Singularities of tangent surfaces in Cartan's split G_2-geometry

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Abstract

In the split G_2-geometry, we study the correspondence found by E. Cartan between the Cartan distribution and the contact distribution with Monge structure on spaces of five variables. Then the generic classification is given on singularities of tangent surfaces to Cartan curves and to Monge curves via the viewpoint of duality. The geometric singularity theory for simple Lie algebras of rank 2, namely, for $A_2, C_2 = B_2$ and G_2 is established.

1 Introduction

In this paper we present a duality of certain singularities appearing in the correspondence for split G_2-geometry found by E. Cartan and formulated by R. L. Bryant [6]. The complex simple Lie algebras are classified by Dynkin diagrams through root systems and in the case of rank 2, there are exactly three cases, namely $A_2, C_2 = B_2$, and G_2.

Figure 1: Dynkin diagrams of types A_2, C_2 and G_2

We associate an explicit pair of fibrations with each type A_2, C_2 or G_2:

$$Y \xleftarrow{\Pi_Y} Z \xrightarrow{\Pi_X} X.$$

The fibration induces canonical geometric structures on the three spaces Z, Y, X in each case. In particular the completely non-integrable plane field $E = \mathrm{Ker}(\Pi_{Y*}) \oplus \mathrm{Ker}(\Pi_{X*})$ on Z is associated. Then parametrized integral curves $f : I \to Z$ of the plane field E project to curves $\Pi_Y \circ f$ and $\Pi_X \circ f$ in Y and X respectively. Moreover each curve $\Pi_Y \circ f$ (resp. $\Pi_X \circ f$) is embedded in a surface ruled by the "tangent lines" $\Pi_Y \Pi_X^{-1} \Pi_X(f(t))$ (resp. $\Pi_X \Pi_Y^{-1} \Pi_Y(f(t))$), $t \in I$, which we call the *tangent surface*. Note that both two curves $\Pi_X^{-1} \Pi_X(f(t_0))$ and f have tangent lines in the plane $E_{f(t_0)}$ at $t = t_0$, therefore $\Pi_Y \Pi_X^{-1} \Pi_X(f(t_0))$ is tangent to $\Pi_Y \circ f$ at $t = t_0$ at least if $\Pi_Y \circ f$ is immersive at t_0. The tangent surfaces are naturally appear in the G_2-geometry and they are regarded as solutions for certain involutive systems of partial differential equations (see [7]). It is classically known that the tangent surfaces necessarily have singularities. However the singularities appearing in such surfaces had never been studied in detail.

In this paper, for the G_2 case, we describe the duality explicitly and provide generic classification results on tangent surfaces, or more exactly, the tangent mappings which parametrize tangent surfaces, under local diffeomorphisms using singularity theory of mapping. Then, as a result, we

have three classes of singularities of tangent surfaces on Y and X respectively. Moreover we observe the manner of appearing generic singularities turns out to be reflected by the underlying geometric structures (Theorem 1.3). Note that, to perform the natural classification by local transformations which preserve associated G_2-Cartan structures, first we must establish the more basic classification by local diffeomorphisms. We do establish it in this paper.

To do exact analysis of singularities, we provide, in this paper, certain local coordinates on Z and local projective coordinates on Y, X which are compatible with the double fibration, so that any fiber of one projection and its another projection become lines in terms of the coordinates. Then, for a curve in a projective space or a space with a flat projective structure [16], we define a strictly increasing sequence of positive integers, called the *type*, using the leading terms in an appropriate system of projective coordinates at each point of the curve. The type is a local projective invariant of the curve and plays an important role as a characteristic describing the singularities we are going to treat in this paper. In fact we classify singularities of tangent surfaces to $\Pi_Y \circ f$ and to $\Pi_X \circ f$, or their parametrizations, for a generic integral curve f.

For A_2, as a real and non-oriented version, we take the flag manifold

$$Z = Z(A_2) := \{(V_1, V_2) \mid V_1 \subset V_2 \subset \mathbf{R}^3, \dim V_1 = 1, \dim V_2 = 2\},$$

which is of dimension 3. The canonical projections $\Pi_Y : Z \to Y = Y(A_2) = P(\mathbf{R}^3)$ and $\Pi_X : Z \to X = X(A_2) = P(\mathbf{R}^{3*})$ form the double fibration

$$P(\mathbf{R}^3) \xleftarrow{\Pi_Y} Z \xrightarrow{\Pi_X} P(\mathbf{R}^{3*}).$$

We set $G = G(A_2) = \mathrm{PGL}(\mathbf{R}^3)$. Then G acts naturally on Z, Y, X transitively and Π_Y, Π_X are G-equivariant. The Π_Y-fibers project by Π_X to projective lines on $P(\mathbf{R}^{3*})$ and the Π_X-fibers project by Π_Y to projective lines on $P(\mathbf{R}^3)$. The canonical contact structure $D(A_2) \subset TZ$ is defined by $E = \mathrm{Ker}(\Pi_{Y*}) \oplus \mathrm{Ker}(\Pi_{X*})$. Then the classical projective duality is well-described in terms of Legendre curves in the contact manifold (Z, E).

We recall the assertion on related singularities to the double fibration:

Theorem 1.1 *For a generic Legendre curve $f : I \to (Z, E)$ in C^∞ topology from an open interval I and for any $t_0 \in I$ the pair of types of curves $\Pi_Y \circ f$ and $\Pi_X \circ f$ at t_0 is given by one of the following:*

$$\mathrm{I} : ((1,2),(1,2)), \quad \mathrm{II} : ((1,3),(2,3)), \quad \mathrm{III} : ((2,3),(1,3)).$$

Moreover, in each case, the pair of diffeomorphism classes of tangent mappings to $\Pi_Y \circ f, \Pi_X \circ f$ is given by

$\mathrm{I} : (\mathrm{fold}, \mathrm{fold}), \quad \mathrm{II} : (\mathrm{beak\text{-}to\text{-}beak}, \mathrm{Whitney's\ cusp}), \quad \mathrm{III} : (\mathrm{Whitney's\ cusp}, \mathrm{beak\text{-}to\text{-}beak}).$

A curve in $P(\mathbf{R}^3)$ or in $P(\mathbf{R}^{3*})$ is called an ordinary point (resp. an inflection point, an cusp point) if it is of type $(1,2)$ (resp. $(1,3), (2,3)$). The tangent lines to a curve form a map-germ $(\mathbf{R}^2, 0) \to (\mathbf{R}^2, 0)$, called the tangent mapping, in terms of projective local coordinates. A map-germ is called a fold (resp. beak-to-beak, Whitney's cusp) if it is diffeomorphic (right-left equivalent) to

$$(x, t) \mapsto (x, t^2 - 2xt), \quad (\text{resp. } (x, 2t^3 - 3xt^2), \ (x, t^3 - 6xt)).$$

See Figure 2.

For C_2, we take the flag manifold, on the symplectic vector space $V = \mathbf{R}^4$,

$$Z = Z(C_2) := \{(V_1, V_2) \mid V_1 \subset V_2 \subset V, \dim V_1 = 1, \dim V_2 = 2, \ V_2 \text{ is a Lagrangian plane}\},$$
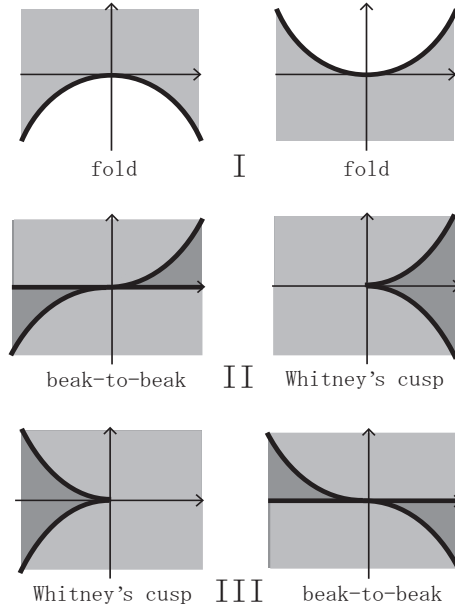
Figure 2: Singularities of tangent mappings associated to the A_2-double fibration

which is of dimension 4. The canonical projections $\Pi_Y : Z \to Y = Y(C_2) = P(\mathbf{R}^4)$ and $\Pi_X : Z \to X = X(C_2) = \mathrm{LG}(\mathbf{R}^4)$ form the double fibration

$$P(\mathbf{R}^4) \xleftarrow{\ \Pi_Y\ } Z \xrightarrow{\ \Pi_X\ } \mathrm{LG}(\mathbf{R}^4).$$

Here $\mathrm{LG}(\mathbf{R}^4)$ is the Lagrange-Grassmann manifold. We set $G = G(C_2) = \mathrm{Sp}(\mathbf{R}^4)$, the symplectic group. Note that G is isomorphic the spinor group $\mathrm{Spin}(\mathbf{R}^{2,3})$. Then G acts naturally on Z, Y, X transitively and Π_Y, Π_X are G-equivariant. Moreover $Y = P(\mathbf{R}^4)$ has the G-invariant canonical contact structure, while $X = \mathrm{LG}(\mathbf{R}^4)$ has the G-invariant quadratic cone structure. We call Π_X-projections of Π_Y-fibers null lines in $\mathrm{LG}(\mathbf{R}^4)$ and Π_Y-projections of Π_X-fibers Legendre lines in $P(\mathbf{R}^4)$. Thus $X = \mathrm{LG}(\mathbf{R}^4)$ is identified with the space of Legendre lines and $Y = P(\mathbf{R}^4)$ is identified with the space of null lines. The canonical Engel structure $E(C_2) \subset TZ$ is defined by $E(C_2) = \mathrm{Ker}(\Pi_{Y*}) \oplus \mathrm{Ker}(\Pi_{X*})$ as well. A curve $f : I \to Z(C_2)$ from an open interval I is called an Engel integral curve if $f_*(TI) \subset E$. Then the curve $\Pi_Y \circ f$ is associated with Legendre tangent lines, while $\Pi_X \circ f$ is associated with null tangent lines. Thus we have the tangent surfaces, which are non-smooth in general, ruled by Legendre tangent lines in Y and by null tangent lines in X respectively. In fact the tangent surfaces are obtained just from the double fibration by considering the set $\Pi_Y \Pi_X^{-1} \Pi_X f(I)$ and $\Pi_X \Pi_Y^{-1} \Pi_Y f(I)$.

An analogous result to the classical projective duality is given in terms of Engel integral curves:

Theorem 1.2 ([13]) *For a generic Engel integral curve $f : I \to Z(C_2)$ in the Lagrange flag manifold $Z(C_2)$, in C^∞ topology, the pair of types of $\Pi_Y \circ f$ and $\Pi_X \circ f$ at any point $t_0 \in I$ is given by one of the following three cases:*

$$\mathrm{I} : ((1,2,3),(1,2,3)), \quad \mathrm{II} : ((1,3,4),(2,3,4)), \quad \mathrm{III} : ((2,3,5),(1,3,5)).$$

The pair of diffeomorphism classes of tangent surfaces to $\Pi_Y \circ f$ and to $\Pi_X \circ f$ is given by one of the following three cases:

I : (cuspidal edge, cuspidal edge),
II : (Mond surface, swallowtail),
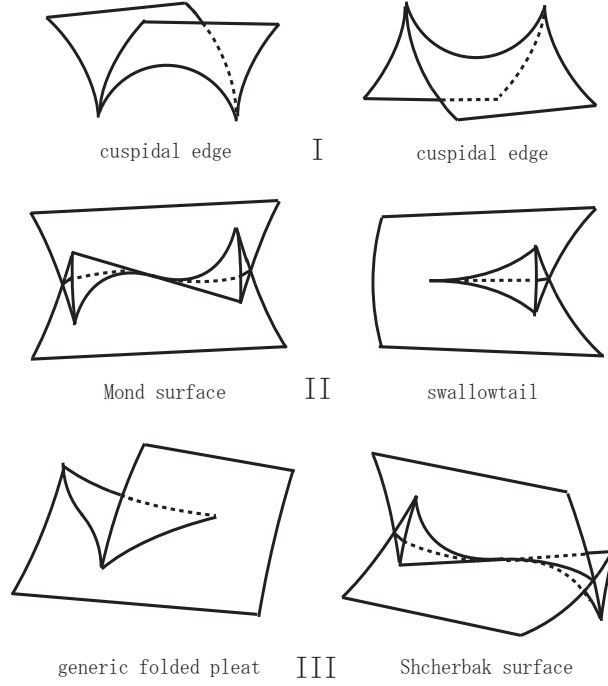III : (generic folded pleat, Shcherbak surface).

3

Figure 3: Singularities of tangent surfaces associated to the C_2-double fibration

A parametrized surface in a 3-dimensional space is called a cuspidal edge (resp. Mond surface, swallowtail, generic folded pleat, Shcherbak surface) if it is locally diffeomorphic to the germ of parametrized surface $(\mathbf{R}^2, 0) \to (\mathbf{R}^3, 0)$ explicitly given by

$$\begin{aligned}
\text{cuspidal edge}: \quad & (x, t) \mapsto (x,\ t^2 - 2xt,\ 2t^3 - 3xt^2), \\
\text{Mond surface}: \quad & (x, t) \mapsto (x,\ 2t^3 - 3xt^2,\ 3t^4 - 4xt^3), \\
\text{swallowtail}: \quad & (x, t) \mapsto (x,\ t^3 - 6xt,\ t^4 - 4xt^2), \\
\text{generic folded pleat}: \quad & (x, t) \mapsto (x,\ 4t^3 + 3t^4 - 12x(2t + t^2),\ 6t^5 + 5t^6 - 5x(4t^3 + 3t^4)), \\
\text{Shcherbak surface}: \quad & (x, t) \mapsto (x,\ 2t^3 - 3xt^2,\ 4t^5 - 5xt^4).
\end{aligned}$$

For the computer aided graphics of those singularities, see Figure 3 and [13].

The main purpose of this paper is to give a precise real model of the double fibration

$$Y(G_2) \xleftarrow{\ \Pi_Y\ } Z(G_2) \xrightarrow{\ \Pi_X\ } X(G_2),$$

for G_2 type, following Bryant's construction [6]. Here $Z(G_2)$ is a kind of flag manifold over the split octonion and it is of dimension 6, while $\dim Y(G_2) = \dim X(G_2) = 5$.

The Engel distribution $E \subset TZ(G_2)$ over $Z(G_2)$ is defined by $E := \mathrm{Ker}(\Pi_{Y*}) \oplus \mathrm{Ker}(\Pi_{X*})$.

In this paper we show the following classification result of singularities:

4

Theorem 1.3 *For a generic Engel integral curve $f : I \to Z$ in the split G_2 flag manifold Z, in C^∞ topology, the pair of types of $\Pi_Y \circ f$ and $\Pi_X \circ f$ at any point $t_0 \in I$ is given by one of the following three cases:*

I : $((1,2,3,4,5),(1,2,3,4,5))$, II : $((1,3,4,5,7),(2,3,4,5,7))$, III : $((2,3,5,7,8),(1,3,5,7,8))$.

The pair of diffeomorphism classes of tangent surfaces to $\Pi_Y \circ f$ and to $\Pi_X \circ f$ is given by one of the following three cases:

I : (cuspidal edge, cuspidal edge),
II : (open Mond surface, open swallowtail),
III : (open generic folded pleat, open Shcherbak surface).

A parametrized surface in a 5-dimensional space is called a cuspidal edge (resp. open Mond surface, open swallowtail, open generic folded pleat, open Shcherbak surface) if it is locally diffeomorphic to the germ of parametrized surface $(\mathbf{R}^2, 0) \to (\mathbf{R}^5, 0)$ explicitly given by

$$\text{cuspidal edge}: \quad (x,t) \mapsto (x,\ t^2 - 2xt,\ 2t^3 - 3xt^2,\ 3t^4 - 4xt^3,\ 4t^5 - 5xt^4),$$
$$\sim (x,t) \mapsto (x,\ t^2 - 2xt,\ 2t^3 - 3xt^2,\ 0,\ 0),$$
$$\text{open Mond surface}: \quad (x,t) \mapsto (x,\ 2t^3 - 3xt^2,\ 3t^4 - 4xt^3,\ 4t^5 - 5xt^4,\ 6t^7 - 7xt^6),$$
$$\sim (x,t) \mapsto (x,\ 2t^3 - 3xt^2,\ 3t^4 - 4xt^3,\ 4t^5 - 5xt^4,\ 0)$$
$$\text{open swallowtail}: \quad (x,t) \mapsto (x,\ t^3 - 6xt,\ t^4 - 4xt^2,\ 3t^5 - 10xt^3,\ 5t^7 - 14xt^5),$$
$$\sim (x,t) \mapsto (x,\ t^3 - 6xt,\ t^4 - 4xt^2,\ 3t^5 - 10xt^3,\ 0),$$
$$\text{open generic folded pleat}: \quad (x,t) \mapsto (x,\ 4t^3 + 3t^4 - 12x(2t + t^2),\ 6t^5 + 5t^6 - 5x(4t^3 + 3t^4),$$
$$120t^7 + 105t^8 - 56x(6t^5 + 5t^6),$$
$$408t^8 + 476t^9 + 102t^{10} - 3x(384t^6 + 408t^7 + 85t^8)),$$
$$\sim (x,t) \mapsto (x,\ 4t^3 + 3t^4 - 12x(2t + t^2),\ 6t^5 + 5t^6 - 5x(4t^3 + 3t^4),$$
$$5t^7 - 14xt^5,\ 3t^8 - 8xt^6)$$
$$\text{open Shcherbak surface}: \quad (x,t) \mapsto (x,\ 2t^3 - 3xt^2,\ 4t^5 - 5xt^4,\ 6t^7 - 7xt^6,\ 7t^8 - 8xt^7),$$
$$\sim (x,t) \mapsto (x,\ 2t^3 - 3xt^2,\ 4t^5 - 5xt^4,\ 6t^7 - 7xt^6,\ 0).$$

Here we use $\sim$ for the diffeomorphism equivalence (right-left equivalence) of map-germs to provide different representatives. See Figure 4 for the illustrations of these singularities.

In §2, we recall the split octonions and G_2, and in §3, we introduce the flag manifold and the double fibration for the split G_2 following [6]. Note that the same construction has been analyzed and utilized in the problem of "rolling balls" ([1][4][3]). From Theorems 1.1, 1.2 and 1.3, we observe that the tangent varieties for G_2 case project to that for C_2 and to that for A_2. We provide an explanation of this observation on the above classification results (Remark 3.7). In §4, we give the explicit description of the double fibration and differential systems associated to it, in order to find exact normal forms of singularities in the following sections. In fact in §5, we provide explicit descriptions of tangent surfaces to $\Pi_Y \circ f$ and to $\Pi_X \circ f$ for any germ of Engel integral curve. Then we show a necessary codimension formula to get the genericity result and the "Cartan-Monge duality" of Engel curves in §6, which is analogous to the ordinal projective duality of plane curve singularities for A_2, and to the "contact-cone, Legendre-null" duality for C_2 observed in [13]. We complete the classification of singularities of tangent mappings in §7 to prove the main Theorem 1.3. In §8, as an appendix, we give a Lie theoretical explanation on the hierarchy of double fibrations associated to simple Lie algebras of rank 2, in terms of root systems of G_2, C_2 and A_2.
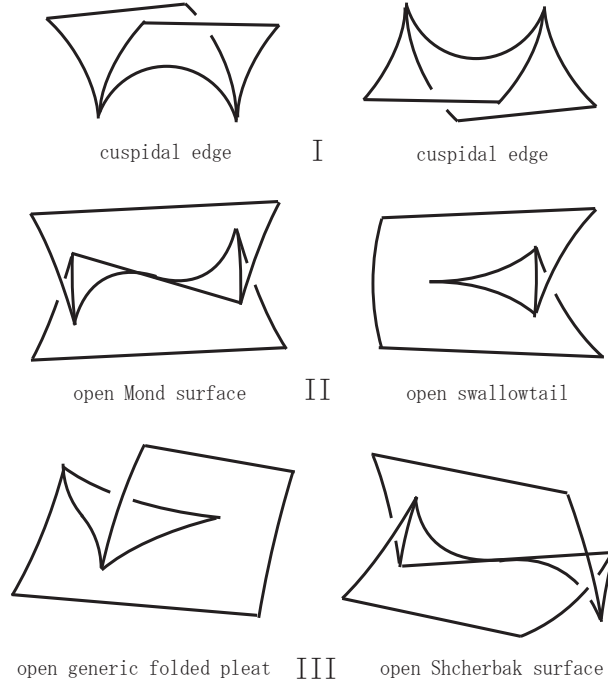
5

cuspidal edge   I   cuspidal edge

open Mond surface   II   open swallowtail

open generic folded pleat   III   open Shcherbak surface

Figure 4: Singularities of tangent surfaces associated to the G_2-double fibration

2 The split octonions and the split G_2

First recall the split octonion algebra $\mathbf{O}' = \mathbf{H}(-)$, following [9]. Let $\mathbf{H} = \{a = x + yi + zj + wk \mid x, y, z, w \in \mathbf{R}\}$ be the Hamilton's quaternion algebra endowed with the operation of conjugation $\bar{a} = x - yi - zj - wk$ and the positive definite inner product $(a|b) = \mathrm{Re}(a\bar{b}) = \frac{1}{2}(a\bar{b} + b\bar{a})$. Setting $\mathbf{O}' = \mathbf{H} \oplus \mathbf{H}$ as a vector space, define the multiplication on $\mathbf{O}'$ by

$$(a, b)(c, d) = (ac + \bar{d}b, da + b\bar{c}).$$

We set $\varepsilon = (0, 1)$ and write $(a, b) = a + b\varepsilon$. Then we have $\varepsilon^2 = 1$ and

$$a(d\varepsilon) = (da)\varepsilon, \ (b\varepsilon)c = (b\bar{c})\varepsilon, \ (b\varepsilon)(d\varepsilon) = \bar{d}b.$$

Remark that the octonion is a non-associative algebra. Moreover we define the conjugation on $\mathbf{O}'$ by $\overline{a + b\varepsilon} = \bar{a} - b\varepsilon$. Then the inner product on $\mathbf{O}'$ is defined by

$$(a + b\varepsilon \,|\, c + d\varepsilon) = \mathrm{Re}((a + b\varepsilon)\overline{(c + d\varepsilon)}) = (a|c) - (b|d),$$

which is of index $(4, 4)$. An element of $\mathbf{O}'$ is uniquely expressed as

$$a + b\varepsilon = a_1 + a_2i + a_3j + a_4k + b_1\varepsilon + b_2i\varepsilon + b_3j\varepsilon + b_4k\varepsilon.$$

We set

$$e_0 = 1, \ e_1 = \tfrac{1}{2}(i + i\varepsilon), \ e_2 = \tfrac{1}{2}(j - j\varepsilon), \ e_3 = \tfrac{1}{2}(k - k\varepsilon),$$
$$e_4 = \varepsilon, \ e_5 = \tfrac{1}{2}(k + k\varepsilon), \ e_6 = \tfrac{1}{2}(j + j\varepsilon), \ e_7 = \tfrac{1}{2}(i - i\varepsilon).$$

Then we have the multiplication table (Table 1) with $e_0e_i = e_ie_0 = e_i$ ($0 \le i \le 7$).

6

	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	0	0	0	e_1	$-e_2$	e_3	$-\frac{1}{2}(e_0 - e_4)$
e_2	0	0	e_1	$-e_2$	0	$-\frac{1}{2}(e_0 + e_4)$	$-e_5$
e_3	0	$-e_1$	0	$-e_3$	$-\frac{1}{2}(e_0 + e_4)$	0	e_6
e_4	$-e_1$	e_2	e_3	e_0	$-e_5$	$-e_6$	e_7
e_5	e_2	0	$-\frac{1}{2}(e_0 - e_4)$	e_5	0	$-e_7$	0
e_6	$-e_3$	$-\frac{1}{2}(e_0 - e_4)$	0	e_6	e_7	0	0
e_7	$-\frac{1}{2}(e_0 + e_4)$	e_5	$-e_6$	$-e_7$	0	0	0

Table 1: Multiplication table of the split octonions

We call any basis of $\mathbf{O}'$ possessing the same multiplication table with $e_0, e_1, e_2, e_3, e_4, e_5, e_6, e_7$ an *admissible basis* of $\mathbf{O}'$.

For the inner product, we have that $(e_i|e_{8-i}) = \frac{1}{2}$ $(1 \le i \le 7, i \ne 4)$, $(e_4|e_4) = -1$, $(e_0|e_0) = 1$ and other pairings are all zero.

If we set $c_1 = \frac{1}{2}(e_0 - e_4), c_2 = \frac{1}{2}(e_0 + e_4), c_3 = e_1, c_4 = e_6, c_5 = e_5, c_6 = e_7, c_7 = e_2, c_8 = e_3$, then the basis $c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8$ of $\mathbf{O}'$ enjoy the multiplication table shown in [10] p.105.

The split G_2 group is defined as the automorphism group of the split octonion algebra $\mathbf{O}'$ and is denoted by G_2':

$$G_2' := \{g \in \mathrm{GL}(\mathbf{O}') \mid g \text{ preserves the multiplication of } \mathbf{O}'\}.$$

Let $V = \mathrm{Im}(\mathbf{O}')$ be the imaginary part of $\mathbf{O}'$. Then G_2' preserves V. For $v \in V$, we have $\overline{v} = -v$. Moreover we have $v^2 = -(v|v)$. In fact, we see $v^2 \in \mathbf{R}$, since $\overline{v^2} = \overline{v}^2 = v^2$. Therefore $(v|v) = \mathrm{Re}(v\overline{v}) = -v^2$. Thus we see G_2' preserves the conjugation and the inner product.

The *associative 3-form* $\phi \in \wedge^3 V^*$ is defined by $\phi(u, v, w) = (uv|w)$. Then G_2' preserves the associative 3-form ϕ. The converse is true by the following result:

Theorem 2.1 ([5][9]) *The group G_2' is represented as*

$$G_2' = \{g \in \mathrm{GL}(V) \mid g^*\phi = \phi\}.$$

Corollary 2.2 *The group G_2' acts transitively on the set of admissible bases of the algebra $\mathbf{O}'$.*

Proof: Let $e_0, e_1, e_2, e_3, e_4, e_5, e_6, e_7$ be any admissible basis of $\mathbf{O}'$. Then $\boldsymbol{e} = (e_1, e_2, e_3, e_4, e_5, e_6, e_7)$ is a basis of V and we have

$$\phi = e_1^* \wedge e_4^* \wedge e_7^* - e_2^* \wedge e_4^* \wedge e_6^* - e_3^* \wedge e_4^* \wedge e_5^* - e_1^* \wedge e_5^* \wedge e_6^* + e_2^* \wedge e_3^* \wedge e_7^*,$$

in terms of the dual basis $\boldsymbol{e}^* = (e_1^*, e_2^*, e_3^*, e_4^*, e_5^*, e_6^*, e_7^*)$ of the dual space V^* to the basis $\boldsymbol{e}$ of V. Let $(f_j)_{0 \le j \le 7}$ be another admissible basis of $\mathbf{O}'$. Then define $g \in \mathrm{GL}(V)$ by $g(f_j) = e_j$ $(1 \le j \le 7)$. Then $g^*\phi = \phi$. Therefore, by Theorem 2.1, $g \in G_2'$. □

3   Flags and Double fibration for the split G_2

Let $\mathbf{O}'$ be the split octonions and G_2' the split G_2 group (§2). Consider the 7-dimensional vector space $V = \mathrm{Im}(\mathbf{O}')$, purely imaginary split octonions.

An element $v \in V$ is called *null* if $v^2 = 0$. This is equivalent to that $(v|v) = 0$. A subspace $W \subset V$ is called a *null subalgebra* if $vw = 0$ for any $v, w \in W$. We consider the flag manifold

$$Z = Z(G_2) := \{(V_1, V_2) \mid V_1 \subset V_2, \ V_1 \text{ and } V_2 \text{ are oriented null subalgebras of } V,$$
$$\dim V_1 = 1, \dim V_2 = 2\}.$$

Moreover we set

$$Y = Y(G_2) := \{V_1 \mid V_1 \text{ is an oriented 1-dimensional null subalgebra in } V\},$$
$$X = X(G_2) := \{V_2 \mid V_2 \text{ is an oriented 2-dimensional null subalgebra in } V\}.$$

We call Z the *null flag manifold*, Y the *null projective space* and X the *null Grassmannian* in this paper. We denote by $\Pi_Y : Z \to Y$ and $\Pi_X : Z \to X$ the canonical projections.

Then we have

Proposition 3.1 *The null flag manifold Z, the null projective space Y and the null Grassmannian X are $6, 5$ and 5 dimensional manifolds respectively. The group G_2' acts transitively on Z, Y, X respectively such that Π_Y, Π_X are G_2'-equivariant. In fact $Z \cong G_2'/B$ as G_2'-manifolds for a Borel subgroup B of G_2', and $Y \cong G_2'/P_1, X \cong G_2'/P_2$ for parabolic subgroups P_1, P_2 of G_2' containing B. Moreover Z (resp. X, Y) is diffeomorphic to $S^3 \times S^3$ (resp. to $S^2 \times S^3$)*

Proposition 3.1 is stated in [6]. For the proof, we refer [2] over algebraically closed fields (see also [8], §23.3). Since we work over $\mathbf{R}$, we give a proof that G_2' acts transitively on Y, X and Z to make sure: First we remark that Y is a connected 5-dimensional manifold. In fact, the inner product on V is of index $(3, 4)$ and we see that Y is diffeomorphic to $S^2 \times S^3$. Since G_2'/P_1 is embedded in Y as a closed set and $\dim(G_2'/P_1) = 5$, we have that $Y \cong G_2'/P_1$ as G_2'-manifolds. Moreover we see that $\Pi_Y : Z \to Y$ is a S^1-fibration. Therefore Z is a connected 6-dimensional manifold. Since G_2'/B is embedded in Z as a closed set and $\dim(G_2'/B) = 6$, we have that $Z \cong G_2'/B$ as G_2'-manifolds. Since $\Pi_X : Z \to X$ is a S^1-bundle, we see X is a connected 5-dimensional manifold. Since G_2'/P_2 is embedded in X as a closed set and $\dim(G_2'/P_2) = 5$, we have that $X \cong G_2'/P_2$ as G_2'-manifolds. Therefore G_2' acts on transitively on the double fibration $Y \leftarrow Z \to X$. In [3], the transitivity of G_2'-action on Z is proved using the notion "null triples". See also [4][1] and §8.

Now we give the concrete description of the double fibration which is needed to obtain the exact and explicit classification results on singularities in the following sections.

Let $\Pi_Y : Z \to Y$ and $\Pi_X : Z \to X$ be natural projections. Note that both Π_Y and Π_X are fibrations with S^1-fibers. Consider the double fibrations

$$Z(G_2)$$
$$\Pi_Y \swarrow \qquad\qquad \searrow \Pi_X$$
$$Y(G_2) \qquad\qquad\qquad X(G_2).$$

For each $U \in X$, we consider a one dimensional submanifold

$$[U] := \widetilde{P}(U) = \{\ell \in Y \mid (\ell, U) \in Z\} = \Pi_Y(\Pi_X^{-1}(U))$$

of Y, $\widetilde{P}(U)$ is the double cover of the projective space $P(U)$. We call $[U] \subset Y$ a *Cartan line* or a *C-line* associated to the null plane U. Later we see that $[U]$ is a projective line for a projective structure on Y. The set of Cartan lines is identified with the null Grassmannian X.

For each $\ell \in Y$, we consider a one dimensional submanifold

$$[\ell] := \{U \in X \mid (\ell, U) \in Z\} = \Pi_X(\Pi_Y^{-1}(\ell))$$

of X. We call $[\ell] \subset X$ a *Monge line* or an *M-line* associated to the null line ℓ. Later we see that $[\ell]$ is a projective line for a projective structure on X. The set of Monge lines is identified with the null projective space Y.

On Z we define the distribution $E \subset TZ$ of rank 2 by

$$E := \operatorname{Ker}(\Pi_{Y*}) \oplus \operatorname{Ker}(\Pi_{X*}).$$

We call E the *Engel distribution*, or G_2-*Engel distribution* to distinguish with the C_2-case.

For each $(\ell, U) \in Z$, we set $[[U]] = T_\ell[U] \subset T_\ell Y$ and $[[\ell]] = T_U[\ell] \subset T_U X$. Then we have

$$E_{(\ell, U)} = \{v \in T_{(\ell, U)}Z \mid \Pi_{Y*}v \in [[U]]\} = T_{(\ell, U)}(\Pi_Y^{-1}(\Pi_Y(\Pi_X^{-1}(U))))$$
$$= \{v \in T_{(\ell, U)}Z \mid \Pi_{X*}v \in [[\ell]]\} = T_{(\ell, U)}(\Pi_X^{-1}(\Pi_X(\Pi_Y^{-1}(\ell)))).$$

The *big* (or *strong*) (resp. the *small* (or *weak*)) derived systems E^i (resp. $E^{(i)}$) of E are defined by $E^1 = E^{(1)} = E$ and

$$\mathcal{E}^i := \mathcal{E}^{i-1} + [\mathcal{E}^{i-1}, \mathcal{E}^{i-1}], \quad (\text{resp.} \quad \mathcal{E}^{(i)} := \mathcal{E}^{(i-1)} + [\mathcal{E}, \mathcal{E}^{(i-1)}]),$$

in terms of sheaves. Note that $\mathcal{E}^{(i)} \subseteq \mathcal{E}^i$. Then, in §4, we will see that $E^{(i)}$ and E^i are subbundles of TZ, $\operatorname{rank}(E^{(i)}) = i + 1, i = 2, 3, 4, 5$, while $\operatorname{rank}(E^2) = 3, \operatorname{rank}(E^3) = 4$ but $\operatorname{rank}(E^4) = 6$. Therefore, for the "vector" counting ranks of small or big derived systems, we obtain:

Lemma 3.2 $E \subset TZ$ *is a* G_2'-*invariant distribution with the small growth vector* $(2, 3, 4, 5, 6)$ *and the big growth vector* $(2, 3, 4, 6)$.

Remark 3.3 The system $E \subset TZ$ is locally isomorphic to the system associated to the Hilbert-Cartan equation (see [19]).

For each $\ell \in Y$, we set

$$H_\ell := \{w \in V \mid vw = 0, \text{ for any } v \in \ell\},$$

Then we see that H_ℓ is a 3-dimensional subspace of V. Moreover any line in H_ℓ is a null line (a 1-dimensional null subalgebra). (See §4.) Therefore the projective $P(H_\ell)$ of $P(V)$ is contained in Y. We define a distribution $D \subset TY$ on Y of rank 2 by

$$D_\ell := T_\ell P(H_\ell) \subset T_\ell Y, \ \ell \in Y.$$

Then we have:

Lemma 3.4 $D \subset TY$ *is a* G_2'-*invariant Cartan distribution with the big and small growth vector* $(2, 3, 5)$.

The derived system D^2 of D is obtained by

$$(D^2)_\ell = T_\ell P(H_\ell^\perp) \subset T_\ell Y, \ \ell \in Y,$$

which is of rank 3. Here $H_\ell^\perp$ is the orthogonal space of H_ℓ:

$$H_\ell^\perp = \{w \in V \mid (v|w) = 0, \text{ for any } v \in H_\ell\}.$$

Then we have, for any $v \in TZ$, $v \in E^2$ if and only if $\Pi_{Y*}(v) \in D$, and $v \in E^3$ if and only if $\Pi_{Y*}(v) \in D^2$.

Remark 3.5 A Cartan line is an abnormal (or a singular) curve of the distribution D on Y. Moreover X is identified with the set of abnormal (or singular) curves of D (see [4][15]).

We define a field of two dimensional cones $C \subset TX$ on X as follows: Let $U \in X$. Consider the Schubert variety

$$S_U := \{U' \in X \mid U' \cap U \neq \{0\}\} = \Pi_X(\Pi_Y^{-1}(\Pi_Y(\Pi_X^{-1}(U)))).$$

Then the cone $C_U \subset T_U X$ is defined as the tangent cone of S_U at U. Moreover we define the contact distribution $D' \subset TX$ as the linear hull of the cone field C. Note that Z is identified with the oriented projective bundle $\widetilde{P}(D)$ and E is identified with the prolongation of $D \subset TY$. On the other hand Z is identified with the set $\widetilde{P}(C)$ of generating oriented lines of the cone field C and E is the "prolongation" or, the (double cover of) "resolution" of C.

In the next section we give the explicit descriptions of E, D, C and D'.

Remark 3.6 As is stated in §1, we observe that we have the Cartan structure on $Y(G_2)$, which is a G_2'-invariant distribution with big and small growth vector $(2,3,5)$, while on $Y(C_2)$ we have a projective contact structure with big and small growth vector $(2,3)$ and on $Y(A_2)$ we have just a projective structure. On $X(G_2)$ we have a G_2'-homogeneous contact structure with big and small growth vector $(4,5)$ and a cubic Lagrange cone field in it, while on $X(C_2)$ we have a Lagrange-Grassmann structure or an indefinite conformal structure, which is given by a quadratic cone field. On $Z(G_2)$ we have the G_2-Engel distribution with small growth vector $(2,3,4,5,6)$ and with big growth vector $(2,3,4,6)$, while on $Z(C_2)$ we have the Engel structure with big and small growth vector $(2,3,4)$ and on $Z(A_2)$ we have a projective contact structure with big and small growth vector $(2,3)$.

Remark 3.7 If we restrict the double fibration $Y(G_2) \leftarrow Z(G_2) \rightarrow X(G_2)$ to local coordinate neighborhoods $O_Y \leftarrow O \rightarrow O_X$ constructed in the next section §4, then there exist submersions $O \rightarrow Z(C_2), O_Y \rightarrow Y(C_2)$ and $O_X \rightarrow X(C_2)$ which are compatible with the double fibration. Similarly if we restrict the double fibration $Y(C_2) \leftarrow Z(C_2) \rightarrow X(C_2)$ to some local coordinate neighborhoods $O_Y' \leftarrow O' \rightarrow O_X'$ constructed in [13], then it is submersed to $Y(A_2) \leftarrow Z(A_2) \rightarrow X(A_2)$.

To the double fibration $(Y, D) \xleftarrow{\pi_Y} (Z, E) \xrightarrow{\pi_X} (X, C)$ in the case G_2, we naturally associate some classes of curves:

Definition 3.8 Let I be an open interval.
A curve $f : I \longrightarrow (Z, E)$ is called an *Engel curve* or an *E-curve* if $f_*(TI) \subset E(\subset TZ)$.
A curve $g : I \longrightarrow (Y, D)$ is called a *Cartan curve* or a *C-curve* if $g_*(TI) \subset D(\subset TY)$.
A curve $h : I \longrightarrow (X, C)$ is called a *Monge curve* or an *M-curve* if $h_*(TI) \subset C(\subset TX)$.

If f is an Engel curve, then $\Pi_Y \circ f$ is a Cartan curve and $\Pi_X \circ f$ is a Monge curve. An Engel curve f is called *transversal* if it is transversal to any Π_Y-fibre $\Pi_Y^{-1}(\ell), \ell \in Y$ and Π_X-fibre $\Pi_X^{-1}(U), U \in X$. Then $\Pi_Y \circ f$ is a Cartan immersion and $\Pi_X \circ f$ is a Monge immersion.

A curve $f : I \rightarrow Z$, $V_1(t) \subset V_2(t)$ $(t \in I)$ is an Engel curve if and only if $V_1'(t) \subset V_2(t), V_2'(t) \subset H_{V_1(t)} = V_3(t)$. Then $V_3'(t) \subset V_3(t)^\perp, V_3''(t) \subset V_2(t)^\perp, V_3'''(t) \subset V_1(t)^\perp$. Here, for instance, $V_1'(t)$ means the subspace generated by the derivative $v'(t)$, with respect to a fixed basis of V, for any C^∞ section $v(t) \in V_1(t)$ together with $V_1(t)$.

A curve $g : I \rightarrow Y, g(t) = V_1(t)$ $(t \in I)$ is a Cartan curve if and only if $V_1'(t) \subset H_{V_1(t)}$. $V_2(t) = V_1(t) + V_1'(t)$ is a null plane in $H_{V_1(t)}$ if g is a Cartan immersion. Then we have $V_2'(t) \subset H_{V_1(t)}$ and g lifts uniquely to an Engel immersion $f : I \rightarrow Z$.

A curve $h : I \rightarrow X, h(t) = V_2(t)$ $(t \in I)$ is a Monge curve if and only if there exists $V_1(t) \subset V_2(t)$ such that $V_1'(t) \subset V_2(t), V_2'(t) \subset H_{V_1(t)}$. Then h lifts to an Engel curve $f : I \rightarrow Z$ uniquely.

4 Explicit description of double fibration and differential systems

We introduce certain charts on X, Z and on Y which are compatible with the double fibration $Y \xleftarrow{\Pi_Y} Z \xrightarrow{\Pi_X} X$.

For a subset $S \subset V$, we define

$$H_S := \{w \in V \mid vw = 0, \text{ for any } v \in S\},$$

$$S^\perp := \{w \in V \mid (v|w) = 0, \text{ for any } v \in S\}.$$

Then, for each $(V_1, V_2) \in Z$, we obtain canonically the complete flag in V:

$$V_1 \subset V_2 \subset V_3 := H_{V_1} \subset V_4 := V_3^\perp \subset V_5 := V_2^\perp \subset V_6 := V_1^\perp \subset V$$

Thus we have an embedding of Z into the complete flag manifold $\mathcal{F}_{1,2,3,4,5,6}(V)$.

Now, fix $(\ell_0, U_0) \in Z$. Then we have the complete flag of V:

$$\ell_0 \subset U_0 \subset H_{\ell_0} \subset H_{\ell_0}^\perp \subset U_0^\perp \subset \ell_0^\perp \subset V.$$

Consider the open subset

$$O = \{(\ell, U) \in Z \mid \ell \cap \ell_0^\perp = \{0\}, \ U \cap U_0^\perp = \{0\}, \ H_\ell \cap H_{\ell_0}^\perp = \{0\}\}$$

of Z. Fix $(\ell_1, U_1) \in O$. Then we have the canonical decomposition of V into the direct sum of lines:

$$V = \ell_1 \oplus (U_1 \cap \ell_0^\perp) \oplus (H_{\ell_1} \cap U_0^\perp) \oplus (H_{\ell_1}^\perp \cap H_{\ell_0}^\perp) \oplus (U_1^\perp \cap H_{\ell_0}) \oplus (\ell_1^\perp \cap U_0) \oplus \ell_0.$$

Since G_2' acts transitively on Z (Proposition 3.1), we can choose a basis $e_1, e_2, e_3, e_4, e_5, e_6, e_7$ of V satisfying

$$e_1 \in \ell_1, e_2 \in U_1 \cap \ell_0^\perp, e_3 \in H_{\ell_1} \cap U_0^\perp, e_4 \in H_{\ell_1}^\perp \cap H_{\ell_0}^\perp, e_5 \in U_1^\perp \cap H_{\ell_0}, e_6 \in \ell_1^\perp \cap U_0, e_7 \in \ell_0$$

and $e_0 = 1, e_1, e_2, e_3, e_4, e_5, e_6, e_7$ form an admissible basis, enjoying the same multiplication table (Table 1) and therefore the same inner product pairing with the standard basis with $e_0 = 1$ (cf. [2]).

Let $(\ell, U) \in O$. Then we fix the basis of U by

$$f_1 = e_1 + ye_3 + xe_4 + ve_5 + ue_6 + ke_7, \quad f_2 = e_2 + ze_3 + \ell e_4 + ae_5 + be_6 + ce_7.$$

Then, from $f_1 f_2 = 0$, we have

$$\ell = y, \quad a = x, \quad b = y^2 - xz, \quad k = x^2 - yv, \quad c = xy - zv - u.$$

Thus we have a system of local coordinates x, y, z, u, v of X near U_1. Then we have a basis of ℓ in the form $f_1 + \lambda f_2$ for some $\lambda \in \mathbf{R}$:

$$f_1 + \lambda f_2 = e_1 + \lambda e_2 + (y + \lambda z)e_3 + (x + \lambda y)e_4 + (v + \lambda x)e_5 + (u + \lambda(y^2 - xz))e_6 + (x^2 - yv + \lambda(xy - zv - u))e_7.$$

Note that (x, y, z, u, v) gives a chart on X.

A chart on Y is given as follows: Let

$$g = e_1 + \lambda e_2 + \nu e_3 + \mu e_4 + \tau e_5 + \sigma e_6 + \rho e_7$$

be a vector in V. Then $(g|g) = \rho + \lambda\sigma + \nu\tau - \mu^2$. Therefore the condition that g is a null vector is given by

$$\rho = -\lambda\sigma - \nu\tau + \mu^2.$$

Hence we can take $(\lambda, \mu, \nu, \tau, \sigma)$ as a chart on Y. Moreover we have a chart (λ, x, y, z, u, v) on $O \subset Z$.

Thus the fibrations Π_Y, Π_X are described via the local coordinates by

$$\Pi_Y(\lambda, x, y, z, u, v) = \left(\lambda,\ x + \lambda y,\ y + \lambda z,\ v + \lambda x,\ u + \lambda(y^2 - xz)\right),$$

and

$$\Pi_X(\lambda, x, y, z, u, v) = (x, y, z, u, v).$$

In particular, the coordinate on Π_Y-fiber is given by z and the coordinate on Π_X-fiber is given λ.

Remark 4.1 As a chart on Z, also we can take $(\lambda, \mu, \nu, \tau, \sigma, z)$. Then the local coordinate transformation for our chart (λ, x, y, z, u, v) is expressed by

$$(\lambda, \mu, \nu, \tau, \sigma, z) \mapsto (\lambda, x, y, z, u, v)$$
$$= (\lambda,\ \mu - \lambda\nu + \lambda^2 z,\ \nu - \lambda z,\ z,\ \sigma - \lambda\nu^2 + (\lambda\mu + \lambda^2\nu)z,\ \tau - \lambda\mu + \lambda^2\nu - \lambda^3 z).$$

We show the explicit local expressions of our differential systems:

Lemma 4.2 *The G_2-Engel differential system E on Z is given by*

$$\alpha_1 := dy + \lambda dz = 0, \quad \alpha_2 := dx - \lambda^2 dz = 0,$$
$$\alpha_3 := dv + \lambda^3 dz = 0, \quad \alpha_4 := du - (\lambda^3 z + 2\lambda^2 y + \lambda x)dz = 0.$$

A local frame (ξ_1, ξ_2) of E is given by

$$\xi_1 = \frac{\partial}{\partial\lambda}, \quad \xi_2 = \frac{\partial}{\partial z} - \lambda\frac{\partial}{\partial y} + \lambda^2\frac{\partial}{\partial x} - \lambda^3\frac{\partial}{\partial v} + (\lambda^3 z + 2\lambda^2 y + \lambda x)\frac{\partial}{\partial u}.$$

Proof: For $\mathbf{v} \in TZ$, take a curve $(V_1(t), V_2(t))$ on Z representing $\mathbf{v}$ at $t = 0$. Suppose $(V_1(t), V_2(t))$ is given by a frame $f_1(t), f_2(t)$. Then the condition that $\mathbf{v} \in E$ is given by $f_1'(0) \in \langle f_1(0), f_2(0) \rangle$. In terms of local coordinates, the condition is given by

$$\lambda' = p, \ (y + \lambda z)' = pz, \ (x + \lambda y)' = py, \ (v + \lambda x)' = px,$$
$$(u + \lambda(y^2 - xz))' = p(y^2 - xz), \ (x^2 - yv + \lambda(xy - zv - u))' = p(xy - zv - u),$$

for some $p \in \mathbf{R}$, at $t = 0$. Then $p = \lambda'$ and

$$(y + \lambda z)' = z\lambda', \quad (x + \lambda y)' = y\lambda', \quad (v + \lambda x)' = x\lambda',$$
$$(u + \lambda(y^2 - xz))' = (y^2 - xz)\lambda', \quad (x^2 - yv + \lambda(xy - zv - u))' = (xy - zv - u)\lambda'.$$

Then the condition is equivalent to $\alpha_1(\mathbf{v}) = \alpha_2(\mathbf{v}) = \alpha_3(\mathbf{v}) = \alpha_4(\mathbf{v}) = \alpha_5(\mathbf{v}) = 0$, where $\alpha_5 = d(x^2 - yv) + \lambda d(xy - zv - u)$. Then we have

$$\alpha_5 = (-v + \lambda x)\alpha_1 + (2x + \lambda y)\alpha_2 - (y + \lambda z)\alpha_3 - \lambda\alpha_4.$$

Thus we obtain the required consequence. $\square$

Remark 4.3 In each system of local coordinates (λ, x, y, z, u, v) of Z, we have the family of G_2-Engel transformations $T = T_{p_0} : (\mathbf{R}^6, p_0) \to (\mathbf{R}^6, 0)$, depending on $p_0 = (\lambda_0, x_0, y_0, z_0, u_0, v_0) \in \mathbf{R}^6$, defined by $(\lambda, x, y, z, u, v) \mapsto (\widetilde{\lambda}, \widetilde{x}, \widetilde{y}, \widetilde{z}, \widetilde{u}, \widetilde{v})$,

$$
\begin{cases}
\widetilde{\lambda} &= \lambda - \lambda_0, \\
\widetilde{z} &= z - z_0, \\
\widetilde{y} &= y - y_0 + \lambda_0(z - z_0), \\
\widetilde{x} &= x - x_0 + 2\lambda_0(y - y_0) + \lambda_0^2(z - z_0), \\
\widetilde{v} &= v - v_0 + 3\lambda_0(x - x_0) + 3\lambda_0^2(y - y_0) + \lambda_0^3(z - z_0), \\
\widetilde{u} &= u - u_0 + x_0(y - y_0) - 2y_0(x - x_0) + z_0(v - v_0) + \lambda_0\{(y - y_0)^2 - (x - x_0)(z - z_0)\}.
\end{cases}
$$

Note that $T_{p_0}(\lambda_0, x_0, y_0, z_0, u_0, v_0) = (0, 0, 0, 0, 0, 0)$. This shows explicitly the local transitivity of G_2'-action on Z (cf. Proposition 3.1). Also note that, if $\lambda_0 \neq 0$, then T is neither a linear nor a projective transformation.

Lemma 4.4 *The Cartan differential system $D \subset TY$ is given, in terms of the system of local projective coordinates $(\lambda, \mu, \nu, \tau, \sigma)$, by $\beta_1 = \beta_2 = \beta_3 = 0$ where*

$$
\beta_1 = -\nu d\lambda + \lambda d\nu + d\mu, \quad \beta_2 = (\lambda\nu - \mu)d\lambda - \lambda^2 d\nu + d\tau, \quad \beta_3 = -\nu^2 d\lambda + (\lambda\nu + \mu)d\nu + d\sigma.
$$

The local frame of D is given by

$$
\eta_1 = \frac{\partial}{\partial\lambda} + \nu\frac{\partial}{\partial\mu} - (\lambda\nu - \mu)\frac{\partial}{\partial\tau} + \nu^2\frac{\partial}{\partial\sigma}, \quad \eta_2 = \frac{\partial}{\partial\nu} - \lambda\frac{\partial}{\partial\mu} + \lambda^2\frac{\partial}{\partial\tau} - (\lambda\nu + \mu)\frac{\partial}{\partial\sigma}.
$$

Proof: Let $r_0 = (\lambda_0, \mu_0, \nu_0, \tau_0, \sigma_0) \in Y$. Then points (x, y, z, u, v) in the Monge line $\Pi_X\Pi_Y^{-1}(r_0)$ are given by the conditions

$$
y = \nu_0 - \lambda_0 z, \; x = \mu_0 - \lambda_0\nu_0 + \lambda_0^2 z, \; v = \tau_0 - \lambda_0\mu_0 + \lambda_0^2\nu_0 - \lambda_0^3 z, \; u = \sigma_0 - \lambda_0\nu_0^2 + \lambda_0(\lambda_0\nu_0 + \mu_0)z.
$$

Then points $(\lambda, M, N, T, \Sigma)$ in $\Pi_Y\Pi_X^{-1}\Pi_X\Pi_Y^{-1}(r_0)$ are given by

$$
M = \mu_0 + \nu_0(\lambda - \lambda_0) - \lambda_0(\lambda - \lambda_0)z, \quad N = \nu_0 + (\lambda - \lambda_0)z,
$$

$$
T = \tau_0 - (\lambda_0\nu_0 - \mu_0)(\lambda - \lambda_0) + \lambda_0^2(\lambda - \lambda_0)z, \quad \Sigma = \sigma_0 + \nu_0^2(\lambda - \lambda_0) - (\lambda_0\nu_0 + \mu_0)(\lambda - \lambda_0)z.
$$

By differentiating by λ, we have a family of tangent lines in $T_{r_0}Y$ with direction vectors

$$
(1, \; \nu_0 - \lambda_0 z, \; z, \; \mu_0 - \lambda_0\nu_0 + \lambda_0^2 z, \; \nu_0^2 - (\lambda_0\nu_0 + \mu_0)z).
$$

Note that the family is linear in z and envelopes the tangent plane to $\Pi_Y\Pi_X^{-1}\Pi_X\Pi_Y^{-1}(r_0)$. We have three independent cotangent vectors

$$
\beta_1 = -\nu_0 d\lambda + \lambda_0 d\nu + d\mu, \; \beta_2 = (\lambda_0\nu_0 - \mu_0)d\lambda - \lambda_0^2 d\nu + d\tau, \; \beta_3 = -\nu_0^2 d\lambda + (\lambda_0\nu_0 + \mu_0)d\nu + d\sigma,
$$

from the condition to annihilate the family of lines, which define the differential system D. The local frame is obtained easily. Note that

$$
[\eta_1, \eta_2] = -2\left(\frac{\partial}{\partial\mu} + 2\lambda\frac{\partial}{\partial\tau} - 2\nu\frac{\partial}{\partial\sigma}\right).
$$

$\square$

Lemma 4.5 *The cone structure $C \subset TX$ is a twisted cubic cone field given, in terms of the system of local projective coordinates (x, y, z, u, v) and symmetric tensors, by*

$$dxdy - dzdv = 0, \ dxdz - (dy)^2 = 0, \ (dx)^2 - dydv = 0, \ du - 2ydx + xdy + zdv = 0.$$

The linear hull of C is a contact structure $D' \subset TX$ given by

$$du - 2ydx + xdy + zdv = 0.$$

Proof: Let $q_0 = (x_0, y_0, z_0, u_0, v_0) \in X$. The Cartan line $\Pi_Y \Pi_X^{-1}(q_0)$ is given by

$$(\lambda, \ y_0 + \lambda z_0, \ x_0 + \lambda y_0, \ v_0 + \lambda x_0, \ u_0 + \lambda(y_0^2 - x_0 z_0)), \quad (\lambda \in \mathbf{R}).$$

Then the condition that a point (x, y, z, u, v) belongs to $\Pi_X \Pi_Y^{-1} \Pi_Y \Pi_X^{-1}(q_0)$ is given by

$$y + \lambda z = y_0 + \lambda z_0, \ x + \lambda y = x_0 + \lambda y_0, \ v + \lambda x = v_0 + \lambda x_0, \ u + \lambda(y^2 - xz) = u_0 + \lambda(y_0^2 - x_0 z_0),$$

for some $\lambda \in \mathbf{R}$. Then

$$y = y_0 - \lambda(z - z_0), x = x_0 + \lambda^2(z - z_0), v = v_0 - \lambda^3(z - z_0).$$

Moreover

$$y^2 - xz = y_0^2 - x_0 z_0 - (\lambda^2 z_0 + 2\lambda^2 y_0 + \lambda x_0)(z - z_0),$$

therefore

$$u - u_0 = (\lambda^3 z_0 + 2\lambda^2 y_0 + \lambda x_0)(z - z_0).$$

Hence the condition is reduced to

$$x - x_0 = \lambda^2(z - z_0), \quad y - y_0 = -\lambda(z - z_0), \quad v - v_0 = -\lambda^3(z - z_0),$$
$$u - u_0 = 2y_0(x - x_0) - x_0(y - y_0) - z_0(v - v_0).$$

Thus we have a family of tangent lines in $T_{q_0}X$ parametrized by λ, which forms a twisted cubic cone:

$$dx = \lambda^2 dz, \ dy = -\lambda dz, \ dv = -\lambda^3 dz,$$

in the hyperplane $\{du = 2y_0 dx - x_0 dy - z_0 dv\} \subset T_{q_0}X$. Eliminating λ, we obtain the equations for C. Moreover we have that the linear hull D'_{q_0} is given by $D'_{q_0} = \{du - 2y_0 dx + x_0 dy + z_0 dv = 0\} \subset T_{q_0}X$. This completes the explicit expression of our geometric structures. □

Remark 4.6 The tangent surfaces of Monge curves, namely integral curves to the cone field C, are Legendre surfaces for the contact structure D'.

5 Explicit descriptions of tangent surfaces

Let $f : (\mathbf{R}, 0) \to Z(G_2)$ be a germ of Engel curve and $f(t) = (\lambda(t), x(t), y(t), z(t), u(t), v(t))$ a local representation of f in local coordinates of Z introduced in §4.

First we give a parametrization of the tangent surface to the curve $\Pi_Y \circ f$ in Y.

Lemma 5.1 *The tangent surface to $\Pi_Y \circ f$ is parametrized by a map-germ $(\mathbf{R}^2, 0) \to (\mathbf{R}^5, 0)$ given by*

$$(r, t) \ \mapsto \ \left(r, \ x(t) + ry(t), \ y(t) + rz(t), \ v(t) + rx(t), \ u(t) + r(y(t)^2 - x(t)z(t))\right).$$

Proof: The curve $\gamma = \Pi_Y \circ f$ is given by

$$\gamma(t) = \left(\lambda(t),\ x(t) + \lambda(t)y(t),\ y(t) + \lambda(t)z(t),\ v(t) + \lambda(t)x(t),\ u(t) + \lambda(t)(y(t)^2 - x(t)z(t))\right).$$

Using the condition that f is an Engel integral curve (Lemma 4.2), we see that the velocity vector of γ is given by

$$\gamma'(t) = \lambda'(t)\left(1,\ y(t),\ z(t),\ x(t),\ y(t)^2 - x(t)z(t)\right).$$

Therefore, for each t, we can take the vector $w(t) = \left(1,\ y(t),\ z(t),\ x(t),\ y(t)^2 - x(t)z(t)\right)$ as a basis of the tangent line to γ at t. Hence the tangent map-germ of γ is given by

$$\begin{aligned}\mathrm{Tan}(\Pi_Y \circ f) &= \mathrm{Tan}(\gamma)(s,t) := \gamma(t) + sw(t)\\ &= \left(\lambda + s,\ x + \lambda y + sy,\ y + \lambda z + sz,\ v + \lambda x + sx,\ u + \lambda(y^2 - xz) + s(y^2 - xz)\right).\end{aligned}$$

If we set $r = \lambda(t) + s$, then we see that the tangent map-germ $F_Y = \mathrm{Tan}(\Pi_Y \circ f) : (\mathbf{R}^2, 0) \to (\mathbf{R}^5, 0)$ is given by

$$(r,t) \ \mapsto\ \left(r,\ x(t) + ry(t),\ y(t) + rz(t),\ v(t) + rx(t),\ u(t) + r(y(t)^2 - x(t)z(t))\right),$$

up to parametrizations. $\qquad\square$

Remark 5.2 Consider the map-germ $G_Y : (\mathbf{R}^2, 0) \to (\mathbf{R}^2, 0)$ define by

$$G_Y(r,t) = (G_1(r,t),\ G_2(r,t)) := (r,\ y(t) + rz(t)).$$

Define the sub-$\mathbf{R}$-algebra $\mathcal{R}_{G_Y}$ of $\mathcal{E}_{\mathbf{R}^2,0} = \mathcal{E}_{r,t}$ by

$$\mathcal{R}_{G_Y} := \left\{h \in \mathcal{E}_{r,t} \mid dh \in \langle dG_1, dG_2\rangle_{\mathcal{E}_{r,t}}\right\} = \left\{h \in \mathcal{E}_{r,t} \ \bigg|\ \tfrac{\partial}{\partial t}h = a\tfrac{\partial}{\partial t}(y(t) + rz(t)) \text{ for some } a \in \mathcal{E}_{r,t}\right\}.$$

Then we see that every components of the germ $\mathrm{Tan}(\Pi_Y \circ f)$ belong to $\mathcal{R}_{G_Y}$. In fact we have

$$\frac{\partial}{\partial t}(x(t) + ry(t)) = -\lambda(t)\frac{\partial}{\partial t}(y(t) + rz(t)), \quad \frac{\partial}{\partial t}(v(t) + rx(t)) = \lambda(t)^2\frac{\partial}{\partial t}(y(t) + rz(t)),$$

$$\frac{\partial}{\partial t}(u(t) + r(y(t)^2 - x(t)z(t)) = -(\lambda(t)^2 z(t) + 2\lambda(t)y(t) + x(t))\frac{\partial}{\partial t}(y(t) + rz(t)).$$

Hence we have $F_Y^*(\mathcal{E}_{\lambda,\mu,\nu,\tau,\sigma}) \subseteq \mathcal{R}_{G_Y}$. Moreover, in fact, the object $\mathcal{R}_G$ is defined for any map-germ $G : (\mathbf{R}^n, 0) \to (\mathbf{R}^m, 0)$ similarly to above and, for any diffeomorphism-germs $\Sigma : (\mathbf{R}^n, 0) \to (\mathbf{R}^n, 0), T : (\mathbf{R}^m, 0) \to (\mathbf{R}^m, 0)$, and for $G' = T \circ G \circ \Sigma$, we have $\mathcal{R}_{G'} = \Sigma^*(\mathcal{R}_G)$.

Second, we give a parametrization of the tangent surface to the curve $\Pi_X \circ f$ in X.

Lemma 5.3 *The tangent surface to $\Pi_X \circ f$ is parametrized by a map-germ $(\mathbf{R}^2, 0) \to (\mathbf{R}^5, 0)$ given by*

$$\begin{aligned}(r,t) \ \mapsto\ &\big(x(t) - \lambda(t)^2 z(t) + r\lambda(t)^2,\ y(t) + \lambda(t)z(t) - r\lambda(t),\ r,\\ &u(t) - \lambda(t)^3 z(t)^2 - 2\lambda(t)^2 y(t)z(t) - \lambda(t)x(t)z(t) + r(\lambda(t)^3 z(t) + 2\lambda(t)^2 y(t) + \lambda(t)x(t)),\\ &v(t) + \lambda(t)^3 z(t) - r\lambda(t)^3\big).\end{aligned}$$

Proof: The curve $\gamma^* = \Pi_X \circ f$ is given by $\gamma^*(t) = (x(t), y(t), z(t), u(t), v(t))$. From Lemma 4.2, the velocity vector of γ^* is given by

$$\gamma^{*\prime}(t) = z'(t)\left(\lambda(t)^2,\ -\lambda(t),\ 1,\ \lambda(t)^3 z(t) + 2\lambda(t)^2 y(t) + \lambda(t)x(t),\ -\lambda(t)^3\right).$$

Thus we have the tangent map-germ of γ^* is given by

$\mathrm{Tan}(\Pi_X \circ f)(s,t) = \mathrm{Tan}(\gamma^*)(s,t) :=$

$\left(x(t) + s\lambda(t)^2,\ y(t) - s\lambda(t),\ z(t) + s,\ u(t) + s(\lambda(t)^3 z(t) + 2\lambda(t)^2 y(t) + \lambda(t)x(t)),\ v(t) - s\lambda(t)^3\right).$

By setting $r = z(t) + s$, we see that $F_X = \mathrm{Tan}(\Pi_X \circ f) : (\mathbf{R}^2, 0) \to (\mathbf{R}^5, 0)$ is given by

$(r, t) \mapsto$

$\left(x - \lambda^2 z + r\lambda^2,\ y + \lambda z - r\lambda,\ r,\ u - \lambda^3 z^2 - 2\lambda^2 yz - \lambda xz + r(\lambda^3 z + 2\lambda^2 y + \lambda x),\ v + \lambda^3 z - r\lambda^3\right),$

up to parametrizations. $\qquad\square$

Remark 5.4 Similarly to Remark 5.2, for the tangent surface to $\Pi_X \circ f$, we set $G_X : (\mathbf{R}^2, 0) \to (\mathbf{R}^2, 0)$ by

$$G_X(r,t) = (G_1(r,t),\ G_2(r,t)) := (r,\ y(t) + \lambda(t)z(t) - r\lambda(t)).$$

Then every components of F_X belong to $\mathcal{R}_{G_X}$. In fact,

$\frac{\partial}{\partial t}(x(t) - \lambda(t)^2 z(t) + r\lambda^2(t)) = -2\lambda(t)\frac{\partial}{\partial t}(y(t) + \lambda(t)z(t) - r\lambda(t)),$

$\frac{\partial}{\partial t}(v(t) + \lambda(t)^3 z(t) - r\lambda(t)^3) = 3\lambda(t)^2 \frac{\partial}{\partial t}(y(t) + \lambda(t)z(t) - r\lambda(t)),$

$\frac{\partial}{\partial t}\left(u(t) - \lambda(t)^3 z(t)^2 - 2\lambda(t)^2 y(t)z(t) - \lambda(t)x(t)z(t) + r(\lambda(t)^3 z(t)2\lambda(t)^2 y(t) + \lambda(t)x(t))\right)$

$= -(3\lambda(t)^2 z(t) + 4\lambda(t)y(t) + x(t))\frac{\partial}{\partial t}(y(t) + \lambda(t)z(t) - r\lambda(t)).$

Moreover we have $F_X^*(\mathcal{E}_{x,y,z,u,v}) \subseteq \mathcal{R}_{G_X}$.

6 Cartan-Monge duality of Engel curves

Let $f : (\mathbf{R}, 0) \to Z(G_2)$ be a germ of Engel curve. Let $f(t) = (\lambda(t), x(t), y(t), z(t), u(t), v(t))$ be a local representation of f in local coordinates centered at $f(0) \in Z$ introduced in §4. Suppose

$$\mathrm{ord}(\lambda) = m, \quad \mathrm{ord}(z) = n$$

at $t = 0$. Here the order means the degree of the leading term at $t = 0$. Then we have that

$$\mathrm{ord}(\lambda, y + \lambda z, x + \lambda y, v + \lambda x, u + \lambda(y^2 - xz)) = (m, m + n, 2m + n, 3m + n, 3m + 2n)$$

and

$$\mathrm{ord}(z, y, x, v, u) = (n, m + n, 2m + n, 3m + n, 3m + 2n).$$

In general, let $\gamma : I \to M^N$ be a C^∞ curve in an N-dimensional manifold M with a flat projective structure. We say that γ is *of finite type* at $t = t_0 \in I$ if there exist a C^∞ coordinate t on I centered at t_0, t takes 0 at t_0, and a local system of projective coordinates $(x_1, \ldots, x_N)$ of M centered at $\gamma(t_0)$ such that

$$x_1 \circ \gamma(t) = t^{a_1} + O(t^{a_1 + 1}), \quad \cdots, \quad x_N \circ \gamma(t) = t^{a_N} + O(t^{a_N + 1}),$$

for some strictly increasing sequence of positive integers $1 \leq a_1 < \cdots < a_N$. Then $(a_1, \ldots, a_N)$ is uniquely determined from the projective class of the germ of γ at $t = t_0$, and we say that γ is *of type* $(a_1, \ldots, a_N)$ at $t = t_0$. If we consider the $(N \times i)$-Wronskian matrices

$$W_i(t) = \left(\gamma'(t), \gamma''(t), \cdots, \gamma^{(i)}(t)\right), \quad i = 1, 2, \ldots,$$

16

regarding $\gamma(t)$ as a column vector, then we have

$$a_1 = \min\{i \mid \mathrm{rank}\, W_i(t_0) = 1\}, \quad \cdots, \quad a_N = \min\{i \mid \mathrm{rank}\, W_i(t_0) = N\}.$$

We will apply the above definition to the case $N = 5$.

We denote by $J^r_E(I, Z)$ the r-jet space of Engel curves (E-integral curves) $I \to (Z, E)$:

$$J^r_E(I, Z) = \{j^r f(t_0) \in J^r(I, Z) \mid t_0 \in I,\ f : (\mathbf{R}, t_0) \to Z \text{ is Engel}\}.$$

Lemma 6.1 $J^r_E(I, Z)$ *is a subbundle of* $J^r(I, Z)$ *for the projection* $\Pi : J^r(I, Z) \to I \times Z$ *of codimension* $4r$.

Proof: It is sufficient to show that

$$J^r_E(1, 6) = \{j^r f(0) \mid f : (\mathbf{R}, 0) \to (\mathbf{R}^6, 0) \text{ is Engel}\}$$

is a submanifold of $J^r(1, 6)$ of codimension $4r$. To show it, define the mapping $\Phi : J^r(1, 6) \to (\Lambda^{r-1}_1)^4$ by

$$\Phi(j^r(\lambda, x, y, z, u, v)(0)) = (j^{r-1}(\alpha_1)(0), j^{r-1}(\alpha_2)(0), j^{r-1}(\alpha_3)(0), j^{r-1}(\alpha_4))(0)),$$

using the four 1-forms $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ in Lemma 4.2 which define $E \subset TZ$. Here Λ^{r-1}_1 denotes the $(r - 1)$-jet space of 1-forms on $(\mathbf{R}, 0)$. Note that $(\Lambda^{r-1}_1)^4$ is diffeomorphic to $\mathbf{R}^{4r}$ Then Φ is a submersion. Therefore $\Phi^{-1}(0) = J^r_E(1, 6)$ is a submanifold of $J^r(1, 6)$ of codimension $4r$. □

Let $\boldsymbol{a} = (a_1, a_2, a_3, a_4, a_5), \boldsymbol{b} = (b_2, b_2, b_3, b_4, b_5)$ be strictly increasing sequences of positive integers. Then we set, for a sufficiently large r,

$$\Sigma^r_{\Pi_Y, \boldsymbol{a}} := \{j^r f(t_0) \in J^r_E(I, Z) \mid \Pi_Y \circ f : I \to Y \text{ is of type } \boldsymbol{a} \text{ at } t_0 \in I\},$$

$$\Sigma^r_{\Pi_X, \boldsymbol{b}} := \{j^r f(t_0) \in J^r_E(I, Z) \mid \Pi_X \circ f : I \to X \text{ is of type } \boldsymbol{b} \text{ at } t_0 \in I\}.$$

From the above calculation of orders, we have

Proposition 6.2 *Let* $\boldsymbol{a} = (a_1, a_2, a_3, a_4, a_5), \boldsymbol{b} = (b_2, b_2, b_3, b_4, b_5)$ *be strictly increasing sequences of positive integers.*
    (1) (Codimension formula I)
$\Sigma^r_{\Pi_Y, \boldsymbol{a}} \neq \emptyset$ *for a sufficiently large* r, *if and only if*

$$a_3 = a_1 + a_2,\ a_4 = 2a_1 + a_2,\ a_5 = a_1 + 2a_2.$$

Then the codimension of $\Sigma^r_{\Pi_Y, \boldsymbol{a}}$ *is equal to* $a_2 - 2$.
    (2) (Codimension formula II)
$\Sigma^r_{\Pi_X, \boldsymbol{b}} \neq \emptyset$ *for a sufficiently large* r, *if and only if*

$$b_3 = -b_1 + 2b_2,\ b_4 = -2b_1 + 3b_2,\ b_5 = -b_1 + 3b_2.$$

Then the codimension of $\Sigma^r_{\Pi_X, \boldsymbol{b}}$ *is equal to* $b_2 - 2$.
    (3) (Duality formula)
$\Sigma^r_{\Pi_Y, \boldsymbol{a}} \cap \Sigma^r_{\Pi_X, \boldsymbol{b}} \neq \emptyset$ *for a sufficiently large* r *if and only if the above conditions* (1)(2) *are satisfied and*

$$(b_1, b_2, b_3, b_4, b_5) = (a_2 - a_1, a_2, a_3, a_4, a_5)$$

or equivalently

$$(a_1, a_2, a_3, a_4, a_5) = (b_2 - b_1, b_2, b_3, b_4, b_5).$$

17

To obtain a generic classification we use the following transversality theorem:

Proposition 6.3 (Engel transversality theorem on open intervals) *Let $I \subset \mathbf{R}$ be an open interval and $Q \subset J_E^r(I, Z)$ a submanifold of Engel jet space $J_E^r(I, Z)$. Then any Engel curve $f : I \to Z$ is approximated in C^∞-topology by an Engel curve $f' : I \to Z$ for which $j^r f' : I \to J_E^r(I, Z)$ is transverse to Q.*

Proof: The proof is achieved by the same method as the proof of [13] Proposition 4.2.

For any open sub-interval $V \subset I$ and for any coordinate neighborhood $O \subset Z$, we define a diffeomorphism

$$\varphi = \varphi_{(V,O)} : J_E^r(V, O) \to V \times O \times J^r(1, 2)$$

by $\varphi(j^r f(t_0)) = (t_0, f(t_0), j^r((\lambda, z) \circ T \circ f(t + t_0))(0))$, using the family of Engel transformations $T = T_{f(t_0)}$ defined in Remark 4.3. Note that $T_{f(t_0)}(f(t_0)) = 0$.

Let $f : I \to Z$ be an Engel curve. Suppose, as a special case, $f(I)$ is in some projective coordinate neighborhood $O \subset Z$. Then, by the ordinary transversality theorem, (λ, z)-components of f are perturbed so that, for a perturbed f', $\varphi \circ j^r f'$ is transverse to $\varphi(Q \cap J_E^r(I, O)) \subset I \times O \times J^r(1, 2)$. Then $j^r f'$ is transverse to Q.

In general case, there is a strictly increasing sequence $\{t_i\}_{i \in \mathbf{Z}}$ of points in I such that $f([t_i, t_{i+1}])$ is contained in some projective coordinate neighborhood O_i. We set $K_i = [t_i, t_{i+1}]$ and take open intervals $W_i \supset K_i$ such that also $f(W_i) \subset O_i$ and that $W_i \cap W_j = \emptyset$ if $|i - j| \geq 2$.

First we perturb f over W_0 into an Engel curve $f_0 : W_0 \to Z$ such that $j^r f_0$ is transverse to Q over W_0. In fact, similarly as in the special case, by the ordinary transversality theorem via $\varphi = \varphi_{(W_0, O_0)}$, (λ, z)-components of $f|_{W_0}$ are perturbed so that, for the perturbed f_0, $\varphi \circ j^r f_0$ is transverse to $\varphi(Q \cap J_E^r(W_0, O_0)) \subset W_0 \times O_0 \times J^r(1, 2)$. Then $j^r f_0$ is transverse to Q over W_0.

Second we perturb f over $W_0 \cup W_1$ into an Engel integral curve $f_1 : W_0 \cup W_1 \to Z$ such that $j^r f_1$ is transverse to Q and $f_1|_{K_0} = f_0|_{K_0}$. This is achieved, under the coordinates on O_1, by

$$x(t) = \int_{t_1}^t \lambda(t)^2 z'(t) dt + x(t_1), \ y(t) = -\int_{t_1}^t \lambda(t) z'(t) dt + y(t_1), \ u(t) = -\int_{t_1}^t \lambda(t)^3 z'(t) dt + u(t_1),$$

$$v(t) = \int_{t_1}^t (\lambda(t)^3 z(t) + 2\lambda(t)^2 y(t) + \lambda(t) x(t)) z'(t) dt + v(t_1),$$

perturbing $\lambda(t), z(t)$ over W_1 just outside of $K_0 \cap W_1$ and setting $f_1(t_1) = f_0(t_1)$.

Third we perturb f over $W_0 \cup W_1 \cup W_2$ into an Engel curve $f_2 : W_0 \cup W_1 \cup W_2 \to \widetilde{\mathcal{F}}$ such that $j^r f_2$ is transverse to Q and $f_2|_{K_0 \cup K_1} = f_1|_{K_0 \cup K_1}$. Thus, by continuing this procedure, we have a perturbation $f' : \cup_{0 \leq i} W_i \to Z$ of f such that $j^r f'$ is transverse to Q.

Finally we perturb f backward to an Engel curve $f'' : I = \cup_{i \in \mathbf{Z}} W_i \to Z$ such that $j^r f''$ is transverse to Q, by perturbing $\lambda(t), z(t)$ and using, for $i \leq 0$,

$$x(t) = -\int_t^{t_i} \lambda(t)^2 z'(t) dt + x(t_i), \ y(t) = \int_t^{t_i} \lambda(t) z'(t) dt + y(t_i), \ u(t) = -\int_t^{t_i} \lambda(t)^3 z'(t) dt + u(t_i),$$

$$v(t) = -\int_t^{t_1} (\lambda(t)^3 z(t) + 2\lambda(t)^2 y(t) + \lambda(t) x(t)) z'(t) dt + v(t_i).$$

Note that, on any compact $K \subset \cup_{i \in \mathbf{Z}} W_i$, the perturbation is achieved just by a finite number of steps. Therefore we can take transversal perturbations of f to Q which are arbitrarily small in C^∞ topology. □

By Proposition 6.2 and by Proposition 6.3, we have

Theorem 6.4 *For a generic Engel curve $f : I \to Z$ in the split G_2-flag manifold (Z, E), the pair of types $(\mathrm{type}(\Pi_Y \circ f)(t), \mathrm{type}(\Pi_X \circ f)(t))$ at any point $t \in I$ is given by one of the following three cases:*

I : $((1, 2, 3, 4, 5), (1, 2, 3, 4, 5))$, II : $((1, 3, 4, 5, 7), (2, 3, 4, 5, 7))$, III : $((2, 3, 5, 7, 8), (1, 3, 5, 7, 8))$.

Proof: We apply Proposition 6.3 to $Q = \Sigma^r_{\Pi_Y, \boldsymbol{a}}$. Then, generically $j^r f^{-1}(Q) \neq \emptyset$ only if $a_2 - 2 \leq 1$, namely if $a_2 \leq 3$, so if $a_2 = 2, 3$. If $a_2 = 2$, then $a_1 = 1$ and $(a_1, a_2, a_3, a_4, a_5) = (1, 2, 3, 4, 5)$, while $(b_1, b_2, b_3, b_4, b_5) = (1, 2, 3, 4, 5)$. If $a_2 = 3$, then $a_1 = 1, 2$. If $a_1 = 1$, then $(a_1, a_2, a_3, a_4, a_5) = (1, 3, 4, 5, 7)$ while $(b_1, b_2, b_3, b_4, b_5) = (2, 3, 4, 5, 7)$. If $a_1 = 2$, then $(a_1, a_2, a_3, a_4, a_5) = (2, 3, 5, 7, 8)$ and $(b_1, b_2, b_3, b_4, b_5) = (1, 3, 5, 7, 8)$. $\qquad\square$

7 Local classification of tangent surface singularities

In this section, first we show

Proposition 7.1 *The diffeomorphism class of tangent surfaces of curves of type* $(1, 2, 3, 4, 5)$ (*resp.* $(1, 3, 4, 5, 7), (2, 3, 4, 5, 7))$ *is uniquely determined. We call it the cuspidal edge* (*resp. the open Mond surface, the open swallowtail*).

Proof: To verify Proposition 7.1, we recall several basic construction from singularity theory (See [12]).

Let $\gamma = \Pi_Y \circ f : I \to Y$ (resp. $\gamma = \Pi_X \circ f : I \to X$), and $t_0 \in I$. Suppose the type of γ at $t = t_0$ is equal to $(a_1, a_2, a_3, a_4, a_5)$. Take a local affine representation $\gamma : (\mathbf{R}, 0) \to (\mathbf{R}^5, 0)$, $\gamma(t) = (x_1(t), x_2(t), x_3(t), x_4(t), x_5(t))$, via a C^∞ coordinate t centered at t_0 of I and some projective coordinates of Y (resp. X) centered at $\gamma(t_0)$ such that

$$x_1(t) = t^{a_1} + O(t^{a_1+1}), \cdots, x_5(t) = t^{a_5} + O(t^{a_5+1}).$$

We may suppose $x_1(t) = t^{a_1}$, by an appropriate parameter t. The tangent surface to γ is parametrized by a mapping $F = \mathrm{Tan}(\gamma)$ defined as

$$F(s, t) = (F_1(s, t), \ldots, F_5(s, t)) = \left(x_i(t) + s \frac{1}{\alpha(t)} x_i'(t) \right)_{1 \leq i \leq 5},$$

where $\alpha(t) = t^{a_1 - 1}$.

We define the map-germ $g' : (\mathbf{R}^2, 0) \to (\mathbf{R}^2, 0)$ by $g' = (F_1, F_2)$. We denote by $\mathcal{E}_n$ the $\mathbf{R}$-algebra of function-germs $(\mathbf{R}^n, 0) \to \mathbf{R}$, and set

$$\mathcal{R}_{g'} := \{ h \in \mathcal{E}_2 \mid dh \in \langle dF_1, dF_2 \rangle_{\mathcal{E}_2} \}.$$

Then we see that $\mathcal{R}_{g'}$ is a module over the algebra $g'^* \mathcal{E}_2$ of composite functions of g'.

It is easy to verify that $F_3, F_4, F_5 \in \mathcal{R}_{g'}$ ([12], Lemma 4.5). Since $F_1(s, t) = x_1(t) + a_1 s$ is a regular function, we set $u = F_1(s, t)$ and regard it as an unfolding parameter. Let $(a_1, a_2) = (1, 2)$ (resp. $(1, 3), (2, 3)$). Then there exist diffeomorphism-germ $\sigma : (\mathbf{R}^2, 0) \to (\mathbf{R}^2, 0)$ and $\tau : (\mathbf{R}^2, 0) \to (\mathbf{R}^2, 0)$ such that σ is of form $\sigma(u, t) = (\sigma_1(u), t\sigma_2(u, t))$ and $g = \tau \circ g' \circ \sigma$ is equal to $(u, t) \mapsto (u, t^2 + ut)$ (resp. $(u, t) \mapsto (u, t^3 + ut^2), (u, t) \mapsto (u, t^3 + ut)$). Then $F_3 \circ \sigma, F_4 \circ \sigma, F_5 \circ \sigma \in \mathcal{R}_g = \sigma^* \mathcal{R}_{g'}$. It is helpful to introduce the notion of openings ([12]). Then $F \circ \sigma$ is a versal opening of g in each of three cases. Therefore the diffeomorphism class of F is unique by Proposition 6.9 or Theorem 7.1 of [12]. This shows Proposition 7.1. $\qquad\square$

Next we show

Proposition 7.2 *The diffeomorphism class of tangent surfaces of curves of type* $(1, 3, 5, 7, 8)$ *is uniquely determined. We call it the open Shcherbak surface.*

To show Proposition 7.2, we need the following:

19

Lemma 7.3 (cf. Lemma 2.4 of [11]) *Let $g : (\mathbf{R}^2, 0) \to (\mathbf{R}^2, 0)$ be the map-germ defined by $g(u, t) = (u, t^3 + ut^2)$. We denote*

$$\mathcal{R}_g^{(2)} := \left\{ h \in t^2 \mathcal{E}_2 \mid dh \in t^2 \langle dg_1, dg_2 \rangle_{\mathcal{E}_2} \right\} = \mathcal{R}_g \cap t^4 \mathcal{E}_2.$$

We put $T(u, t) = t^3 + ut^2, T_i(u, t) = \frac{3}{i+3} t^{i+3} + \frac{2}{i+2} u t^{i+2}, (i = 1, 2, 3, \dots)$. Then we have

(1) *Let $\iota : (\mathbf{R}, 0) \to (\mathbf{R}^2, 0), \iota(t) = (0, t)$. Then $h_1, \dots, h_\ell \in \mathcal{R}_g^{(2)}$ generate $\mathcal{R}_g^{(2)}$ as $\mathcal{E}_2$-module via g^* if and only if $\iota^* h_1, \dots, \iota^* h_\ell$ generate $t^5 \mathcal{E}_1 / t^8 \mathcal{E}_1$ over $\mathbf{R}$.*
(2) *$\mathcal{R}_g^{(2)}$ is a finite $\mathcal{E}_2$-module via $g^* : \mathcal{E}_2 \to \mathcal{E}_2$ generated by T_2, T^2, T_4.*

Proof: (1) is proved in Lemma 2.4 of [11]. Then T_2, T^2, T_4 belong to $\mathcal{R}_g^{(2)}$ and they satisfy the condition of (1). Therefore they generate $\mathcal{R}_g^{(2)}$ as $\mathcal{E}_2$-module via g^*. □

Proof of Proposition 7.2: Let $\gamma : (\mathbf{R}, 0) \to (\mathbf{R}^5, 0)$ be a curve of type $(1, 3, 5, 7, 8)$. Let

$$\gamma(t) = (t, \ t^3 + \varphi(t), \ t^5 + \psi(t), \ t^7 + \rho(t), \ t^8 + \varepsilon(t)),$$

with $\varphi \in \mathfrak{m}_1^4, \psi \in \mathfrak{m}_1^6, \rho \in \mathfrak{m}_1^8, \varepsilon \in \mathfrak{m}_1^9$. Then $F = \mathrm{Tan}(\gamma)$ is given by

$$F(s, t) = \left(t + s, \ t^3 + 3st^2 + \Phi(s, t), \ t^5 + 5st^4 + \Psi(s, t), \ t^7 + 7st^6 + R(s, t), \ t^8 + 8st^7 + E(s, t) \right),$$

where $\Phi(s, t) = \varphi(t) + s\varphi'(t), \Psi(s, t) = \psi(t) + s\psi'(t), R(s, t) = \rho(t) + s\rho'(t)$ and $E(s, t) = \varepsilon(t) + s\varepsilon'(t)$. We set $r = t + s$. Then

$$F(r, t) = \left(r, \ -2t^3 + 3rt^2 + \widetilde{\Phi}, \ -4t^5 + 5rt^4 + \widetilde{\Psi}, \ -6t^7 + 7rt^6 + \widetilde{R}, \ -7t^8 + 8rt^7 + \widetilde{E} \right),$$

where $\widetilde{\Phi}(r, t) = \varphi(t) + (r - t)\varphi'(t), \widetilde{\Psi}(r, t) = \psi(t) + (r - t)\psi'(t), \widetilde{R}(r, t) = \rho(t) + (r - t)\rho'(t)$ and $\widetilde{E}(r, t) = \varepsilon(t) + (r - t)\varepsilon'(t)$.

From the determinacy of tangent varieties to curves of type $(1, 3, 5)$ in $\mathbf{R}^3$ ([17], [11]), there exist diffeomorphism-germ $\sigma : (\mathbf{R}^2, 0) \to (\mathbf{R}^2, 0)$ of form $\sigma(r, t) = (\sigma_1(r), t\sigma_2(r, t))$ and a diffeomorphism-germ $\tau : (\mathbf{R}^5, 0) \to (\mathbf{R}^5, 0)$ such that

$$\tau \circ F \circ \sigma(r, t) = (r, \ T(r, t), \ T_2(r, t), \ T_4(r, t) + S_4(r, t), \ T_5(r, t) + S_5(r, t)),$$

with

$$T(r, t) = t^3 + rt^2, \quad T_2(r, t) = \frac{3}{5} t^5 + \frac{1}{2} rt^4, \quad T_4(r, t) = \frac{3}{7} t^7 + \frac{1}{3} rt^6, \quad T_5(r, t) = \frac{3}{8} t^8 + \frac{2}{7} rt^7,$$

$S_4, S_5 \in \mathcal{R}_g^{(2)}$, $g = (r, \ t^3 + rt^2) = (r, T(r, t))$, $\iota^* S_4 \in \mathfrak{m}_1^8, \iota^* S_5 \in \mathfrak{m}_1^9$. Then we have, by Lemma 7.3,

$$S_4 = (A_4 \circ g) \, T_2 + (B_4 \circ g) \, T^2 + (C_4 \circ g) \, T_4, \quad S_5 = (A_5 \circ g) \, T_2 + (B_5 \circ g) \, T^2 + (C_5 \circ g) \, T_4,$$

for some $A_4, B_4, C_4, A_5, B_5, C_5 \in \mathcal{E}_2$. Comparing the orders of t at $r = 0$, we see $C_4(0, 0) = 0$. Define $\Xi : (\mathbf{R}^5, 0) \to (\mathbf{R}^5, 0)$ by

$$\begin{aligned}
\Xi(x_1, x_2, x_3, x_4, x_5) &= \left(x_1, \ x_2, \ x_3, \ x_4 + A_4(x_1, x_2)x_3 + B_4(x_1, x_2)x_2^2 + C_4(x_1, x_2)x_4, \right. \\
&\qquad \left. x_5 + A_5(x_1, x_2)x_3 + B_5(x_1, x_2)x_2^2 + C_5(x_1, x_2)x_4 \right).
\end{aligned}$$

Then the Jacobi matrix of Ξ at the origin is the unit matrix, so Ξ is a diffeomorphism-germ and we have

$$\Xi^{-1} \circ \tau \circ F \circ \sigma = (r, \ T, \ T_2, \ T_4, \ T_5).$$

Thus we see F is diffeomorphic to the unique normal form. □

For the remaining case, in Theorem 6.4, that the Cartan curve $\Pi_Y \circ f$ is of type $(2, 3, 5, 7, 8)$ on Y, we will give the differential normal form of the tangent map-germ $\mathrm{Tan}(\Pi_Y \circ f)$ under an additional genericity condition:

Proposition 7.4 *Let $f : I \to Z$ be a generic Engel curve. Let $t_0 \in I$ and $f : (\mathbf{R}, 0) \to (\mathbf{R}^6, 0), f(t) = (\lambda(t), x(t), y(t), z(t), u(t), v(t))$ be a local representation of the germ of f at t_0 in terms of coordinates introduced in §4. Suppose $m = \mathrm{ord}(\lambda(t)) = 2$ and $n = \mathrm{ord}(z(t)) = 1$. Then the tangent map-germ $\mathrm{Tan}(\Pi_Y \circ f) : (\mathbf{R}^2, 0) \to (\mathbf{R}^5, 0)$ of the curve $\Pi_Y \circ f$ of type $(2, 3, 5, 7, 8)$ has the unique diffeomorphism class and it is diffeomorphic to the open generic folded pleat given in Theorem 1.3.*

By Lemma 5.1, up to right equivalence, we have

$$\mathrm{Tan}(\Pi_Y \circ f)(r, t) = (\lambda, \mu, \nu, \tau, \sigma) = \left(r, \ x + ry, \ y + rz, \ v + rx, \ u + r(y^2 - xz) \right).$$

Now suppose that

$$\lambda = \frac{1}{2}t^2, \ z = at + \frac{b}{2}t^2 + \kappa(t),$$

for real numbers a, b with $a \neq 0$ and a function $\kappa \in \mathfrak{m}_1^3$. Then

$$y = \int (-\lambda) dz = -\tfrac{a}{6} t^3 - \tfrac{b}{8} t^4 + \varphi(t), \quad x = \int (\lambda^2) dz = \tfrac{a}{20} t^5 + \tfrac{b}{24} t^6 + \psi(t),$$

$$v = \int (-\lambda^3) dz = -\tfrac{a}{56} t^7 - \tfrac{b}{64} t^8 + \rho(t), \quad u = \int (\lambda^3 z + 2\lambda^2 y + \lambda x) dz = \tfrac{a^2}{120} t^8 + \tfrac{7ab}{720} t^9 + \varepsilon(t),$$

$$y^2 - xz = -\tfrac{a^2}{45} t^6 - \tfrac{ab}{40} t^7 + \zeta(t),$$

with $\varphi \in \mathfrak{m}_1^5, \psi \in \mathfrak{m}_1^7, \rho \in \mathfrak{m}_1^9, \varepsilon \in \mathfrak{m}_1^{10}, \zeta \in \mathfrak{m}_1^8$. Thus we have

$$
\begin{aligned}
y + rz &= a(-\tfrac{1}{6} t^3 + rt) + b(-\tfrac{1}{8} t^4 + \tfrac{1}{2} rt^2) + \Phi(r, t), \\
x + ry &= a(\tfrac{1}{20} t^5 - \tfrac{1}{6} rt^3) + b(\tfrac{1}{24} t^6 - \tfrac{1}{8} rt^4) + \Psi(r, t), \\
v + rx &= a(-\tfrac{1}{56} t^7 + \tfrac{1}{20} rt^5) + b(-\tfrac{1}{64} t^8 + \tfrac{1}{24} rt^6) + R(r, t), \\
u + r(y^2 - xz) &= a^2(\tfrac{1}{120} t^8 - \tfrac{1}{45} rt^6) + ab(\tfrac{7}{720} t^9 - \tfrac{1}{40} rt^7) + E(r, t),
\end{aligned}
$$

where $\Phi(r, t) = \varphi(t) + r\kappa(t), \Psi(r, t) = \psi(t) + r\varphi(t), R(r, t) = \rho(t) + r\psi(t), E(r, t) = \varepsilon(t) + r\zeta(t)$.

Now we suppose, as an additional generic condition, that $b \neq 0$. Then, by the linear right-left equivalence,

$$(r, t) \mapsto ((a/b)^2 r, (a/b)t), \quad (\lambda, \mu, \nu, \tau, \sigma) \mapsto ((a/b)^2 \lambda, (a^5/b^4)\mu, (a^4/b^3)\nu, (a^8/b^7)\tau, (a^{10}/b^8)\sigma),$$

we may suppose $a = 1, b = 1$.

Then we put

$$
\begin{aligned}
U(r, t) &= -\tfrac{1}{6} t^3 + rt - \tfrac{1}{8} t^4 + \tfrac{1}{2} rt^2, \\
V(r, t) &= \tfrac{1}{20} t^5 - \tfrac{1}{6} rt^3 + \tfrac{1}{24} t^6 - \tfrac{1}{8} rt^4, \\
W(r, t) &= -\tfrac{1}{56} t^7 + \tfrac{1}{20} rt^5 - \tfrac{1}{64} t^8 + \tfrac{1}{24} rt^6, \\
S(r, t) &= \tfrac{1}{120} t^8 - \tfrac{1}{45} rt^6 + \tfrac{7}{720} t^9 - \tfrac{1}{40} rt^7.
\end{aligned}
$$

Then the tangent map-germ is diffeomorphic to $F_Y : (\mathbf{R}^2, 0) \to (\mathbf{R}^5, 0)$ defined by

$$F_Y(r, t) = (r, \ U(r, t) + \Phi(r, t), \ V(r, t) + \Psi(r, t), \ W(r, t) + R(r, t), \ S(r, t) + E(r, t)).$$

We set $F_Y' : (\mathbf{R}^2, 0) \to (\mathbf{R}^3, 0)$ by $F_Y'(r, t) = (r, \ U(r, t) + \Phi(r, t), \ V(r, t) + \Psi(r, t))$. As is proved in [13] by the infinitesimal method, there exist a diffeomorphism-germ $\Sigma : (\mathbf{R}^2, 0) \to (\mathbf{R}^2, 0)$ of form $\Sigma(r, t) = (\Sigma_1(r), t\Sigma_2(t))$ and a diffeomorphism-germ $T' : (\mathbf{R}^3, 0) \to (\mathbf{R}^3, 0)$ such that

$$T'^{-1} \circ F_Y' \circ \Sigma = (r, \ U(r, t), \ V(r, t)).$$

We define a diffeomorphism-germ $T : (\mathbf{R}^5, 0) \to (\mathbf{R}^5, 0)$ by $T(\lambda, \nu, \mu, \tau, \sigma) = (T'(\lambda, \nu, \mu), \tau, \sigma))$. Then

$$\mathbf{F}_Y = T^{-1} \circ F_Y \circ \Sigma = \left(r, \ U(r,t), \ V(r,t), \ p\widetilde{W}(r,t) + \widetilde{R}(r,t), \ q\widetilde{S}(r,t) + \widetilde{E}(r,t) \right),$$

for some $p, q \in \mathbf{R}, p \neq 0, q \neq 0$, where

$$\widetilde{W}(r,t) := \int t^4 (-\tfrac{1}{2}t^2 + r)dt = -\frac{1}{14}t^7 + \frac{1}{5}rt^5, \quad \widetilde{S}(r,t) := \int t^5(-\tfrac{1}{2}t^2 + r)dt = -\frac{1}{16}t^8 + \frac{1}{6}rt^6,$$

and $\widetilde{R}(r,t), \widetilde{E}(r,t)$ designate remaining higher order functions with respect to the weight $w(r) = 2, w(t) = 1$.

Now we define a map-germ $G : (\mathbf{R}^2, 0) \to (\mathbf{R}^2, 0)$ by $G(r,t) = (r, U(r,t))$. Then we introduce the following key algebraic object:

$$\mathcal{R}_G^{(k)} := \left\{ h \in t^k \mathcal{E}_{r,t} \ \Big| \ \frac{\partial h}{\partial t} \in t^k \frac{\partial U}{\partial t} \mathcal{E}_{r,t} \right\} = \left\{ h \in t^k \mathcal{E}_{r,t} \ \Big| \ \frac{\partial h}{\partial t} \in t^k (-\tfrac{1}{2}t^2 + r)\mathcal{E}_{r,t} \right\},$$

for $k = 1, 2, 3, \ldots$, where $\mathcal{E}_{r,t}$ is the $\mathbf{R}$-algebra of function-germs on (r,t)-plane $(\mathbf{R}^2, 0)$. Then we see that

$$\mathcal{R}_G^{(k)} = \mathcal{R}_G \cap t^{k+1}\mathcal{E}_{r,t}.$$

Note that $\frac{\partial U}{\partial t} = (1+t)(-\tfrac{1}{2}t^2 + r)$. We see that $\mathbf{F}_Y^* \mathcal{E}_{\lambda,\mu,\nu,\tau,\sigma} \subset \mathcal{R}_G$. We have the sequence of $\mathbf{F}_Y^* \mathcal{E}_{\lambda,\mu,\nu,\tau,\sigma}$-modules:

$$\mathcal{E}_{r,t} \supset \mathcal{R}_G \supset \mathcal{R}_G^{(1)} \supset \cdots \supset \mathcal{R}_G^{(k)} \supset \cdots.$$

Moreover we have that

$$-\tfrac{1}{6}t^3 + rt \in \mathcal{R}_G, -\tfrac{1}{8}t^4 + \tfrac{1}{2}rt^2 \in \mathcal{R}_G^{(1)}, \tfrac{1}{20}t^5 - \tfrac{1}{6}rt^3 \in \mathcal{R}_G^{(2)}, \tfrac{1}{24}t^6 - \tfrac{1}{8}rt^4 \in \mathcal{R}_G^{(3)},$$

$$\widetilde{W} \in \mathcal{R}_G^{(4)}, \widetilde{R} \in \mathcal{R}_G^{(5)}, \widetilde{S} \in \mathcal{R}_G^{(5)}, \widetilde{E} \in \mathcal{R}_G^{(6)}.$$

The following is a version of [13], Lemma 6.8 without parameter:

Lemma 7.5 *Let $h_1, \ldots, h_r$ be elements in $\mathcal{R}_G^{(k)}$. Then $h_1, \ldots, h_r$ generate $\mathcal{R}_G^{(k)}$ as $G^* \mathcal{E}_{\lambda,\nu}$-module if and only if their residue classes in $t^{k+3}\mathcal{E}_t / t^{k+7}\mathcal{E}_t$ generate $t^{k+3}\mathcal{E}_t / t^{k+7}\mathcal{E}_t$ via the inclusion $\iota : (\mathbf{R}, 0) \to (\mathbf{R}^2, 0), \iota(t) = (0, t)$.*

By Lemma 7.5, we obtain an explicit system of generators of $\mathcal{R}_G^{(k)}$, for $k = 4$:

Lemma 7.6 *We see that the four elements*

$$\widetilde{W}, \ \widetilde{S}, \ U^3 + \alpha \, r^2 V + \beta \, rUV, \ V^2$$

generate $\mathcal{R}_G^{(4)}$ as $G^* \mathcal{E}_{\lambda,\nu}$-module, for some $\alpha, \beta \in \mathbf{R}$.

Proof: The elements $\widetilde{W}, \widetilde{S}, V^2 \in \mathcal{R}_G^{(4)}$ and $U^3, r^2 V, rUV \in \mathcal{R}_G$. We have

$$U^3 + \alpha \, r^2 V + \beta \, rUV \equiv (1 - \tfrac{\alpha}{6})r^3 t^3 + (3 - \tfrac{\alpha}{8} - \tfrac{\beta}{6})r^3 t^4, \quad (\text{mod. } t^5 \mathcal{E}_{r,t}),$$

Therefore if we set $\alpha = 6, \beta = \frac{27}{2}$, we have that $U^3 + \alpha \, r^2 V + \beta \, rUV$ belongs to $t^5 \mathcal{E}_{r,t}$, hence to $\mathcal{R}_G^{(4)}$. Note that U^3 itself does not belong to $\mathcal{R}_G^{(4)}$. Since

$$\mathrm{ord}(\widetilde{W}(0,t)) = 7, \ \mathrm{ord}(\widetilde{S}(0,t)) = 8, \ \mathrm{ord}((U^3 + \alpha \, r^2 V + \beta \, rUV)(0,t)) = 9, \ \mathrm{ord}(V^2(0,t)) = 10,$$

$\widetilde{W}(0,t), \widetilde{S}(0,t), ((U^3 + \alpha \, r^2 V + \beta \, rUV)(0,t), V^2(0,t)$ generate $t^7 \mathcal{E}_t / t^{11}\mathcal{E}_t$ over $\mathbf{R}$. Therefore by Lemma 7.5, we have the required result. $\qquad \square$

Proof of Proposition 7.4: By Lemma 7.6, we have

$$\widetilde{R} \;=\; (A_4 \circ G)\widetilde{W} + (B_4 \circ G)\widetilde{S} + (C_4 \circ G)(U^3 + \alpha r^2 V + \beta rUV) + (D_4 \circ G)V^2,$$

$$\widetilde{E} \;=\; (A_5 \circ G)\widetilde{W} + (B_5 \circ G)\widetilde{S} + (C_5 \circ G)(U^3 + \alpha r^2 V + \beta rUV) + (D_5 \circ G)V^2,$$

for some $A_4, B_4, C_4, D_4, A_5, B_5, C_5, D_5 \in \mathcal{E}_{\lambda,\nu}$. By setting $r = 0$ and by comparing of orders on r, we see $A_4(0,0) = 0, A_5(0,0) = 0, B_5(0,0) = 0$.

Then define $\Xi : (\mathbf{R}^5, 0) \to (\mathbf{R}^5, 0)$ by

$$\Xi(\lambda,\nu,\mu,\tau,\sigma) := \big(\lambda,\nu,\mu, p\tau + A_4(\lambda,\nu)\tau + B_4(\lambda,\nu)\sigma + C_4(\lambda,\nu)(\nu^3 + \alpha\lambda^2\mu + \beta\lambda\nu\mu) + D_4(\lambda,\nu)\mu^2,$$

$$q\sigma + A_5(\lambda,\nu)\tau + B_5(\lambda,\nu)\sigma + C_5(\lambda,\nu)(\nu^3 + \alpha\lambda^2\mu + \beta\lambda\nu\mu) + D_5(\lambda,\nu)\mu^2\big).$$

We see that Ξ is a diffeomorphism-germ and that

$$\Xi^{-1} \circ \mathbf{F}_Y(r,t) = (r,\ U(r,t),\ V(r,t),\ \widetilde{W}(r,t),\ \widetilde{S}(r,t)).$$

Therefore $\Xi^{-1} \circ T^{-1} \circ F_Y \circ \Sigma = (r, U, V, \widetilde{W}, \widetilde{S})$. Thus we see that the tangent map-germ of $\Pi_X \circ f$ has the unique diffeomorphism type, under the generic condition $b \neq 0$. The first normal form of Theorem 1.3 is obtained by setting $\lambda = \frac{1}{2}t^2$, $z = t + \frac{1}{2}t^2$, namely by setting $a = b = 1, \kappa(t) \equiv 0$, calculating the exact components of $\mathrm{Tan}(\pi_Y \circ f)$, without omitting higher order terms, and by taking a diagonal linear transformation on $\mathbf{R}^5$ to make all coefficients integers. The second normal form is obtained of Theorem 1.3 just from the above normal form $(r,\ U,\ V,\ \widetilde{W},\ \widetilde{S})$ by taking a diagonal linear transformation on $\mathbf{R}^5$. □

8 Appendix: Simple Lie algebras of rank 2

Recall the basic theory of Lie algebras briefly. Let $\mathfrak{g}$ be a semi-simple Lie algebra over $\mathbf{C}$. A Cartan sub-algebra $\mathfrak{h}$ of $\mathfrak{g}$ is a commutative sub-algebra such that the normalizer of $\mathfrak{h}$ coincides with $\mathfrak{h}$ itself. It is known that a Cartan sub-algebra $\mathfrak{h}$ is unique up to inner automorphisms of $\mathfrak{g}$. The rank of $\mathfrak{g}$ is defined as $\dim_{\mathbf{C}} \mathfrak{h}$. Let ℓ be the rank of $\mathfrak{g}$. Fix a Cartan sub-algebra $\mathfrak{h} \subset \mathfrak{g}$. For $\alpha \in \mathfrak{h}^*$, we set

$$\mathfrak{g}^\alpha := \{x \in \mathfrak{g} \mid [h,x] = \alpha(h)x, (h \in \mathfrak{h})\}.$$

α is called a root of $\mathfrak{g}$ if $\alpha \neq 0$ and $\mathfrak{g}^\alpha \neq \{0\}$. Then it is known that $\dim_{\mathbf{C}} \mathfrak{g}^\alpha = 1$. Moreover we have $[\mathfrak{g}^\alpha, \mathfrak{g}^\beta] \subset \mathfrak{g}^{\alpha+\beta}$. The set of roots is called the root system of $\mathfrak{g}$ and is denoted by $R = R(\mathfrak{g})$. We have the root decomposition

$$\mathfrak{g} = \mathfrak{g}^0 \oplus \Big(\bigoplus_{\alpha \in R} \mathfrak{g}^\alpha\Big).$$

Setting $\mathfrak{h}_{\mathbf{R}} = \{h \in \mathfrak{h} \mid \alpha(h) \in \mathbf{R}, (\alpha \in R)\}$, we regard R as a subset of $\mathfrak{h}_{\mathbf{R}}^*$. The Killing form on $\mathfrak{g}$ induces a metric $(\ ,\ )$ on $\mathfrak{h}_{\mathbf{R}}$ and therefore we can regard $R \subset \mathfrak{h}_{\mathbf{R}} \cong \mathbf{R}^\ell$. Each root $\alpha \in R$ defines the reflection $s_\alpha : \mathfrak{h}_{\mathbf{R}} \to \mathfrak{h}_{\mathbf{R}}$ by $s_\alpha(h) := h - \dfrac{2(h,\alpha)}{(\alpha,\alpha)}\alpha, (h \in \mathfrak{h}_{\mathbf{R}})$. The Weyl group is generated by $\{s_\alpha \mid \alpha \in R\}$.

We can choose a basis $(\alpha_1, \ldots, \alpha_\ell)$ of $\mathfrak{h}_{\mathbf{R}}$ from R such that any $\alpha \in R$ is represented as $\alpha = \sum_{i=1}^{\ell} m_i \alpha_i$ with $m_i \in \mathbf{Z}$ and all $m_i \geq 0$ or all $m_i \leq 0$, $(1 \leq i \leq \ell)$. We call $\Pi = \{\alpha_1, \ldots, \alpha_\ell\} \subset R$ a fundamental system (or a simple system) of R. The fundamental system of the root system is unique up to the action of Weyl group on $\mathfrak{h}_{\mathbf{R}}$.

We fix a fundamental system $\Pi = \{\alpha_1, \ldots, \alpha_\ell\} \subset R$. Let $S \subset \Pi$ be any subset. Then the root decomposition of $\mathfrak{g}$ induces a grading $\mathfrak{g} = \bigoplus_{k \in \mathbf{Z}} \mathfrak{g}_k$ of $\mathfrak{g}$, setting $\mathfrak{g}_k = \bigoplus_\alpha \mathfrak{g}^\alpha$, for any non-zero integer k, and $\mathfrak{g}_0 = \mathfrak{g}^0 \oplus \Big(\bigoplus_\alpha \mathfrak{g}^\alpha\Big)$, for $k = 0$. Here the direct sum is taken over all $\alpha \in R$ such that k is equal to the sum of coefficients m_i with $\alpha_i \in S$ of the unique expression $\alpha = \sum_{i=1}^{\ell} m_i \alpha_i$.
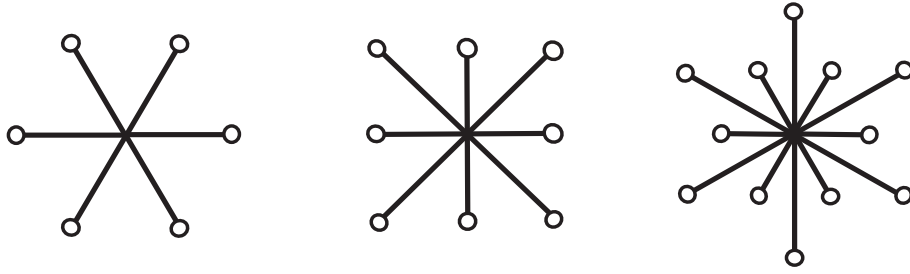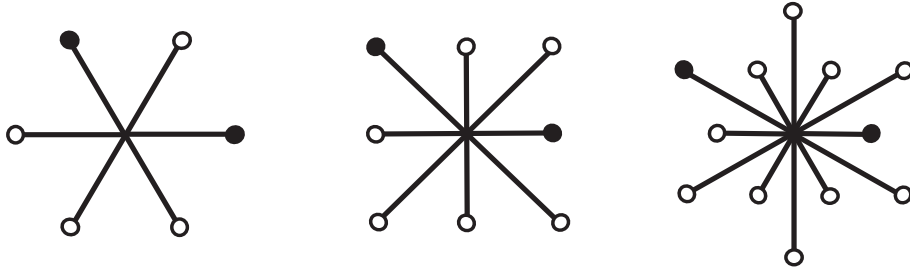
Figure 5: Root systems of types A_2, C_2 and G_2

Figure 6: Fundamental systems for root systems A_2, C_2 and G_2

Let $\mathfrak{g}$ be a Lie algebra of type A_2, C_2 or G_2. Then $\ell = 2$. Then we have three non-trivial gradings of $\mathfrak{g}$ in each case.

In the case A_2, for $S = \Pi$, we have the decomposition

$$\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2,$$

such that the dimensions of components $\mathfrak{g}_k$ are $1, 2, 2, 2, 1$ respectively. The negative part $\mathfrak{g}_- = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$ with the 2-dimensional, bracket generating, subspace $\mathfrak{g}_{-1}$ generates the 3-dimensional homogeneous contact structure. One of its global model is given by the incidence manifold

$$Z = \{([x], [y]) \in \mathbf{P}^2 \times \mathbf{P}^{2*} \mid x \cdot y = 0\} \subset \mathbf{P}^2 \times \mathbf{P}^{2*}$$

of the projective duality between $Y = \mathbf{P}^2$ and $X = \mathbf{P}^{2*}$. The 3-dimensional manifold Z is identified with $PT^*(\mathbf{P}^2)$, as well as with $PT^*(\mathbf{P}^{2*})$, endowed with the canonical contact structure. We can regard Z as a flag manifold, for a 3-dimensional vector space W,

$$Z = \{V_1 \subset V_2 \subset W \mid \dim V_1 = 1, \dim V_2 = 2\}.$$

If S consists of one root from Π, then we have two different gradings of the same type:

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$$

with dimensions of components, $2, 4, 2$. The negative parts of the three gradings of $\mathfrak{g}$ are realized as tangent spaces to the $\mathrm{PGL}(3, \mathbf{C})$-homogeneous double fibration

$$Y = P(W) \xleftarrow{\Pi_Y} Z \xrightarrow{\Pi_X} X = P(W^*),$$

where Π_Y and Π_X are canonical projections. Note that there exists the symmetry exchanging Y and X, realizing the symmetry of A_2 Dynkin diagram.

In the case C_2, we have the grading, by $S = \Pi = \{\alpha_1, \alpha_2\}$,

$$\mathfrak{g} = \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_3,$$

of $\mathfrak{g}$ with dimensions of components $1, 1, 2, 2, 2, 1, 1$. Suppose the square lengths of α_1 and α_2 satisfy $(\alpha_2, \alpha_2) = 2(\alpha_1, \alpha_1)$. The negative part of $\mathfrak{g}$ provides an Engel structure. For $S = \{\alpha_1\}$, then we have the grading

$$\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2,$$

with dimensions of components $1, 2, 4, 2, 1$. The negative part generates $Sp(2, \mathbf{R})$-homogeneous space with an contact structure. For $S = \{\alpha_2\}$, then we have the grading

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1,$$

with dimensions of components $3, 4, 3$. Though the negative part is not endowed with any distribution canonically, however, from the double fibration, the associated homogeneous space enjoys a geometric structure. We provide the double fibration

$$Y = P(V) \xleftarrow{\Pi_Y} Z \xrightarrow{\Pi_X} X = \mathrm{LG}(V),$$

constructed from a symplectic vector space V of dimension 4, for the case C_2 in [13]. In fact, Z is a flag manifold

$$Z = \{V_1 \subset V_2 \subset V \mid \dim V_1 = 1, \dim V_2 = 2, V_2 : \text{ Lagrangian}\}.$$

We have a projective contact structure on Y, while a Lagrange-Grassmann structure (conformal structure) on X, which is given by a quadratic cone field.

In the case G_2, we have three kinds of gradings, the negative nilpotent part $\mathfrak{g}_- = \oplus_{k<0}\mathfrak{g}_k$ of which corresponds to the tangent space to Z, Y and X and of 5-steps, 3-steps, and 2-steps, respectively. See [18][19][20]. For the case $S = R$ corresponding to Z, then we have

$$\begin{aligned}
\mathfrak{g}_- &= \mathfrak{g}_{-5} \oplus \mathfrak{g}_{-4} \oplus \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \\
&= \langle e_6 \rangle \oplus \langle e_5 \rangle \oplus \langle e_4 \rangle \oplus \langle e_3 \rangle \oplus \langle e_1, e_2 \rangle, \\
&[e_1, e_2] = e_3, \ [e_1, e_3] = e_4, \ [e_1, e_4] = e_5; \quad [e_2, e_5] = e_6, \ [e_3, e_4] = e_6.
\end{aligned}$$

By the projection $\Pi_Y : Z \to Y$, the vector e_2 is eliminated and we have the graded Lie algebra of step 3:

$$\begin{aligned}
\mathfrak{g}_- &= \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \\
&= \langle e_5, e_6 \rangle \oplus \langle e_4 \rangle \oplus \langle e_1, e_3 \rangle \\
&[e_1, e_3] = e_4; \quad [e_1, e_4] = e_5, \ [e_3, e_4] = e_6.
\end{aligned}$$

By the projection $\Pi_X : Z \to X$, the vector e_1 is eliminated and we have the graded Lie algebra of step 2:

$$\begin{aligned}
\mathfrak{g}_- &= \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \\
&= \langle e_6 \rangle \oplus \langle e_2, e_3, e_4, e_5 \rangle, \\
&[e_2, e_5] = e_6, \ [e_3, e_4] = e_6.
\end{aligned}$$

For the G_2 case, see also [8].

In particular, we observe that there exist graded Lie algebra epimorphisms

$$\mathfrak{g}_-(G_2) \longrightarrow \mathfrak{g}_-(C_2) \longrightarrow \mathfrak{g}_-(A_2),$$

which gives the Lie theoretic explanation of the local hierarchy of double fibrations.

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