Existence and uniqueness for Legendre curves

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Dedicated to Professor Masahiko Suzuki on the occasion of his 60th birthday

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Abstract

We give a moving frame of a Legendre curve (or, a frontal) in the unite tangent bundle and define a pair of smooth functions of a Legendre curve like as the curvature of a regular plane curve. The existence and uniqueness for Legendre curves are holded like as regular plane curves. It is quite useful to analyse the Legendre curves. As applications, we consider contact between Legendre curves and the arc-length parameter of Legendre immersions in the unite tangent bundle.

1 Introduction

A regular plane curve determines a curvature function, providing valuable geometric information about the original curve by using a moving frame of the curve. The existence and uniqueness results are fundamental theorems for regular plane curves, see below Theorems 1.1 and 1.2.

Let I be an interval or $\mathbb{R}$. Suppose that $\gamma : I \to \mathbb{R}^2$ is a regular curve, that is, $\dot{\gamma}(t) \neq 0$ for any $t \in I$. If s is the arc-length parameter of γ, we denote $\boldsymbol{t}(s)$ by the unite tangent vector $\boldsymbol{t}(s) = \gamma'(s) = d\gamma/ds(s)$ and $\boldsymbol{n}(s)$ by the unite normal vector $\boldsymbol{n}(s) = J(\boldsymbol{t}(s))$ of $\gamma(s)$, where J is the anticlockwise rotation of $\pi/2$. Then we have the Frenet formula as follows:

$$\begin{pmatrix} \boldsymbol{t}'(s) \\ \boldsymbol{n}'(s) \end{pmatrix} = \begin{pmatrix} 0 & \kappa(s) \\ -\kappa(s) & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{t}(s) \\ \boldsymbol{n}(s) \end{pmatrix},$$

where $\kappa(s) = \boldsymbol{t}'(s) \cdot \boldsymbol{n}(s)$ is the curvature of γ and $\cdot$ is the inner product on $\mathbb{R}^2$.

Even if t is not the arc-length parameter, we have the unite tangent vector $\boldsymbol{t}(t) = \dot{\gamma}(t)/|\dot{\gamma}(t)|$, the unite normal vector $\boldsymbol{n}(t) = J(\boldsymbol{t}(t))$ and the Frenet formula

$$\begin{pmatrix} \dot{\boldsymbol{t}}(t) \\ \dot{\boldsymbol{n}}(t) \end{pmatrix} = \begin{pmatrix} 0 & |\dot{\gamma}(t)|\kappa(t) \\ -|\dot{\gamma}(t)|\kappa(t) & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{t}(t) \\ \boldsymbol{n}(t) \end{pmatrix},$$

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where $\dot{\gamma}(t) = d\gamma/dt(t)$, $|\dot{\gamma}(t)| = \sqrt{\dot{\gamma}(t) \cdot \dot{\gamma}(t)}$ and $\kappa(t) = \det(\dot{\gamma}(t), \ddot{\gamma}(t))/|\dot{\gamma}(t)|^3 = \dot{\boldsymbol{t}}(t) \cdot \boldsymbol{n}(t)/|\dot{\gamma}(t)|$. Note that $\kappa(t)$ is independent on the choice of a parametrization.

Let γ and $\widetilde{\gamma} : I \to \mathbb{R}^2$ be regular curves. We say that γ and $\widetilde{\gamma}$ are *congruent* if there exists a congruence C on $\mathbb{R}^2$ such that $\widetilde{\gamma}(t) = C(\gamma(t))$ for all $t \in I$, where the *congruence* C is a composition of a rotation and a translation on $\mathbb{R}^2$.

As well-known results, the existence and uniqueness for regular plane curves are as follows (cf. [5, 6]):

Theorem 1.1 (The Existence Theorem) *Let $\kappa : I \to \mathbb{R}$ be a smooth function. There exists a regular parametrized curve $\gamma : I \to \mathbb{R}^2$ whose associated curvature function is κ.*

Theorem 1.2 (The Uniqueness Theorem) *Let γ and $\widetilde{\gamma} : I \to \mathbb{R}^2$ be regular curves whose speeds $s = |\dot{\gamma}(t)|$ and $\widetilde{s} = |\dot{\widetilde{\gamma}}(t)|$, and also curvatures κ and $\widetilde{\kappa}$ each coincide. Then γ and $\widetilde{\gamma}$ are congruent.*

If γ has a singular point, we can not construct a moving frame of γ. In the analytic category, there is a construction of a moving frame of an analytic curve under a mild condition, see in [9]. However, we can define a moving frame of a frontal for a Legendre curve in the unit tangent bundle in the smooth category. By using the moving frame, we define a pair of smooth functions like as the curvature of a regular curve. We call the pair the *curvature of the Legendre curve*. It is quite useful to analyse the Legendre curves (or, frontals). In this paper, we give an existence and uniqueness for Legendre curves like as regular plane curves, see Theorems 1.4 and 1.5. These results are elementary, however, it might be new results, as far as we know.

We say that $(\gamma, \nu) : I \to \mathbb{R}^2 \times S^1$ is *a Legendre curve* if $(\gamma, \nu)^*\theta = 0$ for all $t \in I$, where θ is a canonical contact form on the unit tangent bundle $T_1\mathbb{R}^2 = \mathbb{R}^2 \times S^1$ (cf. [1, 2]). This condition is equivalent to $\dot{\gamma}(t) \cdot \nu(t) = 0$ for all $t \in I$. Moreover, if (γ, ν) is an immersion, we call (γ, ν) *a Legendre immersion*. We say that $\gamma : I \to \mathbb{R}^2$ is *a frontal* (respectively, *a front* or *a wave front*) if there exists a smooth mapping $\nu : I \to S^1$ such that (γ, ν) is a Legendre curve (respectively, a Legendre immersion).

Let $(\gamma, \nu) : I \to \mathbb{R}^2 \times S^1$ be a Legendre curve. If γ is a regular curve around a point t_0, then we have the Frenet formula of γ. On the other hand, if γ is singular at a point t_0, then we don't define such a frame. However, ν is always defined even if t is a singular point of γ. Therefore, we have the Frenet formula of a frontal γ as follows. We put on $\boldsymbol{\mu}(t) = J(\nu(t))$. We call the pair $\{\nu(t), \boldsymbol{\mu}(t)\}$ a *moving frame of a frontal* $\gamma(t)$ in $\mathbb{R}^2$ and we have the Frenet formula of a frontal (or, Legendre curve) which is given by

$$\begin{pmatrix} \dot{\nu}(t) \\ \dot{\boldsymbol{\mu}}(t) \end{pmatrix} = \begin{pmatrix} 0 & \ell(t) \\ -\ell(t) & 0 \end{pmatrix} \begin{pmatrix} \nu(t) \\ \boldsymbol{\mu}(t) \end{pmatrix}, \tag{1}$$

where $\ell(t) = \dot{\nu}(t) \cdot \boldsymbol{\mu}(t)$. Moreover, if $\dot{\gamma}(t) = \alpha(t)\nu(t) + \beta(t)\boldsymbol{\mu}(t)$ for some smooth functions $\alpha(t), \beta(t)$, then $\alpha(t) = 0$ follows from the condition $\dot{\gamma}(t) \cdot \nu(t) = 0$. Hence, there exists a smooth function $\beta(t)$ such that

$$\dot{\gamma}(t) = \beta(t)\boldsymbol{\mu}(t). \tag{2}$$

The pair (ℓ, β) is an important invariant of Legendre curves (or, frontals). We call the pair $(\ell(t), \beta(t))$ *the curvature of the Legendre curve* (with respect to the parameter t).

Definition 1.3 Let (γ,ν) and $(\widetilde{\gamma},\widetilde{\nu}) : I \to \mathbb{R}^2 \times S^1$ be Legendre curves. We say that (γ,ν) and $(\widetilde{\gamma},\widetilde{\nu})$ are *congruent as Legendre curves* if there exists a congruence C on $\mathbb{R}^2$ such that $\widetilde{\gamma}(t) = C(\gamma(t)) = A(\gamma(t)) + \boldsymbol{b}(t)$ and $\widetilde{\nu}(t) = A(\nu(t))$ for all $t \in I$, where C is given by the rotation A and the translation $\boldsymbol{b}$ on $\mathbb{R}^2$.

The main results in this paper are the existence and uniqueness for Legendre curves in the unite tangent bundle like as regular plane curves, see Theorems 1.1 and 1.2.

Theorem 1.4 (The Existence Theorem) *Let $(\ell,\beta) : I \to \mathbb{R}^2$ be a smooth mapping. There exists a Legendre curve $(\gamma,\nu) : I \to \mathbb{R}^2 \times S^1$ whose associated curvature of the Legendre curve is (ℓ,β).*

Theorem 1.5 (The Uniqueness Theorem) *Let (γ,ν) and $(\widetilde{\gamma},\widetilde{\nu}) : I \to \mathbb{R}^2 \times S^1$ be Legendre curves whose curvatures of Legendre curves (ℓ,β) and $(\widetilde{\ell},\widetilde{\beta})$ coincide. Then (γ,ν) and $(\widetilde{\gamma},\widetilde{\nu})$ are congruent as Legendre curves.*

We shall prove these theorems in §2. Moreover, we consider properties of the curvatures of Legendre curves. As applications, we consider contact between Legendre curves in §3 and give a special parameter, so-called the arc-length parameter, of Legendre immersions in the unite tangent bundle in §4. Further applications, we give the evolute of a front by using the moving frame of a front and the curvature of the Legendre immersion, for more detail in [4].

All maps and manifolds considered here are differential of class C^∞.

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2 Properties of Legendre curves

First we prove the existence theorem (Theorem 1.4).

Proof of Theorem 1.4. Let $\theta : I \to \mathbb{R}$ be any function with the property that $\dot{\theta}(t) = \ell(t)$ for all $t \in I$. Furthermore, let

$$\nu(t) = (\cos\theta(t), \sin\theta(t)),\ \boldsymbol{\mu}(t) = (-\sin\theta(t), \cos\theta(t))$$

be the curves in the unit circle. Define smooth functions $x(t)$ and $y(t)$ with $\dot{x}(t) = -\beta(t)\sin\theta(t)$ and $\dot{y}(t) = \beta(t)\cos\theta(t)$. Then $\gamma : I \to \mathbb{R}^2$ is given by $\gamma(t) = (x(t), y(t))$, that is,

$$\gamma(t) = \left(-\int \left(\beta(t)\sin\int \ell(t)\ dt\right) dt,\ \int \left(\beta(t)\cos\int \ell(t)\ dt\right) dt\right).$$

It follows that $\dot{\gamma}(t) = \beta(t)\boldsymbol{\mu}(t)$, $\dot{\nu}(t) = \ell(t)\boldsymbol{\mu}(t)$ and $\dot{\gamma}(t)\cdot\nu(t) = 0$ for all $t \in I$. Therefore, there exists a Legendre curve $(\gamma,\nu) : I \to \mathbb{R}^2 \times S^1$ whose associated curvature of the Legendre curve is $(\ell(t), \beta(t))$. $\square$

In order to prove the uniqueness theorem (Theorem 1.5), we need two Lemmas.

Lemma 2.1 *Let (γ,ν) and $(\widetilde{\gamma},\widetilde{\nu}) : I \to \mathbb{R}^2 \times S^1$ be congruent as Legendre curves. Then (γ,ν) and $(\widetilde{\gamma},\widetilde{\nu})$ have the same curvatures of Legendre curves (ℓ,β) and $(\widetilde{\ell},\widetilde{\beta})$ respectively.*

Proof. Since (γ,ν) and $(\widetilde{\gamma},\widetilde{\nu})$ are congruent as Legendre curves, there exist a rotation A and a fixed vector $\boldsymbol{b}$ with the property that

$$\widetilde{\gamma}(t)=A(\gamma(t))+\boldsymbol{b},\ \widetilde{\nu}(t)=A(\nu(t))$$

for all $t\in I$. Since the definition of $\boldsymbol{\mu}$ and $JA=AJ$, we have $\widetilde{\boldsymbol{\mu}}(t)=A(\boldsymbol{\mu}(t))$ for all $t\in I$. By $\dot{\gamma}(t)=\beta(t)\boldsymbol{\mu}(t)$ and $\dot{\nu}(t)=\ell(t)\boldsymbol{\mu}(t)$,

$$\begin{aligned}\frac{d}{dt}\widetilde{\gamma}(t)&=A(\dot{\gamma}(t))=A(\beta(t)\boldsymbol{\mu}(t))=\beta(t)A(\boldsymbol{\mu}(t))=\beta(t)\widetilde{\boldsymbol{\mu}}(t),\\ \frac{d}{dt}\widetilde{\nu}(t)&=A(\dot{\nu}(t))=A(\ell(t)\boldsymbol{\mu}(t))=\ell(t)A(\boldsymbol{\mu}(t))=\ell(t)\widetilde{\boldsymbol{\mu}}(t).\end{aligned}$$

Hence we have $\beta(t)=\widetilde{\beta}(t)$ and $\ell(t)=\widetilde{\ell}(t)$. □

Lemma 2.2 *Let (γ,ν) and $(\widetilde{\gamma},\widetilde{\nu}):I\to\mathbb{R}^2\times S^1$ be Legendre curves having equal curvatures of Legendre curves, that is, $(\ell(t),\beta(t))=(\widetilde{\ell}(t),\widetilde{\beta}(t))$ for all $t\in I$. If there exists a parameter $t=t_0$ for which $(\gamma(t_0),\nu(t_0))=(\widetilde{\gamma}(t_0),\widetilde{\nu}(t_0))$, then (γ,ν) and $(\widetilde{\gamma},\widetilde{\nu})$ coincide.*

Proof. Let $f(t)=\nu(t)\cdot\widetilde{\nu}(t)+\boldsymbol{\mu}(t)\cdot\widetilde{\boldsymbol{\mu}}(t)$ be a smooth function on I. Then

$$\begin{aligned}\dot{f}(t)&=\dot{\nu}(t)\cdot\widetilde{\nu}(t)+\nu(t)\cdot\dot{\widetilde{\nu}}(t)+\dot{\boldsymbol{\mu}}(t)\cdot\widetilde{\boldsymbol{\mu}}(t)+\boldsymbol{\mu}(t)\cdot\dot{\widetilde{\boldsymbol{\mu}}}(t)\\ &=(\ell(t)\boldsymbol{\mu}(t))\cdot\widetilde{\nu}(t)+\nu(t)\cdot(\widetilde{\ell}(t)\widetilde{\boldsymbol{\mu}}(t))+(-\ell(t)\nu(t))\cdot\widetilde{\boldsymbol{\mu}}(t)+\boldsymbol{\mu}(t)\cdot(-\widetilde{\ell}(t)\widetilde{\nu}(t))\\ &=(\ell(t)-\widetilde{\ell}(t))\boldsymbol{\mu}(t)\cdot\widetilde{\nu}(t)+(\widetilde{\ell}(t)-\ell(t))\nu(t)\cdot\widetilde{\boldsymbol{\mu}}(t)=0,\end{aligned}$$

since $\ell(t)=\widetilde{\ell}(t)$ by the assumption. It follows that f is constant. Moreover, setting $t=t_0$ and $\nu(t_0)=\widetilde{\nu}(t_0)$, then $\boldsymbol{\mu}(t_0)=\widetilde{\boldsymbol{\mu}}(t_0)$ and hence $f(t_0)=|\nu(t_0)|^2+|\boldsymbol{\mu}(t_0)|^2=2$. The function f is the constant value 2. By the Cauchy-Schwarz inequality, we have

$$\nu(t)\cdot\widetilde{\nu}(t)\le|\nu(t)||\widetilde{\nu}(t)|=1,\ \boldsymbol{\mu}(t)\cdot\widetilde{\boldsymbol{\mu}}(t)\le|\boldsymbol{\mu}(t)||\widetilde{\boldsymbol{\mu}}(t)|=1.$$

If either of these inequalities were strict, the value of $f(t)$ would be less than 2. It follows that both these inequalities are equalities, and we have $\nu(t)\cdot\widetilde{\nu}(t)=1,\ \boldsymbol{\mu}(t)\cdot\widetilde{\boldsymbol{\mu}}(t)=1$ for all $t\in I$. Then we have

$$|\nu(t)-\widetilde{\nu}(t)|^2=\nu(t)\cdot\nu(t)-2\nu(t)\cdot\widetilde{\nu}(t)+\widetilde{\nu}(t)\cdot\widetilde{\nu}(t)=0,$$

and also $|\boldsymbol{\mu}(t)-\widetilde{\boldsymbol{\mu}}(t)|^2=0$. Hence $\nu(t)=\widetilde{\nu}(t)$ and $\boldsymbol{\mu}(t)=\widetilde{\boldsymbol{\mu}}(t)$ for all $t\in I$. Since $\dot{\gamma}(t)=\beta(t)\boldsymbol{\mu}(t)$, $\dot{\widetilde{\gamma}}(t)=\widetilde{\beta}(t)\widetilde{\boldsymbol{\mu}}(t)$ and the assumption $\beta(t)=\widetilde{\beta}(t)$, $(d/dt)(\gamma(t)-\widetilde{\gamma}(t))=0$. It follows that $\gamma(t)-\widetilde{\gamma}(t)$ is constant. By the condition $\gamma(t_0)=\widetilde{\gamma}(t_0)$, we have $\gamma(t)=\widetilde{\gamma}(t)$ for all $t\in I$. □

Proof of Theorem 1.5. Choose any fixed value $t=t_0$ of the parameter. By using a rotation A and a translation $\boldsymbol{b}$, we can assume that $\widetilde{\gamma}(t_0)=A(\gamma(t_0))+\boldsymbol{b}$ and $\widetilde{\nu}(t_0)=A(\nu(t_0))$. By Lemma 2.1, the curvatures of the Legendre curves (γ,ν) and $(A(\gamma(t))+\boldsymbol{b},A(\nu(t)))$ coincide. By Lemme 2.2, $\widetilde{\gamma}(t)=A(\gamma(t))+\boldsymbol{b},\ \widetilde{\nu}(t)=A(\nu(t))$ for all $t\in I$. It follows that (γ,ν) and $(\widetilde{\gamma},\widetilde{\nu})$ are congruent as Legendre curves. □

Remark 2.3 Both Theorems 1.4 and 1.5 can also prove by using the theory of the existence and uniqueness for the system of ordinary differential equations.

Let I and $\overline{I}$ be intervals. A smooth function $s : \overline{I} \to I$ is a *(positive) change of parameter* when s is surjective and has a positive derivative at every point. It follows that s is a diffeomorphism map by calculus.

Let $(\gamma, \nu) : I \to \mathbb{R}^2 \times S^1$ and $(\overline{\gamma}, \overline{\nu}) : \overline{I} \to \mathbb{R}^2 \times S^1$ be Legendre curves whose curvatures of Legendre curves are (ℓ, β) and $(\overline{\ell}, \overline{\beta})$ respectively. Suppose (γ, ν) and $(\overline{\gamma}, \overline{\nu})$ are parametrically equivalent via the change of parameter $s : \overline{I} \to I$. Thus $(\overline{\gamma}(t), \overline{\nu}(t)) = (\gamma(s(t)), \nu(s(t)))$ for all $t \in \overline{I}$. By differentiation, we have

$$\overline{\ell}(t) = \ell(s(t))\dot{s}(t), \ \overline{\beta}(t) = \beta(s(t))\dot{s}(t).$$

Hence the curvature of the Legendre curve is depended on a parametrization. However, for a Legendre immersion $(\gamma, \nu) : I \to \mathbb{R}^2 \times S^1$, we can define the normalized curvature and the arc-length parameter. Then the normalized curvature of the Legendre curve independent on the change of a parametrization, see in §4. Note that (γ, ν) is a Legendre immersion if and only if $(\ell(t), \beta(t)) \neq (0, 0)$ for all $t \in I$.

Remark 2.4 By the definition of the Legendre curve, if (γ, ν) is a Legendre curve, then $(\gamma, -\nu)$ is also. In this case, $\ell(t)$ does not change, but $\beta(t)$ change to $-\beta(t)$.

Now we give examples of Legendre curves.

Example 2.5 One of the typical example of a front (and hence a frontal) is a regular plane curve. Let $\gamma : I \to \mathbb{R}^2$ be a regular plane curve. In this case, we may take $\nu : I \to S^1$ by $\nu(t) = \boldsymbol{n}(t)$. Then it is easy to check that $(\gamma, \nu) : I \to \mathbb{R}^2 \times S^1$ is a Legendre immersion (a Legendre curve).

By a direct calculation, we give a relationship between the curvature of the Legendre curve $(\ell(t), \beta(t))$ and the curvature $\kappa(t)$ if γ is a regular curve.

Proposition 2.6 ([4, Lemma 3.1]) *Under the above notions, if γ is a regular curve, then $\ell(t) = |\beta(t)|\kappa(t)$.*

Example 2.7 Let n, m and k be natural numbers with $m + k$. Let $(\gamma, \nu) : I \to \mathbb{R}^2 \times S^1$ be

$$\gamma(t) = \left(\frac{1}{n}t^n, \frac{1}{m}t^m\right), \ \nu(t) = \frac{1}{\sqrt{t^{2k}+1}}\left(-t^k, 1\right).$$

It is easy to see that (γ, ν) is a Legendre curve, and a Legendre immersion when $k = 1$. We call γ is of *type* (n, m). For example, type $(2, 3)$ is the 3/2-cusp (A_2 singularity), type $(3, 4)$ is the 4/3-cusp (E_6 singularity) and type $(2, 5)$ is the 5/2-cusp (A_4 singularity) (cf. [2, 3, 7]). By definition, we have $\boldsymbol{\mu}(t) = (1/\sqrt{t^{2k}+1})(-1, -t^k)$ and

$$\ell(t) = \frac{kt^{k-1}}{t^{2k}+1}, \ \beta(t) = -t^{n-1}\sqrt{t^{2k}+1}.$$

3 Contact between Legendre curves

In this section, we discuss contact between Legendre curves. Let $(\gamma, \nu) : I \to \mathbb{R}^2 \times S^1; t \mapsto (\gamma(t), \nu(t))$ and $(\widetilde{\gamma}, \widetilde{\nu}) : \widetilde{I} \to \mathbb{R}^2 \times S^1; u \mapsto (\widetilde{\gamma}(u), \widetilde{\nu}(u))$ be Legendre curves, respectively. We say that (γ,ν) *and* $(\widetilde{\gamma},\widetilde{\nu})$ *are* k*-th order contact at* $t=t_0, u=u_0$ if

$$(\gamma,\nu)(t_0)=(\widetilde{\gamma},\widetilde{\nu})(u_0),\ \frac{d}{dt}(\gamma,\nu)(t_0)=\frac{d}{du}(\widetilde{\gamma},\widetilde{\nu})(u_0),\ \cdots,\ \frac{d^{k-1}}{dt^{k-1}}(\gamma,\nu)(t_0)=\frac{d^{k-1}}{du^{k-1}}(\widetilde{\gamma},\widetilde{\nu})(u_0)$$

and

$$\frac{d^k}{dt^k}(\gamma,\nu)(t_0)\neq\frac{d^k}{du^k}(\widetilde{\gamma},\widetilde{\nu})(u_0).$$

Moreover, we say that (γ,ν) *and* $(\widetilde{\gamma},\widetilde{\nu})$ *are at least* k*-th order contact at* $t=t_0, u=u_0$ if

$$(\gamma,\nu)(t_0)=(\widetilde{\gamma},\widetilde{\nu})(u_0),\ \frac{d}{dt}(\gamma,\nu)(t_0)=\frac{d}{du}(\widetilde{\gamma},\widetilde{\nu})(u_0),\ \cdots,\ \frac{d^{k-1}}{dt^{k-1}}(\gamma,\nu)(t_0)=\frac{d^{k-1}}{du^{k-1}}(\widetilde{\gamma},\widetilde{\nu})(u_0).$$

Let $(\gamma,\nu):I\to\mathbb{R}^2\times S^1; t\mapsto(\gamma(t),\nu(t))$ and $(\widetilde{\gamma},\widetilde{\nu}):\widetilde{I}\to\mathbb{R}^2\times S^1; u\mapsto(\widetilde{\gamma}(u),\widetilde{\nu}(u))$ be Legendre curves. In general, we may assume that (γ,ν) and $(\widetilde{\gamma},\widetilde{\nu})$ are at least first order contact at any point $t=t_0, u=u_0$, up to congruent as Legendre curves. We denote the curvatures of the Legendre curves $(\ell(t),\beta(t))$ of (γ,ν) and $(\widetilde{\ell}(u),\widetilde{\beta}(u))$ of $(\widetilde{\gamma},\widetilde{\nu})$, respectively.

Theorem 3.1 *Let* $(\gamma,\nu):I\to\mathbb{R}^2\times S^1; t\mapsto(\gamma(t),\nu(t))$ *and* $(\widetilde{\gamma},\widetilde{\nu}):\widetilde{I}\to\mathbb{R}^2\times S^1; u\mapsto(\widetilde{\gamma}(u),\widetilde{\nu}(u))$ *be Legendre curves. If* (γ,ν) *and* $(\widetilde{\gamma},\widetilde{\nu})$ *are at least* $(k+1)$*-th order contact at* $t=t_0, u=u_0$ *then*

$$(\ell,\beta)(t_0)=(\widetilde{\ell},\widetilde{\beta})(u_0),\ \frac{d}{dt}(\ell,\beta)(t_0)=\frac{d}{du}(\widetilde{\ell},\widetilde{\beta})(u_0),\ \cdots,\ \frac{d^{k-1}}{dt^{k-1}}(\ell,\beta)(t_0)=\frac{d^{k-1}}{du^{k-1}}(\widetilde{\ell},\widetilde{\beta})(u_0).\quad(3)$$

Conversely, if the condition (3) *holds, then* (γ,ν) *and* $(\widetilde{\gamma},\widetilde{\nu})$ *are at least* $(k+1)$*-th order contact at* $t=t_0, u=u_0$*, up to congruent as Legendre curves.*

Proof. Suppose that (γ,ν) and $(\widetilde{\gamma},\widetilde{\nu})$ are at least second order contact at $t=t_0, u=u_0$. Since $\nu(t_0)=\widetilde{\nu}(u_0)$, we have $\boldsymbol{\mu}(t_0)=\widetilde{\boldsymbol{\mu}}(u_0)$. By (1) and (2), $(d/dt)(\gamma,\nu)(t)=(\beta(t)\boldsymbol{\mu}(t),\ell(t)\boldsymbol{\mu}(t))$ and $(d/du)(\widetilde{\gamma},\widetilde{\nu})(u)=(\widetilde{\beta}(u)\widetilde{\boldsymbol{\mu}}(u),\widetilde{\ell}(u)\widetilde{\boldsymbol{\mu}}(u))$. It follows that $\ell(t_0)=\widetilde{\ell}(u_0),\beta(t_0)=\widetilde{\beta}(u_0)$. Hence, the case of $k=1$ holds.

Suppose that (γ,ν) and $(\widetilde{\gamma},\widetilde{\nu})$ are at least $(k+1)$-th order contact at $t=t_0, u=u_0$ and

$$(\ell,\beta)(t_0)=(\widetilde{\ell},\widetilde{\beta})(u_0),\ \frac{d}{dt}(\ell,\beta)(t_0)=\frac{d}{du}(\widetilde{\ell},\widetilde{\beta})(u_0),\ \cdots,\ \frac{d^{k-2}}{dt^{k-2}}(\ell,\beta)(t_0)=\frac{d^{k-2}}{du^{k-2}}(\widetilde{\ell},\widetilde{\beta})(u_0)$$

hold. It follows that $(d^k/dt^k)\gamma(t)$ and $(d^k/dt^k)\nu(t)$ are given by the form

$$\begin{aligned}\frac{d^{k-1}}{dt^{k-1}}\beta(t)\boldsymbol{\mu}(t)\quad+\quad & f_1\left(\beta(t),\ell(t),\dots,\frac{d^{k-2}}{dt^{k-2}}\beta(t),\frac{d^{k-2}}{dt^{k-2}}\ell(t)\right)\nu(t)\\ & +f_2\left(\beta(t),\ell(t),\dots,\frac{d^{k-2}}{dt^{k-2}}\beta(t),\frac{d^{k-2}}{dt^{k-2}}\ell(t)\right)\boldsymbol{\mu}(t)\end{aligned}$$

and

$$\begin{aligned}\frac{d^{k-1}}{dt^{k-1}}\ell(t)\boldsymbol{\mu}(t)\quad+\quad & g_1\left(\beta(t),\ell(t),\dots,\frac{d^{k-2}}{dt^{k-2}}\beta(t),\frac{d^{k-2}}{dt^{k-2}}\ell(t)\right)\nu(t)\\ & +g_2\left(\beta(t),\ell(t),\dots,\frac{d^{k-2}}{dt^{k-2}}\beta(t),\frac{d^{k-2}}{dt^{k-2}}\ell(t)\right)\boldsymbol{\mu}(t)\end{aligned}$$

for some smooth functions f_1, f_2, g_1 and g_2. By the same calculations,

$$
\begin{aligned}
\frac{d^k}{du^k}\widetilde{\gamma}(u) &= \frac{d^{k-1}}{du^{k-1}}\widetilde{\beta}(u)\widetilde{\boldsymbol{\mu}}(u) + f_1\left(\widetilde{\beta}(u), \widetilde{\ell}(u), \dots, \frac{d^{k-2}}{du^{k-2}}\widetilde{\beta}(u), \frac{d^{k-2}}{du^{k-2}}\widetilde{\ell}(u)\right)\widetilde{\nu}(u) \\
&\quad + f_2\left(\widetilde{\beta}(u), \widetilde{\ell}(u), \dots, \frac{d^{k-2}}{du^{k-2}}\widetilde{\beta}(u), \frac{d^{k-2}}{du^{k-2}}\widetilde{\ell}(u)\right)\widetilde{\boldsymbol{\mu}}(u), \\
\frac{d^k}{du^k}\widetilde{\nu}(u) &= \frac{d^{k-1}}{du^{k-1}}\widetilde{\ell}(u)\widetilde{\boldsymbol{\mu}}(u) + g_1\left(\widetilde{\beta}(u), \widetilde{\ell}(u), \dots, \frac{d^{k-2}}{du^{k-2}}\widetilde{\beta}(u), \frac{d^{k-2}}{du^{k-2}}\widetilde{\ell}(u)\right)\widetilde{\nu}(u) \\
&\quad + g_2\left(\widetilde{\beta}(u), \widetilde{\ell}(u), \dots, \frac{d^{k-2}}{du^{k-2}}\widetilde{\beta}(u), \frac{d^{k-2}}{du^{k-2}}\widetilde{\ell}(u)\right)\widetilde{\boldsymbol{\mu}}(u).
\end{aligned}
$$

It follows that $((d^{k-1}/dt^{k-1})\ell, (d^{k-1}/dt^{k-1})\beta)(t_0) = ((d^{k-1}/du^{k-1})\widetilde{\ell}, (d^{k-1}/du^{k-1})\widetilde{\beta})(u_0)$. By the induction, we have the first assertion.

By reversing arguments, we can prove the converse assertion, up to congruent as Legendre curves. $\Box$

Note that if γ is a regular curve, then we also consider a contact between curves (cf. [4]). Let $\gamma : I \to \mathbb{R}^2; t \mapsto \gamma(t)$ and $\widetilde{\gamma} : \widetilde{I} \to \mathbb{R}^2; u \mapsto \widetilde{\gamma}(u)$ be regular plane curves, respectively. We say that *γ and $\widetilde{\gamma}$ are at least k-th order contact at* $t = t_0, u = u_0$ if

$$\gamma(t_0) = \widetilde{\gamma}(u_0),\ \frac{d\gamma}{dt}(t_0) = \frac{d\widetilde{\gamma}}{du}(u_0),\ \cdots,\ \frac{d^k\gamma}{dt^k}(t_0) = \frac{d^k\widetilde{\gamma}}{du^k}(u_0).$$

By example 2.5, we take $\nu : I \to S^1, \nu(t) = \boldsymbol{n}(t)$ and $\widetilde{\nu} : \widetilde{I} \to S^1, \widetilde{\nu}(u) = \widetilde{\boldsymbol{n}}(u)$.

If s be the arc-length parameter of γ, then $\ell(t) = \kappa(t)$ and $|\beta(t)| = 1$ by Proposition 2.6. Therefore, we have following result as a corollary of Theorem 3.1.

Corollary 3.2 *Let $\gamma : I \to \mathbb{R}^2$ and $\widetilde{\gamma} : \widetilde{I} \to \mathbb{R}^2$ be regular curves with the arc-length parameters. Under the above notations, (γ, ν) and $(\widetilde{\gamma}, \widetilde{\nu})$ are Legendre immersions. Then (γ, ν) and $(\widetilde{\gamma}, \widetilde{\nu})$ are at least $(k+1)$-order contact at $t = t_0, u = u_0$ if and only if γ and $\widetilde{\gamma}$ are at least $(k+1)$-order contact at $t = t_0, u = u_0$.*

4 Legendre immersions

In this section, we consider Legendre immersions in the unit tangent bundle. Let $(\gamma, \nu) : I \to \mathbb{R}^2 \times S^1$ be a Legendre immersion. Then the curvature of the Legendre immersion $(\ell(t), \beta(t)) \neq (0, 0)$. In this case, we define *the normalized curvature for the Legendre immersion* by

$$\left(\overline{\ell}(t), \overline{\beta}(t)\right) = \left(\frac{\ell(t)}{\sqrt{\ell(t)^2 + \beta^2(t)}}, \frac{\beta(t)}{\sqrt{\ell(t)^2 + \beta(t)^2}}\right).$$

The normalized curvature $\left(\overline{\ell}(t), \overline{\beta}(t)\right)$ is independent on the choice of a parametrization, see in §2. Moreover, since $\overline{\ell}(t)^2 + \overline{\beta}(t)^2 = 1$, there exists a smooth function $\theta(t)$ such that

$$\overline{\ell}(t) = \cos\theta(t),\ \overline{\beta}(t) = \sin\theta(t).$$

It is helpful to introduce the notion of the arc-length parameter of Legendre immersions. In general, we can not consider the arc-length parameter of the front γ, since γ may have singularities. However, (γ, ν) is an immersion, we introduce the arc-length parameter for the Legendre immersion (γ, ν). The *speed* $s(t)$ of the Legendre immersion at the parameter t is defined to be the length of the tangent vector at t, namely,

$$s(t) = |(\dot{\gamma}(t), \dot{\nu}(t))| = \sqrt{\dot{\gamma}(t) \cdot \dot{\gamma}(t) + \dot{\nu}(t) \cdot \dot{\nu}(t)}.$$

Given scalars $a, b \in I$, we define the arc-length from $t = a$ to $t = b$ to be the integral of the speed,

$$L(\gamma, \nu) = \int_a^b s(t)\, dt.$$

By the same method for the are-length parameter of a regular plane curve, one can prove the following:

Proposition 4.1 *Let $(\gamma, \nu) : I \to \mathbb{R}^2 \times S^1; t \mapsto (\gamma(t), \nu(t))$ be a Legendre immersion, and let t_0 be a fixed choice of parameter. Then (γ, ν) is parametrically equivalent to a unit speed curve*

$$(\overline{\gamma}, \overline{\nu}) : \overline{I} \to \mathbb{R}^2 \times S^1; s \mapsto (\overline{\gamma}(s), \overline{\nu}(s)) = (\gamma \circ u(s), \nu \circ u(s)),$$

under a change of parameter $u : \overline{I} \to I$ with $u(0) = t_0$ and everywhere positive derivative.

We call the above parameter s in Proposition 4.1 *the arc-length parameter for the Legendre immersion.* Let s be the are-length parameter for (γ, ν). By definition, we have $\gamma'(s) \cdot \gamma'(s) + \nu'(s) \cdot \nu'(s) = 1$, where $'$ is the derivation with respect to s. It follows that $\ell(s)^2 + \beta(s)^2 = 1$. Then there exists a smooth function $\theta(s)$ such that

$$\ell(s) = \cos\theta(s),\ \beta(s) = \sin\theta(s).$$

Also, as a corollary of Theorem 3.1, we have the following corollary:

Corollary 4.2 *Let $(\gamma, \nu) : I \to \mathbb{R}^2 \times S^1; t \mapsto (\gamma(t), \nu(t))$ and $(\widetilde{\gamma}, \widetilde{\nu}) : \widetilde{I} \to \mathbb{R}^2 \times S^1; u \mapsto (\widetilde{\gamma}(u), \widetilde{\nu}(u))$ be Legendre immersions with the arc-length parameters. Suppose that $\theta : I \to \mathbb{R}$ and $\widetilde{\theta} : \widetilde{I} \to \mathbb{R}$ are smooth functions with the conditions*

$$\ell(t) = \cos\theta(t),\ \beta(t) = \sin\theta(t),\ \widetilde{\ell}(u) = \cos\widetilde{\theta}(u),\ \widetilde{\beta}(u) = \sin\widetilde{\theta}(u).$$

If (γ, ν) and $(\widetilde{\gamma}, \widetilde{\nu})$ are at least $(k+1)$-th order contact at $t = t_0, u = u_0$, then there exists a integer $n \in \mathbb{Z}$ such that

$$\theta(t_0) = \widetilde{\theta}(u_0) + 2n\pi,\ \frac{d\theta}{dt}(t_0) = \frac{d\widetilde{\theta}}{du}(u_0),\ \cdots,\ \frac{d^{k-1}\theta}{dt^{k-1}}(t_0) = \frac{d^{k-1}\widetilde{\theta}}{du^{k-1}}(u_0). \tag{4}$$

Conversely, if the condition (4) holds, then (γ, ν) and $(\widetilde{\gamma}, \widetilde{\nu})$ are at least $(k+1)$-th order contact at $t = t_0, u = u_0$, up to congruent as Legendre immersions.

Finally, we consider a relation between the curvature of the Legendre immersion and the zigzag number (or Maslov index) (cf. [8]).

Proposition 4.3 *Let $(\gamma, \nu) : [a, b] \to \mathbb{R}^2 \times S^1$ be a closed Legendre immersion with the curvature of the Legendre immersion (ℓ, β). Suppose that (γ, ν) is parametrized by the arc-length parameter t and θ is a smooth function which satisfy $\ell(t) = \cos\theta(t)$ and $\beta(t) = \sin\theta(t)$. Then $(1/2\pi)|\theta(b) - \theta(a)|$ is equal to the zigzag number of the front γ.*

Proof. Let $z(\gamma)$ be the zigzag number of γ, By the definition of the zigzag number (see [8] for example), $z(\gamma) = |\deg([-\dot{\nu}(t), \dot{\gamma}(t)])| = |\deg([-\ell(t), \beta(t)])| = |\deg([-\cos\theta(t), \sin\theta(t)])|$, where $[-\dot{\nu}(t), \dot{\gamma}(t)]$ is a ratio of two vectors, $[-\ell(t), \beta(t)]$ and $[-\cos\theta(t), \sin\theta(t)]$ are ratios of two real numbers, in other words, elements of the real projective line. We consider the real projective line as S^1, then $\deg([-\cos\theta(t), \sin\theta(t)])$ means a rotation number of the map $t \mapsto (-\cos\theta(t), \sin\theta(t)) \in S^1$. Thus, $|\theta(b) - \theta(a)| = 2\pi|\deg([-\cos\theta(t), \sin\theta(t)])|$. Therefore we obtain $(1/2\pi)|\theta(b) - \theta(a)| = z(\gamma)$. $\square$

Remark 4.4 Let $(\gamma, \nu) : [a, b] \to \mathbb{R}^2 \times S^1$ be a closed Legendre immersion with the curvature (ℓ, β). Suppose that (γ, ν) is parametrized by the arc-length parameter t and θ is a smooth function which satisfy $\ell(t) = \cos\theta(t)$ and $\beta(t) = \sin\theta(t)$. Then the curvature of the Legendre immersion $(\gamma, -\nu)$ is equal to $(\ell, -\beta)$ (Remark 2.4). We denote $(\ell(t), -\beta(t)) = (\cos\widetilde{\theta}(t), \sin\widetilde{\theta}(t))$ for a smooth function $\widetilde{\theta}$. Then we obtain simultaneous equations $\cos\theta(t) = \cos\widetilde{\theta}(t)$ and $\sin\theta(t) = -\sin\widetilde{\theta}(t)$. It follows that there exists an integer n such that $\theta(t) = -\widetilde{\theta}(t) + 2n\pi$. Thus $\theta(b) - \theta(a) = -\widetilde{\theta}(b) + \widetilde{\theta}(a)$.

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