Global existence for semilinear wave equations with the blow-up term in high dimensions *

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1 General theory for nonlinear wave equations

First we shall outline the general theory on the initial value problem for fully nonlinear wave equations,

\[
\begin{aligned}
  u_{tt} - \Delta u &= H(u, Du, D_x Du) \quad \text{in } \mathbb{R}^n \times [0, \infty), \\
  u(x, 0) &= \varepsilon f(x), \quad u_t(x, 0) = \varepsilon g(x),
\end{aligned}
\]

where \( u = u(x, t) \) is a scalar unknown function of space-time variables,

\[
Du = (u_{x_0}, u_{x_1}, \cdots, u_{x_n}), \quad x_0 = t, \\
D_x Du = (u_{x_ix_j}; i, j = 0, 1, \cdots, n, \ i + j \geq 1),
\]

\( f, g \in C^\infty_0(\mathbb{R}^n) \) and \( \varepsilon > 0 \) is a “small” parameter. We note that it is impossible to construct a general theory for “large” \( \varepsilon \) due to blow-up results. For example, see Glassey [4], Levine [7], or Sideris [17]. Let

\[\hat{\lambda} = (\lambda; (\lambda_i), i = 0, 1, \cdots, n; (\lambda_{ij}), i, j = 0, 1, \cdots, n, \ i + j \geq 1)\].

Suppose that the nonlinear term \( H = H(\hat{\lambda}) \) is a sufficiently smooth function with

\[H(\hat{\lambda}) = O(|\hat{\lambda}|^{1+\alpha})\]

in a neighborhood of \( \hat{\lambda} = 0 \), where \( \alpha \geq 1 \) is an integer. Let us define the lifespan \( \tilde{T}(\varepsilon) \) of classical solutions of (1) by

\[\tilde{T}(\varepsilon) = \sup\{t > 0 : \exists \text{ classical solution } u(x, t) \text{ of (1)} \text{ for arbitrarily fixed data, } (f, g).\}\].

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When $\widetilde{T}(\varepsilon) = \infty$, the problem (1) admits a global in time solution, while we only have a local in time solution on $t \in [0, \widetilde{T}(\varepsilon))$ when $\widetilde{T}(\varepsilon) < \infty$. For local in time solutions, one can measure the long time stability of a zero solution by orders of $\varepsilon$. Because the uniqueness of the solution of (1) may yield that $\lim_{\varepsilon \to 0^+} \widetilde{T}(\varepsilon) = \infty$. Such an uniqueness theorem can be found in Appendix of John [12] for example.

In Chapter 2 of Li and Chen [9], we have long histories on the estimate for $\widetilde{T}(\varepsilon)$. The lower bounds of $\widetilde{T}(\varepsilon)$ are summarized in the following table. Let $a = a(\varepsilon)$ satisfy

$$a^2 \varepsilon^2 \log(a + 1) = 1$$

(2)

and $c$ stands for a positive constant independent of $\varepsilon$. Then, due to the fact that it is impossible to obtain an $L^2$ estimate for $u$ itself by standard energy methods, we have

<table>
<thead>
<tr>
<th>$\widetilde{T}(\varepsilon) \geq$</th>
<th>$\alpha = 1$</th>
<th>$\alpha = 2$</th>
<th>$\alpha \geq 3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n = 2$</td>
<td>$ca(\varepsilon)$ in general case, $ce^{-1}$ if $\int_{\mathbb{R}^n} g(x)dx = 0$, $ce^{-2}$ if $\partial_x^2 H(0) = 0$</td>
<td>$ce^{-6}$ in general case, $ce^{-18}$ if $\partial_x^3 H(0) = 0$, $c^{1/2}$ if $\partial_x^4 H(0) = \partial_x^1 H(0) = 0$</td>
<td>$\infty$</td>
</tr>
<tr>
<td>$n = 3$</td>
<td>$ce^{-2}$ in general case, $\exp(ce^{-1})$ if $\partial_x^2 H(0) = 0$</td>
<td>$\infty$</td>
<td>$\infty$</td>
</tr>
<tr>
<td>$n = 4$</td>
<td>$\exp(ce^{-2})$ in general case, $\infty$ if $\partial_x^2 H(0) = 0$</td>
<td>$\infty$</td>
<td>$\infty$</td>
</tr>
<tr>
<td>$n \geq 5$</td>
<td>$\infty$</td>
<td>$\infty$</td>
<td>$\infty$</td>
</tr>
</tbody>
</table>

The result for $n = 1$ is that

$$\widetilde{T}(\varepsilon) \geq \begin{cases} 
\varepsilon^{-\alpha/2} & \text{in general case,} \\
\exp(ce^{-1}) & \text{if } \int_{\mathbb{R}^n} g(x)dx = 0, \\
\varepsilon^{-\alpha} & \text{if } \partial_x^1 H(0) = 0 \text{ for } 1 + \alpha \leq \forall \beta \leq 2\alpha.
\end{cases}$$

(3)

For references on these results, see Li and Chen [9]. We shall skip to refer them here. But we note that two parts in this table are different from the one in Li and Chen [9]. One is the general case in $(n, \alpha) = (4, 1)$. In this part, the lower bound of $\widetilde{T}(\varepsilon)$ is $\exp(ce^{-1})$ in Li and Chen [9]. But later, it has been improved by Li and Zhou [10]. Another is the case for $\partial_x^3 H(0) = 0$ in $(n, \alpha) = (2, 2)$. This part is due to Katayama [14]. But it is missing in Li and Chen [9]. Its reason is closely related to the sharpness of results in the general theory. The sharpness is achieved by the fact that there is no possibility to improve the lower bound of $\widetilde{T}(\varepsilon)$ in sense of order of $\varepsilon$ by blow-up results for special equations and special data. It is expressed in the upper bound of $\widetilde{T}(\varepsilon)$ with the same order of $\varepsilon$ as
the lower bound. On this matter, Li and Chen [9] says that all these lower bounds are known to be sharp except for \((n, \alpha) = (4, 1)\). But before this article, Li [8] says that \((n, \alpha) = (2, 2)\) has also open sharpness while the case for \(\partial_t^4 H(0) = 0\) is still missing. Li and Chen [9] might have dropped the open sharpness in \((n, \alpha) = (2, 2)\) by conjecture that \(\partial_t^4 H(0) = 0\) is a technical condition. No one disagrees with this observation because the model case of \(H = u^4\) has a global solution in two space dimensions, \(n = 2\). However, surprisingly, Zhou and Han [23] have obtained this final sharpness in \((n, \alpha) = (2, 2)\) by studying \(H = u^2_t u + u^4\). This puts Katayama’s result into the table after 20 years from Li and Chen [9]. We note that Zhou and Han [22] have also obtained the sharpness of the case for \(\partial_t^3 u H(0) = \partial_t^4 u H(0) = 0\) in \((n, \alpha) = (2, 2)\) by studying \(H = u^3\). This part had been verified by \(H = |u_t|^3\) only.

We now turn back to another open sharpness of the general case in \((n, \alpha) = (4, 1)\). It has been obtained by our previous work, Takamura and Wakasa [19], by studying model case of \(H = u^2\). This part had been open more than 20 years in the analysis on the critical case for model equations, \(u_{tt} - \Delta u = |u|^p (p > 1)\). In this way, the general theory and its optimality have been completed.

2 The final problem and related results

After the completion of the general theory, we are interested in “almost” global existence, namely, the case where \(\bar{T}(\varepsilon)\) has an lower bound of the exponential function of \(\varepsilon\) with a negative power. Such a case appears in \((n, \alpha) = (2, 2), (3, 1), (4, 1)\) in the table of the general theory. It is remarkable that Klainerman [15] and Christodoulou [3] have independently found a special structure on \(H = H(Du, D_x Du)\) in \((n, \alpha) = (3, 1)\) which guarantees the global existence. This algebraic condition on nonlinear terms of derivatives of the unknown function is so-called “null condition”. It has been also established by Godin [5] for \(H = H(Du)\) and Katayama [13] for \(H = H(Du, D_x Du)\) in \((n, \alpha) = (2, 2)\). The null condition has been supposed to be not sufficient for the global existence in \((n, \alpha) = (2, 2)\). Finally Hoshiga [6] and Kubo [16] have independently succeeded to establish “non-positive” condition in this case for \(H = H(Du)\). It might be necessary and sufficient condition to the global existence. On the other hand, the situation in \((n, \alpha) = (4, 1)\) is completely different from \((n, \alpha) = (2, 2), (3, 1)\) because \(H\) has to include \(u^2\).

In the sense of the first section, one of the final open problem on the optimality of the general theory for fully nonlinear wave equations can be established by model problem;

\[
\begin{aligned}
& u_{tt} - \Delta u = u^2 \quad \text{in} \quad \mathbb{R}^4 \times [0, \infty), \\
& u(x, 0) = \varepsilon f(x), \quad u_t(x, 0) = \varepsilon g(x).
\end{aligned}
\]

We note that this is an extended problem of John [11] to high dimensional case which has the “critical” exponent of Strauss’ conjecture [18]. The lifespan \(T(\varepsilon)\) of the solution of (4) should have an estimate of the form;

\[
\exp(c\varepsilon^{-2}) \leq T(\varepsilon) \leq \exp(C\varepsilon^{-2}).
\]

3
This final problem on the upper bound has been solved by our previous work, Takamura and Wakasa [19]. In its proof, the analysis on \( \|u(\cdot, t)\|_{L^2(\mathbb{R}^4)}^2 \) is a key because we cannot use any pointwise estimate of the solution due to so-called derivative loss in fundamental solutions in high dimensions. Therefore one may have questions;

- Do we have any possibility to get \( T(\varepsilon) = \infty \) if the nonlinear term is not single while it includes \( u^2 \)?
- Do we have any possibility to get a pointwise positivity of the solution for some special nonlinear term?

For these questions, we get the following partial answers.

**Theorem 1 (Takamura and Wakasa [20])** Even if the right-hand side of the equation in (4) additionally has integral terms of the form;

\[
- \frac{1}{\pi^2} \int_0^t d\tau \int_{|\xi| \leq 1} (u_t u)(x + (t - \tau) \xi, \tau) d\xi - \frac{\varepsilon^2}{2\pi^2} \int_{|\xi| \leq 1} f(x + t \xi)^2 d\xi - \frac{\varepsilon^2}{2\pi^2} \int_{|\xi| \leq 1} f(x + t \xi)^2 d\xi,
\]

there is no change on the estimate the lifespan (5).

(6) comes from a removal of the derivative loss factor in the nonlinear term of the equivalent integral equation to (4). This observation already appeared in Agemi, Kubota and Takamura [1] in which a global solution is obtained for the “super-critical” case. The proof of this theorem is established by iteration argument in weighted \( L^1 \) space.

In contrast with (6), we have the following.

**Theorem 2 (Takamura and Wakasa [21])** If the right-hand side of the equation in (4) additionally has integral terms of the form;

\[
- \frac{1}{2\pi^2} \int_0^t d\tau \int_{|\omega| = 1} (u_t u)(x + (t - \tau) \omega, \tau) dS_\omega - \frac{\varepsilon^2}{4\pi^2} \int_{|\omega| = 1} (\varepsilon f^2 + \Delta f + 2\omega \cdot \nabla g)(x + t \omega) dS_\omega, 
\]

then, \( T(\varepsilon) = \infty \) holds.

(7) follows from the following fact due to Agemi and Takamura [2]. When \( n \geq 3 \), a classical solution \( u \) of (1) satisfies

\[
(n - 2)\omega_n u(x, t) = \varepsilon \int_{|\omega| = 1} \{ t\omega \cdot \nabla f + (n - 2)f + tg \} (x + t \omega) dS_\omega \\
+ (n - 3) \int_0^t d\tau \int_{|\omega| = 1} u_t (x + (t - \tau) \omega, \tau) dS_\omega \\
+ \int_0^t (t - \tau) d\tau \int_{|\omega| = 1} H(x + (t - \tau) \omega, \tau) dS_\omega, 
\]

where \( \omega_n \) is an area of the unit sphere in \( \mathbb{R}^n \). If we neglect the second term in the right-hand side of (8), we get (7) by replacing \( g \) by \( 2g \) when \( n = 4 \) and \( H = u^2 \). The proof of this theorem is also established by iteration argument in weighted \( L^\infty \) space. But the key estimate is

\[
\left| \int_{|\omega| = 1} (t\omega \cdot \nabla f + 2f + 2tg) (x + t \omega) dS_\omega \right| \leq \frac{C_{f,g}}{(1 + t)^{3/2}}, 
\]

where \( C_{f,g} \) is a positive constant. This is faster than \( (1 + t)^{-3/2} \) which is a decay of a solution of the free equation.
References


