Explicit formulas for the twisted Koecher-Maaß series of the Duke-Imamoglu-Ikeda lift and their applications

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Abstract
We give an explicit formula for the twisted Koecher-Maaß series of the Duke-Imamoglu-Ikeda lift. As an application we prove a certain algebraicity result for the values of twisted Rankin-Selberg series at integers of half-integral weight modular forms.

1 Introduction
It is an interesting problem to give an explicit formula for the Koecher-Maaß series of a Siegel modular form $F$ for the symplectic group $Sp_n(\mathbb{Z})$, and several results have been obtained (cf. Böcherer [B], Ibukiyama and Katsurada [I-K1], [I-K2], [I-K3]). Such explicit formulas are not only interesting in its own right but also have some important applications in the theory of modular forms. For example, we refer to [B-S], [D-I], [Miz]. Now we consider a twist of such a Koecher-Maaß series by a Dirichlet character $\chi$. As for this, in view of Saito [Sai1] for example, we can naturally consider the following Dirichlet series:

$$L^\ast(s, F, \chi) = \sum_T \chi(2^{[n/2]} \det T) c_F(T) e(T)(\det T)^s,$$

where $T$ runs over a complete set of representatives of $SL_n(\mathbb{Z})$-equivalence classes of positive definite half-integral symmetric matrices of degree $n$, $c_F(T)$ is the $T$-th Fourier coefficient of $F$ and $e(T) = \#\{U \in SL_n(\mathbb{Z}); T[U] = T\}$. We will sometimes call $L^\ast(s, F, \chi)$ the twisted Koecher-Maaß series of the second kind.

On the other hand, Choie and Kohnen [C-K] introduced a different type of “twist”. For a positive integer $N$, let $SL_{n,N}(\mathbb{Z}) = \{U \in SL_n(\mathbb{Z}); U \equiv I_n \mod N\}$ and $e_N(T) = \#\{U \in SL_{n,N}(\mathbb{Z}); T[U] = T\}$. For a primitive Dirichlet character $\chi$ mod $N$, the Koecher-Maaß series $L(s, F, \chi)$ of $F$ twisted by $\chi$ is defined to be

$$L(s, F, \chi) = \sum_T \chi(\text{tr}(T)) c_F(T) e_N(T)(\det T)^s.$$
where $T$ runs over a complete set of representatives of $SL_{n,N}(\mathbb{Z})$-equivalence classes of positive definite half-integral symmetric matrices of degree $n$. In [C-K], Choie and Kohnen proved a meromorphy continuation of $L(s, F, \chi)$ to the whole $s$-plane and a functional equation (cf. Theorem 2.1). Moreover they got a result on the algebraicity of its special values (cf. Theorem 2.2.) We shall call $L(s, F, \chi)$ the twist of the first kind.

In this paper we give explicit formulas for the twisted Koecher-Maaß series of the first and second kinds associated with the Duke-Imamoglu-Ikeda lift and apply them to the study of the special values of the Rankin-Selberg series for half-integral weight modular forms. We explain our main results more precisely. Let $k$ and $n$ be positive even integers such that $n \geq 4$ and $2k-n \geq 12$. For a cuspidal Hecke eigenform $h$ in the Kohnen plus subspace of weight $k-n/2+1/2$ for $I_0(4)$, let $I_n(h)$ be the Duke-Imamoglu-Ikeda lift of $h$ to the space of cusp forms of weight $k$ for $Sp_n(\mathbb{Z})$. Moreover let $S(h)$ be the normalized Hecke eigenform of weight $2k-n$ for $SL_2(\mathbb{Z})$ corresponding to $h$ under the Shimura correspondence, and $E_{n/2+1/2}$ be Cohen’s Eisenstein series of weight $n/2+1/2$ for $I_0(4)$. We then give explicit formulas for $L(s, I_n(h), \chi)$ and $L^*(s, I_n(h), \chi)$ in terms of the twisted Rankin-Selberg series $R(s, h, E_{n/2+1/2}, \eta)$ of $h$ and $E_{n/2+1/2}$ and twisted Hecke’s $L$-function $L(s, S(h), \eta')$ of $S(h)$, where $\eta$ and $\eta'$ are Dirichlet characters related with $\chi$. It is relatively easy to get an explicit form of $L^*(s, I_n(h), \chi)$. In fact, by using the same argument as in Ibukiyama and Katsurada [I-K2], we can easily obtain its explicit formula (cf. Theorem 4.1). On the other hand, it seems nontrivial to get that of $L(s, I_n(h), \chi)$ (cf. Theorem 6.1), and we need some explicit formula for a certain character sum associated with a Dirichlet character (cf. Theorem 5.6). Using Theorem 6.1 combined with the result of Choie-Kohnen, we prove certain algebraicity results on $R(s, h, E_{n/2+1/2}, \eta)$ at an integer $s = m$ (cf. Theorems 7.1 and 7.2), which were announced in [Ka]. We note that the algebraicity of the special values of such a Rankin-Selberg series at half-integers was investigated by Shimura [Sh3]. However there are few results on the algebraicity of such values at integers. As an attempt, Mizuno and the author [K-M] proved linear dependency of Rankin-Selberg $L$-values of a cuspidal Hecke eigenform belonging to Kohnen plus subspace of half integral weight and the Zagier’s Eisenstein series of weight $3/2$. Our present result can be regarded as a generalization of our previous result.

**Notation** We denote by $e(x) = \exp(2\pi i x)$ for a complex number $x$. For a commutative ring $R$, we denote by $M_{mn}(R)$ the set of $(m, n)$-matrices with entries in $R$. For an $(m, n)$-matrix $X$ and an $(m, m)$-matrix $A$, we write $A[X] = XAX$, where $X$ denotes the transpose of $X$. Let $a$ be an element of $R$. Then for an element $X$ of $M_{mn}(R)$ we often use the same symbol $X$ to denote the coset $X$ mod $aM_{mn}(R)$. Put $GL_m(R) = \{ A \in M_m(R) \mid \det A \in R^* \}$, and $SL_m(R) = \{ A \in M_m(R) \mid \det A = 1 \}$, where $\det A$ denotes the determinant of a square matrix $A$ and $R^*$ is the unit group of $R$. We denote by $S_n(R)$ the set of symmetric matrices of degree $n$ with entries in $R$. In particular, if $S$ is a subset of $S_n(R)$ with $R$ the field of real numbers, we denote by $S_{\geq 0}$ (resp. $S_{\geq 0}$) the subset of $S$ consisting of positive definite (resp. semi-positive definite)
matrices. The group $SL_n(\mathbb{Z})$ acts on the set $S_n(\mathbb{R})$ in the following way:

$$SL_n(\mathbb{Z}) \times S_n(\mathbb{R}) \ni (g, A) \mapsto {}^t gAg \in S_n(\mathbb{R}).$$

Let $G$ be a subgroup of $GL_n(\mathbb{Z})$. For a subset $\mathcal{B}$ of $S_n(\mathbb{R})$ stable under the action of $G$ we denote by $\mathcal{B}/G$ the set of equivalence classes of $\mathcal{B}$ with respect to $G$. We sometimes identify $\mathcal{B}/G$ with a complete set of representatives of $\mathcal{B}/G$.

Two symmetric matrices $A$ and $A'$ with entries in $\mathbb{R}$ are said to be equivalent with respect to $G$ and write $A \sim_G A'$ if there is an element $X$ of $G$ such that $A' = AXA$. Let $\mathcal{L}_n$ denote the set of half-integral matrices of degree $n$ over $\mathbb{Z}$, that is, $\mathcal{L}_n$ is the set of symmetric matrices of degree $n$ whose $(i, j)$-component belongs to $\mathbb{Z}$ or $\frac{1}{2} \mathbb{Z}$ according as $i = j$ or not.

## 2 Twisted Koecher-Maaß series

Put $J_n = \left( \begin{array}{cc} O_n & -1_n \\ 1_n & O_n \end{array} \right)$, where $1_n$ and $O_n$ denotes the unit matrix and the zero matrix of degree $n$, respectively. Furthermore, put

$$Sp_n(\mathbb{Z}) = \{ M \in GL_{2n}(\mathbb{Z}) \mid J_n[M] = J_n \}.$$

Let $l$ be an integer or a half-integer, and $N$ a positive integer. Let $\Gamma_0^{(n)}(N)$ be the congruence subgroup of $Sp_n(\mathbb{Z})$ consisting of matrices whose left lower $n \times n$ block are congruent to $O_n \mod N$. Moreover let $\chi$ be a Dirichlet character mod $N$. We then denote by $M_l(\Gamma_0^{(n)}(N), \chi)$ the space of modular forms of weight $l$ and character $\chi$ for $\Gamma_0^{(n)}(N)$, and by $E_l(I_0^{(n)}(N), \chi)$ the subspace of $M_l(\Gamma_0^{(n)}(N), \chi)$ consisting of cusp forms. If $\chi$ is the trivial character mod $N$, we simply write $M_l(\Gamma_0^{(n)}(N), \chi)$ and $E_l(I_0^{(n)}(N), \chi)$ as $M_l(\Gamma_0^{(n)}(N))$ and $E_l(I_0^{(n)}(N))$, respectively. Let $k$ be a positive integer, and let $F(Z) \in M_k(Sp_n(\mathbb{Z}))$. Then $F(Z)$ has the Fourier expansion:

$$F(Z) = \sum_{T \in \mathcal{L}_{n \geq 0}} c_F(T)e^{\text{tr}(TZ)},$$

where $\text{tr}(X)$ denotes the trace of a matrix $X$. For $N \in \mathbb{Z}_{>0}$, put $SL_{n,N}(\mathbb{Z}) = \{ U \in SL_n(\mathbb{Z}) \mid U \equiv 1_n \mod N \}$, and for $T \in \mathcal{L}_{n \geq 0}$ put $e_N(T) = \# \{ U \in SL_{n,N}(\mathbb{Z}) \mid T[U] = T \}$. For a primitive Dirichlet character $\chi \mod N$ Let

$$L(s, F; \chi) = \sum_{T \in \mathcal{L}_{n \geq 0}/SL_{n,N}(\mathbb{Z})} \frac{\chi(\text{tr}(T))c_F(T)}{e_N(T)(\det T)^s}$$

be the twisted Koecher-Maaß series of the first kind of $F$ as in Section 1. The following two theorems are due to Choie and Kohnen [C-K].

**Theorem 2.1.** Let $F \in E_k(Sp_n(\mathbb{Z}))$, and $\chi$ a primitive character of conductor $N$. Put

$$\gamma_n(s) = (2\pi)^{-n^2} \prod_{i=1}^{n} \pi^{(i-1)/2} \Gamma(s - (i-1)/2),$$

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and
\[ \Lambda(s, F, \chi) = N^{2s} \tau(\chi)^{-1} \gamma_n(s) L(s, F, \chi) \quad (\text{Re}(s) >> 0), \]
where \( \tau(\chi) \) is the Gauss sum of \( \chi \). Then \( \Lambda(s, F, \chi) \) has an analytic continuation to the whole \( s \)-plane and has the following functional equation:
\[ \Lambda(k - s, F, \chi) = (-1)^{nk/2} \chi(-1) \Lambda(s, F, \chi). \]

**Theorem 2.2.** Let \( F \) and \( \chi \) be as above. Then there exists a finite dimensional \( \overline{Q} \)-vector space \( V_F \) in \( C \) such that
\[ L(m, F, \chi) \pi^{-nm} \in V_F \]
for any primitive character \( \chi \) and any integer \( m \) such that \((n+1)/2 \leq m \leq k - (n+1)/2\).

Now let
\[ L^*(s, F, \chi) = \sum_{T \in \mathcal{L}_{\eta} / SL_2(\mathbb{Z})} \chi(2^{[n/2]} \det T) c_F(T) \]
be the twisted Koecher-Maass series of the second kind of \( F \) as in Section 1. We will discuss a relation between these two Dirichlet series in Section 5.

### 3 Review on the algebraicity of \( L \)-values of elliptic modular forms of integral and half-integral weight

In this section, we review on the special values of \( L \) functions of elliptic modular forms of integral and half-integral weights. For a modular form \( g(z) \) of integral or half-integral weight for a certain congruence subgroup \( \Gamma \) of \( SL_2(\mathbb{Z}) \), let \( \mathbb{Q}(g) \) denote the field generated over \( \mathbb{Q} \) by all the Fourier coefficients of \( g \), and for a Dirichlet character \( \eta \) let \( \mathbb{Q}(\eta) \) denote the field generated over \( \mathbb{Q} \) by all the values of \( \eta \).

First let
\[ f(z) = \sum_{m=1}^{\infty} c_f(m) e(mz) \]
be a normalized Hecke eigenform in \( \mathcal{S}_k(SL_2(\mathbb{Z})) \), and \( \chi \) be a primitive Dirichlet character. Then let us define Hecke’s \( L \)-function \( L(s, f, \chi) \) of \( f \) twisted by \( \chi \) as
\[ L(s, f, \chi) = \sum_{m=1}^{\infty} c_f(m) \chi(m) m^{-s}. \]

Then we have the following result (cf. [Sh2]):
Proposition 3.1. There exist complex numbers \( u_\pm(f) \) uniquely determined up to \( \mathbb{Q}(f)^\times \) multiple such that

\[
\frac{L(m, f, \chi)}{(2\pi\sqrt{-1})^m \tau_\chi u_j(f)} \in \mathbb{Q}(f)\mathbb{Q}(\chi)
\]

for any integer \( 0 < m \leq k - 1 \) and a primitive character \( \chi \), where \( \tau_\chi \) is the Gauss sum of \( \chi \), and \( j = + \) or \(-\) according as \((-1)^m \chi(-1) = 1 \) or \(-1 \).

Corollary. Under the above notation and the assumption, we have

\[
L(m, f, \chi)^{-m} \in \mathbb{Q}u_j(f)
\]

for any integer \( 0 < m \leq k - 1 \) and a primitive character \( \chi \).

We remark that we have \( L(m, f, \chi) \neq 0 \) if \( m \neq k/2 \), and \( L(k/2, f, \chi) \neq 0 \) for infinitely many \( \chi \).

Next let us consider the half-integral weight case. From now on we simply write \( \Gamma_0(4) \) as \( \Gamma_0(M) \).

Let

\[
h_1(z) = \sum_{m=1}^{\infty} c_{h_1}(m)e(mz)
\]

be a Hecke eigenform in \( \mathfrak{S}_{k_1+1/2}(\Gamma_0(4)) \), and

\[
h_2(z) = \sum_{m=0}^{\infty} c_{h_2}(m)e(mz)
\]

be an element of \( \mathfrak{S}_{k_2+1/2}(\Gamma_0(4)) \). For a fundamental discriminant \( D \) let \( \chi_D \) be the Kronecker character corresponding to \( D \). Let \( \chi \) be a primitive character mod \( N \). Then we define

\[
\tilde{R}(s, h_1, h_2, \chi) = L(2s - k_1 - k_2 + 1, \omega) \sum_{m=1}^{\infty} c_{h_1}(m)c_{h_2}(m)\chi(m)m^{-s},
\]

where \( \omega(d) = \chi_{-4}^{k_1-k_2}\chi^2(d) \). We also define \( R(s, h_1, h_2, \chi) \) as

\[
R(s, h_1, h_2, \chi) = L(2s - k_1 - k_2 + 1, \chi^2) \sum_{m=1}^{\infty} c_{h_1}(m)c_{h_2}(m)\chi(m)m^{-s}.
\]

Now let \( S(h_1) \) be the normalized Hecke eigenform in \( \mathfrak{S}_{2k_1}(SL_2(\mathbb{Z})) \) corresponding to \( h_1 \) under the Shimura correspondence. Then the following result is due to Shimura [Sh3].

Proposition 3.2. Assume that \( k_1 > k_2 \). Under the above notation we have

\[
\frac{\tilde{R}(m + 1/2, h_1, h_2, \chi)}{u_-(S(h_1))\tau(\chi^2)\pi^{-k_2+1+2m}\sqrt{-1}} \in \mathbb{Q}(h_1)\mathbb{Q}(h_2)\mathbb{Q}(\chi)
\]

for any integer \( k_2 \leq m \leq k_1 - 1 \) and a primitive character \( \chi \).
Proof. Let $N$ be the conductor of $\chi$. Put

$$h_{2\chi}(z) = \sum_{m=0}^{\infty} c_{h_2}(m) \chi(m) e(mz).$$

Then $h_{2\chi}(z) \in \mathbb{H}_{k_2+1/2}(4N^2, \chi^2)$. We can regard $h_1$ as an element of $\mathfrak{E}_{k_1+1/2}(\Gamma_0(4N^2))$. Then the assertion follows from [[Sh3], Theorem 2].

Corollary. Assume that $c_{h_1}(n), c_{h_2}(n) \in \mathbb{Q}$ for any $n \in \mathbb{Z}_{\geq 0}$. Then there exists a one-dimensional $\mathbb{Q}$-vector space $U_{h_1,h_2}$ in $\mathbb{C}$ such that

$$R(m + 1/2, h_1, h_2, \chi) \pi^{-2m} \in U_{h_1,h_2}$$

for any integer $k_2 \leq m \leq k_1 - 1$ and a primitive character $\chi$.

4 Explicit formulas for the twisted Koecher-Maaß series of the second kind of the Duke-Imamoglu-Ikeda lift

Throughout this section, we assume that $n$ and $k$ are even positive integers. Let $h$ be a Hecke eigenform of weight $k - n/2 + 1/2$ for $\Gamma_0(4)$ belonging to the Kohnen plus space. Then $h$ has the following Fourier expansion:

$$h(z) = \sum_{e} c_{h}(e) e(\varepsilon z),$$

where $e$ runs over all positive integers such that $(-1)^{k-n/2}e \equiv 0, 1 \pmod{4}$. Let

$$S(h)(z) = \sum_{m=1}^{\infty} c_{S(h)}(m) e(mz)$$

be the normalized Hecke eigenform of weight $2k - n$ for $SL_2(\mathbb{Z})$ corresponding to $h$ via the Shimura correspondence (cf. [Ko].) For a prime number $p$ let $\beta_p$ be a nonzero complex number such that $\beta_p + \beta_p^{-1} = p^{-k+n/2+1/2}c_{S(h)}(p)$. For a non-negative integers $l$ and $m$, the Cohen function $H(l, m)$ is given by $H(l, m) = L_{-m}(1 - l)$. Here

$$L_D(s) = \begin{cases} \zeta(2s-1), & D = 0 \\ L(s, \chi_{D_K}) \sum_{a|f} \mu(a) \chi_{D_K}(a)a^{-s}\sigma_{1-2s}(f/a), & D \neq 0, D \equiv 0, 1 \pmod{4} \\ 0, & D \equiv 2, 3 \pmod{4}, \end{cases}$$

$$L_D(s) = \begin{cases} \zeta(2s-1), & D = 0 \\ L(s, \chi_{D_K}) \sum_{a|f} \mu(a) \chi_{D_K}(a)a^{-s}\sigma_{1-2s}(f/a), & D \neq 0, D \equiv 0, 1 \pmod{4} \\ 0, & D \equiv 2, 3 \pmod{4}, \end{cases}$$

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where the positive integer \( f \) is defined by \( D = D_K f^2 \) with the discriminant \( D_K \) of \( K = \mathbb{Q}(\sqrt{D}) \), \( \mu \) is the M"obius function, and \( \sigma_s(n) = \sum_{d|n} d^s \). Furthermore, for an even integer \( l \geq 4 \), we define the Cohen Eisenstein series \( E_{l+1/2}(z) \) by

\[
E_{l+1/2}(z) = \sum_{c=0}^{\infty} H(l, c) e(cz).
\]

It is known that \( E_{l+1/2}(z) \) is a modular form of weight \( l + 1/2 \) for \( \Gamma_0(4) \) belonging to the Kohnen plus space.

For a prime number \( p \) let \( \mathbb{Q}_p \) and \( \mathbb{Z}_p \) be the field of \( p \)-adic numbers, and the ring of \( p \)-adic integers, respectively. We denote by \( \nu_p \) the additive valuation on \( \mathbb{Q}_p \), normalized so that \( \nu_p(p) = 1 \), and by \( e_p \) the continuous homomorphism from the additive group \( \mathbb{Q}_p \) to \( \mathbb{C}^\times \) such that \( e_p(a) = e(a) \) for \( a \in \mathbb{Q} \). For a \( p \)-adic number \( c \) put

\[
\tilde{\zeta}_p(c) = 1, -1 \text{ or } 0
\]

according as \( \mathbb{Q}_p(\sqrt{c}) = \mathbb{Q}_p, \mathbb{Q}_p(\sqrt{c})/\mathbb{Q}_p \) is quadratic unramified, or \( \mathbb{Q}_p(\sqrt{c})/\mathbb{Q}_p \) is quadratic ramified. We note that \( \tilde{\xi}_p(D) = \chi_D(p) \) for a fundamental discriminant \( D \). For a non-degenerate half-integral matrix \( T \) over \( \mathbb{Z}_p \), let

\[
b_p(T, s) = \sum_{R \in S_n(\mathbb{Q}_p)/S_n(\mathbb{Z}_p)} e_p(\text{tr}(TR)) p^{-\nu_p(\mu_p(R)) s}
\]

be the local Siegel series, where \( \mu_p(R) = [R \mathbb{Z}_p^n + \mathbb{Z}_p^n : \mathbb{Z}_p^n] \). Then there exists a polynomial \( F_p(T, X) \) in \( X \) such that

\[
b_p(T, s) = F_p(T, p^{-s})(1 - p^{-s})(1 - \xi_p(T)p^{n/2 - s})^{-1} \prod_{i=1}^{n/2} (1 - p^{2i-2s})
\]

(cf. [Ki1]), where \( \xi_p(T) = \tilde{\zeta}_p((-1)^{n/2} \text{det} T) \). For a positive definite half integral matrix \( T \) of degree \( n \) write \((-1)^{n/2} \text{det}(2T)\) as \((-1)^{n/2} \text{det}(2T) = \nu_T h_T^2 \) with \( h_T \) a fundamental discriminant and \( \nu_T \) a positive integer. We then put

\[
c_{I_n(h)}(T) = c_h([\nu_T]) \prod_{p} (p^{k-n/2-1/2} \beta_p \nu_p(\beta_p)) \beta_p^{-1} F_p(T, p^{-(n+1)/2} \beta_p^{-1}).
\]

We note that \( c_{I_n(h)}(T) \) does not depend on the choice of \( \beta_p \). Define a Fourier series \( I_n(h)(Z) \) by

\[
I_n(h)(Z) = \sum_{T \in \mathcal{L}_{n > 0}} c_{I_n(h)}(T)e(\text{tr}(TZ)).
\]

In [I] Ikeda showed that \( I_n(h)(Z) \) is a Hecke eigenform in \( \mathfrak{S}_h(\text{Sp}_n(\mathbb{Z})) \) and its standard \( L \)-function \( L(s, I_n(h), St) \) is given by

\[
L(s, I_n(h), St) = \zeta(s) \prod_{i=1}^{n} L(s + k - i, S(h)).
\]

We call \( I_n(h) \) the Duke-Imamoglu-Ikeda lift (D-I-I lift) of \( h \).
Theorem 4.1. Let $\chi$ be a primitive Dirichlet character mod $N$. Then we have
\[
L^\ast(s, F, \chi) = c_n R(s, h, E_{n/2+1/2}, \chi) \prod_{j=1}^{n/2-1} L(2s - 2j, S(h), \chi^2)
+ d_n c_h(1) \prod_{j=1}^{n/2} L(2s - 2j + 1, S(h), \chi^2),
\]
where $c_n$ and $d_n$ are nonzero rational numbers depending only on $n$.

To prove Theorem 4.1, we reduce the problem to local computations. For $a, b \in \mathbb{Q}_p^\times$ let $(a, b)_p$ the Hilbert symbol on $\mathbb{Q}_p$. Following Kitaoka [Ki2], we define the Hasse invariant $\varepsilon(A)$ of $A \in S_n(\mathbb{Q}_p)^\times$ by
\[
\varepsilon(A) = \prod_{1 \leq i \leq j \leq n} (a_i, a_j)_p
\]
if $A$ is equivalent to $a_1 \perp \cdots \perp a_n$ over $\mathbb{Q}_p$ with some $a_1, a_2, \ldots, a_n \in \mathbb{Q}_p^\times$. For $T \in S_n(\mathbb{Z}_p)_e$, put $T^{(0)} = 2^{-1} T, F_p^{(0)}(T, X) = F_p(T^{(0)}, X)$, and so on. Then for non-degenerate symmetric matrices $A$ of degree $n$ with entries in $\mathbb{Z}_p$ we define the local density $\alpha_p(A) = \alpha_p(A, A)$ representing $A$ by $A$ as
\[
\alpha_p(A) = 2^{-1} \lim_{a \to \infty} p^{a(-n^2 + n(n+1)/2)} \# A_n(A, A),
\]
where
\[
A_n(A, A) = \{ X \in M_n(\mathbb{Z}_p)/p^n M_n(\mathbb{Z}_p) \mid A[X] - B \in p^n S_n(\mathbb{Z}_p)_e \}.
\]
Furthermore put
\[
M(A) = \frac{1}{\prod_{A' \in \mathcal{G}(A)} \varepsilon(A')}
\]
for a positive definite symmetric matrix $A$ of degree $n$ with entries in $\mathbb{Z}$, where $\mathcal{G}(A)$ denotes the set of $SL_n(\mathbb{Z})$-equivalence classes belonging to the genus of $A$. Then by Siegel's main theorem on the quadratic forms, we obtain
\[
M(A) = \kappa_n 2^{2-n} \det A^{(n+1)/2} \prod_p \alpha_p(A)^{-1}
\]
where $c_n = 1$ or $2$ according as $n = 1$ or not, and $\kappa_n = \prod_{i=1}^{n/2} \Gamma_C(2i)$ (cf. Theorem 6.8.1 in [Ki2]). Put
\[
\mathcal{F}_p = \{ d_0 \in \mathbb{Z}_p \mid \nu_p(d_0) \leq 1 \}
\]
if $p$ is an odd prime, and
\[
\mathcal{F}_2 = \{ d_0 \in \mathbb{Z}_2 \mid d_0 \equiv 1 \mod 4, \text{ or } d_0/4 \equiv -1 \mod 4, \text{ or } \nu_2(d_0) = 3 \}.
\]
For $d \in \mathbb{Z}_p^*$ put

$$S_n(Z_p, d) = \{ T \in S_n(Z_p) \mid (-1)^{n/2} \text{det } T = p^{2i}d \text{ mod } Z_p \text{ with some } i \in \mathbb{Z}, \}$$

and $S_n(Z_p, d)_x = S_n(Z_p, d) \cap S_n(Z_p)_x$ for $x = e$ or $o$. Put $L^{(0)}_{n,p} = S_n(Z_p)_e^*$ and $L^{(0)}_{n,p}(d) = S_n(Z_p, d) \cap L^{(0)}_{n,p}$. Let $\gamma_{n,p}$ be the constant function on $L^{(0)}_{n,p}$ taking the value 1, and $\varepsilon_{n,p}$ the function on $L^{(0)}_{n,p}$ assigning the Hasse invariant of $A$ for $A \in L^{(0)}_{n,p}$. We sometimes drop the suffix and write $\gamma_{n,p}$ as $\gamma_p$ or $\gamma$ and the others if there is no fear of confusion. From now on we sometimes write $\omega = \varepsilon^l$ with $l = 0$ or 1 according as $\omega = \varepsilon$ or $\varepsilon$. For $d_0 \in \mathcal{F}$ and $\omega = \varepsilon^l$ with $l = 0, 1$, we define a formal power series $P^{(0)}_{n,p}(d_0, \omega, X, l)$ in $t$ by

$$P^{(0)}_{n,p}(d_0, \omega, X, l) = \kappa(d_0, n, l)^{-1} \sum_{B \in L^{(0)}_{n,p}(d_0)} \tilde{F}^{(0)}_{p}(B, X) \frac{\alpha_p(B)}{\omega(B)} t^{\nu_p(\text{det } B)},$$

where

$$\kappa(d_0, n, l) = \kappa(d_0, n, l)_p = \{(-1)^{n(n+2)/8}((-1)^{n/2}2d_0)_2\}^{l_2,p}.$$ Let $\mathcal{F}$ denote the set of fundamental discriminants, and for $l = \pm 1$, put

$$\mathcal{F}(l) = \{ d_0 \in \mathcal{F} \mid l d_0 > 0 \}.$$

**Theorem 4.2.** Let the notation and the assumption be as above. Then for $\text{Re}(s) > 0$, we have

$$L^*(s, I_n(h)) = \kappa_n 2^{ns+1-n} \times \left\{ \sum_{d_0 \in \mathcal{F}(l) \cap \mathbb{Z}^*/(\mathbb{Z}^*)^2} c_h([d_0]) |d_0|^{n/4-k/2+1/4} \prod_{p} P^{(0)}_{n,p}(d_0, \gamma_p, \alpha_p, p^{-s+k/2+n/4+1/4}) \chi(p) \right\}$$

$$+ \left\{ (-1)^{n(n+2)/8} (d_0)_2 c_h([d_0]) |d_0|^{n/4-k/2+1/4} \prod_{p} P^{(0)}_{n,p}(d_0, \varepsilon_p, \alpha_p, p^{-s+k/2+n/4+1/4}) \chi(p) \right\}.$$ 

**Proof.** Let $T \in S_n(Z_p)_{e \geq 0}$. Then the $T$-th Fourier coefficient $c_{I_n(h)}(T)$ of $I_n(h)$ is uniquely determined by the genus to which $T$ belongs, and, by definition, it can be expressed as

$$c_{I_n(h)}(T) = c_h([T]) \frac{\zeta^{(0)}(k/2-n/4-1/4)}{\zeta^{(0)}(k/2-n/4-1/4)} \prod_{p} \tilde{F}^{(0)}(T, \alpha_p).$$

We also note that

$$\tilde{F}^{(0)}(k/2-n/4-1/2) = |T^{(0)}|^{-k/2-n/4+1/4} \prod_{p} \tilde{F}^{(0)}(T, \alpha_p)^{k/2-n/4-1/4}.$$
for $T \in S_n(Z_p)_{e>0}$. Hence we have

$$\sum_{T^0 \in \mathcal{G}(T)} \frac{c_{L_n}(b_0)(T^0)}{e(T^0)} = \det T^{k/2+n/4-1/4} \| \varphi^{(0)}_T | k/2-n/4-1/4 \prod_p \frac{\hat{F}^{(0)}_p(T, \alpha_p)}{\alpha_p(T)}. $$

Thus, similarly to [I-K1], Theorem 3.3, (1), and [I-K2], Theorem 3.2, we obtain

$$L(s, I_n(h)) = \kappa_n 2^{n+1-n} \sum_{d_0 \in \mathcal{F}((-1)^{n/2})} c_h([d_0]) |d_0|^{n/4-k+1/4}

\times \left\{ \prod_p P^{(0)}_{n,p}(d_0, t_p, \alpha_p, p^{-s+k/2+n/4+1/4} \chi(p))

+ (-1)^{n(n+2)/8} (-1)^{n/2} (-2, d_0) \prod_p P^{(0)}_{n,p}(d_0, \varepsilon_p, \alpha_p, p^{-s+k/2+n/4+1/4} \chi(p)) \right\}. $$

This proves the assertion. □

**Proposition 4.3.** Let $d_0 \in \mathcal{F}_p$ and $\xi_0 = \hat{\xi}(d_0)$. Then

$$P^{(0)}_n(d_0, t, X, t) = \frac{(p-1) t^\nu \varphi(d_0)}{\phi_n/2-1(p-2)(1-p^{-n/2} \xi_0)}

\times \frac{(1 + t^2 p^{-n/2-3/2})(1 + t^2 p^{-n/2-5/2} \xi_0^2) - \xi_0 t^2 p^{-n/2-2}(X + X^{-1} + p^{1/2-n/2} + p^{-1/2+n/2})}{(1 - p^{-2}Xt^2)(1 - p^{-2}X^{-1}t^2) \prod_{i=1}^{n/2}(1 - t^2 p^{-2i-1}X)(1 - t^2 p^{-2i-1}X^{-1})},$$

and

$$P^{(0)}_n(d_0, \varepsilon, X, t) = \frac{1}{\phi_n/2-1(p-2)(1-p^{-n/2} \xi_0)} \prod_{i=1}^{n/2}(1 - t^2 p^{-2i}X)(1 - t^2 p^{-2i}X^{-1}).$$

**Proof.** Put $H_k = \left( \begin{array}{cc} O & 1_k \\ 1_k & O \end{array} \right)$, and for $d \in Z_p^*$ put

$$D = \{ x \in M_{2k,n}(Z_p) \mid \det(H_k[x]) \in dp^* Z_p^\circ \text{ with some } i \in Z_{\geq 0} \}. $$

We then define $Z_{2k}(u, \varepsilon^l, d)$ as

$$Z_{2k}(u, \varepsilon^l, d) = \int_D \varepsilon^l(H_k[x]) |\det(H_k[x])|_p^{s-k} dx$$

with $u = p^{-s}$, where $| \cdot |_p$ denotes the normalized valuation on $Q_p$, and $dx$ is the measure on $M_{2k,n}(Q_p)$ normalized so that the volume of $M_{2k,n}(Z_p)$ is 1. Moreover put

$$Z_{2k,c}(u, \varepsilon^l, d) = \frac{1}{2}(Z_{2k,n}(u, \varepsilon^l, d) + Z_{2k,n}(-u, \varepsilon^l, d)).$$
and 
\[ Z_{2k,0}(u, \varepsilon^l, d) = \frac{1}{2} (Z_{2k,n}(u, \varepsilon^l, d) - Z_{2k,n}(-u, \varepsilon^l, d)) \.
\]

Then it is well known that 
\[ Z_{2k,x(d_0)}(u, \varepsilon^l, (-1)^{n/2} p^{-\nu_p(d_0)} d_0) = \phi_n(p^{-1}) \sum_{T \in \mathcal{L}_{n,p}(d_0)} \frac{b_p(2^{-\delta_p} T, p^{-k})}{\alpha_p(T)} (p^k t)^{\nu_p(\det(T)} 
\]

for \( d_0 \in \mathcal{F}_p \), where \( x(d_0) = e \) or \( o \) according as \( \nu_p(d_0) \) is even or odd. Recall that 
\[ b_p(2^{-\delta_p} T, p^{-k}) = \frac{(1 - p^{-k}) \prod_{i=1}^{n/2} (1 - p^{-2k+2i})}{1 - \xi(2^{-\delta_p} T)p^{-k+n/2}} F_p^{(0)}(T, p^{-k}) \]

and 
\[ F_p^{(0)}(T, p^{-k}) = p^{(-k/2 + (n+1)/4)\nu_p(\det T) - \nu_p(d_0)} F_p^{(0)}(T, p^{-k+n/2}). \]

Hence we have 
\[ Z_{2k,x(d_0)}(u, \varepsilon^l, (-1)^{n/2} p^{-\nu_p(d_0)} d_0) = \phi_n(p^{-1}) \left( \frac{1 - p^{-k}}{1 - \xi(2^{-\delta_p} T)p^{-k+n/2}} \right) \frac{(1 - p^{-k}) \prod_{i=1}^{n/2} (1 - p^{-2k+2i})}{1 - \xi(2^{-\delta_p} T)p^{-k+n/2}} \times p^{(k/2 - (n+1)/4)\nu_p(d_0)} F_n^{(0)}(d_0, \varepsilon^l, p^{-k+n/2}, \nu_p(d_0)) T(d_0, \varepsilon^l, p^{-k+n/2}, \nu_p(d_0)). \]

Let \( T(d_0, \omega, X, t) \) denote the right-hand side of the formula for \( \omega = \varepsilon^l \) (\( l = 0, 1 \)) in the proposition. Then, by [[Sai2], Theorem 3.4 (2)], we have 
\[ Z_{2k,x(d_0)}(u, \varepsilon^l, (-1)^{n/2} p^{-\nu_p(d_0)} d_0) = \phi_n(p^{-1}) \left( \frac{1 - p^{-k}}{1 - \xi(\nu_0 T)p^{-k+n/2}} \right) \frac{(1 - p^{-k}) \prod_{i=1}^{n/2} (1 - p^{-2k+2i})}{1 - \xi(\nu_0 T)p^{-k+n/2}} \times p^{(k/2 - (n+1)/4)\nu_p(d_0)} T(d_0, \varepsilon^l, p^{-k+n/2}, \nu_p(d_0)). \]

(Remark that there are misprints in [Sai2]: the \((q^{-1})_n \) on page 197, lines 9 and 15 should be \((q^{-1})_r \). Hence we have 
\[ P_n^{(0)}(d_0, \varepsilon^l, p^{-k+n/2}, \nu_p(d_0)) = T(d_0, \varepsilon^l, p^{-k+n/2}, \nu_p(d_0)) \]

for infinitely many positive integers \( k \). Hence we have 
\[ P_n^{(0)}(d_0, \varepsilon^l, X, t) = T(d_0, \varepsilon^l, X, t). \]

\( \square \)

**Proof of Theorem 4.1.**

Put \( \Omega = \{ \omega_p \} \), and let \( d_0 \in \mathcal{F}((-1)^{n/2}) \). Put 
\[ P(s, d_0, \Omega, \chi) = \prod_p P_n^{(0)}(d_0, \varepsilon^l, \alpha_p, p^{-s+k/2+n/4+1/4} \chi(p)). \]
Then by Proposition 4.3, we have

\[ P(s, d_0, \{\varepsilon_p\}, \chi) \]
\[ = |d_0|^{-s+k/2+n/4-3/4} \chi(d_0) \prod_{i=1}^{n/2-1} \zeta(2i) L(n/2, \chi_{d_0}) \prod_{i=0}^{n/2} L(2s + 2i - n, S(h), \chi^2) \]
\[ \times L(2s - n + 1, S(h), \chi^2) \prod_{p} ((1 + p^{-2s+k-1} \chi(p)^2)(1 + \chi_{d_0}(p)^2 p^{-2s+2k-2} \chi(p)^2) \]
\[ - \chi_{d_0}(p)^2 p^{-2s+k-3/2} \chi(p) \beta_p (1 + p^{1/2-n/2} \beta_p^{-1}) (1 + p^{-1/2+n/2} \beta_p^{-1})) \].

We note that \( L(s, h) \) and \( L(s, E_{n/2+1}) \) can be expressed as

\[ L(s, h) = L(2s, S(h)) \sum_{d_0 \in \mathcal{F}((-1)^{n/2})} \frac{c(|d_0|)|d_0|^{-s} \prod_p (1 - \chi_{(n-1)^{s-n/2}}(p)p^{k-n/2-1-2s})}{}, \]

and

\[ L(s, E_{n/2+1}) = \zeta(2s) \zeta(2s - n + 1) \]
\[ \times \sum_{d_0 \in \mathcal{F}((-1)^{n/2})} L(1 - n/2, \chi_{d_0}) |d_0|^{-s} \prod_p (1 - \chi_{d_0}(p)p^{n/2-1-2s}), \]

and therefore, we easily see that \( L(s, h, E_{n/2+1/2}, \chi) \) can be expressed as

\[ L(s, h, E_{n/2+1/2}, \chi) = L(2s, S(h), \chi^2) L(2s - n + 1, S(h), \chi^2) \]
\[ \times \sum_{d_0 \in \mathcal{F}((-1)^{n/2})} |d_0|^{-s} c(|d_0|) \chi(d_0) L(1 - n/2, \chi_{d_0}) \]
\[ \times \prod_{p} ((1 + p^{-2s+k-1} \chi(p)^2)(1 + \chi_{d_0}(p)^2 p^{-2s+2k-2} \chi(p)^2) \]
\[ - \chi_{d_0}(p)^2 p^{-2s+k-3/2} \chi(p) \beta_p (1 + p^{1/2-n/2} \beta_p^{-1}) (1 + p^{-1/2+n/2} \beta_p^{-1})) \]

(cf. [Sh1], Lemma 1.) Thus, by remarking the functional equation

\[ L(1 - n/2, \chi_{d_0}) = 2^{1-n/2} n^{-n/2} \Gamma(n/2) |d_0|^{(n-1)/2} L(n/2, \chi_{d_0}), \]

we have

\[ \sum_{d_0 \in \mathcal{F}((-1)^{n/2})} c_h(|d_0|)|d_0|^{-s+k/2+n/4+1/4} P(s, d_0, \{\varepsilon_p\}, \chi) \]
\[ = \prod_{i=1}^{n/2-1} \zeta(2i) \frac{2^{n/2-1} n^{n/2}}{\Gamma(n/2)} L(s, h, E_{n/2+1/2}; \chi) \prod_{i=1}^{n/2-1} \frac{L(2s - 2i + n, S(h), \chi^2).}{\Gamma(n/2)} \]

On the other hand, if \( d_0 \neq 1 \), by Proposition 4.3, we have

\[ P(s, d_0, \{\varepsilon_p\}, \chi) = 0. \]
Thus if $n \equiv 2 \mod 4$, for any $d_0 \in \mathcal{F}((-1)^{n/2})$,

$$P(s, d_0, \{\varepsilon_p\}, \chi) = 0.$$  

If $n \equiv 0 \mod 4$, by Proposition 4.3, we have

$$P(s, 1, \{\varepsilon_p\}, \chi) = \zeta(n/2) \prod_{i=1}^{n/2-1} \zeta(2i) \prod_{i=1}^{n/2} L(2s - 2i + 1, S(h), \chi^2).$$

Thus the assertion follows from Theorem 4.2. □

5 Relation between twisted K-M series of the first and second kinds

Let $N$ be a positive integer. Let $g$ be a periodic function on $\mathbb{Z}$ with a period $N$ and $\phi$ a polynomial in $t_1, \ldots, t_r$. Then for an element $u = (a_1 \mod N, \ldots, a_r \mod N) \in (\mathbb{Z}/N\mathbb{Z})^r$, the value $g(\phi(a_1, \ldots, a_r))$ does not depend on the choice of the representative of $u$. Therefore we denote this value by $g(\phi(u))$. In particular we sometimes regard a Dirichlet character mod $N$ as a function on $\mathbb{Z}/N\mathbb{Z}$.

For a Dirichlet character $\chi \mod N$ and $A \in \mathcal{L}_{m>0}$, put

$$h(A, \chi) = \sum_{U \in SL_m(\mathbb{Z}/N\mathbb{Z})} \chi(\text{tr}(A[U])).$$

As was shown in [K-M, Proposition 3.3], the twisted Koecher-Maaß series of the first kind of a Siegel modular form can be expressed in terms of $h(A, \chi)$ as stated later. Therefore we shall compute $h(A, \chi)$ in the case where $A$ is an element of $\mathcal{L}_{m>0}$. For $A = (a_{ij})_{m \times m} \in S_m(\mathbb{Z}/N\mathbb{Z})$ and $c \in \mathbb{Z}/N\mathbb{Z}$, put

$$R_N(A, c) = \{X = (x_{ij})_{m \times m} \in M_m(\mathbb{Z}/N\mathbb{Z}) \mid \sum_{i=1}^m \sum_{\alpha, \beta=1}^m a_{\alpha, \beta} x_{\alpha i} x_{i\beta} - c = 0 \text{ and } \det X - 1 = 0\}.$$  

Then we have

$$h(A, \chi) = \sum_{c \in \mathbb{Z}/N\mathbb{Z}} \chi(c) \#(R_N(A, c)).$$

From now on let $p$ be an odd prime number and $F_p$ be the field with $p$-elements. For $S \in S_m(F_p)$ and $T \in S_r(F_p)$ put

$$A(S, T) = \{Y = M_{r,m}(F_p) \mid Y S \downarrow Y = T\}.$$  

For an element $S \in S_m(F_p)$ with $m$ even put $\chi(S) = \left(\frac{(-1)^{m/2} \det S}{p}\right).$
Lemma 5.1. Let $S \in S_m(F_p^\times)$.

(1) Let $T \in S_r(F_p)$ with $m \geq r$.

(1.1) Let $r$ be even. Then
\[
\#A(S, T) = p^{rm-r(r+1)/2} \left( 1 - \chi(S)p^{-m/2} \right) \prod_{m-r+1 \leq s \leq m-1}^{m \text{ even}} (1 - p^{-e})
\]

or
\[
\#A(S, T) = p^{rm-r(r+1)/2} \prod_{m-r+1 \leq s \leq m-1}^{m \text{ even}} (1 - p^{-e})
\]

according as $m$ is even or odd.

(1.2) Let $r$ be odd. Then
\[
\#A(S, T) = p^{rm-r(r+1)/2} \left( 1 + \chi((-S) \perp T)p^{(r-m)/2} \right) \prod_{m-r+1 \leq s \leq m-1}^{m \text{ even}} (1 - p^{-e})
\]

or
\[
\#A(S, T) = p^{rm-r(r+1)/2} \prod_{m-r+1 \leq s \leq m-1}^{m \text{ even}} (1 - p^{-e})
\]

according as $m$ is even or odd. In particular, for $c \in F_p^\times$, we have
\[
\#A(S, c) = p^{m/2-1} \left( \frac{(-1)^{m/2} \det S}{p} \right)
\]

or
\[
\#A(S, c) = p^{(m-1)/2} \left( \frac{(-1)^{(m-1)/2} \det S}{p} \right)
\]

according as $m$ is even or odd.

(2) We have
\[
\#A(S, 0) = p^{m/2-1} \left( \frac{(-1)^{m/2} \det S}{p} \right) + p^{m/2} \left( \frac{(-1)^{m/2} \det S}{p} \right)
\]

or
\[
\#A(S, 0) = p^{m-1}
\]

according as $m$ is even or odd.

Proof. The assertions (1) and (2) follow from [Ki1, Theorem 1.3.2], and [Ki1, Lemma 1.3.1], respectively. \qed

Proposition 5.2. Let $A = a_1 \perp \cdots \perp a_m$ with $a_i \in F_p$. For $c \in F_p^\times$ put
\[
\mathcal{M}_p(A, c) = \{ Z = (z_{ij}) \in S_m(F_p) \mid \det(Z) = 1 \text{ and } c = \sum_{i=1}^m a_i z_{ii} = 0 \},
\]
and
\[ \gamma_{m,p} = p^{m^2-m(m+1)/2}(1-p^{-m/2})^{(m-2)/2} \prod_{e=1}^{(m-1)/2} (1-p^{-2e}) \]
or
\[ \gamma_{m,p} = p^{m^2-m(m+1)/2} \prod_{e=1}^{(m-1)/2} (1-p^{-2e}) \]
according as \( m \) is even or odd. Then we have
\[ \# \mathcal{R}_p(A,c) = \gamma_{m,p} \# \mathcal{M}_p(A,c). \]

**Proof.** Let \( \Phi : GL_m(F_p) \longrightarrow S_m(F_p) \cap GL_m(F_p) \) be the mapping defined by \( \Phi(X) = X^tX \). Then by Lemma 5.1, we have \( \#\Phi^{-1}(Z) = 2\gamma_{m,p} \) for any \( Z \in S_m(F_p) \cap S\mathcal{L}_m(F_p) \). We note that \( \det X = \pm 1 \) for any \( X \in \Phi^{-1}(Z) \). Hence we have \( \#(\Phi^{-1}(Z) \cap S\mathcal{L}_m(F_p)) = \gamma_{m,p} \). Moreover we have
\[ \text{tr}(XAX) = \text{tr}(AX^tX), \]
and hence \( X \in \mathcal{R}_p(A,c) \) if and only if \( \Phi(X) \in \mathcal{M}_p(A,c) \). This proves the assertion.

We rewrite \( \mathcal{M}_p(A,c) \) in more concise form. Let \( p \) be a prime number and \( l \) be a positive integer dividing \( p-1 \). Take an \( l \)-th root of unity \( \zeta_l \) and a prime ideal \( \mathfrak{p} \) of \( \mathbb{Q}(\zeta_l) \) lying above \( p \). Let \( a \) be an integer prime to \( p \). Then we have \( a^{(p-1)/l} \equiv \zeta_l^i \mod \mathfrak{p} \) for some \( i \in \mathbb{Z} \). We then put \( \left( \frac{a}{p} \right)_l = \zeta_l^i \). We call \( \left( \frac{a}{p} \right)_l \) the \( l \)-th power residue symbol mod \( p \). In the case \( l = 2 \), this is the Legendre symbol, and we write it as \( \left( \frac{a}{p} \right)_2 \) as usual. We note that this definition of the power residue symbol is different from the usual one, and depends on the choice of \( \mathfrak{p} \) and \( \zeta_l \) except the case \( l = 2 \). We denote by \( \left( \frac{*}{N} \right) \) the Jacobi symbol for a positive odd integer. Let \( \chi \) be a primitive Dirichlet character of conductor \( N \). We assume that \( N \) is a square free odd integer, and write \( N = p_1 \cdots p_r \) with \( p_1, \cdots, p_r \) prime numbers. Put \( l_j = l_{m,p_j} = \text{GCD}(m,p_j-1) \). For an \( r \)-tuple \( I = (i_1, i_2, \cdots, i_r) \) of integers put
\[ \chi_{(i_1, \cdots, i_r)} = \chi \prod_{j=1}^r \left( \frac{*}{p_j} \right)_{l_j}. \]

For two Dirichlet characters \( \chi \) and \( \eta \mod N \) we define \( J_m(\chi, \eta) \) and \( I_m(\chi, \eta) \)
\[ J_m(\chi, \eta) = \sum_{Z \in S_m(Z/N\mathbb{Z})} \chi(\det Z)\eta(1 - \text{tr}(Z)) \]
and
\[ I_m(\chi, \eta) = \sum_{Z \in S_m(Z/N\mathbb{Z})} \chi(\det Z)\eta(\text{tr}(Z)). \]
By definition, $J_m(\chi, \eta)$ is an algebraic number. We note that $J_1(\chi, \eta)$ is the Jacobi sum $J(\chi, \eta)$ associated with $\chi$ and $\eta$. We also define $J_m(\chi)$ as $J_m(\chi) = J_m(\chi, \chi)$.

**Lemma 5.3.** Let $\eta$ be a primitive character mod $p$. Let $c \in F_p$ and $S \in S(l(F_p))$ of rank $r$. Let $S \sim S_0 \perp O_{l-r}$ with $\det S_0 \neq 0$. Put $$I_{\eta, S, c} = \sum_{w \in F_p} \eta(S[w] + c).$$

Assume that $r$ is odd, and that $\eta^2 \neq 1$. Then $$I_{\eta, S, c} = p^{l-(r+1)/2} J(\eta, \left(\frac{c}{p}\right)) \left(\frac{-1}{p} (r+1)/2 \det S_0 \right) \eta(c) \left(\frac{c}{p}\right).$$

Assume that $r$ is even, and that $\eta \neq 1$. Then $$I_{\eta, S, c} = p^{l-r/2} \left(\frac{-1}{p} r/2 \det S_0 \right) \eta(c).$$

Here we make the convention that $\left(\frac{-1}{p} r/2 \det S_0 \right) = 1$ if $r = 0$.

**Proof.** We have $$I_{\eta, S, c} = p^{l-r} I_{\eta, S_0, c}.$$ Hence we may assume that $r = l$. Then $$I_{\eta, S, c} = \sum_{u \in F_p} \eta(u) \#A(S, u - c).$$

Let $l$ be odd. Then by Lemma 5.1, $$\#A(S, u - c) = p^{(l-1)/2} \left(\frac{-1}{p} (l-1)/2 \det S \right) + \left(\frac{-1}{p} (l-1)/2 (u - c) \det S \right) \eta(c) \left(\frac{c}{p}\right).$$

Hence we have $$I_{\eta, S, c} = p^{(l-1)/2} \left(\frac{-1}{p} (l+1)/2 \det S \right) \sum_{u \in F_p} \eta(u) \left(\frac{u - c}{p}\right).$$

Since $\eta^2$ is nontrivial, we have $I_{\eta, S, c} = 0$ if $c = 0$. If $c \neq 0$, then $$\sum_{u \in F_p} \eta(u) \left(\frac{u - c}{p}\right) = \left(\frac{-c}{p}\right) \sum_{u \in F_p} \eta(u) \left(\frac{1 - c^{-1} u}{p}\right)$$

$$= \eta(c) \left(\frac{-c}{p}\right) \sum_{u \in F_p} \eta(u) \left(\frac{1 - u}{p}\right) = \eta(c) \left(\frac{-c}{p}\right) J(\eta, \left(\frac{1}{p}\right)).$$
Let \( l \) be even. Then
\[
\#A(S, u - c) = \left( p^{l/2} - \frac{(-1)^{l/2} \det S}{p} \right) p^{l/2 - 1} + \frac{(-1)^{l/2} \det S}{p} a_0,
\]
where \( a_0 = 1 \) or 0 according as \( u = c \) or not. Hence
\[
I_{\eta, S, c} = p^{l/2} \left( \frac{(-1)^{l/2} \det S}{p} \right) \eta(c).
\]

Corollary. Let \( d \in F_p^\times \). Then we have
\[
I_{\eta, S, cd} = \eta(d) \left( \frac{d}{p} \right)^r I_{\eta, S, c}.
\]

Proposition 5.4. Let \( \eta \) be a primitive character mod \( p \). For \( Z_1 \in S_{l-1}(F_p) \) and \( z_{il} \in F_p \), put
\[
I(Z_1, z_{il}) = \sum_{w \in M_{l-1}(F_p)} \eta\left( \begin{bmatrix} Z_1 & w \\ t^w & z_{il} \end{bmatrix} \right).
\]

(1) Assume that \( l \) is even, and that \( \eta^2 \neq 1 \). Then
\[
I(Z_1, z_{il}) = p^{(l-2)/2} j(\eta, \frac{\chi}{p}) \left( \frac{(-1)^{l/2} \det Z_1}{p} \right) \eta(\det Z_1 z_{il}) \left( \frac{z_{il}}{p} \right).
\]

(2) Assume that \( l \) is odd, and that \( \eta^2 \neq 1 \). Then
\[
I(Z_1, z_{il}) = p^{(l-1)/2} \left( \frac{(-1)^{(l-1)/2} \det Z_1}{p} \right) \eta(\det Z_1 z_{il}).
\]

Proof. We note that
\[
\det \left( \begin{bmatrix} Z_1 & w \\ t^w & z_{il} \end{bmatrix} \right) = -\text{Adj}(Z_1)[w] + \det Z_1 z_{il},
\]
where \( \text{Adj}(Z_1) \) is the \((l-1) \times (l-1)\) matrix whose \((i, j)\)-th component is the \((j, i)\)-th cofactor of \( Z_1 \). We also note that \( \det(-\text{Adj}(Z_1)) = (-1)^{l-1} (\det Z_1)^{l-2} \). Thus the assertion follows directly from Lemma 5.3 if \( \det Z_1 \neq 0 \). If \( \det Z_1 = 0 \), then \( \text{rank}_{F_p}(Z_1) \leq 1 \), the assertion follows also from Lemma 5.3.

Theorem 5.5. Let \( \chi \) be a primitive character mod \( p \). Let \( l = \text{GCD}(m, p - 1) \), and \( u_0 \) be a primitive \( l \)-th root of unity mod \( p \). Let \( A \in S_m(F_p) \).
(1) If \( \chi(u_0) \neq 1 \), then we have \( h(A, \chi) = 0 \).
(2) Assume that \( \chi(u_0) = 1 \). Fix a character \( \bar{\chi} \) such that \( \bar{\chi}^m = \chi \).
(2.1) Let \( m \) be even. Then
\[
h(A, \chi) = \gamma_{m,p} \sum_{i=0}^{l-1} A_{m,i,p} \bar{\chi(i)}(\det A) J_{m-1}(\chi(i)),
\]
(2.2) Let \( m \) be odd. Then
\[
h(A, \chi) \equiv 0 \pmod{p^{l/2}}.
\]
where $A_{m,i,p} = p^{(m-2)/2}(-1)^{m(p-1)/4} J(\chi(i), \left(\frac{z}{p}\right))$.

(2.2) Let $m$ be odd and assume that $\chi^2 \neq 1$. Then

$$h(A, \chi) = \gamma_{m,p} \sum_{i=0}^{l-1} A_{m,i,p} \tilde{\chi}(i) (\det A) J_{m-1}(\tilde{\chi}(i)),$$

where $p^{(m-1)/2} (-1)^{(m-1)(p-1)/4}$.

Proof. If $A = O_m$ then we have $h(A, \chi) = 0$. Hence we assume that $A \neq O_m$. Then we may assume that $A = a_1 \perp \cdots \perp a_{m-1} \perp d$ with $d \neq 0$. Put

$$\tilde{M}_p(A, c) = \{(Z_1, w) \in S_{m-1}(F_p) \times M_{m-1,1}(F_p) | \det \begin{pmatrix} Z_1 & w \\ t & z \end{pmatrix} d^{-1} (1 - \sum_{i=1}^{m-1} a_i z_i) \} c^m = 1 \}.$$

Write $Z \in S_m(F_p)$ as $Z = \begin{pmatrix} Z_1 & w \\ z & z_m \end{pmatrix}$ with $Z_1 \in S_{m-1}(F_p), w \in M_{m-1,1}(F_p), z \in F_p$. Then the mapping $S_m(F_p) \ni Z \mapsto (c^{-1} Z_1, c^{-1} w) \in S_{m-1}(F_p) \times M_{m-1,1}(F_p)$ induces a bijection from $\tilde{M}_p(A, c)$ to $\tilde{M}_p(A, c)$, and hence $\# \tilde{M}_p(A, c) = \# M_p(A, c)$. Put

$$K(A) = \sum_c \# \tilde{M}_p(A, c) \chi(c).$$

Assume that $\chi(u_0) \neq 1$. Then we have

$$K(A) = \sum_{c \in F_p} \chi(c u_0) \# \tilde{M}_p(A, c u_0).$$

We note that $\tilde{M}_p(A, c u_0) = \tilde{M}_p(A, c)$. Hence we have

$$K(A) = \chi(u_0) K(A).$$

Hence we have $K(A) = 0$.

Assume that $\chi(u_0) = 1$. Then we can take a Dirichlet character $\tilde{\chi}$ such that $\tilde{\chi}^m = \chi$. First assume that $\det A = 0$. Then we may assume that we have $A = A_0 \perp 0$ with $A_0 \in S_{m-1}(F_p)$. Let $P_{m-1,m}$ be the set of $(m-1) \times m$ matrices with entries in $F_p$ of rank $m-1$. Then for each $X_1 \in P_{m-1,m}$ there exist exactly $p^{m-1}$ elements $X_2 \in M_{1,m}(F_p)$ such that $\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \in SL_m(F_p)$. Hence we have

$$h(A, \chi) = p^{m-1} \sum_{X_1 \in P_{m-1,m}} \chi(A_0[X_1]).$$

Let $m$ be even. Then we can take an element $\alpha \in F_p^\times$ such that $\chi(\alpha) \neq 1$. Moreover we can take $U_0 \in GL_m(F_p)$ such that $U_0 U_0 = \alpha 1_m$ in view of (1.1) of Lemma 5.1. Hence

$$h(A, \chi) = p^{m-1} \sum_{X_1 \in P_{m-1,m}} \chi(A_0[X_1 U_0]) = \chi(\alpha) h(A, \chi).$$
Hence we have $h(A, \chi) = 0$. Let $m$ be odd and assume that $\chi^2 \neq 1$. Then we can take an element $\alpha \in (F_p^\times)^2$ such that $\chi(\alpha) \neq 1$. Moreover we can take $U_0 \in GL_m(F_p)$ such that $U_0^tU_0 = \alpha 1_m$ in view of (1.2) of Lemma 5.1. Thus by the same argument as above we have $h(A, \chi) = 0$. This proves the assertion.

Next assume that $\det A \neq 0$. We may assume that

$$A = 1_m - d$$

with $d = \det A$. Then we have

$$K(A) = \sum_c \#\mathfrak{M}_p(A, c) \bar{\chi}(c^m).$$

Hence we have

$$K(A) = \sum_{(Z_1, w)} \bar{\chi}(\det \left( \begin{array}{c|c} Z_1 & w \\ \hline w^t d^{-1} (1 - \tr(Z_1)) \end{array} \right)), $$

where $(Z_1, w)$ runs over elements of $S_{m-1}(F_p) \times M_{m-1,1}(F_p)$ such that

$$(*) \quad \det \left( \begin{array}{c|c} Z_1 & w \\ \hline t_w d^{-1} (1 - \tr(Z_1)) \end{array} \right) = u^m$$

with some $u \in F_p^\times$, and for such a matrix $\left( \begin{array}{c|c} Z_1 & w \\ \hline t_w d^{-1} (1 - \tr(Z_1)) \end{array} \right)$, there exist exactly $l$ elements $u$ of $F_p$ satisfying $(*)$. We have

$$\sum_{i=0}^{l-1} \left( \frac{v}{p} \right)^i_l = l \text{ or } 0$$

according as $v = u^m$ with some $u \in F_p^\times$ or not. Hence we have

$$K(A) = \sum_{i=0}^{l-1} \bar{\chi}(\det \left( \begin{array}{c|c} Z_1 & w \\ \hline w^t d^{-1} (1 - \tr(Z_1)) \end{array} \right))$$

$$\times \left( \frac{\det \left( \begin{array}{c|c} Z_1 & w \\ \hline t_w d^{-1} (1 - \tr(Z_1)) \end{array} \right)}{p} \right)^i_l$$

$$= \sum_{i=0}^{l-1} \bar{\chi}(i)(\det \left( \begin{array}{c|c} Z_1 & w \\ \hline t_w d^{-1} (1 - \tr(Z_1)) \end{array} \right))$$

Put

$$K(A)_i = \sum_{i=0}^{l-1} \bar{\chi}(i)(\det \left( \begin{array}{c|c} Z_1 & w \\ \hline t_w d^{-1} (1 - \tr(Z_1)) \end{array} \right))$$

We note that $\bar{\chi}^2_{(i)} \neq 1$ for any $i$. Hence by Proposition 5.4 we have

$$K(A)_i = A_{m,i,p} \sum_{Z_1 \in S_{m-1}(F_p)} \bar{\chi}_{(i)}(\det A) \bar{\chi}_{(i)}^*(\det Z_1) \bar{\chi}_{(i)}(1 - \tr(Z_1)),$$
Let \( \tilde{\chi}(i) = \frac{i}{p} \). This proves the assertion if \( m \) is odd. Assume that \( m \) is even. Then it is easily seen that the set \( \{ \tilde{\chi}(i) \}_{i=0}^{p-1} \) of Dirichlet characters coincides with \( \{ \tilde{\chi}(i) \}_{i=0}^{p-1} \). Moreover \( \tilde{\chi}^2 \neq 1 \) for any \( i \). This proves the assertion.

**Theorem 5.6.** Let \( N = p_1 \cdots p_r \). Let \( \chi \) be a primitive Dirichlet character mod \( N \). Let \( \nu_1, \ldots, \nu_r \) be primitive \( l_i \)-th root of unity mod \( p_i \). Let \( A \in S_m(F_{p_i}) \).

1. If \( \chi^{(p_1)}(\nu_i) \neq 1 \) for some \( i \). Then we have \( h(A, \chi) = 0 \).
2. Assume that \( \chi^{(p_1)}(\nu_i) = 1 \) for any \( i \). Fix a character \( \bar{\chi} \) such that \( \bar{\chi}^m = \chi \).

(2.1) Let \( m \) be even. Then we have

\[
h(A, \chi) = \prod_{i=1}^{r} (-1)^{m(p_i-1)/2} p_i^{(m-2)/2 \gamma_{m,p_i}} \times \sum_{i_1=0}^{l_1-1} \cdots \sum_{i_r=0}^{l_r-1} \tilde{\chi}(i_1, i_2, \ldots, i_r) \text{det}(A) J_m(\bar{\chi}(i_1, i_2, \ldots, i_r), \left( \frac{\bar{\chi}}{N} \right) J_{m-1}(\bar{\chi}(i_1, i_2, \ldots, i_r)).
\]

(2.2) Let \( m \) be odd, and assume that \( \chi^2 \) is primitive. Then we have

\[
h(A, \chi) = \prod_{i=1}^{r} (-1)^{(m-1)(p_i-1)/2} p_i^{(m-1)/2 \gamma_{m,p_i}} \times \sum_{i_1=0}^{l_1-1} \cdots \sum_{i_r=0}^{l_r-1} \tilde{\chi}(i_1, i_2, \ldots, i_r) \text{det}(A) J_{m-1}(\bar{\chi}(i_1, i_2, \ldots, i_r)).
\]

**Proof.** We note that \( J_m(\eta_1, \eta_2) = \prod_{i=1}^{r} J_m(\eta_i^{(p_i)}, \eta_i^{(p_i)}) \) for primitive characters \( \eta_1 \) and \( \eta_2 \) mod \( N \). Moreover \( \eta_i^{(p_i)} \) is primitive if and only if \( \eta_i^{(p_i)} \neq 1 \) for any \( 1 \leq i \leq r \). Thus the assertion follows from Theorem 5.5 and [[K-M], Lemma 3.2].

Now we give explicit formulas for \( J_m(\chi, \eta) \) and \( I_m(\chi, \eta) \).

**Proposition 5.7.** Let \( \chi \) and \( \eta \) be primitive characters mod \( p \). Assume that \( \chi^2 \neq 1 \). Put \( c_m(\chi, \eta) = 1 \) or 0 according as \( \chi^m \eta = 1 \) or not.

1. Assume that \( m \) is odd. Then

\[
I_m(\chi, \eta) = c_m(\chi, \eta) \left( \frac{-1}{p} \right)^{(m-1)/2} p^{(m-1)/2 (p-1)} J_{m-1}(\chi \left( \frac{\ast}{p} \right), \eta).
\]

2. Assume that \( m \) is even. Then

\[
I_m(\chi, \eta) = c_m(\chi, \eta) \left( \frac{-1}{p} \right)^{m/2} p^{(m-2)/2 (p-1)} \chi(-1) J_{m-1}(\chi \left( \frac{\ast}{p} \right), \eta).
\]
Proof. By Proposition 5.4, we have

\[ I_m(\chi, \eta) = I'_m \times \begin{cases} 
  p^{(m-1)/2} \left( \frac{(-1)^{(m-1)/2}}{p} \right) & \text{if } m \text{ is odd} \\
  p^{(m-2)/2} \left( \frac{(-1)^{(m-2)/2}}{p} \right) J(\chi, \frac{z}{p}) & \text{if } m \text{ is even,}
\end{cases} \]

where

\[ I'_m = \sum_{\substack{z_{mm} \in F_p \\
 \chi \in S_{m-1}(F_p)^{\times}}} \chi(z_{mm}) \chi(\det Z_1) \left( \frac{\det Z_1}{p} \right) \eta(z_{mm} + \text{tr}(Z_1)) \left( \frac{z_{mm}}{p} \right)^{m-1}. \]

Then we have

\[ I'_m = \sum_{\substack{z_{mm} \in F_p^{\times} \\
 \chi \in S_{m-1}(F_p)^{\times}}} \chi(z_{mm}) \eta(z_{mm}) \chi(\det Z_1) \left( \frac{\det Z_1}{p} \right) \eta(1 + z_{mm}^{-1} \text{tr}(Z_1)) \left( \frac{z_{mm}}{p} \right)^{m-1}. \]

Put \( Y_1 = -z_{mm}^{-1} Z_1 \). Then \( \det Y_1 = (-1)^{m-1} z_{mm}^{-1} \det Z_1 \). Hence we have

\[ I'_m = \chi((-1)^{m-1}) \left( \frac{-1}{p} \right)^{m-1} \times \sum_{\substack{z_{mm} \in F_p^{\times} \\
 \chi \in S_{m-1}(F_p)^{\times}}} \chi(z_{mm}) \eta(z_{mm}) \sum_{Y_1 \in S_{m-1}(F_p)^{\times}} \chi(\det Y_1) \left( \frac{\det Y_1}{p} \right) \eta(1 - \text{tr}(Y_1)). \]

We have

\[ \sum_{z_{mm} \in F_p^{\times}} \chi(z_{mm}) \eta(z_{mm}) = \begin{cases} p - 1 & \text{if } \chi^m \eta \text{ is trivial} \\
 0 & \text{if } \chi^m \eta \text{ is not trivial}
\end{cases} \]

according as \( \chi^m \eta \) is trivial or not. This proves the assertion. \( \square \)

**Proposition 5.8.** Let \( \chi \) and \( \eta \) be as in Proposition 5.7.

(1) Assume that \( m \) is odd. Then

\[ J_m(\chi, \eta) = \left( \frac{-1}{p} \right)^{(m-1)/2} p^{(m-1)/2} \times \{ J(\chi, \chi^{m-1} \eta) J_{m-1}(\chi \left( \frac{\chi}{p} \right), \eta) + \eta(-1) I_{m-1}(\chi \left( \frac{\chi}{p} \right), \eta) \}. \]

(2) Assume that \( m \) is even. Then

\[ J_m(\chi, \eta) = \left( \frac{-1}{p} \right)^{m/2} p^{(m-2)/2} J(\chi \left( \frac{\chi}{p} \right)) \times \{ J(\chi \left( \frac{\chi}{p} \right), \chi^{m-1} \left( \frac{\chi}{p} \right) \eta) J_{m-1}(\chi \left( \frac{\chi}{p} \right), \eta) + \eta(-1) I_{m-1}(\chi \left( \frac{\chi}{p} \right), \eta) \}. \]
Proof. By Proposition 5.4, we have

\[
J_m(\chi, \eta) = (J'_m + J''_m) \times \begin{cases} 
p^{(m-1)/2} \left( \frac{(-1)^{(m-1)/2}}{p} \right) J(\chi, \left( \frac{z}{p} \right)) & \text{if } m \text{ is odd} \\
p^{(m-2)/2} \left( \frac{(-1)^{(m-2)/2}}{p} \right) J(\chi, \left( \frac{z}{p} \right)) & \text{if } m \text{ is even},
\end{cases}
\]

where

\[
J'_m = \sum_{\chi_{mm} \in \mathbb{F}_p, \chi_{mm} \neq 1 \atop \tilde{z}_1 \in \mathcal{S}_{m-1}(\mathbb{F}_p)} \left( \frac{\det \tilde{z}_1}{p} \right) \left( \frac{z_{mm}}{p} \right)^{m-1} \chi(z_{mm}) \chi(\det \tilde{z}_1) \eta(1 - z_{mm} - \text{tr}(\tilde{z}_1)),
\]

and

\[
J''_m = \sum_{\tilde{z}_1 \in \mathcal{S}_{m-1}(\mathbb{F}_p)} \left( \frac{\det \tilde{z}_1}{p} \right) \chi(\det \tilde{z}_1) \eta(-\text{tr}(\tilde{z}_1)).
\]

Then we have \(J''_m = \eta(-1) J_{m-1}(\chi \left( \frac{z}{p} \right), \eta).\) Moreover

\[
J'_m = \sum_{\chi_{mm} \in \mathbb{F}_p, \chi_{mm} \neq 1 \atop \tilde{z}_1 \in \mathcal{S}_{m-1}(\mathbb{F}_p)} \chi(z_{mm}) \left( \frac{\det \tilde{z}_1}{p} \right) \left( \frac{z_{mm}}{p} \right)^{m-1} \chi(\det \tilde{z}_1)
\]

\[
\times \eta(1 - z_{mm}) \eta(1 - (1 - z_{mm})^{-1} \text{tr}(\tilde{z}_1)).
\]

Put \(Y_1 = (1 - z_{mm})^{-1} \tilde{z}_1.\) Then \(\det Y_1 = (1 - z_{mm})^{1-m} \det \tilde{z}_1.\) Hence we have

\[
J'_m = \sum_{\chi_{mm} \in \mathbb{F}_p} \chi(z_{mm}) \left( \frac{z_{mm}}{p} \right)^{m-1} \left( \frac{1 - z_{mm}}{p} \right)^{m-1} \chi(1 - z_{mm})^{m-1} \eta(1 - z_{mm})
\]

\[
\times \sum_{Y_1 \in \mathcal{S}_{m-1}(\mathbb{F}_p)} \left( \frac{\det Y_1}{p} \right) \chi(\det Y_1) \eta(1 - \text{tr}(Y_1)).
\]

This proves the assertion. \(\square\)

**Theorem 5.9.** Let \(\chi\) be a primitive character mod \(p.\)

(1) Let \(m\) be odd, and assume that \(\chi^2 \neq 1.\)

(1.1) Assume that \(\chi^m \neq 1.\) Then

\[
J_m(\chi \left( \frac{a}{p} \right)^i, \chi) = \left( \frac{-1}{p} \right)^{(m-1)/2} p^{(m-1)/2} J(\chi \left( \frac{a}{p} \right)^i, \chi^m) J_{m-1}(\chi \left( \frac{a}{p} \right)^{i+1}, \chi).
\]

(1.2) Assume that \(\chi^m = 1.\) Then

\[
J_m(\chi \left( \frac{a}{p} \right)^i, \chi) = p^{m-1} \left( \frac{-1}{p} \right)^{i+1} J(\chi \left( \frac{a}{p} \right)^{i+1}, \chi) J_{m-2}(\chi \left( \frac{a}{p} \right)^i, \chi).
\]
(2) Let \( m \) be even.

(2.1) Assume that \( \chi^m \left( \frac{x}{2} \right)^{i+1} \neq 1 \). Then

\[
J_m(\chi \left( \frac{x}{p} \right)^i, \chi) = \left( \frac{-1}{p} \right)^{(m-2)/2} J(\chi \left( \frac{x}{p} \right)^i, \left( \frac{x}{p} \right))J(\chi \left( \frac{x}{p} \right)^{i+1}, \chi^m \left( \frac{x}{p} \right)^{i+1})J_{m-1}(\chi \left( \frac{x}{p} \right)^i, \chi).
\]

(2.2) Assume that \( \chi^m \left( \frac{x}{2} \right)^{i+1} = 1 \). Then

\[
J_m(\chi \left( \frac{x}{p} \right)^i, \chi) = \chi(-1)p^{m-1} J(\chi \left( \frac{x}{p} \right)^i, \left( \frac{x}{p} \right))J_{m-2}(\chi \left( \frac{x}{p} \right)^i, \chi).
\]

**Proof.** Let \( m \) be odd. Then, by (1) of Proposition 5.8, we have

\[
J_m(\chi \left( \frac{x}{p} \right)^i, \chi) = \left( \frac{-1}{p} \right)^{(m-1)/2} p^{(m-1)/2}
\]

\[
\times \{ J(\chi \left( \frac{x}{p} \right)^i, \chi^m)J_{m-1}(\chi \left( \frac{x}{p} \right)^{i+1}, \chi) + \chi(-1)J_{m-1}(\chi \left( \frac{x}{p} \right)^{i+1}, \chi) \}.
\]

Thus the assertion holds if \( \chi^m \neq 1 \). Assume that \( \chi^m = 1 \). Then by (2) of Proposition 5.8 and (2) of Proposition 5.7 we have

\[
J_{m-1}(\chi \left( \frac{x}{p} \right)^{i+1}, \chi) = \left( \frac{-1}{p} \right)^{(m-1)/2} p^{(m-3)/2} J(\chi \left( \frac{x}{p} \right)^i, \left( \frac{x}{p} \right))
\]

\[
\times J(\chi \left( \frac{x}{p} \right)^i, \chi^m \left( \frac{x}{p} \right)^i)J_{m-2}(\chi \left( \frac{x}{p} \right)^i, \chi).
\]

and

\[
I_{m-1}(\chi \left( \frac{x}{p} \right)^{i+1}, \chi) = \left( \frac{-1}{p} \right)^{(m-3)/2} p^{(m-3)/2} (p-1) \chi(-1) \left( \frac{-1}{p} \right)^{i+1}
\]

\[
\times J(\chi \left( \frac{x}{p} \right)^i, \left( \frac{x}{p} \right))J_{m-2}(\chi \left( \frac{x}{p} \right)^i, \chi).
\]

We note that \( J(\chi \left( \frac{x}{p} \right)^i, \chi^m) = -1, \chi(-1) = 1 \) and

\[
J(\chi \left( \frac{x}{p} \right)^i, \chi^m \left( \frac{x}{p} \right)^i) = J(\chi \left( \frac{x}{p} \right)^i, \chi \left( \frac{x}{p} \right)^i) = \chi(-1) \left( \frac{-1}{p} \right)^i = \left( \frac{-1}{p} \right)^i.
\]

This proves the assertion.

Let \( m \) be even. Then, by (2) of Proposition 5.8, we have

\[
J_m(\chi \left( \frac{x}{p} \right)^i, \chi) = \left( \frac{-1}{p} \right)^{(m-2)/2} p^{(m-2)/2} J(\chi \left( \frac{x}{p} \right)^i, \left( \frac{x}{p} \right)).
\]
Thus the assertion holds if $\chi^m \left( \frac{\lambda}{p} \right)^{i+1} \neq 1$. Assume that $\chi^m \left( \frac{\lambda}{p} \right)^{i+1} = 1$. Then by (1) of Proposition 5.7 and (1) of Proposition 5.8, we have

$$J_{m-1}(\chi \left( \frac{\ast}{p} \right)^{i+1}, \chi) = \left( \frac{-1}{p} \right)^{(m-2)/2} \frac{p^{(m-2)/2}}{\frac{p}{m-2}}$$

and

$$I_{m-1}(\chi \left( \frac{\ast}{p} \right)^{i+1}, \chi) = \left( \frac{-1}{p} \right)^{(m-2)/2} \frac{p^{(m-2)/2}}{\frac{p}{m-2}} J_{m-2}(\chi \left( \frac{\ast}{p} \right)^{i}, \chi).$$

We note that $J(\chi \left( \frac{\ast}{p} \right)^{i}, \chi) = -1, \left( \frac{-1}{p} \right)^{i+1} = 1$ and

$$J(\chi \left( \frac{\ast}{p} \right)^{i+1}, \chi) = J(\chi \left( \frac{\ast}{p} \right)^{i+1}, \chi) \chi^{-1} \chi^{-1} \chi^{-1} = \chi(-1) \left( \frac{-1}{p} \right)^{i+1} = \chi(-1).$$

This proves the assertion.

\[ \square \]

**Corollary.** Let $\chi$ be a primitive character with an odd square free conductor $N$. Assume that $\chi^2$ is primitive. Then the value $J_m(\chi)$ is nonzero.

**Proof.** The assertion follows directly from the above theorem if $N$ is an odd prime. In general case, the assertion can also be proved by remarking that $J_m(\chi) = \prod_{p \mid N} J_m(\chi(p))$ and that $\chi(p)^2 \neq 1$ for any $p \mid N$. \[ \square \]

To compare our present result with the result in [K-M], we give the following:

**Proposition 5.10.** Let $\chi$ be a primitive Dirichlet character mod $p$. Assume that $\chi^2 \neq 1$. Then we have

$$J(\chi, \left( \frac{\ast}{p} \right)) J(\chi, \left( \frac{\ast}{p} \right)) \chi \left( \frac{\ast}{p} \right) = \left( \frac{-1}{p} \right) \bar{\chi}(4)p.$$

**Proof.** Put

$$I = \sum_{(z,w) \in F_p^2} \chi(z(1-z) - w^2).$$

Then by using the same argument as in the proof of Theorem 5.5, we have

$$I = J(\chi, \left( \frac{\ast}{p} \right)) \sum_{z \in F_p} \chi(z(1-z)) \left( \frac{z(1-z)}{p} \right)$$

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\[ J(\chi(\frac{a}{p})), J(\chi(\frac{b}{p}), \chi(\frac{c}{p})). \]

On the other hand, we have

\[ I = \sum_{(y, w) \in F^2_n} \chi(-y^2 - w^2 + 1/4). \]

Hence by Lemma 5.3 we have

\[ I = \mathcal{P}(\frac{-1}{p}) \tilde{\chi}(4). \]

This proves the assertion.

By virtue of the above proposition, we see that Theorem 5.6 coincides with [K-M], Proposition 3.7 in case \( m = 2 \).

Now let

\[ F(Z) = \sum_{A \in \mathcal{L}_{n \geq 0}} c_F(A)e(\operatorname{tr}(AZ)) \]

be an element of \( \mathcal{M}_k(Sp_n(Z)) \) and let \( \chi \) be a Dirichlet character mod \( N \). Assume \( N \neq 2 \). Then by [[K-M], Proposition 3.1], we have

\[ L(s, F, \chi) = \sum_{A \in \mathcal{L}_{n > 0}/SL_n(Z)} \frac{c_F(A)\bar{b}(A, \chi)}{e(A)(\det A)^s}. \]

Thus by Theorem 5.6 we easily obtain:

**Theorem 5.11.** Let \( N, p_i, l_i, u_{0,i} (i = 1, \cdots, r) \) and \( \chi \) be as in Theorem 5.6, and let \( F \) be an element of \( \mathcal{M}_k(Sp_n(Z)) \).

(1) If \( \chi^{(p_i)}(u_{0,i}) \neq 1 \) for some \( i \). Then we have \( L(s, F, \chi) = 0 \).

(2) Assume that \( \chi^{(p_i)}(u_{0,i}) = 1 \) for any \( i \). Fix a character \( \tilde{\chi} \) such that \( \tilde{\chi}^n = \chi \).

(2.1) Let \( n \) be even. Then we have

\[ L(s, F, \chi) = \prod_{i=1}^{r} (-1)^{(n-2)(p_i-1)/4} \gamma_{n, p_i} \]

\[ \times \sum_{i_1=0}^{l_1-1} \cdots \sum_{i_r=0}^{l_r-1} \tilde{\chi}(i_1, \cdots, i_r)(2^n)J_n(\tilde{\chi}(i_1, \cdots, i_r), (\frac{n}{N}))J_{n-1}(\tilde{\chi}(i_1, \cdots, i_r))L^*(s, F, \tilde{\chi}(i_1, i_2, \cdots, i_r)). \]

(2.2) Let \( n \) be odd, and assume that \( \chi^2 \neq 1 \). Then we have

\[ L(s, F, \chi) = \prod_{i=1}^{r} (-1)^{(n-1)(p_i-1)/4} \gamma_{n, p_i} \]

\[ \times \sum_{i_1=0}^{l_1-1} \cdots \sum_{i_r=0}^{l_r-1} \tilde{\chi}(i_1, \cdots, i_r)(2^{n-1})\overline{J_{n-1}(\tilde{\chi}(i_1, i_2, \cdots, i_r))}L^*(s, F, \tilde{\chi}(i_1, i_2, \cdots, i_r)). \]
6 Twisted Koecher-Maaß series of the first kind of the DII lift

By Theorems 4.1 and 5.11, we obtain the following.

**Theorem 6.1.** Let \( k \) and \( n \) be positive even integers such that \( n \geq 4, 2k - n \geq 12 \). Let \( h(z) \) and \( E_{n/2+1/2} \) be as in Section 4. Let \( N \) be a square free odd integer, and \( N = p_1 \cdots p_r \) be the prime decomposition of \( N \). For each \( i = 1, \ldots, r \) let \( l_i = \text{GCD}(n, p_i) - 1 \) and \( u_{i, 0} \in \mathbb{Z} \) be a primitive \( l_i \)-th root of unity mod \( p_i \).

1. Assume \( \chi(p_i)(u_i) \neq 1 \) for some \( i \). Then \( L(s, I_n(h), \chi) = 0 \).
2. Assume \( \chi(p_i)(u_i) = 1 \) for any \( i \). Then

\[
L(s, I_n(h), \chi) = 2^{n} \prod_{i=0}^{l_i-1} J(\breve{\chi}(i_1, \ldots, i_r), \frac{1}{N}) \prod_{i=0}^{r} J_{n-1}(\breve{\chi}(i_1, \ldots, i_r))
\]

\[
\times \left\{ c_{n,N} R(s, h, E_{n/2+1/2}, \breve{\chi}(i_1, \ldots, i_r)) \prod_{j=1}^{n/2-1} L(2s - 2j, S(h), \breve{\chi}(i_1, \ldots, i_r)) \right\}
\]

\[
\left. \right. + d_{n,N} c_{h}(1) \prod_{j=1}^{n/2} L(2s - 2j + 1, S(h), \breve{\chi}(i_1, \ldots, i_r)) \right\},
\]

where \( c_{n,N} \) and \( d_{n,N} \) are nonzero rational numbers depending only on \( n \) and \( N \), and \( \breve{\chi} \) is a character s.t. \( \breve{\chi}^n = \chi \).

**Remark.** In the case \( n = 2 \), an explicit formula for \( L(s, I_2(h), \chi) \) was given by Katsurada-Mizuno [K-M].

7 Applications

Let \( h_1 \) and \( h_2 \) be modular forms of weight \( k_1 + 1/2 \) and \( k_2 + 1/2 \), respectively, and \( \chi \) be a Dirichlet character. In Section 2, we reviewed on the algebraicity of the values \( \breve{R}(m, h_1, h_2, \chi) \) at half integers. We then naturally ask the following question:

**Question.** What can one say about the algebraicity of \( \breve{R}(m, h_1, h_2, \chi) \) with \( m \) an integer?

As an application of Theorem 6.1, we give a partial answer to this question. We note that

\[
\breve{R}(s, h_1, h_2, \chi) = (1 - 2^{-2s+k_1+k_2-1} \chi^2(2))^{-1} \breve{R}(s, h_1, h_2, \chi)
\]

if the conductor of \( \chi \) is odd. Hence it suffices to consider the above question for \( \breve{R}(m, h_1, h_2, \chi) \) with integer \( m \) if \( k_1 + k_2 \) is even.
Let $k$ and $n$ be positive even integers such that $n \geq 4$, $2k - n \geq 12$. Let $h(z)$ and $E_{n/2+1/2}$ be as in Section 4. For a Dirichlet character $\chi$ of odd square free conductor $N = p_1 \cdots p_r$, we define

$$R(\chi)(s, h, E_{n/2+1/2}) = \sum_{i_1=0}^{l_1-1} \cdots \sum_{i_r=0}^{l_r-1} J(\chi(i_1, \ldots, i_r), \left(\frac{\ast}{N}\right))J_{n-1}(\chi(i_1, \ldots, i_r))$$

$$\times R(s, h, E_{n/2+1/2}, \chi_{(i_1, \ldots, i_r)}) \prod_{j=1}^{n/2-1} L(2s - 2j, S(h), \chi_{(i_1, \ldots, i_r)}^2),$$

where $l_i = \text{GCD}(n, p_i - 1)$ as in Theorem 6.1.

**Theorem 7.1.** There exists a finite dimensional $\mathbb{Q}$-vector space $W_{h,E_{n/2+1/2}}$ in $\mathbb{C}$ such that

$$\frac{R(\chi)(m, h, E_{n/2+1/2})}{\pi^{mn}} \in W_{h,E_{n/2+1/2}}$$

for any integer $n/2 + 1 \leq m \leq k - n/2 - 1$ and a character $\chi$ of odd square free conductor such that $\chi^n$ is primitive.

**Proof.** Put

$$M(\chi)(s, S(h)) = \sum_{i_1=0}^{l_1-1} \cdots \sum_{i_r=0}^{l_r-1} J(\tilde{\chi}(i_1, \ldots, i_r), \left(\frac{\ast}{N}\right))J_{n-1}(\chi(i_1, \ldots, i_r))$$

$$\times \prod_{j=1}^{n/2} L(2s - 2j + 1, S(h), \chi_{(i_1, \ldots, i_r)}^2).$$

Then by Corollary to Proposition 3.1, we have

$$\frac{M(\chi)(m, S(h))}{\pi^{mn}} \in \mathbb{Q} u_{-}(S(h))^n/2 \pi^{-n^2/4}.$$

By Theorem 6.1, we have

$$L(m, I_n(h), \chi^n) = 2^{m} \chi(2^n) \{c_nN R(\chi)(m, h, E_{n/2+1/2}) + d_{n,N}c(h)(1)M(\chi)(m, S(h))\}.$$

Hence by Theorem 2.2, we have

$$\frac{R(\chi)(m, h, E_{n/2+1/2})}{\pi^{mn}} \in \mathbb{Q} u_1 \otimes \mathbb{Q} V_{I_n(h)} + \mathbb{Q} u_2$$

with some complex numbers $u_1$ and $u_2$, where $V_{I_n(h)}$ is the $\mathbb{Q}$-vector space associated with $I_n(h)$ in Theorem 2.2. This proves the assertion.

By the above theorem, we immediately obtain the following:
Theorem 7.2. Let $d > \dim \mathcal{Q} W_h, E_{n/2+1/2}$. Let $m_1, m_2, \ldots, m_d$ be integers such that $n/2+1 \leq m_1, m_2, \ldots, m_d \leq k - n/2 - 1$ and $\chi_1, \chi_2, \ldots, \chi_d$ be Dirichlet characters of odd square free conductors $N_1, N_2, \ldots, N_d$, respectively such that $\chi_i^n$ is primitive for any $i = 1, 2, \ldots, d$. Then the values $\frac{R^{(\chi_i)}(m_1, h, E_{n/2+1/2})}{\pi^m}$, $\ldots$, $\frac{R^{(\chi_d)}(m_d, h, E_{n/2+1/2})}{\pi^m}$ are linearly dependent over $\mathbb{Q}$.

Corollary. In addition to the notation and the assumption as above, assume that $n \equiv 2 \mod 4$. Write $N_i = \prod_{p} p_i$ with $p_i$ an odd prime number, and let $l_{ij} = \gcd(p_{ij} - 1, n)$. Then the values $\left\{ \frac{R(m_i, h, E_{n/2+1/2}, \chi(a_1, \ldots, a_{r_i}))}{\pi^{2m_i}} \right\}_{1 \leq i \leq d, 0 \leq t_{i1}, \ldots, t_{ir_i} \leq l_{ij}}$ are linearly dependent over $\overline{\mathbb{Q}}$. In particular, if $\chi_1, \chi_2, \ldots, \chi_d$ are Dirichlet characters of odd prime conductors $p_1, p_2, \ldots, p_d$, respectively such that $\chi_i^n$ is primitive for any $i = 1, 2, \ldots, d$, then the values $\left\{ \frac{R(m_i, h, E_{n/2+1/2}, \chi^{(\frac{a_i}{n})^{\frac{1}{l_{ir_i}}}}(\frac{a_i}{n}^{\frac{1}{l_{ir_i}}}))}{\pi^{2m_i}} \right\}_{1 \leq i \leq d, 0 \leq a_i \leq l_{ir_i} - 1}$ are linearly dependent over $\overline{\mathbb{Q}}$, where $l_i = \gcd(n, p_i - 1)$ for $i = 1, \ldots, d$.

Proof. By Theorem 1.1, the value $\frac{L_n(n, S(h), \chi_{i_1, \ldots, i_r})}{\pi^{2n}}$ belongs to $\overline{\mathbb{Q}}[\pi^{n/2}]$ and in particular if $n \equiv 2 \mod 4$, then it is nonzero for any $\chi_i$. Moreover, by Corollary to Theorem 5.10, $J(\chi_{i_1, \ldots, i_r}, (\frac{a_i}{n}^{\frac{1}{l_{ir_i}}}))$ is non-zero and belongs to $\overline{\mathbb{Q}}$. Thus the assertion holds.

As another application of Theorem 7.1, we also have a functional equation for $R^{(\chi)}(s, h, E_{n/2+1/2})$. Namely, by Theorem 3.1 we obtain:

Theorem 7.3. Let $h$ be as above. Let $\chi$ be a primitive character of odd square free conductor $N$. Assume that $n \equiv 2 \mod 4$, and that $\chi^n$ is primitive. Put $\mathcal{R}^{(\chi)}(s, h, E_{n/2+1/2}) = N^{2s} \tau(\chi^{-1})^{-1} \gamma_n(s) R^{(\chi)}(s, h, E_{n/2+1/2})$, where $\tau(\chi^n)$ is the Gauss sum, and

$$\gamma_n(s) = (2\pi)^{-ns} \prod_{i=1}^{n} \pi^{(i-1)/2} \Gamma(s - (i - 1)/2).$$

Then $\mathcal{R}^{(\chi)}(s, h, E_{n/2+1/2})$ has an analytic continuation to the whole $s$-plane, and has the following functional equation:

$$\mathcal{R}^{(\chi)}(s) = \mathcal{R}^{(\chi)}(s - k, h, E_{n/2+1/2}).$$

Remark. (1) As functions of $s$, the Dirichlet series $\{R(s, h, E_{n/2+1/2}, \chi_{i(j)})\}_{1 \leq i \leq r, 0 \leq j \leq l_{ij} - 1}$ are linearly independent over $\mathbb{C}$.

(2) In the case of $n = 2$, this type of result was given for $R(m, h, E_3/2)$ with $E_{3/2}$ Zagier’s Eisenstein series of weight $3/2$ by [K-M].

(3) The meromorphy of this type of series was derived in [Sh3] by using so called the Rankin-Selberg integral expression in more general setting, but we don’t know whether the functional equation of the above type can be directly proved without using the above method.
References


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