CONVERGENCE OF THE ALLEN-CAHN EQUATION WITH NEUMANN BOUNDARY CONDITIONS

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ABSTRACT. We study a singular limit problem of the Allen-Cahn equation with Neumann boundary conditions and general initial data of uniformly bounded energy. We prove that the time-parametrized family of limit energy measures is Brakke's mean curvature flow with a generalized right angle condition on the boundary.

1. INTRODUCTION

We consider the following Allen-Cahn equation:

$$\begin{cases} \partial_t u^\varepsilon = \Delta u^\varepsilon - \frac{W'(u^\varepsilon)}{\varepsilon^2}, & t>0,\ x\in\Omega, \\ \left.\frac{\partial u^\varepsilon}{\partial \nu}\right|_{\partial\Omega} = 0, & t>0, \\ u^\varepsilon(x,0) = u_0^\varepsilon(x), & x\in\Omega, \end{cases} \tag{1.1}$$

where $\Omega\subset\mathbb{R}^n$ is a bounded domain with smooth boundary, $\varepsilon>0$ is a small positive parameter, ν is the outer unit normal vector field on $\partial\Omega$ and W is a bi-stable potential with two equal wells at ±1. $W(u)=\frac14(1-u^2)^2$ is a typical example. The equation (1.1) is a gradient flow of

$$E^\varepsilon[u] := \int_\Omega \left(\frac{\varepsilon}{2}|\nabla u|^2 + \frac{W(u)}{\varepsilon}\right) dx$$

as one may check easily that $\frac{dE^\varepsilon}{dt}\le 0$. Under the assumption that a given family $\{u_0^\varepsilon\}_{0<\varepsilon<1}$ satisfies

$$\sup_{0<\varepsilon<1} E^\varepsilon[u_0^\varepsilon] < \infty,$$

it is interesting to study the limiting behavior of the solution u^ε of (1.1) as $\varepsilon\to0$. Heuristically, one expects that the finiteness assumption for $E^\varepsilon[u^\varepsilon(\cdot,t)]$ for very small ε implies a 'phase separation', i.e., Ω is mostly divided into two regions where $u^\varepsilon(\cdot,t)$ is close to 1 on one of them and to -1 on the other, with thin 'transition layer' of order ε thickness separating these two regions. With this heuristic picture, one may also expect that the following measures μ_t^ε defined by

$$d\mu_t^\varepsilon := \left(\frac{\varepsilon}{2}|\nabla u^\varepsilon(x,t)|^2 + \frac{W(u^\varepsilon(x,t))}{\varepsilon}\right) dx \tag{1.2}$$

behave more or less like surface measures of moving phase boundaries. It is thus interesting and natural to study $\lim_{\varepsilon\to0}\mu_t^\varepsilon$. By the well-known heuristic argument using the signed distance functions to the moving phase boundaries composed with the one-dimensional standing wave solution of $\varepsilon^2u''=W'(u)$, one may also expect that the motion of the phase boundaries is the

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mean curvature flow. The rigorous proof of this in the most general setting, on the other hand, requires extensive use of tools from geometric measure theory.

The singular limit of (1.1) without boundary is studied by many researchers with different settings and assumptions. The most relevant among them to the present paper is Ilmanen's work [16], which showed that the limit measures of μ_t^ε are the mean curvature flow in the sense of Brakke [4] (where $\Omega = \mathbb{R}^n$). There was a technical assumption in [16] on the initial condition, which was removed by Soner [30]. The second author observed that Ilmanen's work can be extended to bounded domains, and showed that the limit measures have integer densities a.e. modulo division by a constant [37]. If the densities are equal to one a.e., the support of the measures is smooth a.e. as well [4, 17, 38]. By these works, interior behavior of the limit measures has been rigorously characterized as the mean curvature flow in Brakke's formulation. There are numerous earlier and relevant results on (1.1) and we additionally mention [5, 8, 10, 11, 25, 26, 27, 29, 31, 35] which is by no means an exhaustive listing.

Now turning to the attention to the boundary behavior, due to the Neumann boundary condition, one may heuristically expect that the limit phase boundaries intersect $\partial\Omega$ with 90 degree angle. This may be indeed correct if the sequence of initial conditions are carefully chosen and at least for a short time interval. If we consider general initial condition, on the other hand, this heuristic picture may not be always correct. There is a family of stationary solutions of (1.1) such that the corresponding μ_t^ε converges to a constant multiple of surface measure on $\partial\Omega$ as $\varepsilon \to 0$ (for example see [22, 23] and further discussion in Section 8). It is thus worthwhile to investigate what can be said in general about the limit measure near $\partial\Omega$ along the line of [16]. The analysis may give some insight on the mean curvature flow in Brakke's formulation with an angle condition. In this paper, we prove that the limit measures μ_t defined on $\overline{\Omega}$ for all $t \geq 0$ are $n-1$-rectifiable and are the mean curvature flow with suitable modification on the boundary measure, which will be explained in the next section. We make an assumption that Ω is strictly convex, even though some generalization is possible (see Section 8). The proof uses various ideas developed through [16, 37, 35]. In those paper, the Huisken/Ilmanen monotonicity formula played a central role and the situation is the same in this paper as well. We first prove up to the boundary monotonicity formula by a boundary reflection method, and this leads us to similar estimates as in the interior case. We need to be concerned with measures concentrated on $\partial\Omega$ as well as the limit of 'boundary measures of phase boundary'. All those quantities are incorporated in the final formulation appearing in Theorem 2.6.

The paper is organized as follows. We explain notation and main results in Section 2. In Section 3 we obtain up to the boundary monotonicity formula. The formula is not useful until we obtain an ε-independent estimate on the so-called discrepancy in Section 4. Section 5 shows the existence of converging subsequence for all time, and Section 6 shows the vanishing of the discrepancy which is the key to show the main result. Combining all the ingredients, Section 7 finally proves the main results of the paper.

2. Preliminaries and main results

2.1. **Basic notation.** Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{R}^+ := \{x \geq 0\}$. For $0 < r < \infty$ and $a \in \mathbb{R}^k$, define $B_r^k(a) := \{x \in \mathbb{R}^k \,:\, |x-a| < r\}$. When $k = n$, we omit writing k and we write $B_r := B_r^n(0)$. The Lebesgue measure is denoted by $\mathcal{L}^n$ and the k-dimensional Hausdorff measure is denoted by $\mathcal{H}^k$. Let $\omega_n := \mathcal{L}^n(B_1)$.

For any Radon measure μ on $\mathbb{R}^n$ and $\phi \in C_c(\mathbb{R}^n)$ we often write $\mu(\phi)$ for $\int \phi \, d\mu$. We write $\operatorname{spt} \mu$ for the support of μ. Thus $x \in \operatorname{spt} \mu$ if $\forall r > 0$, $\mu(B_r(x)) > 0$. We use the standard notation for the Sobolev spaces such as $W^{1,p}(\Omega)$ from [12].

For $A, B \in \operatorname{Hom}(\mathbb{R}^n; \mathbb{R}^n)$ which we identify with $n \times n$ matrices, we define

$$A \cdot B := \sum_{i,j} A_{ij} B_{ij}.$$

The identity of $\operatorname{Hom}(\mathbb{R}^n; \mathbb{R}^n)$ is denoted by I. For $k \in \mathbb{N}$ with $k < n$, let $\mathbf{G}(n, k)$ be the space of k-dimensional subspaces of $\mathbb{R}^n$. For $S \in \mathbf{G}(n, k)$, we identify S with the corresponding orthogonal projection of $\mathbb{R}^n$ onto S and its matrix representation. For $a \in \mathbb{R}^n$, $a \otimes a \in \operatorname{Hom}(\mathbb{R}^n; \mathbb{R}^n)$ is the matrix with the entries $a_i a_j$ $(1 \le i, j \le n)$. For any unit vector $a \in \mathbb{R}^n$, $I - a \otimes a \in \mathbf{G}(n, n-1)$. For $x, y \in \mathbb{R}^n$ and $t < s$, define

$$\rho_{(y,s)}(x,t) := \frac{1}{(4\pi(s-t))^{\frac{n-1}{2}}} e^{-\frac{|x-y|^2}{4(s-t)}}. \tag{2.1}$$

2.2. **Varifold.** We recall some definitions related to varifold and refer to [2, 28] for more details. In this paper, for a bounded open set $\Omega \subset \mathbb{R}^n$, we need to consider various objects on $\overline{\Omega}$ instead of Ω. For this reason, let $X \subset \mathbb{R}^n$ be either open or compact in the following. Let $G_k(X) := X \times \mathbf{G}(n, k)$. A general k-varifold in X is a Radon measure on $G_k(X)$. We denote the set of all general k-varifold in X by $\mathbf{V}_k(X)$. For $V \in \mathbf{V}_k(X)$, let $\|V\|$ be the weight measure of V, namely,

$$\|V\|(\phi) := \int_{G_k(X)} \phi(x) \, dV(x, S), \ \ \forall \phi \in C_c(X).$$

We say $V \in \mathbf{V}_k(X)$ is rectifiable if there exist a $\mathcal{H}^k$ measurable countably k-rectifiable set $M \subset X$ and a locally $\mathcal{H}^k$ integrable function θ defined on M such that

$$V(\phi) = \int_M \phi(x, \operatorname{Tan}_x M) \theta(x) \, d\mathcal{H}^k \tag{2.2}$$

for $\phi \in C_c(G_k(X))$. Here $\operatorname{Tan}_x M$ is the approximate tangent space of M at x which exists $\mathcal{H}^k$ a.e. on M. Rectifiable k-varifold is uniquely determined by its weight measure through the formula (2.2). For this reason, we naturally say a Radon measure μ on X is rectifiable if there exists a rectifiable varifold such that the weight measure is equal to μ. If in addition that $\theta \in \mathbb{N}$ $\mathcal{H}^k$ a.e. on M, we say V is integral. The set of all rectifiable (resp. integral) k-varifolds in X is denoted by $\mathbf{RV}_k(X)$ (resp. $\mathbf{IV}_k(X)$). If $\theta = 1$ $\mathcal{H}^k$ a.e. on M, we say V is a unit density k-varifold.

For $V \in \mathbf{V}_k(X)$ let δV be the first variation of V, namely,

$$\delta V(g) := \int_{G_k(X)} \nabla g(x) \cdot S \, dV(x, S) \tag{2.3}$$

for $g \in C_c^1(X; \mathbb{R}^n)$. If the total variation $\|\delta V\|$ of δV is locally bounded (note in the case of $X = \overline{\Omega}$, this means $\|\delta V\|(\overline{\Omega}) < \infty$), we may apply the Radon-Nikodym theorem to δV with respect to $\|V\|$. Writing the singular part of $\|\delta V\|$ with respect to $\|V\|$ as $\|\delta V\|_{\text{sing}}$, we have $\|V\|$ measurable $h(V, \cdot)$, $\|\delta V\|$ measurable ν_{sing} with $|\nu_{\text{sing}}| = 1$ $\|\delta V\|$ a.e., and a Borel set $Z \subset X$ such that $\|V\|(Z) = 0$ with,

$$\delta V(g) = -\int_X h(V, \cdot) \cdot g \, d\|V\| + \int_Z \nu_{\text{sing}} \cdot g \, d\|\delta V\|_{\text{sing}}$$

for all $g \in C_c^1(X;\mathbb{R}^n)$. We say $h(V,\cdot)$ is the generalized mean curvature vector of V, $\nu_{\rm sing}$ is the (outer-pointing) generalized co-normal of V and Z is the generalized boundary of V.

2.3. **Setting of the problem.** Suppose that $n \geq 2$ and

$$\Omega \subset \mathbb{R}^n \text{ is a bounded, strictly convex domain with smooth boundary } \partial\Omega. \tag{2.4}$$

Here the strict convexity means that the principal curvatures of $\partial\Omega$ are all positive. Suppose that $W : \mathbb{R} \to \mathbb{R}$ is a C^3 function with $W(\pm 1) = 0$, $W(u) \geq 0$ for all $u \in \mathbb{R}$,

$$\text{for some } -1 < \gamma < 1,\ W' < 0 \text{ on } (\gamma, 1) \text{ and } W' > 0 \text{ on } (-1,\gamma), \tag{2.5}$$

$$\text{for some } 0 < \alpha < 1 \text{ and } \kappa > 0,\ W''(u) \geq \kappa \text{ for all } \alpha \leq |u| \leq 1. \tag{2.6}$$

A typical example of such W is $(1-u^2)^2/4$, for which we may set $\gamma = 0$, $\alpha = \sqrt{2/3}$ and $\kappa = 1$. For a given sequence of positive numbers $\{\varepsilon_i\}_{i=1}^\infty$ with $\lim_{i\to\infty}\varepsilon_i = 0$, suppose that $u_0^{\varepsilon_i} \in W^{1,2}(\Omega)$ satisfies

$$\|u_0^{\varepsilon_i}\|_{L^\infty(\Omega)} \leq 1 \tag{2.7}$$

and

$$\sup_i E^{\varepsilon_i}[u_0^{\varepsilon_i}] \leq c_1. \tag{2.8}$$

The condition (2.7) may be dropped if we assume a suitable growth rate upper bound on W which is suitable for the existence of solution for (1.1). A typical example of sequence of $u_0^{\varepsilon_i}$ may be given as in [24]. We include the detail for the convenience of the reader. Let $U \subset \mathbb{R}^n$ be any domain with C^1 boundary $M = \partial U$, and let Φ be a solution of ODE $\Phi'' = W'(\Phi)$ with $\Phi(\pm\infty) = \pm 1$ and $\Phi(0) = 0$. Note that such a solution exists uniquely, and Φ also satisfies $\Phi' = \sqrt{2W(\Phi)}$. Let d be the signed distance function to M so that it is positive inside of U. Define $u_0^{\varepsilon_i}(x) := \Phi(d(x)/\varepsilon_i)$ for $x \in \Omega$. Then one can check that, using $\Phi' = \sqrt{2W(\Phi)}$ and $|\nabla d| = 1$ a.e.,

$$E^{\varepsilon_i}[u_0^{\varepsilon_i}] = \int_\Omega \varepsilon_i^{-1}(\Phi')^2\,dx = \int_\Omega \varepsilon_i^{-1}\Phi'\sqrt{2W(\Phi)}|\nabla d|\,dx. \tag{2.9}$$

By the co-area formula, then,

$$E^{\varepsilon_i}[u_0^{\varepsilon_i}] = \int_{-\infty}^{\infty}\int_{\Omega\cap\{d=\varepsilon_i s\}} \Phi'(s)\sqrt{2W(\Phi(s))}\,d\mathcal{H}^{n-1}ds. \tag{2.10}$$

If M is transverse to $\partial\Omega$, $\mathcal{H}^{n-1}(\Omega\cap\{d = \varepsilon_i s\}) \approx \mathcal{H}^{n-1}(M\cap\Omega)$ for small ε_i and (2.10) shows

$$\lim_{i\to\infty} E^{\varepsilon_i}[u_0^{\varepsilon_i}] = \sigma\mathcal{H}^{n-1}(\Omega\cap M), \quad \sigma := \int_{-1}^{1}\sqrt{2W(u)}\,du. \tag{2.11}$$

Thus in this case, we may take $c_1 = \sigma\mathcal{H}^{n-1}(M\cap\Omega) + 1$, for example.

We next solve the problem (1.1) with ε_i and $u_0^{\varepsilon_i}$ satisfying (2.7) and (2.8). By the standard parabolic existence and regularity theory, for each i, there exists a unique solution u^{ε_i} with

$$u^{\varepsilon_i} \in L^2_{loc}([0,\infty); W^{2,2}(\Omega)) \cap C^\infty(\overline{\Omega}\times(0,\infty)), \quad \partial_t u^{\varepsilon_i} \in L^2([0,\infty); L^2(\Omega)). \tag{2.12}$$

By the maximum principle and (2.7),

$$\sup_{x\in\overline{\Omega},\ t>0} |u^{\varepsilon_i}(x,t)| \leq 1, \tag{2.13}$$

and due to the gradient structure and (2.8), we also have

$$E^{\varepsilon_i}[u^{\varepsilon_i}(\cdot,T)] + \int_0^T \int_\Omega \varepsilon_i\big(\Delta u^{\varepsilon_i} - \frac{W'}{\varepsilon_i^2}\big)^2\,dxdt = E^{\varepsilon_i}[u^{\varepsilon_i}(\cdot,0)] \le c_1 \tag{2.14}$$

for any $T > 0$. Thus, for each i through (1.2), we have a family $\{\mu_t^{\varepsilon_i}\}_{t\in[0,\infty)}$ of uniformly bounded Radon measures.

2.4. **Main results.** The following sequence of theorems and definitions constitutes the main results of the present paper.

Theorem 2.1. *Under the assumptions* (2.4)-(2.8)*, let u^{ε_i} be the solution of* (1.1)*. Define $\mu_t^{\varepsilon_i}$ as in* (1.2)*. Then there exists a subsequence (denoted by the same index) and a family of Radon measures $\{\mu_t\}_{t\ge 0}$ on $\overline{\Omega}$ such that for all $t \ge 0$, $\mu_t^{\varepsilon_i} \rightharpoonup \mu_t$ as $i \to \infty$ on $\overline{\Omega}$. Moreover, for a.e. $t \ge 0$, μ_t is rectifiable on $\overline{\Omega}$.*

Due to Theorem 2.1, we may define rectifiable varifolds as follows.

Definition 2.2. For a.e. $t \ge 0$, let $V_t \in \mathbf{RV}_{n-1}(\overline{\Omega})$ be the unique rectifiable varifold such that $\|V_t\| = \mu_t$ on $\overline{\Omega}$. For any t such that μ_t is not rectifiable, define $V_t \in \mathbf{V}_{n-1}(\overline{\Omega})$ to be an arbitrary varifold with $\|V_t\| = \mu_t$ (for example $V_t(\phi) := \int_{\overline{\Omega}} \phi(\cdot, \mathbb{R}^{n-1}\times\{0\})\,d\mu_t$ for $\phi \in C(G_{n-1}(\overline{\Omega}))$).

Theorem 2.3. *Let V_t be defined as above. Then the following property holds.*

(1) *For a.e. $t \ge 0$, $\sigma^{-1}V_t\lfloor_\Omega \in \mathbf{IV}_{n-1}(\Omega)$.*
(2) *For a.e. $t \ge 0$, $\|\delta V_t\|(\overline{\Omega}) < \infty$ and $\int_0^T \|\delta V_t\|(\overline{\Omega})\,dt < \infty$ for all $T > 0$.*

We next define the tangential component of the first variation δV_t on $\partial\Omega$.

Definition 2.4. For a.e. $t \ge 0$ such that $\|\delta V_t\|(\overline{\Omega}) < \infty$, define

$$\delta V_t\lfloor_{\partial\Omega}^\top(g) := \delta V_t\lfloor_{\partial\Omega}(g - (g\cdot\nu)\nu) \quad \text{for } g \in C(\partial\Omega;\mathbb{R}^n) \tag{2.15}$$

where ν is the unit outward-pointing normal vector field on $\partial\Omega$.

We have the following absolute continuity result.

Theorem 2.5. *For a.e. $t \ge 0$, we have $\|\delta V_t\lfloor_{\partial\Omega}^\top + \delta V_t\lfloor_\Omega\| \ll \|V_t\|$, and there exists $h_b = h_b(t) \in L^2(\|V_t\|)$ such that*

$$\delta V_t\lfloor_{\partial\Omega}^\top + \delta V_t\lfloor_\Omega = -h_b(t)\|V_t\|. \tag{2.16}$$

Moreover,

$$\int_0^\infty \int_{\overline{\Omega}} |h_b|^2\,d\|V_t\|dt \le c_1. \tag{2.17}$$

Note that $h_b = h(V_t,\cdot)$ in Ω. Finally, using the above quantities, we have

Theorem 2.6. *For $\phi \in C^1(\overline{\Omega}\times[0,\infty)\,;\mathbb{R}^+)$ with $\nabla\phi(\cdot,t)\cdot\nu = 0$ on $\partial\Omega$ and for any $0 \le t_1 < t_2 < \infty$, we have*

$$\int_{\overline{\Omega}} \phi(\cdot,t)\,d\|V_t\|\Big|_{t=t_1}^{t_2} \le \int_{t_1}^{t_2}\int_{\overline{\Omega}} \big(-\phi|h_b|^2 + \nabla\phi\cdot h_b + \partial_t\phi\big)\,d\|V_t\|dt. \tag{2.18}$$

If $\phi(\cdot,t)$ has a compact support in Ω, (2.18) is Brakke's inequality [4] in an integral form. If we have a situation that $\|V_t\|(\partial\Omega) = 0$, then Theorem 2.5 shows $\delta V_t\lfloor_{\partial\Omega}^\top = 0$ and $\delta V_t\lfloor_{\partial\Omega}$ is singular with respect to $\|V_t\|$. It is parallel to ν for $\|\delta V_t\|$ a.e. which would, if $\operatorname{spt}\|V_t\|$ is smooth up to the boundary, correspond to 90 degree angle of intersection. The reader is referred to Section 8 for further remarks on the above formulation.

3. Boundary monotonicity formula

The first task of our problem is to establish some up-to the boundary monotonicity formula of Huisken/Ilmanen type. Define c_2 by

$$c_2 := (\|\text{principal curvatures of } \partial\Omega\|_{L^\infty(\partial\Omega)})^{-1}.$$

Since $\partial\Omega$ is assumed to be smooth and compact, $0 < c_2 < \infty$. For $r \le c_2$, let us denote by N_r the interior tubular neighborhood of $\partial\Omega$, namely

$$N_r := \{x - \lambda\nu(x) : x \in \partial\Omega,\ 0 \le \lambda < r\},$$

where ν is the unit outer-pointing normal vector field to $\partial\Omega$. For $x \in N_{c_2}$, there exists a unique point $\zeta(x) \in \partial\Omega$ such that $\mathrm{dist}(x, \partial\Omega) = |x - \zeta(x)|$. We define the reflection point $\tilde{x}$ of x with respect to $\partial\Omega$ as $\tilde{x} := 2\zeta(x) - x$ (see Figure 1). We also fix a radially symmetric function $\eta \in C^\infty(\mathbb{R}^n)$ such that

$$0 \le \eta \le 1,\ \ \frac{\partial\eta}{\partial r} \le 0,\ \ \mathrm{spt}\,\eta \subset B_{c_2/2},\ \ \eta = 1 \text{ on } B_{c_2/4}. \tag{3.1}$$

For $s > t > 0$ and $x, y \in N_{c_2}$, we define the $(n-1)$-dimensional reflected backward heat kernel denoted by $\tilde{\rho}_{(y,s)}(x,t)$ as

$$\tilde{\rho}_{(y,s)}(x,t) := \rho_{(y,s)}(\tilde{x}, t), \tag{3.2}$$

where $\rho_{(y,s)}$ is defined as in (2.1). For $x, y \in N_{c_2}$, we define truncated versions of $\rho_{(y,s)}$ and $\tilde{\rho}_{(y,s)}$ as

$$\rho_1 = \rho_1(x,t) = \eta(x-y)\rho_{(y,s)}(x,t) \quad \text{and} \quad \rho_2 = \rho_2(x,t) = \eta(\tilde{x}-y)\tilde{\rho}_{(y,s)}(x,t). \tag{3.3}$$

For $x \in N_{c_2} \setminus N_{c_2/2}$ and $y \in N_{c_2/2}$, we have $|\tilde{x} - y| > c_2/2$. Thus we may smoothly define $\rho_2 = 0$ for $x \in \Omega \setminus N_{c_2/2}$ and $y \in N_{c_2/2}$. We also define a (signed) measure

$$d\xi_t^\varepsilon = \left(\frac{\varepsilon}{2}|\nabla u^\varepsilon|^2 - \frac{W(u^\varepsilon)}{\varepsilon}\right) dx \tag{3.4}$$

where the right-hand side is evaluated at time t.

Proposition 3.1 (Boundary monotonicity formula)**.** *There exist $0 < c_3, c_4 < \infty$ depending only on n, c_1 and c_2 such that*

$$\frac{d}{dt}\big(e^{c_3(s-t)^{\frac14}} \int_\Omega (\rho_1 + \rho_2)\, d\mu_t^\varepsilon(x)\big) \le e^{c_3(s-t)^{\frac14}} \big(c_4 + \int_\Omega \frac{\rho_1 + \rho_2}{2(s-t)}\, d\xi_t^\varepsilon(x)\big) \tag{3.5}$$

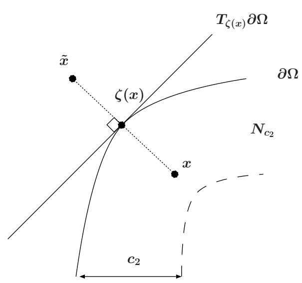

Figure 1. The interior tubular neighbourhood and the reflection point $\tilde{x}$

for $s > t > 0$ and $y \in N_{c_2/2}$. For $s > t > 0$ and $y \in \Omega \setminus N_{c_2/2}$, we have

$$\frac{d}{dt}\int_\Omega \rho_1\, d\mu_t^\varepsilon(x) \le c_4 + \int_\Omega \frac{\rho_1}{2(s-t)}\, d\xi_t^\varepsilon(x). \tag{3.6}$$

Above monotonicity formula is an analogue of Ilmanen's monotonicity formula in $\mathbb{R}^n$ without boundary [16], which is the 'Allen-Cahn equation version' of Huisken's monotonicity formula for the mean curvature flow [14]. Huisken's monotonicity formula for the mean curvature flow with the 90 degree angle boundary condition is derived by Stahl [32, 33], Buckland [6] and Koeller [18]. For stationary case of (1.1), the second author derived a boundary monotonicity formula using the reflection argument [36], and just as in the case of mean curvature flow, it is a 'diffuse interface version' of a boundary monotonicity formula for stationary varifold derived by Grüter-Jost [13].

To derive Huisken's as well as Ilmanen's monotonicity formula,

$$\frac{(a\cdot\nabla\rho)^2}{\rho} + \big((I - a\otimes a)\cdot\nabla^2\rho\big) + \partial_t\rho = 0 \tag{3.7}$$

is the crucial identity. Here, $\rho = \rho_{(y,s)}(x,t)$ and $a = (a_j)$ is any unit vector. Before proving the boundary monotonicity formula, we derive a similar identity for the reflected backward heat kernel $\tilde\rho_{(y,s)}$.

Lemma 3.2. *For a with $|a| = 1$ and $\tilde\rho = \tilde\rho_{(y,s)}(x,t)$, we have*

$$\frac{(a\cdot\nabla\tilde\rho)^2}{\tilde\rho} + ((I - a\otimes a)\cdot\nabla^2\tilde\rho) + \partial_t\tilde\rho = \sum_{i,j,k=1}^n \left(\frac{(\delta_{ij} - a_i a_j)\nabla_{x_j}(\nu_i\nu_k)(\tilde x_k - y_k)}{s-t}\right)\tilde\rho \tag{3.8}$$

for $0 < t < s$ and $x, y \in N_{c_2}$ where $\nu = (\nu_i) = (\nu_i(\zeta(x)))$ is the unit outer-pointing normal to $\partial\Omega$ and $(\delta_{ij}) = I$.

Proof of Lemma 3.2. Since $\nabla\zeta(x) = I - \nu\otimes\nu$ and $\tilde x = 2\zeta(x) - x$, we have

$$\nabla|\tilde x - y|^2 = 2(I - 2\nu\otimes\nu)(\tilde x - y),\quad \nabla^2_{ij}|\tilde x - y|^2 = 2\delta_{ij} - 4\sum_k(\partial_{x_j}(\nu_i\nu_k))(\tilde x_k - y_k). \tag{3.9}$$

By direct calculation and (3.9), we have

$$\begin{aligned}
&\partial_t\tilde\rho = \left(\frac{n-1}{2(s-t)} - \frac{|\tilde x - y|^2}{4(s-t)^2}\right)\tilde\rho,\quad \nabla\tilde\rho = -\frac{\nabla|\tilde x - y|^2}{4(s-t)}\tilde\rho,\\
&\nabla^2\tilde\rho = \left(\frac{\nabla|\tilde x - y|^2\otimes\nabla|\tilde x - y|^2}{16(s-t)^2} - \frac{I}{2(s-t)} + \left(\sum_k \frac{\partial_{x_j}(\nu_i\nu_k)(\tilde x_k - y_k)}{s-t}\right)_{i,j}\right)\tilde\rho.
\end{aligned} \tag{3.10}$$

Using (3.10) and noticing that $|\nabla|\tilde x - y|^2|^2 = 4|\tilde x - y|^2$, we obtain (3.8). □

Proof of Proposition 3.1. By integration by part and using (1.1) and denoting $f^\varepsilon := -\varepsilon\Delta u^\varepsilon + \frac{W'(u^\varepsilon)}{\varepsilon}$, we may obtain for each $i = 1, 2$

$$\begin{aligned}
\frac{d}{dt}\int_\Omega \rho_i\, d\mu_t^\varepsilon &= -\frac{1}{\varepsilon}\int_\Omega (f^\varepsilon)^2\rho_i\, dx + \int_\Omega f^\varepsilon\nabla\rho_i\cdot\nabla u^\varepsilon\, dx + \int_\Omega \partial_t\rho_i\, d\mu_t^\varepsilon\\
&= \int_\Omega -\frac{1}{\varepsilon}\left(f^\varepsilon - \frac{\varepsilon\nabla u^\varepsilon\cdot\nabla\rho_i}{\rho_i}\right)^2\rho_i + \varepsilon\frac{(\nabla u^\varepsilon\cdot\nabla\rho_i)^2}{\rho_i}\, dx\\
&\quad - \int_\Omega f^\varepsilon\nabla\rho_i\cdot\nabla u^\varepsilon\, dx + \int_\Omega \partial_t\rho_i\, d\mu_t^\varepsilon.
\end{aligned} \tag{3.11}$$

By integration by parts again, we have

$$
\begin{aligned}
(3.12)\qquad -\int_\Omega f^\varepsilon \nabla\rho_i\cdot\nabla u^\varepsilon\,dx &= \int_\Omega \Delta\rho_i\,d\mu_t^\varepsilon - \int_\Omega \varepsilon(\nabla u^\varepsilon\otimes\nabla u^\varepsilon\cdot\nabla^2\rho_i)\,dx\\
&\quad - \int_{\partial\Omega}\nabla\rho_i\cdot\nu\left(\frac{\varepsilon}{2}|\nabla u^\varepsilon|^2+\frac{W(u^\varepsilon)}{\varepsilon}\right)d\mathcal{H}^{n-1}.
\end{aligned}
$$

In the following, denote $a^\varepsilon = \frac{\nabla u^\varepsilon}{|\nabla u^\varepsilon|}$. For $x\in\partial\Omega$, $x=\tilde{x}$ and one can check that $\nabla|\tilde{x}-y|^2\cdot\nu + \nabla|x-y|^2\cdot\nu = 0$, which implies $\nabla(\rho_1+\rho_2)\cdot\nu\big|_{\partial\Omega}\equiv 0$. Therefore we may obtain (using also $\mu_t^\varepsilon = \varepsilon|\nabla u^\varepsilon|^2 - \xi_t^\varepsilon$)

$$
\begin{aligned}
\frac{d}{dt}\int_\Omega \rho_1+\rho_2\,d\mu_t^\varepsilon &\le \sum_{i=1,2}\int_\Omega\left(\frac{(a^\varepsilon\cdot\nabla\rho_i)^2}{\rho_i}+\big((I-a^\varepsilon\otimes a^\varepsilon)\cdot\nabla^2\rho_i\big)+\partial_t\rho_i\right)\varepsilon|\nabla u^\varepsilon|^2\,dx\\
&\quad -\sum_{i=1,2}\int_\Omega(\partial_t\rho_i+\Delta\rho_i)\,d\xi_t^\varepsilon.
\end{aligned}
$$

Note that ρ_i is bounded uniformly on $\{|\nabla\eta|\neq 0\}$. Using this fact, (3.7) and (3.8) we may obtain

$$
(3.13)\qquad \frac{(a^\varepsilon\cdot\nabla\rho_1)^2}{\rho_1}+\big((I-a^\varepsilon\otimes a^\varepsilon)\cdot\nabla^2\rho_1\big)+\partial_t\rho_1\le c_4
$$

and

$$
\begin{aligned}
(3.14)\qquad &\frac{(a^\varepsilon\cdot\nabla\rho_2)^2}{\rho_2}+\big((I-a^\varepsilon\otimes a^\varepsilon)\cdot\nabla^2\rho_2\big)+\partial_t\rho_2\\
&\le \sum_{i,j,k=1}^n\left(\frac{(\delta_{ij}-n_in_j)\partial_{x_j}(\nu_i\nu_k)(\tilde{x}_k-y_k)}{2(s-t)}\right)\rho_2+c_4\le\frac{c_3|\tilde{x}-y|}{s-t}\rho_2+c_4
\end{aligned}
$$

for some constant $c_3,\,c_4>0$ depending only on n and c_2. In the following c_3 and c_4 may be different constants which depend only on n, c_1, c_2. To compute the integration of (3.14), we decompose the integration as

$$
\begin{aligned}
\int_\Omega\frac{c_3|\tilde{x}-y|}{s-t}\rho_2\varepsilon|\nabla u^\varepsilon|^2\,dx &\le c_3\int_{\Omega\cap\{|\tilde{x}-y|\le(s-t)^{\frac14}\}}\frac{|\tilde{x}-y|}{s-t}\rho_2\,d\mu_t^\varepsilon\\
&\quad + c_3\int_{\Omega\cap\{|\tilde{x}-y|\ge(s-t)^{\frac14}\}}\frac{|\tilde{x}-y|}{s-t}\rho_2\,d\mu_t^\varepsilon =: I_1+I_2.
\end{aligned}
$$

I_1 is estimated by

$$
(3.15)\qquad I_1\le c_3(s-t)^{-\frac34}\int_{\Omega\cap\{|\tilde{x}-y|<(s-t)^{\frac14}\}}\rho_2\,d\mu_t^\varepsilon\le c_3(s-t)^{-\frac34}\int_\Omega\rho_2\,d\mu_t^\varepsilon.
$$

We may estimate I_2 as

$$
(3.16)\qquad I_2\le\frac{c_3}{(s-t)^{1+\frac{n-1}{2}}}e^{\frac{-1}{4\sqrt{s-t}}}\int_{\Omega\cap\operatorname{spt}\rho_2}|\tilde{x}-y|\,d\mu_t^\varepsilon\le c_4
$$

with an appropriately chosen c_4. Therefore from (3.15) and (3.16) we obtain

$$\int_\Omega \Big(\frac{(a^\varepsilon \cdot \nabla \rho_2)^2}{\rho_2} + \big((I - a^\varepsilon \otimes a^\varepsilon) \cdot \nabla^2 \rho_2\big) + \partial_t \rho_2 \Big) \varepsilon |\nabla u^\varepsilon|^2 \, dx$$
$$\leq c_3 (s-t)^{-\frac{3}{4}} \int_\Omega \rho_2 \, d\mu_t^\varepsilon + c_4.$$

Almost a similar calculation shows that

$$-\int_\Omega (\partial_t \rho_1 + \Delta \rho_1) \, d\xi_t^\varepsilon \leq \int_\Omega \frac{\rho_1}{2(s-t)} \, d\xi_t^\varepsilon + c_4$$

and

$$-\int_\Omega (\partial_t \rho_2 + \Delta \rho_2) \, d\xi_t^\varepsilon \leq \int_\Omega \frac{\rho_2}{2(s-t)} \, d\xi_t^\varepsilon + c_3 (s-t)^{-\frac{3}{4}} \int_\Omega \rho_2 \, d\mu_t^\varepsilon + c_4.$$

Therefore, we have

$$\frac{d}{dt} \int_\Omega (\rho_1 + \rho_2) \, d\mu_t^\varepsilon \leq \frac{c_3}{(s-t)^{\frac{3}{4}}} \int_\Omega (\rho_1 + \rho_2) \, d\mu_t^\varepsilon + c_4 + \int_\Omega \frac{\rho_1 + \rho_2}{2(s-t)} \, d\xi_t^\varepsilon.$$

This leads to (3.5). The inequality (3.6) can be obtained by observing that $\operatorname{spt} \rho_1 \subset \Omega$ for $y \in \Omega \setminus N_{c_2/2}$ and by following the same but simpler computation with $\rho_2 \equiv 0$. □

We use the following estimate later.

Proposition 3.3. *There exists a constant c_5 depending only on n, c_1 and c_2 with*

$$\int_t^{t+1} \int_{\partial\Omega} \Big(\frac{\varepsilon}{2} |\nabla u^\varepsilon|^2 + \frac{W(u^\varepsilon)}{\varepsilon} \Big) \, d\mathcal{H}^{n-1} dt \leq c_5 \tag{3.17}$$

for any $t \geq 0$.

Proof of Proposition 3.3. Let $\phi \in C^2(\overline{\Omega})$ be a non-negative function so that, near $\partial\Omega$, $\phi(x) = \operatorname{dist}(x, \partial\Omega)$, and smoothly becomes a constant function on $\Omega \setminus N_{c_2/2}$. We may construct such function so that $\|\phi\|_{C^2(\overline{\Omega})}$ is bounded only in terms of c_2 and n. Below, we use $\nabla\phi \cdot \nu = -1$ on $\partial\Omega$. We then compute as in the first line of (3.11) and (3.12) with ρ_i there replaced by ϕ. By (2.14) and dropping a negative term on the right-hand side, we obtain

$$\frac{d}{dt} \int_\Omega \phi \, d\mu_t^\varepsilon \leq c_1 \|\phi\|_{C^2} + \int_{\partial\Omega} \nabla\phi \cdot \nu \Big(\frac{\varepsilon}{2} |\nabla u^\varepsilon|^2 + \frac{W(u^\varepsilon)}{\varepsilon} \Big) \, d\mathcal{H}^{n-1}.$$

By integrating over $[t, t+1]$ and again using (2.14), we obtain the desired estimate. □

4. Estimate on ξ_t^ε from above

In this section we prove that ξ_t^ε may be estimated from above by the sup norm for any positive time. One can prove the desired estimate by modifying the similar estimate in [15, 35] combined with the boundary behavior of $|\nabla u^\varepsilon|^2$ when Ω is strictly convex. It is here that the assumption of strict convexity is essential.

Proposition 4.1 (Negativity of the discrepancy)**.** *For any $0 < T < \infty$, $0 < \varepsilon < 1$, there exists c_6 depending only on T such that*

$$\sup_{x \in \Omega,\, t \in [T, \infty)} \xi_t^\varepsilon \leq c_6. \tag{4.1}$$

To show Proposition 4.1, we use the following identities which gives a relationship between the normal derivative of $|\nabla u|^2$ and the second fundamental form of the boundary. Though it is a simple observation and has been used in a number of papers (see for example [7, 21, 34]), we include the proof for the convenience of the reader.

Lemma 4.2. *Let B_x be the second fundamental form of $\partial\Omega$ at $x \in \partial\Omega$. Suppose that $u \in C^2(\overline{\Omega})$ satisfies $\nabla u \cdot \nu = 0$ on $\partial\Omega$. Then at $x \in \partial\Omega$, we have*

$$\frac{\partial}{\partial \nu}|\nabla u|^2 = 2B_x(\nabla u, \nabla u). \tag{4.2}$$

Remark 4.3. In particular, when Ω is convex, the right-hand side of (4.2) is ≤ 0. Furthermore, when Ω is strictly convex, (4.2) is $= 0$ if and only if $\nabla u = 0$ at x.

Proof of Lemma 4.2. Without loss of generality by translation and rotation we may assume that $\partial\Omega$ is a graph near $x = 0 \in \partial\Omega$, namely there exists a function $f = f(x_1, \dots, x_{n-1})$ such that $\partial\Omega \cap B_r$ is included in the graph of f for some $r > 0$ and

$$0 = f(0, \cdots, 0), \quad \nabla_{\mathbb{R}^{n-1}} f(0, \cdots, 0) = 0, \quad \frac{\partial^2 f}{\partial x_i \partial x_j}(0, \cdots, 0) = \kappa_j \delta_{ij},$$

where $\kappa_1, \dots, \kappa_{n-1}$ are the principal curvatures at $x = 0$. We remark that

$$B_0(X, Y) = \sum_{i,j=1}^{n-1} \frac{\partial^2 f}{\partial x_i \partial x_j}(0) X_i Y_j = \kappa_j X_j Y_j$$

for $X = (X_i), Y = (Y_i) \in T_0\partial\Omega$. The outer unit normal vector is given by

$$\nu = \frac{1}{\sqrt{1 + |\nabla_{\mathbb{R}^{n-1}} f|^2}}(-\nabla_{\mathbb{R}^{n-1}} f, 1).$$

By the boundary condition of u we have

$$0 = \frac{\partial u}{\partial \nu} = \frac{1}{\sqrt{1 + |\nabla_{\mathbb{R}^{n-1}} f|^2}} \left(-\sum_{i=1}^{n-1} \frac{\partial f}{\partial x_i} \frac{\partial u}{\partial x_i} + \frac{\partial u}{\partial x_n} \right).$$

Differentiating with respect to x_j again and plugging in $x = 0$, we have

$$\frac{\partial^2 u}{\partial x_n \partial x_j} = \kappa_j \frac{\partial u}{\partial x_j}$$

for $j = 1, \dots, n-1$. By the boundary condition again, we may compute

$$\frac{\partial}{\partial \nu}|\nabla u|^2 = 2\sum_{j=1}^{n-1} \frac{\partial u}{\partial x_j} \frac{\partial^2 u}{\partial x_n \partial x_j} = 2\sum_{j=1}^{n-1} \kappa_j \frac{\partial u}{\partial x_j} \frac{\partial u}{\partial x_j} = 2B_0(\nabla u, \nabla u).$$

□

In the proof of Lemma 4.2, we also need the following.

Lemma 4.4. *There exists a constant $c_7 > 0$ depending only on Ω such that*

$$\sup_{\Omega \times [\varepsilon^2, \infty)} \varepsilon |\nabla u^\varepsilon| \leq c_7$$

for all $1 > \varepsilon > 0$.

Proof of Lemma 4.4. After the parabolic re-scaling, the interior estimates for ∇u^ε can be obtained by the standard argument (see Ladyženskaja-Solonnikov-Ural'ceva [19]). To show the boundary estimates for ∇u^ε, we use the reflection argument on the tubular neighborhood of the boundary. A reflection of u^ε satisfies a parabolic equation on the tubular neighborhood hence we may apply the interior estimates. $\square$

Proof of Proposition 4.1. Under the parabolic change of variables $x \mapsto \frac{x}{\varepsilon}$ and $t \mapsto \frac{t}{\varepsilon^2}$, we continue to use the same notation u^ε which we denote by u in the following. For G to be chosen, define

$$\xi := \frac{|\nabla u|^2}{2} - W(u) - G(u). \tag{4.3}$$

We compute $\partial_t \xi - \Delta\xi$ and obtain

$$\begin{aligned} \partial_t \xi - \Delta\xi =& \nabla u\cdot\nabla\partial_t u - (W'+G')\partial_t u - |\nabla^2 u|^2 - \nabla u\cdot\nabla\Delta u \\ &+ (W'+G')\Delta u + (W''+G'')|\nabla u|^2. \end{aligned} \tag{4.4}$$

Differentiate the equation (1.1) after the change of variables with respect to x_j, multiply $\partial_{x_j} u$ and sum over j to obtain

$$\nabla u\cdot\nabla\partial_t u = \nabla u\cdot\nabla\Delta u - W''|\nabla u|^2. \tag{4.5}$$

By (1.1), (4.4) and (4.5), we obtain

$$\partial_t\xi - \Delta\xi = W'(W'+G') - |\nabla^2 u|^2 + G''|\nabla u|^2. \tag{4.6}$$

Differentiating (4.3) with respect to x_j and by using the Cauchy-Schwarz inequality we have

$$\begin{aligned} \sum_{j=1}^n \Big(\sum_{i=1}^n \partial_{x_i} u\, \partial_{x_i x_j} u\Big)^2 &= \sum_{j=1}^n (\partial_{x_j}\xi + (W'+G')\partial_{x_j} u)^2 \\ &= |\nabla\xi|^2 + 2(W'+G')\nabla\xi\cdot\nabla u + (W'+G')^2|\nabla u|^2 \le |\nabla u|^2|\nabla^2 u|^2. \end{aligned} \tag{4.7}$$

On $\{|\nabla u| \neq 0\}$, divide (4.7) by $|\nabla u|^2$ and substitute into (4.6) to obtain

$$\partial_t\xi - \Delta\xi \le -(G')^2 - W'G' - \frac{2(W'+G')}{|\nabla u|^2}\nabla\xi\cdot\nabla u + G''|\nabla u|^2. \tag{4.8}$$

Given $T > 0$, by Lemma 4.4, we have a uniform estimate on $M := \sup_{t\ge\varepsilon^{-2}T/2,\, x\in\varepsilon^{-1}\Omega} \frac{|\nabla u|^2}{2}$ depending only on T but independent of $0 < \varepsilon < 1$. Let ϕ be a smooth function of t such that $\phi(t) = M$ for $t \le \varepsilon^{-2}T/2$, $\phi(t) = 0$ for $t \ge \varepsilon^{-2}T$, $0 \le \phi(t) \le M$ for all t and $|\phi'| \le 4T^{-1}\varepsilon^2 M$. Let

$$\tilde\xi := \xi - \phi \ \text{ and }\ G(u) := \varepsilon\Big(1 - \frac{1}{8}(u-\gamma)^2\Big), \tag{4.9}$$

where γ is as in (2.5). Due to the choice of G, we have

$$0 < G < \varepsilon,\ \ G'W' \ge 0,\ \ G'' = -\frac{\varepsilon}{4} \tag{4.10}$$

for $|u| \le 1$. Now consider the maximum point of $\tilde\xi$ on $\varepsilon^{-1}\Omega\times[\varepsilon^{-2}T/2, \tilde T]$ for any large $\tilde T$. Due to the choice of M and ϕ, $\tilde\xi \le 0$ for $t = \varepsilon^{-2}T/2$. Suppose for a contradiction that

$$\max_{x\in\varepsilon^{-1}\overline{\Omega},\, t\in[\varepsilon^{-2}T, \tilde T]} \xi \ge C\varepsilon \tag{4.11}$$

for some C to be chosen. Since $\phi = 0$ for $t \geq \varepsilon^{-2}T$, (4.11) implies

$$\max_{x\in\varepsilon^{-1}\overline{\Omega},\, t\in[\varepsilon^{-2}T/2,\tilde{T}]} \tilde{\xi} \geq C\varepsilon. \tag{4.12}$$

Consider a maximum point $(\hat{x},\hat{t})$ of $\tilde{\xi}$ of (4.12). Note that $\hat{x} \notin \partial\Omega$. Because, if $\hat{x} \in \partial\Omega$, $\frac{\partial\xi}{\partial\nu} \geq 0$ while Lemma 4.2 and Remark 4.3 show $\nabla u = 0$. But then $\xi < 0$ there and we have a contradiction. Thus $\hat{x}$ is an interior point. Furthermore, $\hat{t} > \varepsilon^{-2}T/2$ and thus we have

$$\partial_t\tilde{\xi} \geq 0, \quad \nabla\tilde{\xi} = \nabla\xi = 0, \quad \text{and } \Delta\tilde{\xi} = \Delta\xi \leq 0 \tag{4.13}$$

at $(\hat{x},\hat{t})$. By evaluating (4.8) at this point, and using (4.10) and (4.13), we obtain

$$-4T^{-1}\varepsilon^2 M \leq G''|\nabla u|^2 < -\frac{\varepsilon}{4}2C\varepsilon \tag{4.14}$$

where the last inequality follows from $|\nabla u|^2 \geq 2\tilde{\xi}$. Thus choosing C sufficiently large depending only on T and M which ultimately depends only on T, we obtain a contradiction. Thus we proved that

$$\max_{x\in\varepsilon^{-1}\overline{\Omega},\, t\in[\varepsilon^{-2}T/2,\tilde{T}]} \tilde{\xi} \leq C\varepsilon. \tag{4.15}$$

Note that $\phi = 0$ for $t \geq \varepsilon^{-2}T$ and $\tilde{T}$ is arbitrary, and since $G \leq \varepsilon$, we obtained the desired inequality (4.1) by choosing $c_6 := C + 1$. ☐

Corollary 4.5. *There exists* $0 < D_0 < \infty$ *depending only on* c_1, c_2 *and* T *such that*

$$\mu_t^\varepsilon(B_r(y)\cap\Omega) \leq D_0 r^{n-1} \tag{4.16}$$

for all $y \in \overline{\Omega}$, $0 < r \leq c_4/4$, $0 < \varepsilon < 1$ *and* $t \geq T$.

Proof. Let c_6 be the constant in Proposition 4.1 corresponding to $T/2$. Suppose $y \in N_{c_2/2}$. For $\hat{t} \geq T$ and $0 < r \leq c_2/4$, set $s := \hat{t} + r^2$ in the formulas of ρ_1 and ρ_2. We then integrate (3.5) over $t \in [\hat{t} - \frac{T}{2}, \hat{t}]$ to obtain

$$e^{c_3(s-t)^{\frac{1}{4}}} \int_\Omega (\rho_1+\rho_2)\, d\mu_t^\varepsilon(x)\Big|_{t=\hat{t}-\frac{T}{2}}^{\hat{t}} \leq e^{c_3(r^2+T/2)^{\frac{1}{4}}} \Big(\frac{c_4 T}{2} + c_6\sqrt{4\pi}\int_{\hat{t}-\frac{T}{2}}^{\hat{t}} \frac{dt}{\sqrt{s-t}}\Big) \tag{4.17}$$

where we have used (4.1) and $\int_{\mathbb{R}^n} \frac{\rho_i}{\sqrt{4\pi(s-t)}}\, dx \leq 1$. The right-hand side of (4.17) may be estimated in terms of a constant depending only on c_1, c_2 and T. Using $\eta(x-y) = 1$ for $x \in B_r(y)$, we have

$$\begin{aligned}\frac{e^{-\frac{1}{4}}}{(4\pi)^{\frac{n-1}{2}} r^{n-1}} \mu_{\hat{t}}^\varepsilon(B_r(y)\cap\Omega) &\leq \int_{B_r(y)\cap\Omega} \frac{e^{-\frac{1}{4}}}{(4\pi r^2)^{\frac{n-1}{2}}}\, d\mu_{\hat{t}}^\varepsilon(x) \\ &\leq \int_{B_r(y)\cap\Omega} \frac{e^{-\frac{|x-y|^2}{4r^2}}}{(4\pi r^2)^{\frac{n-1}{2}}}\, d\mu_{\hat{t}}^\varepsilon(x) \leq e^{c_3(s-\hat{t})^{\frac{1}{4}}} \int_\Omega \rho_1\, d\mu_{\hat{t}}^\varepsilon.\end{aligned} \tag{4.18}$$

On the other hand,

$$\begin{aligned} e^{c_3(s-\hat{t}+\frac{T}{2})^{\frac{1}{4}}} \int_\Omega (\rho_1+\rho_2)\, d\mu_{\hat{t}-\frac{T}{2}}^\varepsilon(x) &\leq e^{c_3(r^2+\frac{T}{2})^{\frac{1}{4}}} \int_\Omega \frac{2}{(4\pi(r^2+\frac{T}{2}))^{\frac{n-1}{2}}}\, d\mu_{\hat{t}-\frac{T}{2}}^\varepsilon(x) \\ &\leq 2e^{c_3(c_4^2+T)^{\frac{1}{4}}}(2\pi T)^{\frac{1-n}{2}} c_1.\end{aligned} \tag{4.19}$$

Combining (4.17)-(4.19), we obtain (4.16) with an appropriate constant D_0 depending only on c_1, c_2 and T. The case of $y \in \Omega \setminus N_{c_2/2}$ can be proved using (3.6). $\square$

5. Convergence of the energy measures

In this section we prove that there exists a family of Radon measures $\{\mu_t\}_{t\geq 0}$ such that after taking some subsequence, $\mu_t^{\varepsilon_i} \rightharpoonup \mu_t$ as $i \to \infty$ for all $t \geq 0$ on $\overline{\Omega}$. Note that we want consider up to the boundary convergence of μ_t^ε, so we take a test function which does not vanish near $\partial\Omega$ in general.

Lemma 5.1 (Semidecreasing properties). *For all $\phi \in C^2(\overline{\Omega})$ with $\phi \geq 0$ on $\overline{\Omega}$, we have*

$$\int_\Omega \phi \, d\mu_t^\varepsilon - c_1 \|\phi\|_{C^2(\overline{\Omega})} t$$

is monotone decreasing with respect to $t \geq 0$ for all $0 < \varepsilon < 1$.

Proof of Lemma 5.1. For ϕ with the given assumptions, using the Neumann condition of u^ε, we have

$$\begin{aligned}
\frac{d}{dt} \int_\Omega \phi \, d\mu_t^\varepsilon &= -\int_\Omega \varepsilon \nabla\phi \cdot \nabla u^\varepsilon \partial_t u^\varepsilon \, dx - \int_\Omega \varepsilon \phi (\partial_t u^\varepsilon)^2 \, dx \\
&= \int_\Omega \frac{\varepsilon (\nabla\phi \cdot \nabla u^\varepsilon)^2}{4\phi} \, dx - \int_\Omega \varepsilon\phi \left(\partial_t u^\varepsilon + \frac{\nabla\phi \cdot \nabla u^\varepsilon}{2\phi} \right)^2 dx \\
&\leq \int_\Omega \frac{|\nabla\phi|^2}{2\phi} \, d\mu_t^\varepsilon \leq \|\phi\|_{C^2(\overline{\Omega})} c_1
\end{aligned}$$

by (2.14). $\square$

Proposition 5.2. *There exist a family of Radon measures $\{\mu_t\}_{t\geq 0}$ and a subsequence such that $\mu_t^{\varepsilon_i} \rightharpoonup \mu_t$ as $i \to \infty$ for all $t \geq 0$ on $\overline{\Omega}$.*

Proof of Proposition 5.2. Since we aim to obtain convergence of measures on $\overline{\Omega}$, we may define μ_t^ε to be zero measure on $\mathbb{R}^n \setminus \overline{\Omega}$ and we may regard μ_t^ε to be a measure on $\mathbb{R}^n$. Let $B_0 \subset [0, \infty)$ be a countable, dense subset. Then by the compactness of Radon measures and the diagonal argument, there exist a family of Radon measures $\{\mu_t\}_{t\in B_0}$ and a subsequence such that $\mu_t^{\varepsilon_i} \rightharpoonup \mu_t$ as $i \to \infty$ for $t \in B_0$ on $\mathbb{R}^n$. Obviously, μ_t has a support in $\overline{\Omega}$ and note that it may be possible that $\mu_t(\partial\Omega) > 0$ in general.

Let $\{\phi_k\}_{k=1}^\infty \subset C^2(\overline{\Omega})$ be a dense subset in $C(\overline{\Omega})$. Then for each $k \in \mathbb{N}$, there is a countable set $B_k \subset [0, \infty)$ such that $\mu_t(\phi_k)$ has continuous extension with respect to $t \in [0, \infty) \setminus B_k$ by the semidecreasing property of $\mu_t(\phi_k)$. Therefore letting $B = \cup_{k=1}^\infty B_k$, which is countable, $\mu_t(\phi_k)$ is continuous extension with respect to $t \in [0, \infty) \setminus B$, namely for $s \in [0, \infty) \setminus B$, we may define

$$\lim_{\substack{t\uparrow s \\ t\in B_0}} \mu_t(\phi_k) = \lim_{\substack{t\downarrow s \\ t\in B_0}} \mu_t(\phi_k) =: \mu_s(\phi_k). \tag{5.1}$$

Let $s \in [0, \infty) \setminus B$ and let $\{\varepsilon_{i_j}\}_{j=1}^\infty$ be any subsequence satisfying

$$\mu_s^{\varepsilon_{i_j}} \rightharpoonup \tilde{\mu}_s \quad \text{as} \quad j \to \infty \tag{5.2}$$

for some Radon measure $\tilde{\mu}_s$. Then for any $t, t' \in B_0$ with $t < s < t'$ and for any $k \in \mathbb{N}$, we have

$$\mu_t^{\varepsilon_{i_j}}(\phi_k) - c_1\|\phi_k\|_{C^2}(t-s) \geq \mu_s^{\varepsilon_{i_j}}(\phi_k) \geq \mu_{t'}^{\varepsilon_{i_j}}(\phi_k) - c_1\|\phi_k\|_{C^2}(t'-s).$$

From (5.1) and (5.2), we have

$$\mu_t(\phi_k) - c_1\|\phi_k\|_{C^2}(t-s) \geq \tilde{\mu}_s(\phi_k) \geq \mu_{t'}(\phi_k) - c_1\|\phi_k\|_{C^2}(t'-s)$$

hence taking $t \uparrow s$ and $t' \downarrow s$, we find $\tilde{\mu}_s(\phi_k) = \mu_s(\phi_k)$. Therefore $\mu_s^{\varepsilon_i}(\phi_k)$ converges to $\mu_s(\phi_k)$ as $i \to \infty$ for all $s \in [0,\infty) \setminus B$. Since $\{\phi_k\}_{k=1}^\infty$ is a dense subset in $C(\overline{\Omega})$, $\mu_s^{\varepsilon_i} \rightharpoonup \mu_s$ as $i \to \infty$ for all $s \in [0,\infty) \setminus B$.

Finally since B is countable, we may choose a further subsequence (denoted by same index) such that $\mu_t^{\varepsilon_i}$ converges to some Radon measure μ_t for all $t \geq 0$ by the diagonal argument. $\square$

6. Vanishing of the discrepancy

In this section, we prove the vanishing of L^1 limit of $|\xi_t^{\varepsilon_i}|$ as a sequence of functions on $\overline{\Omega} \times (0,\infty)$. Note that, due to (2.14) and the weak compactness theorem of Radon measures, we may choose a subsequence (denoted by the same index) such that $|\xi_t^{\varepsilon_i}|\,dxdt$ converges to a Radon measure on $\overline{\Omega} \times [0,\infty)$ locally in time. We show that the limit measure denoted by $|\xi|$ is identically 0, which will prove the L^1 vanishing. We also define $d\mu^\varepsilon := d\mu_t^\varepsilon dt$ and the subsequence limit μ on $\overline{\Omega} \times [0,\infty)$.

Lemma 6.1. *For any $(x',t') \in \operatorname{spt}\mu$ with $t' > 0$ and $x' \in \overline{\Omega}$, there exist a sequence $\{(x_i,t_i)\}_{i=1}^\infty$ and a subsequence ε_i (denoted by same index) such that $t_i > 0$, $x_i \in \Omega$, $(x_i,t_i) \to (x',t')$ as $i \to \infty$ and $|u^{\varepsilon_i}(x_i,t_i)| < \alpha$ for all $i \in \mathbb{N}$.*

Proof of Lemma 6.1. For simplicity we omit the subscript i. For a contradiction, assume that there exists $0 < r_0 < \sqrt{t'}$ such that

$$\inf_{(B_{r_0}(x')\cap\Omega)\times(t'-r_0^2,t'+r_0^2)} |u^\varepsilon| > \alpha \tag{6.1}$$

for all sufficiently small $\varepsilon > 0$. Differentiating (1.1) with respect to x_j, we have

$$\varepsilon\partial_t(\partial_{x_j}u^\varepsilon) = \varepsilon\Delta(\partial_{x_j}u^\varepsilon) - \frac{W''(u^\varepsilon)}{\varepsilon}\partial_{x_j}u^\varepsilon. \tag{6.2}$$

Fix $\phi \in C_c^2(B_{r_0}(x'))$ such that

$$|\nabla\phi| \leq \frac{3}{r_0}, \quad \phi\big|_{B_{\frac{r_0}{2}}(x')} \equiv 1.$$

Then testing $\partial_{x_j}u^\varepsilon\phi^2$ to (6.2), we have

$$\begin{aligned}\frac{d}{dt}\int_\Omega \frac{\varepsilon}{2}|\nabla u^\varepsilon|^2\phi^2\,dx = &-\varepsilon\int_\Omega |\nabla^2 u^\varepsilon|^2\phi^2\,dx - 2\varepsilon\int_\Omega \partial_{x_j}u^\varepsilon\phi(\nabla\partial_{x_j}u^\varepsilon\cdot\nabla\phi)\,dx \\ &+ \varepsilon\int_{\partial\Omega}\partial_{x_j}u^\varepsilon\phi^2(\nabla\partial_{x_j}u^\varepsilon\cdot\nu)\,d\sigma - \frac{1}{\varepsilon}\int_\Omega W''(u^\varepsilon)|\nabla u^\varepsilon|^2\phi^2\,dx.\end{aligned}$$

By the Hölder and Young inequalities, Lemma 4.2 and the convexity of Ω, we have

$$\frac{d}{dt}\int_\Omega \frac{\varepsilon}{2}|\nabla u^\varepsilon|^2\phi^2\,dx \leq -\frac{1}{\varepsilon}\int_\Omega W''(u^\varepsilon)|\nabla u^\varepsilon|^2\phi^2\,dx + \varepsilon\int_\Omega |\nabla u^\varepsilon|^2|\nabla\phi|^2\,dx.$$

Using (6.1) and (2.6), we have

$$\frac{d}{dt}\int_\Omega \frac{\varepsilon}{2}|\nabla u^\varepsilon|^2\phi^2\,dx \leq -\frac{2\kappa}{\varepsilon^2}\int_\Omega \frac{\varepsilon}{2}|\nabla u^\varepsilon|^2\phi^2\,dx + \frac{18}{r_0^2}c_1. \tag{6.3}$$

Applying the Gronwall inequality to (6.3), we obtain

$$\int_{\Omega} \frac{\varepsilon}{2}|\nabla u^{\varepsilon}|^2 \phi^2 \, dx \leq \left(\exp\left(\frac{2\kappa}{\varepsilon^2}(t' - r_0^2 - t) \right) + \frac{9\varepsilon^2}{r_0^2 \kappa} \right) c_1$$

for $t' - r_0^2 < t < t' + r_0^2$ hence

$$\int_{t'-r_0^2}^{t'+r_0^2} dt \int_{\Omega} \frac{\varepsilon}{2} |\nabla u^{\varepsilon}|^2 \phi^2 \, dx \to 0 \quad \text{as } \varepsilon \downarrow 0. \tag{6.4}$$

By the continuity of u^{ε} and (6.1), we may assume $\alpha \leq u^{\varepsilon} \leq 1$ on $(B_{r_0}(x') \cap \Omega) \times (t' - r_0^2, t' + r_0^2)$ without loss of generality. Otherwise we have $-1 \leq u^{\varepsilon} \leq -\alpha$ and we may argue similarly. Testing $(u^{\varepsilon} - 1)\phi^2$ on $\Omega \times (t' - r_0^2, t' + r_0^2)$ to (1.1) we have

$$\begin{aligned}
&\frac{\varepsilon}{2} \int_{\Omega} (u^{\varepsilon} - 1)^2 \phi^2 \, dx \bigg|_{t = t' + r_0^2} - \frac{\varepsilon}{2} \int_{\Omega} (u^{\varepsilon} - 1)^2 \phi^2 \, dx \bigg|_{t = t' - r_0^2} \\
&\leq \varepsilon \int_{t'-r_0^2}^{t'+r_0^2} dt \int_{\Omega} (u^{\varepsilon} - 1)^2 |\nabla \phi|^2 \, dx - \frac{1}{\varepsilon} \int_{t'-r_0^2}^{t'+r_0^2} dt \int_{\Omega} W'(u^{\varepsilon})(u^{\varepsilon} - 1)\phi^2 \, dx
\end{aligned}$$

hence

$$\begin{aligned}
\int_{t'-r_0^2}^{t'+r_0^2} dt \int_{\Omega} \frac{W'(u^{\varepsilon})}{\varepsilon}(u^{\varepsilon} - 1)\phi^2 \, dx \leq{}& \varepsilon \int_{t'-r_0^2}^{t'+r_0^2} dt \int_{\Omega} (u^{\varepsilon} - 1)^2 |\nabla \phi|^2 \, dx \\
&+ \frac{\varepsilon}{2} \int_{\Omega} (u^{\varepsilon} - 1)^2 \phi^2 \, dx \bigg|_{t = t' - r_0^2}.
\end{aligned}$$

Using

$$W'(s)(s-1) \geq \kappa (s-1)^2 \geq c W(s)$$

for some constant $c > 0$ if $\alpha \leq s \leq 1$, we may obtain

$$\begin{aligned}
\int_{t'-r_0^2}^{t'+r_0^2} dt \int_{\Omega} \frac{W(u^{\varepsilon})}{\varepsilon} \phi^2 \, dx \leq{}& \frac{\varepsilon}{c} \int_{t'-r_0^2}^{t'+r_0^2} dt \int_{\Omega} (u^{\varepsilon} - 1)^2 |\nabla \phi|^2 \, dx \\
&+ \frac{\varepsilon}{2c} \int_{\Omega} (u^{\varepsilon} - 1)^2 \phi^2 \, dx \bigg|_{t = t' - r_0^2}
\end{aligned}$$

hence

$$\int_{t'-r_0^2}^{t'+r_0^2} dt \int_{\Omega} \frac{W(u^{\varepsilon})}{\varepsilon} \phi^2 \, dx \to 0 \quad \text{as } \varepsilon \downarrow 0. \tag{6.5}$$

Thus we have by (6.4) and (6.5)

$$\int_{t'-r_0^2}^{t'+r_0^2} \left(\int_{\Omega} \phi^2 \, d\mu_t^{\varepsilon} \right) dt \to 0 \quad \text{as } \varepsilon \downarrow 0.$$

This shows that $(x', t') \notin \operatorname{spt} \mu$, which is contradiction. $\square$

Lemma 6.2. *There exist δ_0, r_0, $\gamma_0 > 0$ depending only on κ, W and $T > 0$ such that the following holds: If*

$$\int_{\overline{\Omega}} \eta(x - y) \rho_{(y,s)}(x, t) \, d\mu_s(y) < \delta_0 \tag{6.6}$$

for some $T < t < s < t + \frac{r_0^2}{2}$ and $x \in \overline{\Omega}$, then $(x', t') \notin \operatorname{spt} \mu$ for all $x' \in B_{\gamma_0 r}(x) \cap \overline{\Omega}$, where $t' = 2s - t$ and $r = \sqrt{2(s-t)}$.

Proof of Lemma 6.2. In the following we assume $x' \in N_{c_2/2}$. The proof for the case $x' \in \Omega \setminus N_{c_2/2}$ may be carried out using (3.6) in place of (3.5). Let us assume $(x', t') \in \operatorname{spt} \mu$ for a contradiction. Then by Lemma 6.1 there exists a sequence $\{(x_i, t_i)\}_{i=1}^{\infty}$ such that $(x_i, t_i) \to (x', t')$ as $i \to \infty$ and $|u^{\varepsilon_i}(x_i, t_i)| < \alpha$ for all $i \in \mathbb{N}$. Put $r_i := \gamma_0 \varepsilon_i$, where $\gamma_0 > 0$ will be chosen later, and $T_i := t_i + r_i^2$. Then

$$
\begin{aligned}
&\int_{B_{r_i}(x_i)} \eta(y - x_i) \rho_{(x_i, T_i)}(y, t_i)\, d\mu_{t_i}^{\varepsilon_i}(y) \\
&\geq \frac{1}{(4\pi r_i^2)^{\frac{n-1}{2}}} \int_{B_{r_i}(x_i)} \eta(y - x_i) \exp\left(-\frac{|y - x_i|^2}{4r_i^2}\right) \frac{W(u^{\varepsilon_i}(y, t_i))}{\varepsilon_i}\, dy.
\end{aligned} \tag{6.7}
$$

For $y \in B_{r_i}(x_i)$,

$$
|u^{\varepsilon_i}(y, t_i)| \leq \gamma_0 \sup_{x \in \Omega} \varepsilon_i |\nabla u^{\varepsilon_i}(x, t_i)| + |u^{\varepsilon_i}(x_i, t_i)| \leq c_7 \gamma_0 + \alpha,
$$

where c_7 is a constant given by Lemma 4.4. Thus for sufficiently small $\gamma_0 > 0$ and $y \in B_{r_i}(x_i)$, we have $W(u^{\varepsilon_i}(y, t_i)) \geq c$ for some $c > 0$. Thus for all sufficiently large i, we may obtain from (6.7)

$$
\int_{B_{r_i}(x_i)} \eta(y - x_i) \rho_{(x_i, T_i)}(y, t_i)\, d\mu_{t_i}^{\varepsilon_i}(y) \geq \frac{c}{(4\pi \gamma_0^2)^{\frac{n-1}{2}} \varepsilon_i^n} \int_{B_{r_i}(x_i)} \exp\left(-\frac{|y - x_i|^2}{4r_i^2}\right) dy \geq c_8
$$

for some constant $c_8 > 0$. By (3.5) and (4.1) we have

$$
\begin{aligned}
c_8 &\leq \int_{\Omega} (\eta(y - x_i) \rho_{(x_i, T_i)}(y, t_i) + \eta(\tilde{y} - x_i) \tilde{\rho}_{(x_i, T_i)}(y, t_i))\, d\mu_{t_i}^{\varepsilon_i}(y) \\
&\leq e^{c_3 (T_i - s)^{\frac{1}{4}}} \int_{\Omega} (\eta(y - x_i) \rho_{(x_i, T_i)}(y, s) + \eta(\tilde{y} - x_i) \tilde{\rho}_{(x_i, T_i)}(y, s))\, d\mu_s^{\varepsilon_i}(y) \\
&\quad + \int_s^{t_i} e^{c_3 (T_i - \tau)^{\frac{1}{4}}} \left(c_4 + \frac{\sqrt{4\pi} c_6}{\sqrt{T_i - \tau}}\right) d\tau.
\end{aligned}
$$

Letting $i \to \infty$, we have

$$
\begin{aligned}
c_8 &\leq e^{c_3 (t' - s)^{\frac{1}{4}}} \int_{\overline{\Omega}} (\eta(y - x') \rho_{(x', t')}(y, s) + \eta(\tilde{y} - x') \tilde{\rho}_{(x', t')}(y, s))\, d\mu_s(y) \\
&\quad + \int_s^{t'} e^{c_3 (t' - \tau)^{\frac{1}{4}}} \left(c_4 + \frac{\sqrt{4\pi} c_6}{\sqrt{t' - \tau}}\right) d\tau.
\end{aligned} \tag{6.8}
$$

Since $t' - s = s - t = \frac{r^2}{2}$, we may choose sufficiently small r_0 such that $s - t < r_0^2/2$ implies

$$
\int_s^{t'} e^{c_3 (t' - \tau)^{\frac{1}{4}}} \left(c_4 + \frac{\sqrt{4\pi} c_6}{\sqrt{t' - \tau}}\right) d\tau \leq \frac{c_8}{2}, \quad e^{c_3 (t' - s)^{\frac{1}{4}}} \leq 2. \tag{6.9}
$$

By the convexity of Ω, we have $|y - x'| \leq |\tilde{y} - x'|$ for $y, \tilde{y} \in B_{c_2/2}(x') \subset N_{c_2}$, thus considering (3.1) as well, we have

$$
\eta(\tilde{y} - x') \tilde{\rho}_{(x', t')}(y, s) \leq \eta(y - x') \rho_{(x', t')}(y, s). \tag{6.10}
$$

Combining (6.8)-(6.10) and putting $\delta_0 := \frac{c_8}{32}$, we have

$$4\delta_0 \le \int_{\overline{\Omega}} \eta(y - x')\rho_{(x',t')}(y,s)\, d\mu_s(y). \tag{6.11}$$

Now we assume (6.6). Then for any $\delta > 0$ we may take $\gamma_1 > 0$ as in Lemma 9.1 (note also Corollary 4.5) such that

$$\begin{aligned}
\int_{\overline{\Omega}} \eta(y - x')\rho_{(x',t')}(y,s)\, d\mu_s(y) &= \int_{\overline{\Omega}} \eta(y - x')\rho^r_{x'}(y)\, d\mu_s(y) \\
&\le (1+\delta) \int_{\overline{\Omega}} \eta(y - x)\rho^r_{x}(y)\, d\mu_s(y) + \delta D_0 \\
&= (1+\delta) \int_{\overline{\Omega}} \eta(y - x)\rho_{(y,s)}(x,t)\, d\mu_s(y) + \delta D_0 \\
&\le \delta_0(1+\delta) + \delta D_0.
\end{aligned}$$

Choose $\delta > 0$ such that $\delta_0(1+\delta) + \delta D_0 \le 2\delta_0$. Then we have from (6.11)

$$4\delta_0 \le \int_{\overline{\Omega}} \eta(y - x')\rho_{(x',t')}(y,s)\, d\mu_s(y) \le 2\delta_0,$$

which is contradiction. Hence we have $(x',t') \notin \operatorname{spt}\mu$. $\square$

Lemma 6.3 (Forward density lower bounds)**.** *For $T > 0$, let $\delta_0(T) > 0$ be a constant given in Lemma 6.2. Then we have $\mu(Z^-(T)) = 0$, where*

$$Z^-(T) := \left\{ (x,t) \in \operatorname{spt}\mu : \limsup_{s \downarrow t} \int_{\overline{\Omega}} \eta(y - x)\rho_{(y,s)}(x,t)\, d\mu_s(y) < \delta_0(T), \;\; t > T \right\}.$$

Proof of Lemma 6.3. We do not write out the dependence on T in the following for simplicity, where we assume $t > T$. Corresponding to T, let δ_0, γ_0 and r_0 be constants given by Lemma 6.2. For $0 < \tau < \frac{r_0^2}{2}$ define

$$Z^\tau := \left\{ (x,t) \in \operatorname{spt}\mu : \int_{\overline{\Omega}} \eta(y - x)\rho_{(y,s)}(x,t)\, d\mu_s(y) < \delta_0 \quad \text{for } t < s < t + \tau \right\}.$$

If we take a sequence $\tau_m > 0$ with $\lim_{m\to\infty} \tau_m = 0$, then $Z^- \subset \cup_{m=1}^\infty Z^{\tau_m}$. Hence we only need to show $\mu(Z^\tau) = 0$.

Let $(x,t) \in Z^\tau$ be fixed and we define

$$P(x,t) := \left\{ (x',t') \in \overline{\Omega} \times (0,\infty) : \gamma_0^{-2}|x' - x|^2 < |t' - t| < \tau \right\}.$$

We claim that $P(x,t) \cap Z^\tau = \emptyset$. Indeed, suppose for a contradiction that $(x',t') \in P(x,t) \cap Z^\tau$. Assume $t' > t$ and put $s = \frac{1}{2}(t+t')$. Then $t < s \le t+\tau$, $|x - x'| < \gamma_0\sqrt{|t'-t|} = \gamma_0\sqrt{2(s-t)}$ and

$$\int_{\overline{\Omega}} \eta(y - x)\rho_{(y,s)}(x,t)\, d\mu_s(y) < \delta_0.$$

Hence by Lemma 6.2, $(x',t') \notin \operatorname{spt}\mu$, which contradicts $(x',t') \in Z^\tau$. If $t' < t$, by the similar argument, we obtain $(x,t) \notin \operatorname{spt}\mu$ which is a contradiction. This proves $P(x,t) \cap Z^\tau = \emptyset$.

For a fixed $(x_0,t_0) \in \overline{\Omega} \times (T,\infty)$, define

$$Z^{\tau,x_0,t_0} := Z^\tau \cap \left(B_{\frac{\gamma_0}{2}\sqrt{\tau}}(x_0) \times (t_0 - \frac{\tau}{2}, t_0 + \frac{\tau}{2}) \right).$$

Then Z^τ is a countable union of Z^{τ,x_m,t_m} with (x_m,t_m) spaced appropriately. Hence we only need to show that $\mu(Z^{\tau,x_0,t_0})=0$. Denote Z^{τ,x_0,t_0} by Z'. For $0<\rho\leq 1$, we may find a covering of $\pi_\Omega(Z') := \{x\in\overline{\Omega} \,:\, (x,t)\in Z'\}$ by a collection of balls $\{B_{r_i}(x_i)\}_{i=1}^\infty$, where $(x_i,t_i)\in Z'$, $r_i\leq\rho$, so that

$$\sum_{i=1}^\infty \omega_n r_i^n \leq c(n)\mathscr{L}^n(B_{\frac{\gamma_0}{2}\sqrt{\tau}}(x_0))). \tag{6.12}$$

For for such covering, we find

$$Z' \subset \bigcup_{i=1}^\infty B_{r_i}(x_i)\times\left(t_i - r_i^2\gamma_0^{-2}, t_i + r_i^2\gamma_0^{-2}\right).$$

Indeed, if $(x,t)\in Z'$, then $x\in B_{r_i}(x_i)$ for some $i\in\mathbb{N}$. Since $P(x_i,t_i)\cap Z^\tau=\emptyset$, we have

$$|t-t_i| \leq |x-x_i|^2\gamma_0^{-2} < r_i^2\gamma_0^{-2}.$$

Therefore we obtain by (6.12)

$$\begin{aligned}\mu(Z') &\leq \sum_{i=1}^\infty \mu(B_{r_i}(x_i)\times(t_i-r_i^2\gamma_0^{-2},t_i+r_i^2\gamma_0^{-2})) \leq \sum_{i=1}^\infty 2r_i^2\gamma_0^{-2}D_0 r_i^{n-1}\\ &\leq 2\rho\gamma_0^{-2}D_0\omega_n^{-1}c(n)\mathscr{L}^n(B_{\frac{\gamma_0}{2}\sqrt{\tau}}(x_0)).\end{aligned}$$

Since ρ is arbitrary, we have $\mu(Z')=0$. This concludes the proof. □

Proposition 6.4 (Vanishing of discrepancy)**.** *We have* $|\xi|=0$ *on* $\overline{\Omega}\times(0,\infty)$.

Proof of Proposition 6.4. Due to (2.8), it is enough to prove $|\xi|=0$ on $\overline{\Omega}\times(T_1,T_2)$ for all $0<T_1<T_2<\infty$. In the following we fix T_1 and T_2. For $y\in N_{c_2/2}$ and $T_2>s>t>T_1$, by (3.5) and (4.1) we obtain (c_6 corresponding to T_1)

$$\frac{d}{dt}\Big(e^{c_3(s-t)^{\frac14}}\int_\Omega(\rho_1+\rho_2)\,d\mu_t^{\varepsilon_i}\Big) + e^{c_3(s-t)^{\frac14}}\int_\Omega\frac{\rho_1+\rho_2}{2(s-t)}\,d|\xi_t^\varepsilon| \leq e^{c_3(s-t)^{\frac14}}\big(c_4+\frac{2c_6\sqrt{4\pi}}{(s-t)^{\frac12}}\big).$$

Integrating over $t\in(T_1,s)$ and taking $i\to\infty$, we obtain

$$\iint_{\overline{\Omega}\times(T_1,s)}\frac{\rho_1+\rho_2}{2(s-t)}d|\xi| \leq e^{c_3 s^{\frac14}}\int_{\overline{\Omega}}(\rho_1+\rho_2)\,d\mu_{T_1} + \int_{T_1}^s e^{c_3(s-t)^{\frac14}}\big(c_4+\frac{2c_6\sqrt{4\pi}}{(s-t)^{\frac12}}\big)\,dt. \tag{6.13}$$

Note that the right-hand side of (6.13) is uniformly bounded for $(y,s)\in N_{c_2/2}\times(T_1,T_2)$ once T_1 and T_2 are fixed. For $y\in\Omega\setminus N_{c_2/2}$, the similar argument using (3.6) in place of (3.5) gives the similar estimate (with $\rho_2=0$). Since the right-hand side of (6.13) is bounded uniformly on $\overline{\Omega}\times(T_1,T_2)$, integration of (6.13) over $(y,s)\in\overline{\Omega}\times(T_1,T_2)$ with respect to $d\mu_s ds$ shows that

$$\int_{T_1}^{T_2} ds\int_{\overline{\Omega}} d\mu_s(y)\iint_{\overline{\Omega}\times(T_1,s)}\frac{\rho_1+\rho_2}{2(s-t)}d|\xi|(x,t) \tag{6.14}$$

is finite. By the Fubini theorem, (6.14) is turned into

$$\iint_{\overline{\Omega}\times(T_1,T_2)} d|\xi|(x,t)\int_t^{T_2} ds\int_{\overline{\Omega}}\frac{\rho_1+\rho_2}{2(s-t)}\,d\mu_s(y).$$

Thus we have

$$\int_t^{T_2}\frac{1}{2(s-t)}\,ds\int_{\overline{\Omega}}\rho_1+\rho_2\,d\mu_s(y)<\infty \tag{6.15}$$

for $|\xi|$-almost all $(x,t) \in \overline{\Omega} \times (T_1, T_2)$. We next prove that for $|\xi|$-almost all (x,t),

$$\lim_{s \downarrow t} \int_{\overline{\Omega}} \rho_1 \, d\mu_s(y) = 0. \tag{6.16}$$

For $t < s$, we define $\beta := \log(s-t)$ and

$$h(s) := \int_{\overline{\Omega}} \rho_1 \, d\mu_s(y).$$

Then (6.15) is translated into

$$\int_{-\infty}^{\log(T_2 - s)} h(t + e^{\beta}) \, d\beta < \infty. \tag{6.17}$$

Let $0 < \theta < 1$ be arbitrary for the moment. Due to (6.17), we may choose a decreasing sequence $\{\beta_i\}_{i=1}^{\infty}$ such that $\beta_i \to -\infty$, $\beta_i - \beta_{i+1} < \theta$ and

$$h(t + e^{\beta_i}) < \theta$$

for all i. For any $-\infty < \beta < \beta_1$ fixed, we may choose $i \geq 2$ such that $\beta_i \leq \beta < \beta_{i-1}$. We use $\rho_{(y,t+\varepsilon^{\beta})}(x,t) = \rho_{(x,t+2\varepsilon^{\beta})}(y, t+\varepsilon^{\beta})$ and use (3.5) and (4.1) to obtain

(6.18)

$$\begin{aligned}
h(t+e^{\beta}) &= \int_{\overline{\Omega}} \eta(y-x)\rho_{(y,t+e^{\beta})}(x,t) \, d\mu_{t+e^{\beta}}(y) \\
&\leq \int_{\overline{\Omega}} \eta(x-y)\rho_{(x,t+2e^{\beta})}(y, t+e^{\beta}) + \eta(x-\tilde{y})\tilde{\rho}_{(x,t+2e^{\beta})}(y, t+e^{\beta}) \, d\mu_{t+e^{\beta}}(y) \\
&\leq e^{c_3(2e^{\beta} - e^{\beta_i})^{\frac{1}{4}}} \int_{\overline{\Omega}} \eta(x-y)\rho_{(x,t+2e^{\beta})}(y, t+e^{\beta_i}) + \eta(x-\tilde{y})\tilde{\rho}_{(x,t+2e^{\beta})}(y, t+e^{\beta_i}) \, d\mu_{t+e^{\beta_i}}(y) \\
&+ \int_{t+e^{\beta_i}}^{t+e^{\beta}} e^{c_3(t+2e^{\beta}-\tau)^{\frac{1}{4}}} \big(c_4 + \frac{c_6\sqrt{4\pi}}{(t+2e^{\beta}-\tau)^{\frac{1}{2}}} \big) \, d\tau.
\end{aligned}$$

Let us denote the last integral of (6.18) as $c(i)$. Note that $c(i)$ can be made uniformly small (with respect to i) if θ is chosen small. By the convexity of Ω, we have $|x - \tilde{y}| \geq |x - y|$ for $x \in \overline{\Omega}$ and $y \in N_{c_2/2}$, thus

$$\eta(x-\tilde{y})\tilde{\rho}_{(x,t+2e^{\beta})}(y, t+e^{\beta_i}) \leq \eta(x-y)\rho_{(x,t+2e^{\beta})}(y, t+e^{\beta_i}).$$

Hence we obtain

$$h(t+e^{\beta}) \leq 2e^{c_3(2R_i^2)^{\frac{1}{4}}} \int_{\overline{\Omega}} \eta(x-y)\rho_x^{R_i}(y) \, d\mu_{t+e^{\beta_i}}(y) + c(i) \tag{6.19}$$

where $2R_i^2 = 2e^{\beta} - e^{\beta_i}$.

We next show the lower bound of $h(t+e^{\beta_i})$. By the assumption of β_i, we have

$$\begin{aligned}
\theta \geq h(t+e^{\beta_i}) &= \int_{\overline{\Omega}} \eta(x-y)\rho_{(y,t+e^{\beta_i})}(x,t) \, d\mu_{t+e^{\beta_i}}(y) \\
&= \int_{\overline{\Omega}} \eta(x-y)\rho_x^{r_i}(y) \, d\mu_{t+e^{\beta_i}}(y),
\end{aligned} \tag{6.20}$$

where $2r_i^2 = e^{\beta_i}$. Since $\beta \geq \beta_i$, we have $R_i \geq r_i$. Also $\beta - \beta_i < \beta_{i-1} - \beta_i < \theta$ implies $R_i^2/r_i^2 < 2e^{\theta} - 1$ which can be made arbitrarily close to 1 by restricting θ to be small. For arbitrary $\delta > 0$, we restrict θ to be sufficiently small using Lemma 9.1 so that $\frac{R_i}{r_i} < 1 + \gamma_2$, where $\gamma_2 > 0$ is given by Lemma 9.1 corresponding to $\delta > 0$. Then we obtain

$$\int_{\overline{\Omega}} \eta(x-y)\rho_x^{R_i}(y)\,d\mu_{t+e^{\beta_i}}(y) \le (1+\delta)\int_{\overline{\Omega}} \eta(x-y)\rho_x^{r_i}(y)\,d\mu_{t+e^{\beta_i}}(y) + \delta D_0 \tag{6.21}$$

hence from (6.19), (6.20) and (6.21) we have

$$\begin{aligned} h(t+e^{\beta}) &\le 2e^{c_3(2R_i^2)^{\frac{1}{4}}}\Big((1+\delta)\int_{\overline{\Omega}} \eta(x-y)\rho_x^{r_i}(y)\,d\mu_{t+e^{\beta_i}}(y) + \delta D_0\Big) + c(i) \\ &\le 2e^{c_3(2R_i^2)^{\frac{1}{4}}}((1+\delta)\theta + \delta D_0) + c(i). \end{aligned}$$

Since δ and θ are arbitrary, above estimate shows

$$\limsup_{\beta\to-\infty} h(t+e^{\beta}) = 0 \quad |\xi|\text{-almost all } (x,t) \in \overline{\Omega}\times(T_1,T_2)$$

as well as (6.16). This proves that $|\xi|((\overline{\Omega}\times(T_1,T_2))\setminus Z^-(T_1)) = 0$, since otherwise, we have $\limsup_{\beta\to-\infty} h(t+e^{\beta}) \ge \delta_0$ on a set of positive measure with respect to $|\xi|$. Lemma 6.3 shows $\mu(Z^-(T_1)) = 0$, and since $|\xi| \le \mu$ by the definitions of these measures, we have $|\xi|(\overline{\Omega}\times(T_1,T_2)) = 0$. ☐

7. Proof of main theorems

In Section 5, we have seen that there exists a subsequence such that $\mu_t^{\varepsilon_i}$ converges to μ_t for all $t \ge 0$. In this section we prove that the first variation of the limit varifold is bounded and rectifiable for a.e. $t \ge 0$. On the boundary $\partial\Omega$, we show that the tangential component of the first variation is absolutely continuous with respect to μ_t and prove at the end the desired limiting inequality (2.18).

For each u^{ε_i}, we associate a varifold as follows.

Definition 7.1. For $\phi \in C(G_{n-1}(\overline{\Omega}))$, define

$$V_t^{\varepsilon_i}(\phi) := \int_{\Omega\cap\{|\nabla u^{\varepsilon_i}(t,\cdot)|\neq 0\}} \phi(x, I - a^{\varepsilon_i}\otimes a^{\varepsilon_i})\,d\mu_t^{\varepsilon_i}(x). \tag{7.1}$$

Here, $a^{\varepsilon_i} = \frac{\nabla u^{\varepsilon_i}}{|\nabla u^{\varepsilon_i}|}$.

Note that we have $\|V_t^{\varepsilon_i}\| = \mu_t^{\varepsilon_i}$. We then derive a formula for the first variation of $V_t^{\varepsilon_i}$ up to the boundary.

Lemma 7.2. *For $g \in C^1(\overline{\Omega};\mathbb{R}^n)$, we have*

$$\begin{aligned} \delta V_t^{\varepsilon_i}(g) &= \int_{\Omega} (g\cdot\nabla u^{\varepsilon_i})\big(\varepsilon_i\Delta u^{\varepsilon_i} - \frac{W'}{\varepsilon_i}\big)\,dx + \int_{\Omega\cap\{|\nabla u^{\varepsilon_i}|\neq 0\}} \nabla g\cdot(a^{\varepsilon_i}\otimes a^{\varepsilon_i})\xi^{\varepsilon_i}\,dx \\ &\quad + \int_{\partial\Omega} (g\cdot\nu)\big(\frac{\varepsilon_i|\nabla u^{\varepsilon_i}|^2}{2} + \frac{W}{\varepsilon_i}\big) - \int_{\Omega\cap\{|\nabla u^{\varepsilon_i}|=0\}} \nabla g\cdot I\,\frac{W}{\varepsilon_i}\,dx. \end{aligned} \tag{7.2}$$

Proof. Omit the sub-index i. We have

$$\delta V_t^{\varepsilon}(g) = \int_{\Omega\cap\{|\nabla u^{\varepsilon}|\neq 0\}} \nabla g(x)\cdot(I - a^{\varepsilon}\otimes a^{\varepsilon})\,d\mu_t^{\varepsilon}. \tag{7.3}$$

Using the boundary condition $\nabla u^\varepsilon \cdot \nu = 0$ on $\partial\Omega$ and integration by parts, we have

$$(7.4)\quad \int_\Omega \nabla g \cdot I \, \frac{|\nabla u^\varepsilon|^2}{2}\, dx = \int_{\partial\Omega} (g\cdot\nu)\frac{|\nabla u^\varepsilon|^2}{2} + \int_\Omega \nabla g \cdot (\nabla u^\varepsilon \otimes \nabla u^\varepsilon) + (g\cdot \nabla u^\varepsilon)\Delta u^\varepsilon \, dx.$$

Also by integration by parts,

$$(7.5)\quad \int_{\Omega\cap\{|\nabla u^\varepsilon|\neq 0\}} W\nabla g\cdot I\, dx = -\int_{\Omega\cap\{|\nabla u^\varepsilon|=0\}} W\nabla g\cdot I\, dx - \int_\Omega (g\cdot\nabla u^\varepsilon)W'\, dx + \int_{\partial\Omega}(g\cdot\nu)W.$$

Substituting (7.4) and (7.5) into (7.3) and recalling the definition of ξ^ε, we obtain (7.2). ☐

Proposition 7.3. *For a.e. $t \geq 0$, μ_t is rectifiable on $\overline{\Omega}$, and any convergent subsequence $\{V_t^{\varepsilon_{i_j}}\}_{j=1}^\infty$ with*

$$(7.6)\qquad \liminf_{j\to\infty} \Big\{ \Big(\int_\Omega \varepsilon_{i_j}\big(\Delta u^{\varepsilon_{i_j}} - \frac{W'}{\varepsilon_{i_j}^2}\big)^2 dx\Big)^{\frac{1}{2}} + \int_{\partial\Omega}\Big(\frac{\varepsilon_{i_j}|\nabla u^{\varepsilon_{i_j}}|^2}{2} + \frac{W}{\varepsilon_{i_j}}\Big)\Big\} < \infty$$

(evaluated at t) converges to the unique varifold V_t associated with μ_t. Moreover we have

$$(7.7)\qquad \|\delta V_t\|(\overline{\Omega}) < \infty$$

and

$$(7.8)\qquad \int_0^T \|\delta V_t\|(\overline{\Omega})\, dt < \infty$$

for all $T < \infty$.

Proof. Due to the energy inequality, (3.17) and Fatou's lemma, we have (7.6) for a.e. $t \geq 0$ for the full sequence. Also for a.e. $t \geq 0$, we have

$$(7.9)\qquad \lim_{i\to\infty} \int_\Omega |\xi^{\varepsilon_i}(t,\cdot)|\, dx = 0$$

by Proposition 6.4 and the dominated convergence theorem. For such $t \geq 0$, there exists a converging subsequence $\{V_t^{\varepsilon_{i_j}}\}_{j=1}^\infty$ and a limit V_t with (7.6) satisfied. Then by (7.2), (7.6) and (7.9), we obtain for $g \in C^1(\overline{\Omega};\mathbb{R}^n)$

$$(7.10)\qquad \lim_{j\to\infty} |\delta V_t^{\varepsilon_{i_j}}(g)| \leq c(t)(c_1+1)\max_{\overline{\Omega}}|g|,$$

where we set $c(t)$ be the quantity (7.6). By the definition of varifold convergence, we have

$$(7.11)\qquad |\delta V_t(g)| = \lim_{j\to\infty} |\delta V_t^{\varepsilon_{i_j}}(g)| \leq c(t)(c_1+1)\sup_{\overline{\Omega}}|g|.$$

This shows that the total variation $\|\delta V_t\|$ is a Radon measure, showing (7.7). Since $\|V_t^{\varepsilon_{i_j}}\| = \mu_t^{\varepsilon_{i_j}}$, we have $\|V_t\| = \mu_t$ which is uniquely determined. A covering argument using the monotonicity formula (see the proof of [35, Cor. 6.6]) shows

$$(7.12)\qquad \mathcal{H}^{n-1}(\operatorname{spt}\mu_t) < \infty.$$

By (7.12) (for more detail, see [35, Prop. 6.11]) and (7.11), Allard's rectifiability theorem shows that V_t is a rectifiable varifold, and in particular, V_t is determined uniquely by $\|V_t\| = \mu_t$. This proves μ_t is rectifiable for a.e. $t \geq 0$. The argument up to this point applies equally to any converging subsequence with (7.6) and (7.9), thus the uniqueness of the limit varifold follows. Since $c(t)$ is locally uniformly integrable, Fatou's lemma shows (7.8). ☐

We comment that the $\sigma^{-1}V_t\lfloor_{\Omega}\in \mathbf{IV}_{n-1}(\Omega)$ follows from the interior argument of [37] or [35]. Thus, up to this point, we proved Theorem 2.1 and 2.3. We next prove Theorem 2.5.

Proposition 7.4. *For a.e. $t\geq 0$ such that the claim of Proposition 7.3 holds, define $\delta V_t\lfloor_{\partial\Omega}^{\top}$ as in* (2.15)*. Then we have*

$$\|\delta V_t\lfloor_{\partial\Omega}^{\top}+\delta V_t\lfloor_{\Omega}\|\ll\|V_t\| \tag{7.13}$$

and writing the Radon-Nikodym derivative as

$$h_b(t):=\begin{cases}-\frac{\delta V_t\lfloor_{\partial\Omega}^{\top}}{\|V_t\|} & \text{on } \partial\Omega,\\ -\frac{\delta V_t\lfloor_{\Omega}}{\|V_t\|} & \text{on } \Omega,\end{cases} \tag{7.14}$$

we have (2.17) *and*

$$\int_{\overline{\Omega}}\phi|h_b|^2\,d\|V_t\|\leq\liminf_{i\to\infty}\int_{\Omega}\varepsilon_i\big(\Delta u^{\varepsilon_i}-\frac{W'}{\varepsilon_i^2}\big)^2\phi\,dx \tag{7.15}$$

for $\phi\in C(\overline{\Omega};\mathbb{R}^+)$.

Proof. Let $V_t^{\varepsilon_{i_j}}$ be a subsequence converging to V_t. For any $g\in C^1_c(\Omega;\mathbb{R}^n)$, we may prove from (7.2) that

$$|\delta V_t(g)|=\lim_{j\to\infty}|\delta V_t^{\varepsilon_{i_j}}(g)|\leq\big(\int_{\Omega}|g|^2\,d\|V_t\|\big)^{\frac12}\liminf_{j\to\infty}\big(\int_{\Omega}\varepsilon_{i_j}\big(\Delta u^{\varepsilon_{i_j}}-\frac{W'}{\varepsilon_{i_j}^2}\big)^2\,dx\big)^{\frac12}. \tag{7.16}$$

This shows that $\|\delta V_t\lfloor_{\Omega}\|\ll\|V_t\|$ and $\delta V_t\lfloor_{\Omega}=-h(V_t,\cdot)\|V_t\|$ for $h(V_t,\cdot)\in L^2(\|V_t\|)$. Next, given arbitrary $\epsilon>0$, let $\nu^{\epsilon}\in C^1(\overline{\Omega};\mathbb{R}^n)$ be such that $\nu^{\epsilon}\lfloor_{\partial\Omega}=\nu$, $|\nu^{\epsilon}|\leq 1$ and $\mathrm{spt}\,\nu^{\epsilon}\subset N_{\epsilon}$. For $g\in C^1(\overline{\Omega};\mathbb{R}^n)$, define $\tilde g:=g-(\nu^{\epsilon}\cdot g)\nu^{\epsilon}$. Then, we have $\tilde g\cdot\nu=0$ on $\partial\Omega$ thus $\delta V_t\lfloor_{\partial\Omega}^{\top}(g)=\delta V_t\lfloor_{\partial\Omega}^{\top}(\tilde g)$. To prove (7.13), we note

$$\begin{aligned}\delta V_t\lfloor_{\partial\Omega}^{\top}(g)+\delta V_t\lfloor_{\Omega}(g)&=\delta V_t\lfloor_{\partial\Omega}(\tilde g)+\delta V_t\lfloor_{\Omega}(\tilde g)+\delta V_t\lfloor_{\Omega}(g-\tilde g)\\&=\delta V_t(\tilde g)+\delta V_t\lfloor_{\Omega}(g-\tilde g)\\&=\lim_{j\to\infty}\delta V_t^{\varepsilon_{i_j}}(\tilde g)+\delta V_t\lfloor_{\Omega}(g-\tilde g)\\&=\lim_{j\to\infty}\int_{\Omega}(\tilde g\cdot\nabla u^{\varepsilon_{i_j}})(\varepsilon_{i_j}\Delta u^{\varepsilon_{i_j}}-\frac{W'}{\varepsilon_{i_j}})\,dx+\delta V_t\lfloor_{\Omega}(g-\tilde g)\\&\leq\big(\int_{\Omega}|g|^2\,d\|V_t\|\big)^{\frac12}\liminf_{j\to\infty}\big(\int_{\Omega}\varepsilon_{i_j}\big(\Delta u^{\varepsilon_{i_j}}-\frac{W'}{\varepsilon_{i_j}^2}\big)^2\,dx\big)^{\frac12}+\delta V_t\lfloor_{\Omega}(g-\tilde g)\end{aligned} \tag{7.17}$$

where we used (7.2), (7.9) and $|\tilde g|\leq|g|$. Since $\mathrm{spt}\,\nu^{\epsilon}\subset N_{\epsilon}$, we have

$$|\delta V_t\lfloor_{\Omega}(g-\tilde g)|=\big|\int_{\Omega}h(V_t,\cdot)\cdot(g-\tilde g)\,d\|V_t\|\big|\leq\sup|g|\int_{N_{\epsilon}}|h(V_t,\cdot)|\,d\|V_t\|\to 0 \tag{7.18}$$

as $\epsilon\to 0$. Then (7.17) and (7.18) show (7.15) with $\phi=1$. For general $\phi\in C(\overline{\Omega};\mathbb{R}^+)$, we may carry out an approximation argument to obtain (7.15) (see [35, Prop. 8.2] for the detail). ☐

The inequality (2.17) follows from (7.15) with $\phi=1$ and (2.14).

Proposition 7.5. *Let $t\geq 0$ and $\{V_t^{\varepsilon_{i_j}}\}_{j=1}^{\infty}$ be as in Proposition 7.3, and define $h_b(t)$ as in* (7.14)*. Then we have*

$$\lim_{j\to\infty}\int_{\Omega}(g\cdot\nabla u^{\varepsilon_{i_j}})\big(\varepsilon_{i_j}\Delta u^{\varepsilon_{i_j}}-\frac{W'}{\varepsilon_{i_j}}\big)\,dx=-\int_{\overline{\Omega}}g\cdot h_b(t)\,d\|V_t\| \tag{7.19}$$

for $g \in C^1(\overline{\Omega};\mathbb{R}^n)$ with $g\cdot\nu = 0$ on $\partial\Omega$.

Proof. Since $V_t^{\varepsilon_{i_j}}$ converges to V_t as varifold, we have $\lim_{j\to\infty}\delta V_t^{\varepsilon_{i_j}}(g) = \delta V_t(g)$. On the right-hand side of (7.2), since $g\cdot\nu = 0$ on $\partial\Omega$, the third boundary integral term vanishes. Then we have (7.19) from (7.2), (7.9) and $\delta V_t(g) = \delta V_t\lfloor_{\Omega}(g) + \delta V_t\lfloor^{\top}_{\partial\Omega}(g)$. □

Finally we give

Proof of Theorem 2.6. It is enough to prove (2.18) for $\phi \in C^2(\overline{\Omega}\times[0,\infty)\,;\mathbb{R}^+)$ with $\nabla\phi(\cdot,t)\cdot\nu = 0$ on $\partial\Omega$. By writing $f^{\varepsilon_i} := -\varepsilon_i\Delta u^{\varepsilon_i} + \frac{W'}{\varepsilon_i}$, we have from (1.1)

$$\int_\Omega \phi\, d\mu_t^{\varepsilon_i}\Big|_{t=t_1}^{t_2} = \int_{t_1}^{t_2}\Big(\int_\Omega -\frac{1}{\varepsilon_i}(f^{\varepsilon_i})^2\phi + f^{\varepsilon_i}\nabla\phi\cdot\nabla u^{\varepsilon_i}\,dx + \int_\Omega \partial_t\phi\, d\mu_t^{\varepsilon_i}\Big)\,dt. \tag{7.20}$$

Since we already know that $\mu_t^{\varepsilon_i} \rightharpoonup \|V_t\|$ for all $t\geq 0$, the left-hand side of (7.20) converges to that of (2.18), and so is the last term of the right-hand side. So we only need to consider the first and second terms of the right-hand side. Just as in the proof of Lemma 5.1, $\int_\Omega(\varepsilon_i^{-1}(f^{\varepsilon_i})^2\phi - f^{\varepsilon_i}\nabla\phi\cdot\nabla u^{\varepsilon_i})\,dx \geq -c_1\|\phi\|_{C^2}$. Thus by Fatou's lemma,

$$\lim_{i\to\infty}\int_{t_1}^{t_2}\int_\Omega \frac{1}{\varepsilon_i}(f^{\varepsilon_i})^2\phi - f^{\varepsilon_i}\nabla\phi\cdot\nabla u^{\varepsilon_i}\,dxdt \geq \int_{t_1}^{t_2}\liminf_{i\to\infty}\int_\Omega\frac{1}{\varepsilon_i}(f^{\varepsilon_i})^2\phi - f^{\varepsilon_i}\nabla\phi\cdot\nabla u^{\varepsilon_i}\,dxdt. \tag{7.21}$$

Thus from (7.20) and (7.21), we will finish the proof if we prove

$$\liminf_{i\to\infty}\int_\Omega\frac{1}{\varepsilon_i}(f^{\varepsilon_i})^2\phi - f^{\varepsilon_i}\nabla\phi\cdot\nabla u^{\varepsilon_i}\,dx \geq \int_{\overline{\Omega}}\phi|h_b|^2 - h_b\cdot\nabla\phi\,d\|V_t\| \tag{7.22}$$

for a.e. $t\in[t_1,t_2]$. For a.e. t where the assumption of Proposition 7.3 is satisfied, we have already proved (7.15) and (7.19). But this shows precisely (7.22). This ends the proof of (2.18). □

8. Final remarks

It seems likely that, if $\|V_0\|(\partial\Omega) = 0$, then $\|V_t\|(\partial\Omega) = 0$ holds for all $t > 0$. Intuitively, due to the strict convexity of the domain and the Neumann boundary condition (which should intuitively imply 90 degree angle of intersection), interior of moving hypersurfaces should not touch $\partial\Omega$. Due to the maximum principle, this cannot happen if the hypersurfaces are smooth up to the boundary. But within the general framework of this paper, we do not know how to prove such statement or if it is indeed true.

Though it may first appear counter intuitive in view of the connection to the mean curvature flow, if we have $\|V_0\|(\partial\Omega) > 0$, then it is possible to have $\|V_t\|(\partial\Omega) > 0$ for all $t > 0$. An example can be provided by a limit of time-independent solutions of (1.1) where $\mu^\varepsilon \rightharpoonup c\mathcal{H}^{n-1}\lfloor_{\partial\Omega}$ on $\overline{\Omega}$ as $\varepsilon\to 0$, where $c > 0$ is some constant. One can obtain such family of solutions u^ε by considering $\Omega = B_1$ and a mountain path solution connecting two constant functions 1 and -1 within a class of radially symmetric functions. There are uniform positive lower and upper bounds of $E^\varepsilon(u^\varepsilon)$ and the limiting varifold V is non-trivial. On the other hand, if $\|V\|(B_1) > 0$, due to [15], spt $\|V\|$ has to be a minimal surface, which contradicts the radially symmetry. Thus $\|V\|$ is concentrated only on ∂B_1 and is non-trivial. In this particular case, note that $\delta V = -\frac{x}{|x|}\mathcal{H}^{n-1}\lfloor_{\partial B_1}$ and the tangential component $\delta V\lfloor^{\top}_{\partial B_1}$ is 0. Using more explicit and sophisticated method, Malchiodi-Ni-Wei [23] constructed a family of solutions with multiple layers whose energy concentrates on ∂B_1 with $\|V\|(\partial B_1) = N\sigma\mathcal{H}^{n-1}$, $N\in\mathbb{N}$. N may be arbitrarily chosen. Furthermore, for general strictly mean convex domain Ω, Malchiodi-Wei

[22] constructed a family of single layered solutions whose limit energy concentrates on $\partial\Omega$. Even though such limit measures are not certainly the mean curvature flow in $\mathbb{R}^n$ in the usual sense (it should shrink), such time independent measures satisfies (2.18) trivially since $h_b = 0$. This is the reason that we need to decompose the first variation on $\partial\Omega$ to accommodate such cases in general.

The existence result of the present paper suggests a reasonable setting for proving the boundary regularity of mean curvature flow. It is interesting to extend interior regularity theorem (see [4, 17, 38]) to the corresponding boundary regularity theorem. For the time-independent case, interior regularity [1] has been extended to boundary regularity [2, 3, 13].

It is worthwhile to comment on the strict convexity assumption. The places the condition played any role are in the proof of Proposition 4.1 via Lemma 4.2, and in some computations such as (6.10) and before (6.19). Even without strict convexity on the whole of Ω, one can in fact localize these arguments. Namely, for general bounded domain with smooth boundary Ω, let $\Gamma \subset \partial\Omega$ be a set of points with some non-positive principal curvature. Then one can carry out the argument of this paper for $\overline{\Omega} \setminus (\Gamma)_\epsilon$, where $(\Gamma)_\epsilon$ is the ϵ-neighborhood of Γ. All the statements in Section 2.4 hold with $\overline{\Omega} \setminus \Gamma$ in place of $\overline{\Omega}$. We did not write the paper in this generality to avoid further notational complications.

9. Appendix

We include a lemma which appeared in [16] for reader's convenience.

$$\rho_y^r(x) := \frac{1}{(\sqrt{2\pi}r)^{n-1}} \exp\left(-\frac{|x-y|^2}{2r^2}\right).$$

Then, $\rho_{(y,s)} = \rho_y^r$ when $r^2 = 2(s-t)$.

Lemma 9.1. *Let μ be a Radon measure on $\mathbb{R}^n$ satisfying for some $D > 0$*

$$\frac{\mu(B_R(x))}{\omega_{n-1}R^{n-1}} \le D \tag{9.1}$$

for $R > 0$ and $x \in \mathbb{R}^n$. Then we obtain the following:

(1) For $r > 0$ and for $x \in \mathbb{R}^n$,

$$\int_{\mathbb{R}^n} \rho_x^r \, d\mu \le D$$

(2) For $r, R > 0$ and for $x \in \mathbb{R}^n$,

$$\int_{\mathbb{R}^n \setminus B_R(x)} \rho_x^r \, d\mu \le 2^{n-1} e^{-\frac{3R^2}{8r^2}} D$$

(3) For $\delta > 0$ there is $\gamma_1 > 0$ depending only on n and δ such that for $x, x_0 \in \mathbb{R}^n$ and $r > 0$ satisfying $|x - x_0| < \gamma_1 r$ we have

$$\int_{\mathbb{R}^n} \rho_{x_0}^r \, d\mu \le (1+\delta) \int_{\mathbb{R}^n} \rho_x^r \, d\mu + \delta D.$$

(4) For $\delta > 0$ there is $\gamma_2 > 0$ depending only on n and δ such that for $x \in \mathbb{R}^n$ and $r, R > 0$ satisfying $1 \le \frac{R}{r} \le 1 + \gamma_2$ we have

$$\int_{\mathbb{R}^n} \rho_x^R \, d\mu \le (1+\delta) \int_{\mathbb{R}^n} \rho_x^r \, d\mu + \delta D.$$

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