^{\frac{N}{p}-1}_{p,1}).$$

Zheng [24] used the linearized formulation to (1.1) and solve the system by the way of integral equations. The key idea is to consider the Poisson term as the linear term and he introduced the semi-group

$$\frac{d}{dt}\begin{pmatrix} a \\ u \end{pmatrix} = \begin{pmatrix} 0 & \mathrm{div} \\ -|\nabla| - \nabla(-\Delta)^{-1} & \mathcal{L} \end{pmatrix}\begin{pmatrix} a \\ u \end{pmatrix}. \tag{1.5}$$

Establishing the L^p-L^q type estimate of the semi-group generated by the above operator he constructed a global solution for small data to (1.1) in the critical Besov space $\rho_0 - 1 \in \dot{B}^{\frac{n}{2}-2}_{2,1} \oplus \dot{B}^{\frac{n}{p}}_{p,1}$, $u_0 \in \dot{B}^{\frac{n}{2}-2}_{2,1} \oplus \dot{B}^{\frac{n}{p}-1}_{p,1}$. However the critical case $p = n$ was not treated, since the product formula necessarily required in the case $p = n$ such as

$$\|fg\|_{\dot{B}^{n/p-1}_{p,1}} \le C\|f\|_{\dot{B}^{n/p-1}_{p,1}}\|g\|_{\dot{B}^{n/p+1}_{p,1}}$$

failes in general.

We now recover the local existence result in the critical hybrid Besov spaces.

Theorem 1.1 ([3])**.** *Let $n = 3$, $1 < p \le 3$. $\mu > 0$ with $\mu + 2\lambda > 0$. For any $\rho_0 - 1 \in \dot{B}^{\frac{n}{p}-2}_{p,1}(\mathbb{R}^3) \oplus \dot{B}^{\frac{n}{p}}_{p,1}(\mathbb{R}^3)$, $u_0 \in \dot{B}^{\frac{n}{p}-2}_{p,1}(\mathbb{R}^3) \oplus \dot{B}^{\frac{n}{p}-1}_{p,1}(\mathbb{R}^3)$. Then there exists a weak solution to* (1.1) *such that for some $T > 0$ with $I = [0, T)$, (ρ, u, ψ): a solution of* (1.1) *satisfying*

$$\begin{aligned} \rho - 1 &\in C\big(I; \dot{B}^{\frac{n}{p}-1}_{p,1} \oplus \dot{B}^{\frac{n}{p}}_{p,1}\big), \\ u &\in \big(C(I; \dot{B}^{\frac{n}{p}-2}_{p,1} \oplus \dot{B}^{\frac{n}{p}-1}_{p,1}) \cap L^1(I; \dot{B}^{\frac{n}{p}+1}_{p,1})\big)^N, \\ \psi &\in C\big(I; \dot{B}^{\frac{n}{p}+1}_{p,1} \oplus \dot{B}^{\frac{n}{p}+2}_{p,1}\big). \end{aligned} \tag{1.6}$$

2. Key estimates

2.1. The mass conservation equation. To prove the case $p = n = 3$, we employ the following proposition. Let (a, u) solves the following equation.

$$\begin{cases} \partial_t a + u \cdot \nabla a = -(1 + a)\mathrm{div}\ u, & (t, x) \in I \times \mathbb{R}^n, \\ a(0, x) = a_0(x), & x \in \mathbb{R}^N, \end{cases} \tag{2.1}$$

where $I = [0, T)$.

Proposition 2.1. *Let $a_0 \in \dot{B}^{-1}_{3,1}(\mathbb{R}^3)$, $u \in L^\infty(I; \dot{B}^0_{3,1}(\mathbb{R}^3)) \cap L^1(I; \dot{B}^2_{3,1}(\mathbb{R}^3))$ and $U(t) := \int_0^t \|\nabla u(\tau)\|_{\dot{B}^1_{3,1}} d\tau$. Suppose that $a \in L^\infty(I; \dot{B}^{-1}_{3,1}(\mathbb{R}^3) \cap \dot{B}^1_{3,1}(\mathbb{R}^3))$ solves the equation* (2.1)*. Then there exists a constant $C > 0$ depending on n and p such that the following inequality holds.*

$$\|a\|_{L^\infty_t(I;\dot{B}^{-1}_{3,1})} \le e^{U(t)}\left[\|a_0\|_{\dot{B}^{-1}_{3,1}} + C\int_0^t e^{-U(\tau)}(1 + \|a\|_{\dot{B}^1_{3,1}})\|u\|_{\dot{B}^0_{3,1}} d\tau\right], \tag{2.2}$$

for $t \in [0, T]$.

In view of the above estimate (2.2), it is required that the velocity field has to have maximal regularity in L^1 in time variable. This is the key point to show the main theorem.

2.2. **Maximal L^1 Regularity.** Let u_0 be the initial data, and a and h be given functions. The momentum governed by the following linearized parabolic equation.

$$\begin{cases} \partial_t u - \mu\Delta u - (\lambda+\mu)\nabla(\text{div } u)) = F, & (t,x) \in I \times \mathbb{R}^n, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^n. \end{cases} \tag{2.3}$$

It is known that the Cauchy problem for the heat equation (2.3) has maximal regularity in such non-reflexive function spaces. One of the general result can be seen in [21].

Proposition 2.2 ([21]). *Let $1 < \rho, \sigma \le \infty$ and $I = [0,T)$ be an interval with $T \le \infty$. For $f \in L^\rho(I; \dot{B}^0_{1,\rho}(\mathbb{R}^n))$ and $u_0 \in \dot{B}^{2(1-1/\rho)}_{1,\rho}(\mathbb{R}^n)$, let u be a solution of the Cauchy problem of the heat equation (2.3). Then we have*

$$\|\partial_t u\|_{L^\rho(I;\dot{B}^0_{1,\rho})} + \left\|\nabla^2 u\right\|_{L^\rho(I;\dot{B}^0_{1,\rho})} \le C(\|u_0\|_{\dot{B}^{2(1-1/\rho)}_{1,\rho}} + \|F\|_{L^\rho(I;\dot{B}^0_{1,\rho})}). \tag{2.4}$$

The above result does not cover the end-point exponent $\rho = 1$. In general, the end-point case $p = 1$ is eliminated in the abstract theory and we need to develop the each cases. Danchin [6] (see also Haspot [14]) obtained maximal regularity in the homogeneous Besov space for the case $\rho = 1$.

Theorem 2.3 ([8], [22]). *Let $1 \le p \le \infty$. For $F \in L^1(\mathbb{R}_+; \dot{B}^0_{p,1}(\mathbb{R}^n))$ and $u_0 \in \dot{B}^0_{p,1}(\mathbb{R}^n)$ there exists a unique solution u to (2.3) which satisfies the estimate:*

$$\begin{aligned} &\|\partial_t u\|_{L^1(\mathbb{R}_+;\dot{B}^0_{p,1})} + \left\|\nabla^2 u\right\|_{L^1(\mathbb{R}_+;\dot{B}^0_{p,1})} \\ &\qquad \le C\left(\|u_0\|_{\dot{B}^0_{p,1}} + \|F\|_{L^1(\mathbb{R}_+;\dot{B}^0_{p,1})}\right), \end{aligned} \tag{2.5}$$

where constant C is depending only on n. Moreover the estimate is optimal for the class of initial data. Namely if the data is $u_0 \in L^p(\mathbb{R}^N)$ or $\dot{F}^0_{p,1}(\mathbb{R}^N)$ the above estimate fails.

Remark 2.1. The upper estimate of (2.5) was obtained by Danchin- Mucha [9, Proposition 5] with $1 < p < \infty$. For $p = 1$, Danchin essentialy obtained the same estimate even for the variable coefficient case. Giga-Saal considered time L^1 maximal regularity in some space [12].

If we replace $u_0 \in \dot{B}^0_{p,1}(\mathbb{R}^n)$ into $u_0 \in \dot{B}^0_{p,\sigma}(\mathbb{R}^n)$ for $1 < \sigma \le \infty$, then maximal regularity *fails* since the lower bound by the initial data and the strict inclusion result for the subsufix σ as $\dot{B}^0_{p,1}(\mathbb{R}^n) \subsetneq \dot{B}^0_{p,\sigma}(\mathbb{R}^n)$.

To avoid the difficulty on using the limiting case of the bi-linear estimate in the homogeneous Besov spaces, we employ the following bi-linear estimate to treat the nonlinear term of the equation (2.1).

Lemma 2.4. *For $u \in \dot{B}^0_{3,1} \cap \dot{B}^2_{3,1}$ and $\tilde{a} \in \dot{B}^{-1}_{3,1} \cap \dot{B}^1_{3,1}$ it holds that*

$$\|\text{div } (\tilde{a}u)\|_{\dot{B}^{-1}_{3,1}} \le C\big(\|u\|_{\dot{B}^0_{3,1}}\|\tilde{a}\|_{\dot{B}^1_{3,1}} + \|u\|_{\dot{B}^2_{3,1}}\|\tilde{a}\|_{\dot{B}^{-1}_{3,1}}\big).$$

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