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Almost global solutions of semilinear wave equations with the critical exponent in high dimensions *

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1 General theory for nonlinear wave equations

First we consider the initial value problem for fully nonlinear wave equations,

$$\begin{cases} u_{tt} - \Delta u = H(u, Du, D_x Du) \quad \text{in} \quad \mathbf{R}^n \times [0, \infty), \\ u(x,0) = \varepsilon f(x), \ u_t(x,0) = \varepsilon g(x), \end{cases} \tag{1}$$

where $u = u(x,t)$ is a scalar unknown function of space-time variables,

$$\begin{array}{l} Du = (u_{x_0}, u_{x_1}, \cdots, u_{x_n}), \ x_0 = t, \\ D_x Du = (u_{x_i x_j}, \ i, j = 0, 1, \cdots, n, \ i + j \geq 1), \end{array}$$

$f, g \in C_0^\infty(\mathbf{R}^n)$ and $\varepsilon > 0$ is a "small" parameter. Set

$$\widehat{\lambda} = (\lambda; \ (\lambda_i), i = 0, 1, \cdots, n; \ (\lambda_{ij}), i, j = 0, 1, \cdots, n, \ i + j \geq 1).$$

Suppose that the nonlinear term $H = H(\widehat{\lambda})$ is a sufficiently smooth function with

$$H(\widehat{\lambda}) = O\left(|\widehat{\lambda}|^{1+\alpha}\right) \quad \text{in a neighbourhood of } \widehat{\lambda} = 0,$$

where $\alpha \geq 1$ is an integer. Let us define the lifespan $T(\varepsilon)$ by

$$T(\varepsilon) = \sup\{t > 0 \ : \ \exists \text{classical solution } u \text{ of (1) for arbitrarily fixed } (f, g).\}.$$

In this talk we assume $n \geq 2$ for the simplicity.

According to Chapter 2 of Li and Chen [3], we have long histories on the estimate for $T(\varepsilon)$. All the lower bounds of $T(\varepsilon)$ are summarized in the following table.

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$T(\varepsilon) \geq$	$\alpha = 1$	$\alpha = 2$	$\alpha \geq 3$
$n = 2$	$ca(\varepsilon)$ in general case, $c\varepsilon^{-1}$ if $\int_{\mathbf{R}^2} g(x)dx = 0$, $c\varepsilon^{-2}$ if $\partial_u^2 H(0) = 0$	$c\varepsilon^{-6}$ in general case, $\exp(c\varepsilon^{-2})$ if $\partial_u^b H(0) = 0$ $(b = 3, 4)$	∞
$n = 3$	$c\varepsilon^{-2}$ in general case, $\exp(c\varepsilon^{-1})$ if $\partial_u^2 H(0) = 0$	∞	∞
$n = 4$	$\exp(c\varepsilon^{-2})$ in general case, ∞ if $\partial_u^2 H(0) = 0$	∞	∞
$n \geq 5$	∞	∞	∞

Here c stands for a positive constant independent of ε and $a = a(\varepsilon)$ is a number satisfying $a^2\varepsilon^2 \log(a+1) = 1$. We note that the result in the case where $n = 4$ and $\alpha = 1$ is $\exp(c\varepsilon^{-1})$ for general case in [3] in which we can find all the references on the whole history. But later, Li and Zhou [4] improved this part. It is also remarkable that [3] states "all these lower bounds are known to be sharp except for $(n, \alpha) = (4, 1)$" by studying model equations $H = |u|^p$, or $|u_t|^p$. See also another table in Li [2] in which the author notes that the case where $(n, \alpha) = (2, 2)$ has an open sharpness. For this case, it seems that $b = 4$ is a technical condition which may be removed. Recently, Zhou and Han [8] have obtained the sharpness for $b = 3$ in $(n, \alpha) = (2, 2)$ by studying $H = u_t^3$.

2 The final problem and related result

In the sense of the first section, the final open problem on the optimality of the general theory for fully nonlinear wave equations can be established by model problem;

$$\left\{ \begin{array}{l} u_{tt} - \Delta u = u^2 \quad \text{in} \quad \mathbf{R}^4 \times [0, \infty), \\ u(x, 0) = \varepsilon f(x), \ u_t(x, 0) = \varepsilon g(x). \end{array} \right. \tag{2}$$

We note that this is an extended problem of John [5] to high dimensional case which has the "critical" exponent of Strauss' conjecture [6]. The lifespan $T(\varepsilon)$ of the solution of (2) should have an estimate of the form;

$$\exp(c\varepsilon^{-2}) \leq T(\varepsilon) \leq \exp(C\varepsilon^{-2}). \tag{3}$$

This final problem on the upper bound was solved by our previous work, Takamura and Wakasa [7]. In its proof, the analysis on $\|u(\cdot, t)\|^2_{L^2(\mathbf{R}^4)}$ is a key because we cannot use any pointwise esimate of the solution due to so-called derivative loss in fundamental solutions in high dimensions.

Therefore one may have questions;

- Do we have any possibility to get $T(\varepsilon) = \infty$ if the nonlinear term is not single while it includes u^2?
- Do we have any possibility to get a pointwise positivity of the solution for some special nonlinear term?

For these questions, we get the following partial answer.

Theorem *Even if the right-hand side of the equation in* (2) *additionally has an integral term of the form;*

$$-\frac{1}{\pi^2}\int_0^t d\tau \int_{|\xi|\le 1} \frac{(u_t u)(x+(t-\tau)\xi,\tau)}{\sqrt{1-|\xi|^2}}d\xi, \tag{4}$$

there is no change on the estimate the lifespan (3).

In the proof of this theorem, the key is the analysis on $u(x,t)$ itself. This observation already appeared in Agemi, Kubota and Takamura [1] in which a global solution is obtained for the "super-critical" case.

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ASYMPTOTIC EXPANSION OF SOLUTIONS TO THE DISSIPATIVE EQUATION WITH ANOMALOUS DIFFUSION

Masakazu Yamamoto[1]

1. Introduction

The following Cauchy problem for the linear dissipative equation is studied by many authors:

$$\begin{cases} \partial_t u - \Delta u + a(t,x)u = 0, & t>0,\ x\in\mathbb{R}^n, \\ u(0,x) = u_0(x), & x\in\mathbb{R}^n, \end{cases} \tag{1.1}$$

where $n\in\mathbb{N}$ and the coefficient $a:(0,\infty)\times\mathbb{R}^n\to\mathbb{R}$ and the initial data $u_0:\mathbb{R}^n\to\mathbb{R}$ are given functions. Upon the suitable condition for $a(t,x)$ and $u_0(x)$, the well-posedness, the global in time existence and the decay of solutions to (1.1) are shown. Moreover the asymptotic expansion of the solution to (1.1) as $t\to\infty$ is derived (cf [6]). Here we consider those problems when the dissipative effect on the equation is provided by "the anomalous diffusion". In this manuscript, we define the Fourier transform and the Fourier inverse transform by

$$\mathcal{F}[\varphi](\xi) := (2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{-ix\cdot\xi}\varphi(x)dx \quad\text{and}\quad \mathcal{F}^{-1}[\varphi](x) := (2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{ix\cdot\xi}\varphi(\xi)d\xi.$$

Then, for $\theta>0$, the fractional Laplacian is given by

$$(-\Delta)^{\theta/2}\varphi(x) = \mathcal{F}^{-1}\left[|\xi|^\theta\mathcal{F}[\varphi]\right](x).$$

The fractional Laplacian with $\theta=2$ is the positive Laplacian. On the other hand, when $1<\theta<2$, this operator provides the anomalous diffusion on dissipative equations (see [2, 8]). Namely, for the fundamental solution of $\partial_t u + (-\Delta)^{\theta/2}u = 0$, we see the following property.

Lemma 1.1 ([1])**.** *Let* $n\in\mathbb{Z}$, $\theta>0$, $C_\theta := \theta 2^{\theta-1}\pi^{-\frac{n}{2}-1}\sin\frac{\theta\pi}{2}\Gamma\left(\frac{n+\theta}{2}\right)\Gamma\left(\frac{\theta}{2}\right)$ *and*

$$G_\theta(t,x) := \mathcal{F}^{-1}[e^{-t|\xi|^\theta}](x). \tag{1.2}$$

Then the following property holds:

$$|x|^{n+\theta}G_\theta(t,x)\to C_\theta t \quad \text{as} \quad |x|\to\infty$$

for any $t>0$.

Here we remark that $G_\theta(t,x)$ is the fundamental solution of $\partial_t u + (-\Delta)^{\theta/2}u = 0$. When $\theta=2$, the fundamental solution of $\partial_t u - \Delta u = 0$ is given by the Gaussian $G(t,x) = (4\pi t)^{-n/2}e^{-|x|^2/(4t)}$. This satisfies

$$|x|^M G(t,x)\to 0 \quad \text{as} \quad |x|\to\infty$$

for any $t>0$ and $M>0$. This property and Lemma 1.1 are not contradictory. Indeed, when $\theta=2$, we see that $C_\theta=0$ in Lemma 1.1. When $1<\theta<2$, Lemma 1.1 immediately gives that

$$\|x^\alpha G_\theta(t)\|_1 = +\infty$$

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Uniqueness and non-degeneracy of ground states of quasilinear Schrödinger equations

Tatsuya Watanabe (Kyoto Sangyo University) [1]

1 Introduction

We consider the following quasilinear elliptic problem:

$$-\Delta u + \lambda u - \kappa \Delta(|u|^{\alpha})|u|^{\alpha-2}u = |u|^{p-1}u \text{ in } \mathbb{R}^N, \tag{1}$$

where $\lambda > 0$, $\kappa > 0$, $\alpha > 1$, $p > 1$ and $N \geq 1$. Equation (1) can be obtained as a stationary problem of the following modified Schrödinger equation:

$$i\frac{\partial z}{\partial t} = -\Delta z - \kappa \Delta(|z|^{\alpha})|z|^{\alpha-2}z - |z|^{p-1}z, \ (t,x) \in (0,\infty) \times \mathbb{R}^N. \tag{2}$$

Equation (2) appears in the study of plasma physics. See [7], [10] for the derivation and the background. Especially if we consider the standing wave of (2) of the form $z(t,x) = u(x)e^{i\lambda t}$, then $u(x)$ satisfies (1).

Equation (1) has a variational structure, that is, one can obtain solutions of (1) as critical points of the associated functional I defined by

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2(1 + \alpha\kappa|u|^{2\alpha-2}) + \lambda u^2 \, dx - \frac{1}{p+1}\int_{\mathbb{R}^N} |u|^{p+1} \, dx.$$

We remark that nonlinear functional $\int_{\mathbb{R}^N} |\nabla u|^2|u|^{2\alpha-2} \, dx$ is not defined on all $H^1(\mathbb{R}^N)$ except for $N = 1$. Thus the natural function space for $N \geq 2$ is given by

$$X := \{u \in H^1(\mathbb{R}^N); \int_{\mathbb{R}^N} |\nabla u|^2|u|^{2\alpha-2} \, dx < \infty\}.$$

Existence of a solution of (1) has been studied in [1], [8], [11], [12], [14]. We are interested in the *ground state* of (1). We define the ground state energy level and the set of ground states by

$$m := \inf\{I(u); \, I'(u) = 0, \ u \in X \setminus \{0\}\},$$

$$\mathcal{G} := \{u \in X \setminus \{0\}; \, I(u) = m, \ I'(u) = 0\}.$$

As to the existence of a ground state, we have the following result.

[1] This talk is based on joint works [2], [3], [4] with Shinji Adachi (Shizuoka University) and Masataka Shibata (Tokyo Institute of Technology).

Theorem 1.1. *([2], [9]) Let $\lambda > 0$, $\kappa > 0$, $\alpha > 1$ and $1 < p < \frac{(2\alpha-1)N+2}{N-2}$ for $N \geq 3$, $1 < p < \infty$ for $N = 1, 2$. Then $\mathcal{G} \neq \emptyset$. Moreover any ground state $w \in \mathcal{G}$ is of the class $C^2(\mathbb{R}^N)$, positive, radially symmetric and decreasing with respect to $r = |x|$ (up to translation).*

We note that the ground state of (1) exists even if p is H^1-supercritical because $\frac{(2\alpha-1)N+2}{N-2} > \frac{N+2}{N-2}$. We can also see that $p = \frac{(2\alpha-1)N+2}{N-2}$ is the critical exponent for (1) by the Pohozaev type identity.

Remark 1.2. *As to the existence of a ground state, we have more general result. More precisely, we consider the following equation:*

$$-\Delta u - \kappa\Delta(|u|^\alpha)|u|^{\alpha-2}u = g(u) \text{ in } \mathbb{R}^N. \tag{3}$$

We impose the following conditions on the nonlinear term g:
(g1) $g(s)$ is real-valued and locally Hölder continuous on $[0, \infty)$.
(g2) $-\infty < \liminf_{s\to 0} \frac{g(s)}{s} \leq \limsup_{s\to 0} \frac{g(s)}{s} = -\lambda < 0$ for some $\lambda > 0$.
(g3) $\lim_{s\to\infty} \frac{|g(s)|}{s^{\frac{(2\alpha-1)N+2}{N-2}}} = 0$.
(g4) There exists $s_0 > 0$ such that $G(s_0) > 0$, where $G(s) = \int_0^s g(t)\,dt$.

Under (g1)-(g4), we can prove the existence of a ground state of (3).

On the other hand, the uniqueness and the non-degeneracy of the ground state are less investigated. When $N = 1$, Ambrosetti and Wang [5] showed that there exists $\kappa^* > 0$ such that the non-degeneracy holds for any $\kappa > -\kappa^*$, $\lambda > 0$ and $p > 1$. In [9], the authors studied the case $N = 1, \kappa = 1$ and proved that the uniqueness holds for any $\lambda > 0$ and $p > 1$. Their argument is based on the ODE analysis. The aim of this talk is to give the uniqueness and non-degeneracy in the higher dimensional case. We believe it is important for applications, for example, the stability of the standing wave.

2 Main results

Theorem 2.1. *(Uniqueness for large κ)*

Suppose $N \geq 3$, $\alpha > 1$ and $1 < p < \frac{(2\alpha-1)N+2}{N-2}$ if $1 < \alpha \leq 2$, $\alpha - 1 \leq p < \frac{(2\alpha-1)N+2}{N-2}$ if $\alpha > 2$. There exists $c_0 = c_0(p, \alpha) > 0$ such that if $\kappa\lambda^{\frac{2\alpha-2}{p-1}} \geq c_0$, then (1) has at most one positive radial solution w and hence the ground state of (1) is unique up to translation. In other words, it follows that

$$\mathcal{G} = \{w(\cdot - y); y \in \mathbb{R}^N\}.$$

Remark 2.2. *For a solution u of (1), we rescale $\tilde{u}(x)$ as $u(x) = \lambda^{\frac{1}{p-1}}\tilde{u}(\lambda^{\frac{1}{2}}x)$. Then we can see that (1) is reduced to*

$$-\Delta\tilde{u} + \tilde{u} - \kappa\lambda^{\frac{2\alpha-2}{p-1}}\Delta(|\tilde{u}|^{\alpha})|\tilde{u}|^{\alpha-2}\tilde{u} = |\tilde{u}|^{p-1}\tilde{u} \quad \text{in } \mathbb{R}^N.$$

Thus it seem to be natural to describe the condition for the uniqueness in terms of $\kappa\lambda^{\frac{2\alpha-2}{p-1}}$.

Theorem 2.3. *Suppose $N = 2$, $\alpha > 1$ and $2\alpha - 1 \le p < \infty$. There exists $c_1 = c_1(p, \alpha) > 0$ such that if $\kappa\lambda^{\frac{2\alpha-2}{p-1}} \ge c_1$, then the ground state of (1) is unique up to translation.*

Theorem 2.4. *(Non-degeneracy for large κ)*
Suppose $N \ge 3$, $\alpha > 1$ and $2\alpha - 1 \le p < \frac{(2\alpha-1)N+2}{N-2}$. Assume further $\kappa\lambda^{\frac{2\alpha-2}{p-1}} \ge c_0$ where c_0 is given in Theorem 2.1. Then w is non-degenerate in $H^1_{rad}(\mathbb{R}^N)$, that is, if $L_w(\phi) = 0$ in $\mathbb{R}^N$ and $\phi \in H^1_{rad}(\mathbb{R}^N)$, then $\phi \equiv 0$.
Here L_w is the linearized operator of (1) defined by

$$\begin{aligned} L_w(\phi) = &-\Delta\phi + \lambda\phi - pw^{p-1}\phi - \kappa \operatorname{div}(\alpha w_\kappa^{2\alpha-2}\nabla\phi) \\ &- \kappa(2\alpha(\alpha-1)w_\kappa^{2\alpha-3}\Delta w_\kappa + \alpha(\alpha-1)(2\alpha-3)w_\kappa^{2\alpha-4}|\nabla w_\kappa|^2)\phi. \end{aligned}$$

Theorem 2.5. *(Uniqueness and non-degeneracy for small κ)*
Suppose $N \ge 2$, $\alpha > 1$ and $1 < p < \frac{N+2}{N-2}$ if $N \ge 3$, $1 < p < \infty$ if $N = 2$. There exists $c_2(p, \alpha) > 0$ such that if $0 < \kappa\lambda^{\frac{2\alpha-2}{p-1}} \le c_2$, then the ground state of (1) is unique up to translation and non-degenerate in $H^1_{rad}(\mathbb{R}^N)$.

Here we briefly explain the ideas of the proof. Firstly we adapt the *dual variational formulation.* Let f be a unique solution of the following ODE:

$$f'(s) = \frac{1}{\sqrt{1 + \alpha\kappa f(s)^{2\alpha-2}}} \quad \text{on } s \in [0, \infty), \quad f(0) = 0.$$

Using the function f, we consider the following semilinear problem:

$$-\Delta v + \lambda f(v)f'(v) = |f(v)|^{p-1}f(v)f'(v) \quad \text{in } \mathbb{R}^N. \tag{4}$$

The functional associated to (4) is defined by

$$J(v) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2 + \lambda f(v)^2\,dx - \frac{1}{p+1}\int_{\mathbb{R}^N} |f(v)|^{p+1}\,dx.$$

Lemma 2.6. *It follows*
(i) $X = f(H^1(\mathbb{R}^N))$, that is, $X = \{f(v); v \in H^1(\mathbb{R}^N)\}$.
(ii) For any $v \in H^1(\mathbb{R}^N)$, we put $u = f(v)$. Then it follows

$$I(u) = J(v), I'(u)u = J'(v)\frac{f(v)}{f'(v)}.$$

By Lemma 2.6, we can see that the set of ground states $\mathcal{G}$ has one-to-one correspondence to that of the semilinear problem (4). This enables us to apply the uniqueness and non-degeneracy result [6], [13], [15] for semilinear elliptic equations. We require that $\kappa\lambda^{\frac{2\alpha-2}{p-1}}$ is large in order to guarantee some monotonicity condition on the nonlinear term.

On the other hand if we fix λ and put $\kappa = 0$, (1) becomes

$$-\Delta u + \lambda u = |u|^{p-1}u \text{ in } \mathbb{R}^N. \tag{5}$$

Then for $1 < p < \frac{N+2}{N-2}$, it is well-known that the ground state is unique up to translation. Moreover the corresponding linearized operator $L_0 = -\Delta + \lambda - pu^{p-1}$ satisfies Ker $L_0 = \text{span}\left\{\frac{\partial u}{\partial x_i}\right\}$. The uniqueness and the non-degeneracy for small κ follows by applying the implicit function theorem if we could treat the linearized operator L_w as a perturbation of L_0. To this aim, we have to show L^∞-norm of the ground state is uniformly bounded with respect to κ. The proof of uniform boundedness is based on the Moser type iteration. We also need to show the following uniform estimate whose proof is given by the ODE analysis.

Lemma 2.7. *Suppose $N \geq 2$, $\alpha > 1$ and $1 < p < \frac{N+2}{N-2}$ if $N \geq 3$, $1 < p < \infty$ if $N = 2$. Let $\lambda > 0$ be given. There exist $\kappa_0 > 0$ and $C > 0$ independent of $\kappa \in (0, \kappa_0)$ such that*

$$\|\nabla(\log w)\|_{L^\infty(\mathbb{R}^N)} = \left\|\frac{\nabla w}{w}\right\|_{L^\infty(\mathbb{R}^N)} \leq C \text{ for all } \kappa \in (0, \kappa_0).$$

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A spectral theory of linear operators on Gelfand triplets and its applications to infinite dimensional dynamical systems

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Abstract

The dynamics of systems of large populations of coupled oscillators have been of great interest because collective synchronization phenomena are observed in a variety of areas. The Kuramoto model is often used to investigate such phenomena, which is a system of differential equations of the form

$$\frac{d\theta_k}{dt} = \omega_k + \frac{K}{N}\sum_{j=1}^{N} f(\theta_j - \theta_k),\ \ k = 1, \cdots, N. \tag{1}$$

In this talk, an infinite dimensional Kuramoto model is considered, and the Kuramoto's conjecture on a bifurcation diagram of the system, which is open since 1985, will be proved.

It is well known that the spectrum (eigenvalues) of a linear operator determines a local dynamics of a system of differential equations. Unfortunately, the infinite dimensional Kuramoto model has the continuous spectrum on the imaginary axis, so that the usual spectral theory does not say anything about the dynamics. To handle such continuous spectra, a new spectral theory of linear operators based on Gelfand triplets is developed. Basic notions in the usual spectral theory, such as eigenspaces, algebraic multiplicities, point/continuous/residual spectra, Riesz projections are extended to those defined on a Gelfand triplet. They prove to have the same properties as those of the usual spectral theory.

The results are applied to the Kuramoto model to prove the Kuramoto's conjecture. A center manifold theorem will be given with the aid of the Gelfand triplet and the generalized spectrum. Even if there exists the continuous spectrum on the imaginary axis, it is proved that there exists a finite dimensional center manifold on a certain space of distributions. This determines a bifurcation diagram of the Kuramoto model.

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POSITIVE p-HARMONIC FUNCTIONS WITH ZERO BOUNDARY DATA ON CONE DOMAINS

TSUBASA ITOH

1. Introduction

Let $1 < p < \infty$ and let D be a domain in $\mathbb{C}$. The Euler-Lagrange equation for the problem of minimizing the p-Dirichlet integral $\int_D |\nabla u|^p dx$ over a suitable function class is written in weak form as

$$\int_D |\nabla u|^{p-2} \nabla u \cdot \nabla \eta = 0, \tag{1.1}$$

which must hold at least for all $\eta \in C_0^\infty(D)$. If $u \in C^2(D)$, this implies that

$$\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u) = 0 \tag{1.2}$$

in D. This equation is equivalent to

$$(p-2) \sum_{i,j=1}^{2} u_{x_i} u_{x_j} u_{x_i x_j} + |\nabla u|^2 \Delta u = 0. \tag{1.3}$$

Either of the three equations is called the p-harmonic equation and the solutions are called p-harmonic functions.

Let $0 < \phi < \pi$. We denote a cone of aperture ϕ by

$$D_\phi = \{z \in \mathbb{C} : |\arg z| < \phi\}.$$

In this paper we find positive p-harmonic functions $u(z)$ on D_ϕ with the boundary condition,

$$u(z) = \begin{cases} 0 & \text{for } |\arg z| = \phi \text{ and } z = 0, \\ \infty & \text{for } z = \infty, \end{cases} \tag{1.4}$$

or

$$u(z) = \begin{cases} 0 & \text{for } |\arg z| = \phi \text{ and } z = \infty, \\ \infty & \text{for } z = 0. \end{cases} \tag{1.5}$$

We consider the form $u(z) = r^k f(\theta)$ for $z = re^{i\theta}$, $k \neq 0$. Aronsson [1] determined all p-harmonic functions in $\mathbb{C}$ of the form $u(z) = r^k f(\theta)$, assuming $p > 2$. Here, for $p > 1$, we determine all positive p-harmonic functions

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in D_ϕ of the form $u(z) = r^k f(\theta)$ satisfying the boundary condition (1.4) or (1.5).

If $u(z)$ satisfy the boundary condition (1.4), then $k > 0$. This k is denoted by k_+^p. If $u(z)$ satisfy the boundary condition (1.5), then $k < 0$. This k is denoted by k_-^p. Let $\beta = \pi/(2\phi)$. For $p = 2$, it is easy to calculate k_+^2, k_-^2, and $f(\theta)$. We see that

$$\begin{cases} k_+^2 = \beta, \\ k_-^2 = -\beta. \end{cases}$$

and

$$f(\theta) = C\cos\beta\theta,$$

where C is a arbitrary positive constant. For general $p > 1$, we obtain the following theorems.

Theorem 1.1. *Let $\alpha = (p-2)/(p-1)$ and $\beta = \pi/(2\phi)$. If*

$$k_+^p = \frac{2\beta^2 - \alpha(\beta-1)^2 + (\beta-1)\sqrt{4\beta^2 - 4\alpha\beta^2 + \alpha^2(\beta-1)^2}}{2(2\beta-1)},$$

then there exists $f(\theta)$ such that $u(z) = r^{k_+^p} f(\theta)$ is p-harmonic in D_ϕ and satisfy the boundary condition (1.4).

Theorem 1.2. *Let $\alpha = (p-2)/(p-1)$ and $\beta = \pi/(2\phi)$. If*

$$k_-^p = \frac{-2\beta^2 + \alpha(\beta+1)^2 - (\beta+1)\sqrt{4\beta^2 - 4\alpha\beta^2 + \alpha^2(\beta+1)^2}}{2(2\beta+1)},$$

then there exists $f(\theta)$ such that $u(z) = r^{k_-^p} f(\theta)$ is p-harmonic in D_ϕ and satisfy the boundary condition (1.5).

These theorems are main results of this paper.

2. Separation equation

In this section we give the representation formula for $f(\theta)$. See [1] for these accounts.

We observe that $u(z) = r^k f(\theta)$ satisfies (1.3) if and only if $f(\theta)$ satisfies the separation equation

(2.1)
$$[(p-1)(f')^2 + k^2 f^2]f'' + (2kp - 3k - p + 2)kf(f')^2 + (kp - k - p + 2)k^3 f^3 = 0.$$

Hence we find $f(\theta)$ satisfying the separation equation (2.1) with the condition

$$\begin{cases} f(\theta) > 0 & \text{for } -\phi < \theta < \phi, \\ f(\pm\phi) = 0. \end{cases} \tag{2.2}$$

Lemma 2.1. *Let I be an open interval and $f(\theta) \in C^2(I)$. Assume that $f(\theta) > 0$ and $f'(\theta) \neq 0$ on I. Put $\alpha = (p-2)/(p-1)$ and $g(\theta) = f'(\theta)^2 + (k-\alpha)kf(\theta)^2$.*
(1) *If $f(\theta)$ satisfies the separation equation (2.1) on I, then either (i) or (ii) holds:*

(i) $g \neq 0$ on I, and there is a constant $C_1 > 0$ such that

$$[(f')^2 + k^2 f^2]^k = C_1^2 |g|^{k-1}. \tag{2.3}$$

(ii) $g \equiv 0$ on I. Further, $f(\theta) = Ce^{\pm\mu\theta}$ where $\mu = \sqrt{(\alpha - k)k}$.
(2) *Conversely, if either (i) or (ii) holds, then $f(\theta)$ satisfies the separation equation (2.1) on I.*

Proof. Put $s = k^2 f(\theta)^2 > 0$. Let $J = s(I)$. We consider the inverse mapping $F : J \ni s \mapsto \theta \in I$. Obviously, $F \in C^2(J)$. Define a function $w(s)$ for $s \in J$ by

$$w(s) = \frac{f'(F(s))^2}{s} + 1.$$

We observe that $w(s) \in C^1(J)$ and

$$(w-1) + s\frac{dw}{ds} = \frac{f''}{k^2 f}.$$

Hence, $f(\theta)$ satisfies the separation equation (2.1) if and only if $w(s)$ satisfies the ordinary differential equation

$$\Big(w - \frac{\alpha}{k}\Big)w = -s(w-\alpha)\frac{dw}{ds},$$

where $\alpha = (p-2)/(p-1)$. If $f(\theta)$ satisfies the separation equation (2.1), then $w - \frac{\alpha}{k}$ is $\neq 0$ or $\equiv 0$ on J. On the other hand, we have

$$w - \frac{\alpha}{k} = \frac{f'(\theta)^2 + k^2 f(\theta)^2}{k^2 f(\theta)^2} - \frac{\alpha}{k} = \frac{g(\theta)}{k^2 f(\theta)^2}.$$

Hence g is $\neq 0$ or $\equiv 0$ on I. Let us consider three cases.

Case 1: $g(\theta) > 0$. The separation equation (2.1) is equivalent to

$$\frac{dw}{ds}\Big(\frac{k}{w} - \frac{k-1}{w - \frac{\alpha}{k}}\Big) + \frac{1}{s} = 0, \quad \text{or}$$

$$\frac{d}{ds}\Big[\log w^k - \log\Big(w - \frac{\alpha}{k}\Big)^{k-1} + \log s\Big] = 0.$$

This holds if and only if

$$w^k s = C_1^2\Big(w - \frac{\alpha}{k}\Big)^{k-1}$$

for all $s \in J$, for some $C_1 > 0$. Thus we obtain

$$[(f')^2 + k^2 f^2]^k = C_1^2 g^{k-1}.$$

Case 2: $g(\theta) < 0$. The separation equation (2.1) is equivalent to

$$\frac{dw}{ds}\left(\frac{k}{w} - \frac{1-k}{\frac{\alpha}{k} - w}\right) + \frac{1}{s} = 0, \qquad \text{or}$$
$$\frac{d}{ds}\left[\log w^k + \log\left(\frac{\alpha}{k} - w\right)^{1-k} + \log s\right] = 0.$$

This holds if and only if

$$w^k s = C_1^2\left(\frac{\alpha}{k} - w\right)^{k-1}$$

for all $s \in J$, for some $C_1 > 0$. Thus we obtain

$$[(f')^2 + k^2 f^2]^k = C_1^2(-g)^{k-1}.$$

Case 3: $g(\theta) \equiv 0$. Then we have

$$f'(\theta)^2 + (k-\alpha)kf(\theta)^2 \equiv 0.$$

Since $f'(\theta) \neq 0$, we see $(k-\alpha)k < 0$. Put $\mu = \sqrt{(\alpha - k)k}$. Then we have $f'(\theta) = \pm\mu f(\theta)$. Thus, $f(\theta) = Ce^{\pm\mu\theta}$. Conversely, if $(k-\alpha)k < 0$ and $f(\theta) = Ce^{\pm\mu\theta}$ where $\mu = \sqrt{(\alpha-k)k}$, then $f(\theta)$,obviously, satisfies the separation equation (2.1). □

Lemma 2.2. *Let I be an open interval and $f(\theta) \in C^2(I)$. Put $\alpha = (p-2)/(p-1)$ and $g(\theta) = f'(\theta)^2 + (k-\alpha)kf(\theta)^2$. Assume that $f(\theta) > 0$, $f'(\theta) \neq 0$, and $g(\theta) \neq 0$ on I. If there is a constant $C_1 > 0$ satisfying (2.3), then $f(\theta)$ has a parametric representation, given by*

$$\begin{cases} f(t) = \frac{C_1}{k}\left|1 - \frac{\alpha}{k}\cos^2 t\right|^{\frac{k-1}{2}} \cdot \cos t, \\ \theta(t) = \theta^* + \int_{t^*}^{t} \frac{1-\alpha\cos^2 t'}{k-\alpha\cos^2 t'}dt'. \end{cases}$$

Proof. Assume that $g(\theta) > 0$. We introduce polar coordinates in the plane:

$$\begin{cases} kf = \rho\cos t, \\ -f' = \rho\sin t \qquad (\neq 0). \end{cases} \tag{2.4}$$

We see that $\rho = \rho(\theta)$ and $t = t(\theta)$ are in $C^1(I)$. The equation (2.3) gives

$$\rho^{2k} = C_1^2\left[\rho^2\left(1 - \frac{\alpha}{k}\cos^2 t\right)\right]^{k-1}.$$

Then

$$\rho = C_1\left(1 - \frac{\alpha}{k}\cos^2 t\right)^{(k-1)/2}. \tag{2.5}$$

Thus we have

$$f = \frac{C_1}{k}\left(1 - \frac{\alpha}{k}\cos^2 t\right)^{\frac{k-1}{2}} \cdot \cos t$$

Next we give a representation of $\theta = \theta(t)$. Since $kf = \rho\cos t$ and $f'(\theta) \neq 0$, we see that $\theta = \theta(t) \in C^1$. By (2.4), we have

$$k = \frac{dt}{d\theta} - \frac{1}{\rho\tan t}\frac{d\rho}{d\theta}.$$

Then

$$\frac{dt}{d\theta}\left(1 - \frac{1}{\tan t}\frac{d(\log\rho)}{dt}\right) = k.$$

By (2.5), we get

$$\frac{d(\log\rho)}{dt} = (k-1)\frac{\alpha\sin t\cos t}{k - \alpha\cos^2 t}.$$

Then

$$\frac{d\theta}{dt} = \frac{1-\alpha\cos^2 t}{k-\alpha\cos^2 t}.$$

This implies the representation formula in the case $g(\theta) > 0$.

In the case $g(\theta) < 0$, the representation formula follows by a similar argument. Thus the lemma is proved. $\square$

The following lemma is proved by easy computations. See [1].

Lemma 2.3. *Let I be a maximal open interval such that $\alpha\cos^2 t \neq k$ for $t \in I$. We consider the mapping $t \mapsto (f, \theta)$ defined by*

$$\begin{cases} f(t) = \left|1 - \frac{\alpha}{k}\cos^2 t\right|^{\frac{k-1}{2}} \cdot \cos t \\ \theta(t) = \int_{t^*}^{t} \frac{1-\alpha\cos^2 t'}{k-\alpha\cos^2 t'}dt' \end{cases}$$

for $t \in I$. Then $f(\theta)$ satisfies the separation equation (2.1).

3. Proof of Theorem 1.1 and Theorem 1.2

In this section we prove Theorem 1.1 and Theorem 1.2. Assume that $p \neq 2$. Let us cinsider the following four cases:

(i) $p > 2$ and $k > 0$,
(ii) $1 < p < 2$ and $k > 0$,
(iii) $p > 2$ and $k < 0$,
(iv) $1 < p < 2$ and $k < 0$.

Put $\alpha = (p-2)/(p-1)$ and $\beta = \pi/(2\phi)$. For simplicity, we let

$$\lambda = \frac{\sqrt{|k-\alpha|}}{\sqrt{|k|} + \sqrt{|\alpha|}}$$

and

$$\mu = \frac{\sqrt{|k|}}{\sqrt{|\alpha| + |k|}}.$$

3.1. **The case $p > 2$ and $k > 0$.** We observe that if $k \le \alpha$, then there is no function $f(\theta)$ satisfying the separation equation (2.1) with the condition (2.2) (see [1]). Hence we assume that $k > \alpha$. Then $g(\theta) = f'(\theta)^2 + (k-\alpha)kf(\theta)^2 > 0$. Since Lemma 2.2, all solutions of the separation equation (2.1) are given by

$$\begin{cases} f(t) = \frac{C}{k}(1 - \frac{\alpha}{k}\cos^2 t)^{\frac{k-1}{2}} \cdot \cos t \\ \theta(t) = t - t^* + (1-k)\int_0^t \frac{dt'}{k - \alpha\cos^2 t'} \end{cases}$$

for $-\infty < t < \infty$. We see that $\theta(t)$ is strictly increasing. By the condition (2.2), we have $-\pi/2 \le t \le \pi/2$, $t^* = 0$, and $\theta(\pi/2) = \phi$. Easy computations gives

$$\theta(t) = t - \frac{k-1}{\sqrt{(k-\alpha)k}}[\arctan(\lambda \tan\frac{t}{2}) + \arctan(\lambda^{-1}\tan\frac{t}{2})]$$

for $-\pi/2 \le t \le \pi/2$. Since $\theta(\pi/2) = \phi$, we have

$$\frac{\pi}{2} - \frac{k-1}{\sqrt{(k-\alpha)k}} \cdot \frac{\pi}{2} = \phi. \tag{3.1}$$

If $\phi = \pi/2$, then $k = 1$. We assume that $\phi \ne \pi/2$. Squaring and rewriting gives

$$(2\beta - 1)k^2 - [2\beta^2 - \alpha(\beta-1)^2]k + \beta^2 = 0.$$

The roots of this equation are

$$k_1 = \frac{2\beta^2 - \alpha(\beta-1)^2 + |\beta - 1|\sqrt{4\beta^2 - 4\alpha\beta^2 + \alpha^2(\beta-1)^2}}{2(2\beta-1)}$$

and

$$k_2 = \frac{2\beta^2 - \alpha(\beta-1)^2 - |\beta - 1|\sqrt{4\beta^2 - 4\alpha\beta^2 + \alpha^2(\beta-1)^2}}{2(2\beta-1)}.$$

We observe that $\alpha < k_2 < 1 < k_1$ and (3.1) has only one root. If $0 < \phi < \pi/2$, then $\beta < 1$ and only k_2 satisfies (3.1). If $\pi/2 < \phi < \pi$, then $\beta > 1$ and only k_1 satisfies (3.1). Thus, the following theorem is obtained.

Theorem 3.1. *Let $p > 2$. Put*

$$k_+^p = \frac{2\beta^2 - \alpha(\beta-1)^2 + (\beta - 1)\sqrt{4\beta^2 - 4\alpha\beta^2 + \alpha^2(\beta-1)^2}}{2(2\beta-1)}.$$

Let $f(\theta)$ be a function given by

$$\begin{cases} f(t) = C(1 - \frac{\alpha}{k_+^p}\cos^2 t)^{\frac{k_+^p-1}{2}} \cos t \\ \theta(t) = t - \frac{k_+^p - 1}{\sqrt{(k_+^p-\alpha)k_+^p}}[\arctan(\lambda\tan\frac{t}{2}) + \arctan(\lambda^{-1}\tan\frac{t}{2})] \end{cases}$$

for $-\pi/2 < t < \pi/2$, where C is a arbitrary positive constant. Then $f(\theta)$ satisfies the separation equation (2.1) with the condition (2.2).

3.2. **The case $1 < p < 2$ and $k > 0$.** We obtain the following theorem by a similar argument of the case $p > 2$ and $k > 0$.

Theorem 3.2. *Let $1 < p < 2$. Put*

$$k_+^p = \frac{2\beta^2 - \alpha(\beta-1)^2 - (\beta-1)\sqrt{4\beta^2 - 4\alpha\beta^2 + \alpha^2(\beta-1)^2}}{2(2\beta-1)}.$$

Let $f(\theta)$ be a function given by

$$\begin{cases} f(t) = C(1 - \frac{\alpha}{k_+^p}\cos^2 t)^{\frac{k_+^p - 1}{2}} \cos t \\ \theta(t) = t - \frac{k_+^p - 1}{\sqrt{(k_+^p - \alpha)k_+^p}} \tan^{-1}(\mu \tan t) \end{cases}$$

for $-\pi/2 < t < \pi/2$, where C is a arbitrary positive constant. Then $f(\theta)$ satisfies the separation equation (2.1) with the condition (2.2).

Thus Theorem 3.1 and Theorem 3.2 imply Theorem 1.1.

Remark 3.3. If $\phi = \pi/2$, then $k_+^p = 1$ and $f(\theta) = C\cos\theta$ for all $1 < p < \infty$. In fact, $u(z) = x$ for $z = x + iy$ is a positive p-harmonic function in D_ϕ and satisfy the boundary condition (1.4).

3.3. **The case $p > 2$ and $k < 0$.** Then $g(\theta) = f'(\theta)^2 + (k-\alpha)kf(\theta)^2 > 0$. Since Lemma 2.2, all solutions of the separation equation (2.1) are given by

$$\begin{cases} f(t) = \frac{C}{k}(1 - \frac{\alpha}{k}\cos^2 t)^{\frac{k-1}{2}} \cdot \cos t \\ \theta(t) = t - t^* + (1-k)\int_0^t \frac{dt'}{k - \alpha\cos^2 t'} \end{cases}$$

for $-\infty < t < \infty$. We see that $\theta(t)$ is strictly decreasing. By the condition (2.2), we have $-\pi/2 \le t \le \pi/2$, $t^* = 0$, and $\theta(\pi/2) = -\phi$. Easy computations gives

$$\theta(t) = t - \frac{1-k}{\sqrt{(k-\alpha)k}} \arctan(\mu \tan t)$$

for $-\pi/2 \le t \le \pi/2$. Since $\theta(\pi/2) = -\phi$, we have

$$\frac{\pi}{2} - \frac{1-k}{\sqrt{(k-\alpha)k}} \cdot \frac{\pi}{2} = -\phi. \tag{3.2}$$

Squaring and rewriting gives

$$(2\beta+1)k^2 + [2\beta^2 - \alpha(\beta+1)^2]k - \beta^2 = 0.$$

The roots of this equation are

$$k_1 = \frac{-2\beta^2 + \alpha(\beta+1)^2 - (\beta+1)\sqrt{4\beta^2 - 4\alpha\beta^2 + \alpha^2(\beta+1)^2}}{2(2\beta+1)}$$

and

$$k_2 = \frac{-2\beta^2 + \alpha(\beta+1)^2 - (\beta+1)\sqrt{4\beta^2 - 4\alpha\beta^2 + \alpha^2(\beta+1)^2}}{2(2\beta+1)}$$

We see that $\alpha < k_2 \leq 1 \leq k_1$ and only k_2 satisfies (3.2). Thus, the following theorem is obtained.

Theorem 3.4. *Let $p > 2$. Put*

$$k_-^p = \frac{-2\beta^2 + \alpha(\beta+1)^2 - (\beta+1)\sqrt{4\beta^2 - 4\alpha\beta^2 + \alpha^2(\beta+1)^2}}{2(2\beta+1)}.$$

Let $f(\theta)$ be a function given by

$$\begin{cases} f(t) = C(1 - \frac{\alpha}{k_-^p}\cos^2 t)^{\frac{k_-^p - 1}{2}} \cos t \\ \theta(t) = t - \frac{1-k_-^p}{\sqrt{(k_-^p - \alpha)k_-^p}} \tan^{-1}(\mu \tan t) \end{cases}$$

for $-\pi/2 < t < \pi/2$, where C is a arbitrary positive constant. Then $f(\theta)$ satisfies the separation equation (2.1) with the condition (2.2).

3.4. **The case $1 < p < 2$ and $k < 0$.** We obtain the following theorem by a similar argument of the case $p > 2$ and $k < 0$.

Theorem 3.5. *Let $1 < p < 2$. Put*

$$k_-^p = \frac{-2\beta^2 + \alpha(\beta+1)^2 - (\beta+1)\sqrt{4\beta^2 - 4\alpha\beta^2 + \alpha^2(\beta+1)^2}}{2(2\beta+1)}.$$

Let $f(\theta)$ be a function given by

$$\begin{cases} f(t) = C(1 - \frac{\alpha}{k_-^p}\cos^2 t)^{\frac{k_-^p - 1}{2}} \cos t \\ \theta(t) = t - \frac{1-k_-^p}{\sqrt{(k_-^p - \alpha)k_-^p}} [\arctan(\lambda \tan \frac{t}{2}) + \arctan(\lambda^{-1} \tan \frac{t}{2})] \end{cases}$$

for $-\pi/2 < t < \pi/2$, where C is a arbitrary positive constant. Then $f(\theta)$ satisfies the separation equation (2.1) with the condition (2.2).

Thus Theorem 3.4 and Theorem 3.5 imply Theorem 1.2.

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ON WELL-POSEDNESS OF INCOMPRESSIBLE TWO-PHASE FLOWS WITH PHASE TRANSITIONS

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1. The model

In this talk, we consider a free boundary problem of incompressible two-phase flows with phase transitions in the framework of L_p-theory with nearly flat interface represented as a graph over $\mathbb{R}^{n-1}$, namely in the regions

$$\Omega_\pm(t) = \{(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : \; x_n \gtrless h(t,x), \; t \geq 0\}$$

with interface

$$\Gamma(t) = \{(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : \; x_n = h(t,x), \; t \geq 0\}.$$

We set $\Omega_0 = \Omega_+(0) \cup \Omega_-(0)$ and ν_0 be the outer normal of $\Omega_-(0)$.

Let u denote the velocity field, π the pressure field, $T(u,\pi,\theta)$ the stress tensor, $D(u) = (\nabla u + [\nabla u]^{\mathsf{T}})/2$ the rate of deformation tensor, θ the (absolute) temperature field, ν_Γ the outer normal of $\Omega_-(t)$, u_Γ the interface velocity, $V_\Gamma = u_\Gamma \cdot \nu_\Gamma$ the normal velocity of $\Gamma(t)$, $H_\Gamma = H(\Gamma(t)) = -\mathrm{div}_\Gamma \nu_\Gamma$ the curvature of $\Gamma(t)$, j the phase flux, and

$$[\![v]\!] = \left(v|_{\Omega_+(t)} - v|_{\Omega_-(t)}\right)\big|_{\Gamma(t)}$$

the jump of a quantity v across $\Gamma(t)$.

Let $\rho_\pm > 0$ denote the densities of $\Omega_\pm(t)$. In order to economize our notation, we set

$$\rho = \rho_+ \chi_{\Omega_+(t)} + \rho_- \chi_{\Omega_-(t)},$$

where χ_D denotes the indicator function of a set D, and this notation is employed for μ, κ, d, etc. as well. We just keep in mind that the coefficients depend on the phases.

By an *Incompressible Two-Phase Flow with Phase Transition* we mean the following problem: Find a family of closed hypersurfaces $\{\Gamma(t)\}_{t\geq 0}$ and appropriately

This talk is based on a joint work with Jan Prüss (Institut für Mathematik Martin-Luther-Universität Halle-Wittenberg, Germany).

smooth functions $u : \mathbb{R}_+ \times \bar{\Omega} \to \mathbb{R}^n$, and $\pi, \theta : \mathbb{R}_+ \times \bar{\Omega} \to \mathbb{R}$ satisfy (1.1) - (1.3):

$$\left.\begin{aligned} \rho(\partial_t u + u \cdot \nabla u) - \operatorname{div} T(u,\pi,\theta) = 0 \qquad & \text{in } \Omega(t),\ t>0, \\ T(u,\pi,\theta) = 2\mu(\theta)D(u) - \pi I, \quad \operatorname{div} u = 0 \qquad & \text{in } \Omega(t),\ t>0, \\ [\![\frac{1}{\rho}]\!] j^2 \nu_\Gamma - [\![T(u,\pi,\theta)\nu_\Gamma]\!] - \sigma H_\Gamma \nu_\Gamma = 0 \qquad & \text{on } \Gamma(t),\ t>0, \\ [\![u]\!] - [\![\frac{1}{\rho}]\!] j \nu_\Gamma = 0 \qquad & \text{on } \Gamma(t),\ t>0, \\ u(0) = u_0 \qquad & \text{in } \Omega_0, \end{aligned}\right\} \tag{1.1}$$

$$\left.\begin{aligned} \rho\kappa(\theta)(\partial_t\theta + u\cdot\nabla\theta) - \operatorname{div}(d(\theta)\nabla\theta) - 2\mu(\theta)|D(u)|_2^2 = 0 \qquad & \text{in } \Omega(t),\ t>0, \\ l(\theta)j + [\![d(\theta)\partial_{\nu_\Gamma}\theta]\!] = 0 \qquad & \text{on } \Gamma(t),\ t>0, \\ [\![\theta]\!] = 0 \qquad & \text{on } \Gamma(t),\ t>0, \\ \theta(0) = \theta_0 \qquad & \text{in } \Omega_0, \end{aligned}\right\} \tag{1.2}$$

$$\left.\begin{aligned} [\![\psi(\theta)]\!] + [\![\frac{1}{2\rho^2}]\!] j^2 - [\![\frac{T(u,\pi,\theta)\nu_\Gamma\cdot\nu_\Gamma}{\rho}]\!] = 0 \qquad & \text{on } \Gamma(t),\ t>0, \\ V_\Gamma - u\cdot\nu_\Gamma + \frac{1}{\rho} j = 0 \qquad & \text{on } \Gamma(t),\ t>0, \\ \Gamma(0) = \{x \in \mathbb{R}^n \mid x_n = h_0(x)\}. & \end{aligned}\right\} \tag{1.3}$$

Several quantities are derived from the specific *free energy* $\psi_\pm(\theta)$ in $\Omega_\pm(t)$ as follows.

- $\epsilon_\pm(\theta) := \psi_\pm(\theta) + \theta\eta_\pm(\theta)$ the internal energy,
- $\eta_\pm(\theta) := -\psi'_\pm(\theta)$ the entropy,
- $\kappa_\pm(\theta) := \epsilon'_\pm(\theta) = -\theta\psi''_\pm(\theta) > 0$ the heat capacity,
- $l(\theta) := \theta[\![\psi'(\theta)]\!] = -\theta[\![\eta(\theta)]\!]$ the latent heat.

Further $d_\pm(\theta) > 0$ denotes the coefficient of heat conduction in Fourier's law, $\mu_\pm(\theta) > 0$ the viscosity in Newton's law, and $\sigma > 0$ the constant coefficient of surface tension.

Concerning the second equation of (1.3), we remind that balance of mass across $\Gamma(t)$ requires $[\![\rho(u - u_\Gamma)]\!] \cdot \nu_\Gamma = 0$, which implies

$$j = \rho_+(u_+ - u_\Gamma)\cdot\nu_\Gamma = \rho_-(u_- - u_\Gamma)\cdot\nu_\Gamma,$$

and so

$$u_+\cdot\nu_\Gamma - \frac{1}{\rho_+} j = u_-\cdot\nu_\Gamma - \frac{1}{\rho_-} j.$$

Therefore this equation is well-defined on $\Gamma(t)$.

This model is derived from balance of mass, balance of momentum, balance of energy under the assumption of no entropy production on the interface and of constitutive laws, which is explained in more detail in [12]. It has been recently proposed by Anderson et al. [1], see also the monographs by Ishii [7] and Ishii and Takashi [8], and it is thermodynamically consistent in the sense that in absence of exterior forces and heat sources, the total energy is preserved and the total entropy is nondecreasing, see [12]. It is in some sense the simplest sharp interface model for incompressible Newtonian two-phase flows taking into account phase transitions driven by temperature.

Note that in the case of equal densities, since $[\![\rho]\!] = [\![1/\rho]\!] = 0$, the phase flux j does not enter (1.1). So in this case we obtain essentially a Stefan problem with surface tension, which is only weakly coupled to the standard two-phase Navier-Stokes problem via temperature dependent viscosities. We call this case *temperature dominated*. But in the case of different densities, the phase flux j causes a jump in the velocity field on the interface, which leads to so called Stefan currents which are convections driven by phase transitions. In this situation it turns out that the heat problem (1.2) is only weakly coupled to (1.1) and (1.3), we call this case *velocity dominated*. The resulting two-phase Navier-Stokes problem is non-standard, therefore it requires a new analysis.

The analytical properties of the problem appear to be different in these two cases. The spaces for well-posedness are not the same. In the temperature dominated case the phase flux j can be eliminated by solving the second equation in (1.2) for j. This yields

$$j = -[\![d(\theta)\partial_\nu\theta]\!]/l(\theta),$$

as long as $l(\theta) \neq 0$; this is the essential well-posedness condition in this case. Then the equation describing the evolution of the interface becomes

$$V_\Gamma = u_\Gamma \cdot \nu_\Gamma + [\![d(\theta)\partial_\nu\theta]\!]/\rho l(\theta).$$

On the other hand, in the velocity determined case $[\![\rho]\!] \neq 0$ we can eliminate j by taking the inner product of the fourth equation in (1.1) with ν_Γ to the result

$$j = [\![u \cdot \nu_\Gamma]\!]/[\![1/\rho]\!].$$

In this case the equation for V_Γ becomes

$$V_\Gamma = [\![\rho u \cdot \nu_\Gamma]\!]/[\![\rho]\!],$$

which does not contain temperature, in contrast to the first case. Therefore the analysis for these two cases necessarily is different, too.

There is a large literature on isothermal incompressible Newtonian two-phase flows without phase transitions, and also on the two-phase Stefan problem with surface tension modeling temperature driven phase transitions. On the other hand, mathematical work on two-phase flow problems including phase transitions are rare. In this direction, we only know the papers by Hoffmann and Starovoitov [5, 6] dealing with a simplified two-phase flow model, and Kusaka and Tani [10, 11] which is two-phase for temperature but only one phase is moving. The papers of Di Benedetto and Friedman [2] and Di Benedetto and O'Leary[3] deal with weak solutions of conduction-convection problems with phase change. However, none of these papers considers models which are consistent with thermodynamics.

2. Results

2.1. The case of equal densities. In this case, the equilibrium state is

$$u_\infty = 0,\ \theta_\infty = \text{const.},\ \pi_\infty = \text{const.},\ [\![\pi_\infty]\!] = 0,\ j = 0,$$
$$[\![\psi(\theta_\infty)]\!] = 0,\ \Gamma_\infty = \{x \in \mathbb{R}^n \mid x_n = 0\}.$$

The main result in the case of equal densities is the local well-posedness of (1.1)-(1.3) close to the equilibrium state.

Theorem 2.1. *Let $p > n+2$, $\sigma > 0$, $\rho_+ = \rho_- > 0$ be constants, and suppose $\psi_\pm \in C^3(0,\infty)$, $\mu_\pm$, $d_\pm \in C^2(0,\infty)$ such that*

$$\kappa_\pm(s) = -s\psi_\pm''(s) > 0, \quad \mu_\pm(s) > 0, \quad d_\pm(s) > 0, \quad s \in (0,\infty).$$

Let the initial interface Γ_0 be given by a graph $x \mapsto (x, h_0(x))$ and let θ_∞ be the constant temperature at infinity.

Then given any finite interval $J = [0,T]$, there exists $\eta > 0$ such that (1.1)-(1.3) *admits a unique L_p-solution on J provided the smallness conditions:*

$$\|u_0\|_{W_p^{2-2/p}(\Omega_0)} + \|\theta_0 - \theta_\infty\|_{W_p^{2-2/p}(\Omega_0)} + \|h_0\|_{W_p^{4-3/p}(\mathbb{R}^{n-1})} \le \eta,$$

the compatibility conditions[1]*:*

$$\begin{aligned} \operatorname{div} u_0 &= 0 \quad \text{in} \quad \Omega_0,\\ [\![u_0]\!] = [\![\theta_0]\!] = 0, \quad P_{\Gamma_0}[\![\mu(\theta_0)D(u_0)\nu_0]\!] &= 0 \quad \text{on} \quad \Gamma_0,\\ [\![\psi(\theta_0)]\!] + \sigma H_{\Gamma_0} &= 0 \quad \text{on} \quad \Gamma_0,\\ [\![d(\theta_0)\partial_{\nu_0}\theta_0]\!] &\in W_p^{2-6/p}(\Gamma_0), \end{aligned}$$

and the well-posedness conditions:

$$l(\theta_\infty) \neq 0 \ \ \textit{on} \ \ \Gamma_0 \qquad \textit{and} \qquad \theta_\infty > 0 \ \ \textit{on} \ \ \mathbb{R}^n$$

are satisfied.

For a proof of this result we show maximal regularity for the linear part of the problem and finally employ the contraction mapping principle to solve the nonlinear problem.

We set $\dot{\mathbb{R}}^n = \mathbb{R}^n_+ \cup \mathbb{R}^n_-$ and $\nu = e_n = (0,\ldots,0,1)^\mathsf{T}$. The principal part of the linearization of (1.1)-(1.3) reads as follows.

$$\left.\begin{aligned} \rho\partial_t u - \mu_\infty\Delta u + \nabla\pi &= f_u \quad \text{in} \quad \dot{\mathbb{R}}^n, \quad t>0,\\ \operatorname{div} u &= f_d \quad \text{in} \quad \dot{\mathbb{R}}^n, \quad t>0,\\ -2[\![\mu_\infty D(u)\nu]\!] + [\![\pi]\!]\nu - \sigma(\Delta_{x'}h)\nu &= g_u \quad \text{on} \quad \mathbb{R}^{n-1},\ t>0,\\ [\![u]\!] &= 0 \quad \text{on} \quad \mathbb{R}^{n-1},\ t>0\\ u(0) &= u_0 \quad \text{in} \quad \dot{\mathbb{R}}^n, \end{aligned}\right\} \tag{2.1}$$

[1] $P_{\Gamma_0} = I - \nu_0 \otimes \nu_0$ denotes the projection onto the tangent bundle of Γ_0.

$$\left.\begin{aligned} \rho\kappa_\infty\partial_t\theta - d_\infty\Delta\theta &= f_\theta \quad \text{in} \quad \dot{\mathbb{R}}^n, \quad\ \ t>0,\\ [\![\theta]\!] &= 0 \quad\ \text{on} \quad \mathbb{R}^{n-1},\ t>0,\\ \theta(0) &= \theta_0 \quad \text{in} \quad \dot{\mathbb{R}}^n, \end{aligned}\right\} \tag{2.2}$$

$$\left.\begin{aligned} (l_\infty/\theta_\infty)\theta + (\sigma/\rho)\Delta_{x'}h &= g_\theta \quad \text{on} \quad \mathbb{R}^{n-1},\ t>0\\ \partial_t h - [\![d_\infty\partial_n\theta]\!]/\rho l_\infty &= g_h \quad \text{on} \quad \mathbb{R}^{n-1},\ t>0,\\ h(0) &= h_0 \quad \text{on} \quad \mathbb{R}^{n-1}. \end{aligned}\right\} \tag{2.3}$$

Here $\mu_{\infty\pm} = \mu_\pm(\theta_\infty)$, $d_{\infty\pm} = d_\pm(\theta_\infty)$, $\kappa_{\infty\pm} = \kappa_\pm(\theta_\infty)$, $l_\infty = l(\theta_\infty)$ are constants. Observe that the term $u\cdot\nu_\Gamma$ in the equation for h is of lower order as it enjoys more regularity than the trace of θ on $\mathbb{R}^{n-1}$. Since u does neither appear in (2.2) nor in (2.3), (2.1) decouples from the remaining problem. (2.1) is the two-phase Stokes problem with surface tension which was studied by Prüss and Simonett [14, 15] and Shibata and Shimizu [17]. The latter system comprises the linearized Stefan problem with surface tension which has been studied by Prüss, Simonett and Zacher [16]. Therefore the linearized problem (2.1)-(2.3) has the property of maximal L_p-regularity.

Before stating maximal regularity results of linear problems, let us introduce the relevant function spaces. Let $\Omega\subset\mathbb{R}^m$ be open and X be an arbitrary Banach space.By $L_p(\Omega;X)$ and $H_p^s(\Omega;X)$, for $1\le p\le\infty$, $s\in\mathbb{R}$, we denote the X-valued Lebesgue and the X-valued Bessel potential spaces of order s, respectively. We will also make use of the fractional Sobolev-Slobodeckij spaces $W_p^s(\Omega;X)$, $1\le p<\infty$, $s>0, s\notin\mathbb{N}$ with norm

$$\|g\|_{W_p^s(\Omega;X)} = \|g\|_{W_p^{[s]}(\Omega;X)} + \sum_{|\alpha|=[s]}\left(\int_\Omega\int_\Omega \frac{\|\partial^\alpha g(x)-\partial^\alpha g(y)\|_X^p}{|x-y|^{m+(s-[s])p}}\,dxdy\right)^{1/p},$$

where $[s]$ denotes the largest integer smaller than s. We remind that $H_p^k = W_p^k$ for $k\in\mathbb{N}$ and $1<p<\infty$, and that $W_p^s = B_{pp}^s$ for $s>0, s\notin\mathbb{N}$.

For $s\in\mathbb{R}$ and $1<p<\infty$, $\dot{H}_p^s(\mathbb{R}^n)$ denotes the homogeneous Bessel-potentianl spaces. For $s\in\mathbb{R}\setminus\mathbb{Z}$, the homogeneous Sobolev-Slobodeckij spaces $\dot{W}_p^s(\mathbb{R}^n)$ of fractional order can be obtained by real interpolation as

$$\dot{W}_p^s(\mathbb{R}^n) := (\dot{H}_p^k(\mathbb{R}^n), \dot{H}_p^{k+1}(\mathbb{R}^n))_{s-k,p}, \quad k<s<k+1,$$

where $(\cdot,\cdot)_{\theta,p}$ is the real interpolation functor.

To state the result we introduce appropriate function spaces. We set

$$\begin{aligned} \mathbb{E}_u(J) &= H_p^1(J;L_p(\mathbb{R}^n))^n \cap L_p(J;H_p^2(\dot{\mathbb{R}}^n))^n\\ &\qquad \cap \{[\![u_n]\!]\in H_p^1(J;\dot{W}_p^{-1/p}(\mathbb{R}^{n-1}))\}\cap\{[\![u]\!]=0\},\\ \mathbb{E}_\pi(J) &= L_p(J;\dot{H}_p^1(\dot{\mathbb{R}}^n)),\\ \mathbb{E}_{\gamma\pi}(J) &= W_p^{1/2-1/2p}(J;L_p(\mathbb{R}^{n-1}))^2\cap L_p(J;W_p^{1-1/p}(\mathbb{R}^{n-1}))^2,\\ \mathbb{E}_\theta(J) &= H_p^1(J;L_p(\mathbb{R}^n))\cap L_p(J;H_p^2(\dot{\mathbb{R}}^n))\cap\{[\![\theta]\!]=0\}, \end{aligned}$$

$$\mathbb{E}_h(J) = W_p^{3/2-1/2p}(J; L_p(\mathbb{R}^{n-1})) \cap W_p^{1-1/2p}(J; H_p^2(\mathbb{R}^{n-1})) \\ \cap L_p(J; W_p^{4-1/p}(\mathbb{R}^{n-1})),$$

and define the solution space for (2.1)-(2.3) as

$$\mathbb{E}(J) = \mathbb{E}_u(J) \times \mathbb{E}_\pi(J) \times \mathbb{E}_{\gamma\pi}(J) \times \mathbb{E}_\theta(J) \times \mathbb{E}_h(J).$$

We denote by $\gamma\pi$ the two one-sided traces of π on $\mathbb{R}^{n-1}$. $\mathbb{E}(J)$ is a Banach space with its natural norm, and the generic elements of $\mathbb{E}(J)$ are functions $(u, \pi, \gamma\pi, \theta, h)$. Moreover we set

$$\begin{aligned}
&\mathbb{F}_u(J) := L_p(J; L_p(\mathbb{R}^n))^n,\\
&\mathbb{F}_d(J) := H_p^1(J; \dot{H}_p^{-1}(\mathbb{R}^n)) \cap L_p(J; H_p^1(\dot{\mathbb{R}}^n)),\\
&\mathbb{G}_u(J) := W_p^{1/2-1/2p}(J; L_p(\mathbb{R}^{n-1})^n \cap L_p(J; W_p^{1-1/p}(\mathbb{R}^{n-1}))^n,\\
&\mathbb{F}_\theta(J) := L_p(J; L_p(\mathbb{R}^n)),\\
&\mathbb{G}_\theta(J) := W_p^{1-1/2p}(J; L_p(\mathbb{R}^{n-1})) \cap L_p(J; W_p^{2-1/p}(\mathbb{R}^{n-1})),\\
&\mathbb{G}_h(J) := W_p^{1/2-1/2p}(J; L_p(\mathbb{R}^{n-1})) \cap L_p(J; W_p^{1-1/p}(\mathbb{R}^{n-1})),
\end{aligned}$$

and define the regularity of the data space for (2.4)-(2.6) as

$$\mathbb{F}(J) := \mathbb{F}_u(J) \times \mathbb{F}_d(J) \times \mathbb{G}_u(J) \times \mathbb{F}_\theta(J) \times \mathbb{G}_\theta(J) \times \mathbb{G}_h(J).$$

$\mathbb{F}(J)$ is a Banach space with its natural norm, and the generic elements of $\mathbb{F}(J)$ are functions $(f_u, f_d, g_u, f_\theta, g_\theta, g_h)$. Finally, we define the time trace space X_γ of $\mathbb{E}(J)$ as

$$X_\gamma := W_p^{2-2/p}(\dot{\mathbb{R}}^n)^n \times W_p^{2-2/p}(\dot{\mathbb{R}}^n) \times W_p^{4-3/p}(\mathbb{R}^{n-1}).$$

Then the main result on the linearized problem (2.1)-(2.3) is stated as

Theorem 2.2. *Let $1 < p < \infty$, $p \neq 3/2, 3$ and assume that $\sigma > 0$, $\rho_+ = \rho_- > 0$, $\mu_{\infty\pm}$, $d_{\infty\pm}$ $\kappa_{\infty\pm} > 0$. Then the linear problem (2.1)-(2.3) admits a unique solution $(u, \pi, \gamma\pi, \theta, h) \in \mathbb{E}(J)$ if and only if the data (u_0, θ_0, h_0) and $(f_u, f_d, g_u, f_\theta, g_\theta, g_h)$ satisfy the regularity conditions:*

$$(u_0, \theta_0, h_0) \in X_\gamma, \quad (f_u, f_d, g_u, f_\theta, g_\theta, g_h) \in \mathbb{F}(J),$$

and the compatibility conditions:

$$\begin{aligned}
\operatorname{div} u_0 &= f_d(0) && \text{in} \quad \dot{\mathbb{R}}^n,\\
[\![u_0]\!] = 0, \quad -P_{\mathbb{R}^{n-1}}[\![\mu_\infty(\nabla u_0 + [\nabla u_0]^{\mathsf{T}})\nu]\!] &= P_{\mathbb{R}^{n-1}} g_u(0), && \text{on} \quad \mathbb{R}^{n-1},\\
[\![\theta_0]\!] = 0, \quad (l_\infty/\theta_\infty)\theta_0 + \sigma\Delta_{x'} h_0 &= g_\theta(0) && \text{on} \quad \mathbb{R}^{n-1}\\
g_h(0) + [\![d_\infty \partial_n \theta_0]\!]/\rho l_\infty &\in W_p^{2-6/p}(\mathbb{R}^{n-1}),
\end{aligned}$$

and the well-posedness conditions:

$$l_\infty \neq 0 \quad \text{on } \mathbb{R}^{n-1} \quad \text{and} \quad \theta_\infty > 0 \quad \text{on } \mathbb{R}^n.$$

The solution map $[(u_0, \theta_0, h_0, f_u, f_d, g_u, f_\theta, g_\theta, g_h) \mapsto (u, \pi, \gamma\pi, \theta, h)]$ is continuous between the corresponding spaces.

2.2. **The case of non-equal densities.** The equilibrium state is the same as the equal density case except

$$[\![\psi(\theta_\infty)]\!] + [\![\pi_\infty/\rho]\!] = 0$$

replaces $[\![\psi(\theta_\infty)]\!] = 0$.

The main result in the case of non-equal densities is the local well-posedness of (1.1)-(1.3) close to the equilibrium state.

Theorem 2.3. *Let $p > n+2$, $\rho_+, \rho_-, \sigma > 0$ be constant, $\rho_+ \neq \rho_-$, and suppose $\psi_\pm \in C^3(0,\infty)$, $\mu_\pm$, $d_\pm \in C^2(0,\infty)$ are such that*

$$\kappa_\pm(s) = -s\psi_\pm''(s) > 0, \quad \mu_\pm(s) > 0, \quad d_\pm(s) > 0 \quad s \in (0,\infty).$$

Let the initial interface Γ_0 be given by a graph $x \mapsto (x', h_0(x'))$, and let $\theta_\infty > 0$ be the constant temperature at infinity.

Then given any finite interval $J = [0,T]$, there exists $\eta > 0$ such that (1.1)-(1.3) *admits a unique L_p-solution on J provided the smallness conditions:*

$$\|u_0\|_{W_p^{2-2/p}(\Omega_0)} + \|\theta_0 - \theta_\infty\|_{W_p^{2-2/p}(\Omega_0)} + \|h_0\|_{W_p^{3-2/p}(\mathbb{R}^{n-1})} \leq \eta,$$

and the compatibility conditions:

$$\begin{aligned} \operatorname{div} u_0 = 0 \quad &\text{in } \Omega_0,\\ P_{\Gamma_0}[\![\mu(\theta_0)D(u_0)\nu_0]\!] = 0, \quad P_{\Gamma_0}[\![u_0]\!] = 0 \quad &\text{on } \Gamma_0,\\ [\![\theta_0]\!] = 0, \quad (l(\theta_0)/[\![1/\rho]\!])[\![u_0\cdot\nu_0]\!] + [\![d(\theta_0)\partial_{\nu_0}\theta_0]\!] = 0 \quad &\text{on } \Gamma_0, \end{aligned}$$

are satisfied.

For a proof of this result we show maximal regularity for the linear part of the problem and finally employ the contraction mapping principle to solve the nonlinear problem.

The principal part of the linearized problem in the case of a nearly flat initial interface reads as follows

$$\left.\begin{aligned} \rho\partial_t u - \mu_\infty\Delta u + \nabla\pi = f_u \quad &\text{in } \dot{\mathbb{R}}^n, && t>0,\\ \operatorname{div} u = f_d \quad &\text{in } \dot{\mathbb{R}}^n, && t>0,\\ -2[\![\mu_\infty D(u)\nu]\!] + [\![\pi]\!]\nu - \sigma(\Delta_x h)\nu = g_u \quad &\text{on } \mathbb{R}^{n-1}, && t>0,\\ [\![u']\!] = g_j \quad &\text{on } \mathbb{R}^{n-1}, && t>0,\\ u(0) = u_0 \quad &\text{in } \dot{\mathbb{R}}^n, \end{aligned}\right\} \tag{2.4}$$

$$\left.\begin{aligned} \rho\kappa_\infty\partial_t\theta - d_\infty\Delta\theta = f_\theta \quad &\text{in } \dot{\mathbb{R}}^n, && t>0,\\ -[\![d_\infty\partial_n\theta]\!] = g_\theta \quad &\text{on } \mathbb{R}^{n-1}, && t>0,\\ [\![\theta]\!] = 0 \quad &\text{on } \mathbb{R}^{n-1}, && t>0,\\ \theta(0) = \theta_0 \quad &\text{in } \dot{\mathbb{R}}^n, \end{aligned}\right\} \tag{2.5}$$

$$\left.\begin{aligned} -2[\![\frac{\mu_\infty D(u)\nu\cdot\nu}{\rho}]\!] + [\![\frac{\pi}{\rho}]\!] &= g_\pi && \text{on } \mathbb{R}^{n-1}, \quad t>0, \\ \partial_t h - [\![\rho u_n]\!]/[\![\rho]\!] &= g_h && \text{on } \mathbb{R}^{n-1}, \quad t>0, \\ h(0) &= h_0 && \text{on } \mathbb{R}^{n-1}, \end{aligned}\right\} \tag{2.6}$$

where μ_∞, d_∞, κ_∞, are positive constants and $\nu = e_n$. We assume in this subsection $[\![\rho]\!] = \rho_+ - \rho_- \neq 0$. Apparently, (2.5) decouples from the remaining problem and it is well-known that this problem has maximal L_p-regularity (cf. Escher, Prüss and Simonett [4]), we concentrate on the remaining one. The resulting two-phase Navier-Stokes problem is non-standard, it requires a new analysis. Therefore the analysis of the coupled system (2.4) and (2.6) is the most important part through this work.

To state the result we introduce appropriate function spaces. We set

$$\begin{aligned}
\mathbb{E}_u(J) &= H_p^1(J; L_p(\mathbb{R}^n))^n \cap L_p(J; H_p^2(\dot{\mathbb{R}}^n))^n \cap \{[\![u_n]\!] \in H_p^1(J; \dot{W}_p^{-1/p}(\mathbb{R}^{n-1}))\}, \\
\mathbb{E}_\pi(J) &= L_p(J; \dot{H}_p^1(\dot{\mathbb{R}}^n)), \\
\mathbb{E}_{\gamma\pi}(J) &= W_p^{1/2-1/2p}(J; L_p(\mathbb{R}^{n-1}))^2 \cap L_p(J; W_p^{1-1/p}(\mathbb{R}^{n-1}))^2, \\
\mathbb{E}_\theta(J) &= H_p(J; L_p(\mathbb{R}^n) \cap L_p(J; H_p^2(\dot{\mathbb{R}}^n)) \cap \{[\![\theta]\!] = 0\}, \\
\mathbb{E}_h(J) &= W_p^{2-1/2p}(J; L_p(\mathbb{R}^{n-1})) \cap H_p^1(J; W_p^{2-1/p}(\mathbb{R}^{n-1})) \\
&\quad \cap L_p(J; W_p^{3-1/p}(\mathbb{R}^{n-1})),
\end{aligned}$$

and define the solution space for (2.4)-(2.6) as

$$\mathbb{E}(J) := \mathbb{E}_u(J) \times \mathbb{E}_\pi(J) \times \mathbb{E}_{\gamma\pi}(J) \times \mathbb{E}_\theta(J) \times \mathbb{E}_h(J).$$

We denote by $\gamma\pi$ the two one-sided traces of π on $\mathbb{R}^{n-1}$. $\mathbb{E}(J)$ is a Banach space with its natural norm, and the generic elements of $\mathbb{E}(J)$ are functions $(u, \pi, \gamma\pi, \theta, h)$. Moreover we set

$$\begin{aligned}
\mathbb{F}_u(J) &:= L_p(J; L_p(\mathbb{R}^n))^n, \\
\mathbb{F}_d(J) &:= H_p^1(J; \dot{H}_p^{-1}(\mathbb{R}^n)) \cap L_p(J; H_p^1(\dot{\mathbb{R}}^n)), \\
\mathbb{G}_u(J) &:= W_p^{1/2-1/2p}(J; L_p(\mathbb{R}^{n-1})^n \cap L_p(J; W_p^{1-1/p}(\mathbb{R}^{n-1}))^n, \\
\mathbb{G}_j(J) &:= W_p^{1-1/2p}(J; L_p(\mathbb{R}^{n-1}))^{n-1} \cap L_p(J; W_p^{2-1/p}(\mathbb{R}^{n-1}))^{n-1}, \\
\mathbb{F}_\theta(J) &:= L_p(J; L_p(\mathbb{R}^n)), \\
\mathbb{G}_\theta(J) &:= W_p^{1/2-1/2p}(J; L_p(\mathbb{R}^{n-1})) \cap L_p(J; W_p^{1-1/p}(\mathbb{R}^{n-1})), \\
\mathbb{G}_\pi(J) &:= \mathbb{G}_\theta(J), \\
\mathbb{G}_h(J) &:= W_p^{1-1/2p}(J; L_p(\mathbb{R}^{n-1})) \cap L_p(J; W_p^{2-1/p}(\mathbb{R}^{n-1})),
\end{aligned}$$

and define the regularity of the data space for (2.4)-(2.6) as

$$\mathbb{F}(J) := \mathbb{F}_u(J) \times \mathbb{F}_d(J) \times \mathbb{G}_u(J) \times \mathbb{G}_j(J) \times \mathbb{F}_\theta(J) \times \mathbb{G}_\theta(J) \times \mathbb{G}_\pi(J) \times \mathbb{G}_h(J).$$

$\mathbb{F}(J)$ is a Banach space with its natural norm, and the generic elements of $\mathbb{F}(J)$ are functions $(f_u, f_d, g_u, g_j, f_\theta, g_\theta, g_\pi, g_h)$. Finally, we define the time trace space

X_γ of $\mathbb{E}(J)$ as

$$X_\gamma := W_p^{2-2/p}(\dot{\mathbb{R}}^n)^n \times W_p^{2-2/p}(\dot{\mathbb{R}}^n) \times W_p^{3-2/p}(\mathbb{R}^{n-1}).$$

We obtain the maximal L_p regularity results of (2.4) and (2.6) as the following way. Let λ and ξ' be dual parameters of the Laplace transform w.r.t. t and of the Fourier transform w.r.t. x', respectively. Solving the coupled system (2.4) and (2.6), we express the second equation of (2.6) by

$$s(\lambda, |\xi'|)\hat{h} = \hat{g}_h.$$

We set $\tau = |\xi'|$. The boundary symbol $s(\lambda, \tau)$ is written by

$$s(\lambda, \tau) = \lambda + \frac{\sigma\tau}{[\![\rho]\!]^2} m(z)$$

with $z = \lambda/\tau^2$, where the holomorphic function $m(z)$ satisfies

$$|m(z)| \le \frac{M}{1+|z|}, \quad z \in \Sigma_\phi \cup B_r(0)$$

for each $\phi \le \pi/2 + \eta$ and some $r > 0$. If λ_0 is chosen large enough, the boundary symbol is estimated as

$$c_\eta(|\lambda| + |\tau|) \le |s(\lambda, \tau)| \le C_\eta(|\lambda| + |\tau|), \quad \lambda \in \Sigma_{\pi/2+\eta},\ \tau \in \Sigma_\eta,\ |\lambda| \ge \lambda_0.$$

By this estimte, the operator-valued $\mathcal{H}^\infty$-calculus allows for an application of the Kalton and Weis theorem [9, Theorem 4.4], which shows $Sh = g_h$ has a unique solution in the right regularity class.

The main result on the linearized problem (2.4)-(2.6) now can be stated as

Theorem 2.4. *Let $1 < p < \infty$ be fixed, $p \ne 3/2, 3$, and assume that $\rho_+ \ne \rho_-$ and $\mu_{\infty\pm}$, $\kappa_{\infty\pm}$, $d_{\infty\pm} > 0$. Then the linear problem (2.4)-(2.6) admits a unique solution $(u, \pi, \gamma\pi, \theta, h) \in \mathbb{E}(J)$ if and only if the data (u_0, θ_0, h_0) and $(f_u, f_d, g_u, g_j, f_\theta, g_\theta, g_\pi, g_h)$ satisfy the regularity conditions:*

$$(u_0, \theta_0, h_0) \in X_\gamma, \quad (f_u, f_d, g_u, g_j, f_\theta, g_\theta, g_\pi, g_h) \in \mathbb{F}(J),$$

and the compatibility conditions:

$$\begin{aligned} \operatorname{div} u_0 = f_d(0) \quad &\text{in } \dot{\mathbb{R}}^n, \\ -P_{\mathbb{R}^{n-1}}[\![\mu_\infty(\nabla u_0 + [\nabla u_0]^\mathsf{T})\nu]\!] = P_{\mathbb{R}^{n-1}} g_u(0), \quad [\![u_0']\!] = g_j(0) \quad &\text{on } \mathbb{R}^{n-1}, \\ [\![\theta_0]\!] = 0, \quad -[\![d_\infty \partial_n \theta_0]\!] = g_\theta(0) \quad &\text{on } \mathbb{R}^{n-1}. \end{aligned}$$

The solution map $[(f_u, f_d, g_u, g_j, f_\theta, g_\theta, g_\pi, g_h, u_0, \theta_0, h_0) \mapsto (u, \pi, \gamma\pi, \theta, h)]$ is continuous between the corresponding spaces.

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On the solution of nonlinear elliptic equation in a thin domain

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1 Introduction

We consider the following Neumann boundary value problem for a semilinear elliptic equation:

$$\begin{cases} -\varepsilon^2\Delta u + u - u^p = 0 \text{ in } \Omega_\varepsilon, \\ \dfrac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega_\varepsilon, \end{cases} \tag{1}$$

where $\varepsilon > 0$ and $p > 1$, Ω_ε is a (bounded or unbounded) domain in $\mathbb{R}^2$ which may depend on ε. We assume that the boundary $\partial\Omega_\varepsilon$ is smooth, and ν denotes the outer unit normal to $\partial\Omega_\varepsilon$. If Ω_ε is bounded, then this problem is related to a stationary problem for activator-inhibitor systems modeling biological pattern formation which was proposed by Gierer and Meinhardt [4]. There are already many papers which studies (1) for the case Ω_ε is independent of ε, which have revealed the richness of the solution set when ε is sufficiently small. For example, the existence of solutions with spike-layers on the boundary, as well as in the interior of the domain (see, e.g., [8, 9, 12, 6]). In particular, in [8, 9] it is proved that for ε sufficiently small there exist "least-energy solutions" and the least-energy solution has only one local maximum point $P_\varepsilon \in \partial\Omega_\varepsilon$. Moreover, the mean curvature of $\partial\Omega_\omega$ at P_ε approaches its maximum over $\partial\Omega_\varepsilon$ as $\varepsilon \to 0$. In [12] Wei proved that if $P \in \partial\Omega_\varepsilon$ is a nondegenerate critical point of the mean curvature, then there exists a solution which has its peak near P for ε sufficiently small. These results concerning the least-energy solutions are based on the properties of function w, where w is a solution of the boundary value problem

$$\begin{cases} \Delta w - w + w^p = 0 \quad \text{and } w > 0 \quad \text{in } \mathbb{R}^n, \\ \lim_{|x|\to+\infty} w(x) = 0, \quad w(0) = \max_{x\in\mathbb{R}^n} w(x). \end{cases} \tag{2}$$

It is well-known that the problem (2) has a unique solution ([5, 7]) and the solution w is spherically symmetric about the origin and decays exponentially in $r = |x|$ ([5]). On the other hand, very little is known about solutions of (2) which don't decay in all directions. Dancer [3] first showed that there are solutions which are periodic in one direction and decay in all the other directions.

All these results which we have denoted above are concerns with domains which are independent of the diffusion coefficient ε. In this paper, we explore the case where the domain Ω_ε depends on ε. In such a case, we would like to understand what governs the location of the peak. Concerning the case where Ω_ε depends on ε, Berestyski and Wei [1] considered (1) in a thin domain in $\mathbb{R}^2$:

$$\Omega_\varepsilon = \mathbb{R} \times (0, \varepsilon L),$$

where $L > 0$. They have shown that there exists $L^* > 0$ satisfying the following: (i) if $L \leqslant L^*$, then the least energy solution $u_\varepsilon(x, y)$ of (1) is independent of y. More precisely, $u_\varepsilon(x, y) = \Phi(x/\varepsilon)$, where $\Phi(x)$ is a unique solution of

$$\begin{cases} -\Phi'' + \Phi - \Phi^p = 0 \quad \text{and} \quad \Phi > 0 \quad \text{in } \mathbb{R}, \\ \lim_{x^2\to+\infty} \Phi(x) = 0, \quad \Phi(0) = \max_{x\in R} \Phi(x), \end{cases} \tag{3}$$

and (ii) if $L > L^*$, then the least energy solution depends on y. Therefore, if the domain depends on ε, the least energy solution does not always concentrate on a single point. See also a related result by Terracini, Tzvetkov and Visciglia [11] for the case $\mathbb{R}^n \times M$, where M compact Riemannian manifold. On the other hand, in [10], it was considered the case where a domain $\Omega_\varepsilon \subset \mathbb{R}^2$ shrinks to a Jordan curve as $\varepsilon \downarrow 0$ more slowly, that is, Ω_ε is a domain of constant width $\varepsilon\ell_\varepsilon$, where ℓ_ε is a smooth function of ε and satisfies

$$\lim_{\varepsilon\to 0} \varepsilon\ell_\varepsilon = 0, \quad \lim_{\varepsilon\to 0} \ell_\varepsilon = +\infty, \quad \limsup_{\varepsilon\to 0} \frac{3\sqrt{2}}{\ell_\varepsilon} \log\frac{1}{\varepsilon} < 1.$$

Then, it was shown in [10] that solutions have the same properties as those in the case where Ω is not depend on ε. Especially, the least-energy solution has only one maximum point P_ε which lies on the boundary, and the mean curvature of $\partial\Omega_\varepsilon$ at P_ε approaches its maximum over $\partial\Omega_\varepsilon$ as $\varepsilon \downarrow 0$.

Motivated by these facts, we study (1) in a domain $\Omega_\varepsilon \subset \mathbb{R}^2$ which shrinks quickly to a straight line in the plane as $\varepsilon \downarrow 0$. We prove the existence of the least-energy solution, and investigate the asymptotic form of the least-energy solution in such a domain. By [1], it is natural to think that the least energy solution converges to Φ in some sense. However it is not clear that where the "concentration line" locates. We will show that the least energy solution concentrates to the shortest line which goes across the shrinking domain. Further, if there are several shortest line, we show the least energy solution concentrates at the "flattest" place.

This paper is organized as follows. In section 2, we state our result. In section 3, we introduce a change of coordinates which is essential for the proof of Theorem 2 which is given in section 2. In section 4, we give an upper bound of the energy of the least energy solution which implies the convergence to Φ.

2 Statement of main results

Let $f(X) \in C^\infty(\mathbb{R};\mathbb{R})$ be a function satisfying the following conditions:

$$f(X+L) = f(X), \ \min_{X\in\mathbb{R}} f(X) = f(0) = 1, \ \min_{f(X)=1} f''(X) = f''(0), \tag{4}$$

for some $L > 0$. For $\delta > 0$, we define a domain $\widetilde{\Omega}_\delta$ as

$$\widetilde{\Omega}_\delta = \{(X,Y) \in \mathbb{R}^2 \mid X \in \mathbb{R}, \ 0 < Y < \delta f(X)\}. \tag{5}$$

We consider the following Neumann problem:

$$\begin{cases} -\varepsilon^2 \Delta u + u - u^p = 0 \text{ in } \widetilde{\Omega}_{\varepsilon l_\varepsilon}, \\ \dfrac{\partial u}{\partial \nu} = 0 \text{ on } \partial\widetilde{\Omega}_{\varepsilon l_\varepsilon}, \end{cases} \tag{6}$$

where $\varepsilon > 0$, $p > 1$, $l_\varepsilon > 0$ and $l_\varepsilon \to 0$ as $\varepsilon \to 0$. For $u \in H^1(\widetilde{\Omega}_{\varepsilon l_\varepsilon})$, we define functionals $\widetilde{J}_\varepsilon$ and $\widetilde{I}_\varepsilon$ by

$$\begin{aligned} \widetilde{J}_\varepsilon(u) &:= \frac{1}{2}\int_{\widetilde{\Omega}_{\varepsilon l_\varepsilon}} \left(\varepsilon^2|\nabla u|^2 + |u|^2\right) dXdY - \frac{1}{p+1}\int_{\widetilde{\Omega}_{\varepsilon l_\varepsilon}} |u|^{p+1}\, dXdY, \\ \widetilde{I}_\varepsilon(u) &:= \int_{\widetilde{\Omega}_{\varepsilon l_\varepsilon}} \left(\varepsilon^2|\nabla u|^2 + |u|^2\right) dXdY - \int_{\widetilde{\Omega}_{\varepsilon l_\varepsilon}} |u|^{p+1}\, dXdY. \end{aligned}$$

Definition 1 (Least energy solution)**.** *We call u_ε a least-energy solution of* (6) *if u_ε is positive, solve* (6) *and has the smallest energy $\widetilde{J}_\varepsilon$ among all the positive solutions to* (6)*. The critical value $c_\varepsilon = \widetilde{J}_\varepsilon(u_\varepsilon)$ is called the least-energy for the Neumann problem* (6)*.*

Since a solution u_ε of (6) satisfies $\widetilde{I}_\varepsilon(u_\varepsilon) = 0$, we can obtain a least-energy solution by the following minimizing problem:

$$\text{Minimize}\{\widetilde{J}_\varepsilon(v) \mid v \in H^1_N(\widetilde{\Omega}_{\varepsilon l_\varepsilon}) \setminus \{0\},\ \widetilde{I}_\varepsilon(v) = 0\},$$

where

$$H^1_N(\Omega) := \overline{\{u \in C^\infty(\bar{\Omega}) \cap H^1(\Omega) \mid \partial u/\partial\nu = 0 \text{ on } \partial\Omega\}}^{H^1}.$$

For the case where $f \equiv 1$, it was shown in [1] that, for $0 < \varepsilon \ll 1$, the least-energy solution u_ε satisfies $u_\varepsilon(x, y) = \Phi(x/\varepsilon)$, where Φ is the solution of (3).

The existence of the least-energy solution of (6) can proved by a standard argument using concentration compactness lemma by Lions.

Theorem 1. *For $\varepsilon > 0$ sufficiently small, there exists a least-energy solution of* (6)*.*

Next, we study the asymptotic form of the least-energy solution as $\varepsilon \downarrow 0$. Rescale the coordinate system as follows: $(X, Y) \mapsto (\varepsilon x, \varepsilon y)$. Then, the problem (6) becomes

$$\begin{cases} -\Delta u + u - u^p = 0 \text{ in } \Omega_{l_\varepsilon}, \\ \dfrac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega_{l_\varepsilon}, \end{cases} \tag{7}$$

where

$$\Omega_{l_\varepsilon} := \{(x, y) \in \mathbb{R}^2 \mid x \in \mathbb{R},\ 0 < y < l_\varepsilon f(\varepsilon x)\}.$$

Furthermore, the rescaled energy J_ε and Nehari function I_ε becomes

$$J_\varepsilon(u) := \frac{1}{2}\int_{\Omega_{l_\varepsilon}} \left(|\nabla u|^2 + |u|^2\right) dxdy - \frac{1}{p+1}\int_{\Omega_{l_\varepsilon}} |u|^{p+1}\, dxdy, \tag{8}$$

$$I_\varepsilon(u) := \int_{\Omega_{l_\varepsilon}} \left(|\nabla u|^2 + |u|^2\right) dxdy - \int_{\Omega_{l_\varepsilon}} |u|^{p+1}\, dxdy. \tag{9}$$

We denote the set of least-energy solutions of the rescaled problem (7) by G_ε.

For $u, v \in H^1(\Omega_{l_\varepsilon})$, we set

$$\langle u, v\rangle = \int_{\mathbb{R}}\int_0^{l_\varepsilon f(\varepsilon x)} (\nabla u \cdot \nabla v + uv)\, dxdy,$$

and $\|u\|^2_{H^1(\Omega_{l_\varepsilon})} = \langle u, u\rangle$. Our second result is the following.

Theorem 2. *There exists $x_\varepsilon \in \mathbb{R}$ such that the least-energy solution $u_\varepsilon(x,y)$ of* (7) *satisfies*

$$u_\varepsilon(x - x_\varepsilon, y) < Ce^{-\lambda|x|},$$

where $C > 0$ and $\lambda > 0$ are independent of ε. Furthermore, we have

$$\lim_{\varepsilon \to 0} l_\varepsilon^{-1} ||u - \Phi(\cdot - x_\varepsilon)||^2_{H^1(\Omega_{l_\varepsilon})} = 0.$$

The concentration line x_ε converges to the set

$$\Sigma = \left\{ x \in \mathbb{R} \mid f(x) = 1,\ f''(x) = \min_{f(\tilde{x})=1} f''(\tilde{x}) \right\}.$$

3 Local coordinate system

In this section we make a change of coordinates to make the domain Ω_ε to be straight. We will be careful to preserve the Neumann boundary condition after the change of coordinates.

3.1 New coordinates in the macro scale

Let $\delta > 0$ and let $\widetilde{\Omega}_\delta$ be the domain given in (5). Let $G(Y; \tilde{\xi})$ be the solution of the following initial value problem:

$$\partial_Y G(Y; \tilde{\xi}) = -Y F(G(Y; \tilde{\xi})),\ G(0; \tilde{\xi}) = \tilde{\xi}, \tag{10}$$

where $F(u) = f'(u)/f(u)$. By the local wellposedness of (10), we have the following.

Proposition 1. *Let $Y(\tilde{\xi}) > 0$ be the maximal existence time of the solution $G(Y; \tilde{\xi})$ of* (10)*. Then, there exists $\delta_0 > 0$ such that for $\delta \in (0, \delta_0)$, $\inf_{\tilde{\xi} \in \mathbb{R}} Y(\tilde{\xi}) > \delta \sup_{X \in \mathbb{R}} f(X)$.*

Since for $\delta \in (0, \delta_0)$ the domain $\widetilde{\Omega}_\delta$ is written in the form

$$\widetilde{\Omega}_\delta = \{(X, \tilde{\eta} f(X)) \mid X \in \mathbb{R},\ \tilde{\eta} \in (0, \delta)\},$$

we see

$$\begin{aligned} \widetilde{\Omega}_\delta &\subset \{(X, Y) \in \mathbb{R}^2 \mid \exists \tilde{\xi} \in \mathbb{R} \text{ s.t. } 0 < Y < Y(\tilde{\xi}),\ X = G(Y; \tilde{\xi})\} \\ &= \{(G(Y; \tilde{\xi}), Y) \mid \tilde{\xi} \in \mathbb{R},\ 0 < Y < Y(\tilde{\xi})\}. \end{aligned}$$

Using the above property of $\widetilde{\Omega}_\delta$, we introduce a new coordinate system. For $(X, Y) \in \widetilde{\Omega}_\delta$, we define $(\tilde{\eta}, \tilde{\xi})$ as

(i) $\tilde{\eta} = \tilde{\eta}(X, Y) = Y/f(X)$,

(ii) $\tilde{\xi} = \tilde{\xi}(X, Y)$ is an initial data of a solution of (10), and

$$X = G(Y; \tilde{\xi}) \qquad \text{for} \quad 0 < Y < Y(\tilde{\xi}).$$

The definition of $\tilde{\xi}$ is implicit. However it is well-defined because we have the uniqueness of the solution to (10) and the construction of the domain $\widetilde{\Omega}_\delta$. The transformation $(X, Y) \mapsto (\tilde{\xi}, \tilde{\eta})$ maps the region $\widetilde{\Omega}_\delta$ to

$$V_\delta = \{(\tilde{\xi}, \tilde{\eta}) \in \mathbb{R}^2 \mid \tilde{\xi} \in \mathbb{R},\ \tilde{\eta} \in (0, \delta)\}. \tag{11}$$

Now, we investigate several properties of $G(Y; \tilde{\xi})$. By direct computation, we have the following estimates.

Lemma 1. *There exists $\delta_1 \in (0, \delta_0)$ such that we see, for $0 < Y < \delta_1$, that*

$$\sup_{\tilde{\xi} \in \mathbb{R}} |\partial_\xi G(Y; \tilde{\xi}) - 1| \leq CY^2,$$
$$\sup_{\tilde{\xi} \in \mathbb{R}} |\partial_\xi \partial_Y G(Y; \tilde{\xi})| \leq CY$$
$$\sup_{\tilde{\xi} \in \mathbb{R}} |\partial_\xi^2 G(Y; \tilde{\xi})| \leq CY^2.$$

where C is a positive constants.

We show that the new coordinate system $(\tilde{\xi}, \tilde{\eta})$ is useful to consider the Neumann problem. From (11), the boundary $\partial\widetilde{\Omega}_\delta$ is transformed to $\partial V_\delta = \partial_0 V_\delta \cup \partial_\delta V_\delta$, where $\partial_0 V_\delta = \{\tilde{\eta} = 0\}$ and $\partial_\delta V_\delta = \{\tilde{\eta} = \delta\}$.

Proposition 2. *The unit outer normal vector to $\partial_0 V_\delta$ becomes $(0, -1)$, and the unit outer normal to $\partial_\delta V_\delta$ becomes $(0, 1)$.*

3.2 New coordinate in the micro scale

For $(X, Y) \in \widetilde{\Omega}_{\varepsilon l_\varepsilon}$, we put

$$X = \varepsilon x, \qquad Y = \varepsilon y. \tag{12}$$

Then, the transformation (12) maps $\widetilde{\Omega}_{\varepsilon l_\varepsilon}$ to the folowing domain:

$$\Omega_{l_\varepsilon} = \{(x, y) \in \mathbb{R}^2 \mid x \in \mathbb{R},\ y \in (0,\ l_\varepsilon f(\varepsilon x))\}.$$

On the other hand, for $(\tilde{\xi}, \tilde{\eta}) \in V_\delta$, we define (ξ, η) as

$$\tilde{\xi} = \varepsilon \xi, \qquad \tilde{\eta} = \varepsilon \eta. \tag{13}$$

Since we have a map from (X, Y) to $(\tilde{\xi}, \tilde{\eta})$, we see

(i) $\eta = y/f(\varepsilon x)$,

(ii) $\varepsilon x = G(\varepsilon y; \varepsilon\xi)$, that is, $G(\varepsilon y; \varepsilon\xi)/\varepsilon$ is a solution of the following problem and ξ is its initial data:

$$\partial_y g_\varepsilon(y;\xi) = -\varepsilon y \frac{f'(\varepsilon g_\varepsilon(y;\xi))}{f(\varepsilon g_\varepsilon(y;\xi))}, \qquad g_\varepsilon(0,\xi) = \xi. \tag{14}$$

We note that the solution $g_\varepsilon(y;\xi)$ of (14) satisfies

$$(1,\ \varepsilon\eta f'(\varepsilon x)) \cdot (\partial_y g_\varepsilon(y;\xi),\ 1) = 0. \tag{15}$$

It is easy to see that (ξ,η) is in

$$V_{l_\varepsilon} = \{(\xi,\eta) \in \mathbb{R}^2 \mid \xi \in \mathbb{R},\ \eta \in (0, l_\varepsilon)\}.$$

The Jachobian and the Laplacian in the new coordinates can be expressed as follows.

$$\begin{aligned}
&\det \frac{\partial(x,y)}{\partial(\xi,\eta)} = f(\varepsilon x)(1+\varepsilon^2\eta^2 f'(\varepsilon x)^2)^{-1} \partial_\xi G(\varepsilon y, \varepsilon\eta),\\
&\Delta = \left(1+\varepsilon^2\eta^2 f'(\varepsilon x)^2\right) \left(\partial_\xi G(\varepsilon y,\varepsilon\xi)^{-2}\partial_\xi^2 + \frac{1}{f(\varepsilon x)^2}\partial_\eta^2\right)\\
&+\varepsilon \partial_\xi G(\varepsilon y,\varepsilon\xi)^{-3}\left(\frac{f'(\varepsilon x)}{f(\varepsilon x)}\partial_\xi G(\varepsilon y,\varepsilon\xi)^2 - \partial_\xi^2 G(\varepsilon t,\varepsilon\xi)\left(1+\varepsilon^2\eta^2 f'(\varepsilon x)^2\right)\right.\\
&\left.-\varepsilon\eta f'(\varepsilon x)\partial_\xi G(\varepsilon y,\varepsilon\xi)\partial_Y \partial_\xi G(\varepsilon y,\varepsilon\xi)\right)\partial_\xi\\
&+\varepsilon^2\eta\left(2\frac{f'(\varepsilon x)^2}{f(\varepsilon x)^2} - \frac{f''(\varepsilon x)}{f(\varepsilon x)}\right)\partial_\eta
\end{aligned}$$

Further, using Lemma 1, we can show the difference between x and ξ are "small". That is, we can show

$$|\varepsilon\xi - \varepsilon x| \le C\varepsilon^2 l_\varepsilon^2,$$

for some constant $C > 0$.

4 Upper bound of the energy of least energy solution

First, (without proof), we claim that it is enough to compute all quantities up to $\varepsilon^2 l_\varepsilon$ order. So, we ignore the $o(\varepsilon^2 l_\varepsilon)$ terms. If we do so, then the

Jachobian and Laplacian will be

$$\det\frac{\partial(x,y)}{\partial(\xi,\eta)} \sim f(\varepsilon\xi), \tag{16}$$

$$\Delta \sim \partial_\xi^2 + \frac{1}{f(\varepsilon\xi)^2}\partial_\eta^2 + \varepsilon\frac{f'(\varepsilon\xi)}{f(\varepsilon\xi)}\partial_\xi. \tag{17}$$

For $u \in H^1(\mathbb{R})$, we set

$$J_0(u) = \frac{1}{2}\int_{\mathbb{R}} \left(|\nabla u|^2 + |u|^2\right)\, dx - \frac{1}{p+1}\int_{\mathbb{R}} |u|^{p+1}\, dx.$$

Lemma 2. *Let u_ε be a least-energy solution of* (7)*. Then, we have*

$$J_\varepsilon(u_\varepsilon) \le l_\varepsilon J_0(\Phi) + C_p f''(0) l_\varepsilon \varepsilon^2 + o(l_\varepsilon\varepsilon^2),$$

where C_p is an absolute constant which depends only on p. In particular

$$C_p = \frac{1}{4}\left(\int_{\mathbb{R}} \Phi^p(\xi)\, d\xi + \frac{p-1}{p+1}\int_{\mathbb{R}} \Phi^{p+1}(\xi)\xi^2\, d\xi\right)$$

Proof. We use the transformation from (x,y) to (ξ,η), where we note

$$(\xi,\eta) \in V_{l_\varepsilon} = \left\{(\xi,\eta) \in \mathbb{R}^2 \mid \xi \in \mathbb{R},\ \eta \in (0,l_\varepsilon)\right\}.$$

We first calculate $I_\varepsilon(\Phi(\xi))$, where Φ is a solution of the problem (3), and the definition of $I_\varepsilon(\cdot)$ is (9). The function Φ is spherically symmetric about the origin and decays exponentially in $r = |\xi|$. It follows from (16) and (17) that

$$I_\varepsilon(\Phi(\xi)) = \int_0^{l_\varepsilon}\int_{\mathbb{R}} \left(-\left(\partial_\xi^2 + \frac{1}{f(\varepsilon\xi)^2}\partial_\eta^2 + \varepsilon\frac{f'(\varepsilon\xi)}{f(\varepsilon\xi)}\partial_\xi\right)\Phi + \Phi - \Phi^p\right)\Phi\, f(\varepsilon\xi) \\ + o(\varepsilon^2 l_\varepsilon).$$

Since Φ is the solution of (3) and do not depend on η, we obtain

$$I_\varepsilon(\Phi(\xi)) = -\varepsilon^2 l_\varepsilon \frac{f''(0)}{2} M_p + o(\varepsilon^2 l_\varepsilon).$$

Here, we put $M_p = \int_{\mathbb{R}} \Phi(\xi)^2\, d\xi$. Now, we find $t_\varepsilon > 0$ which satisfies $I_\varepsilon(t_\varepsilon\Phi(\xi)) = 0$. Then, such t_ε is given by

$$t_\varepsilon = 1 + \varepsilon^2\frac{f''(0)}{2(p-1)}\frac{M_p}{N_p} + o(\varepsilon^2), \tag{18}$$

where $N_p = \int_{\mathbb{R}} \Phi(\xi)^{p+1}\, d\xi$. Since u_ε is the least-energy solution, it is clear that $J_\varepsilon(u_\varepsilon) \leqslant J_\varepsilon(t_\varepsilon \Phi)$. We calculate $J_\varepsilon(t_\varepsilon\Phi)$ and have

$$\begin{aligned} J_\varepsilon(t_\varepsilon\Phi) &= I_\varepsilon(t_\varepsilon\Phi) + \left(\frac{1}{2} - \frac{1}{p+1}\right) t_\varepsilon^{p+1} \int_0^{l_\varepsilon} \int_{\mathbb{R}} \Phi^{p+1}(\xi) f(\varepsilon\xi)\, d\xi d\eta \\ &= \frac{p-1}{2(p+1)} l_\varepsilon t_\varepsilon^{p+1} \int_{\mathbb{R}} \Phi^{p+1}(\xi) \left(1 + \frac{1}{2} f''(0)\varepsilon^2\xi^2\right) d\xi + o(\varepsilon^2 l_\varepsilon). \end{aligned}$$

Substituting (18), we have

$$\begin{aligned} J_\varepsilon(t_\varepsilon\Phi) &= \frac{p-1}{2(p+1)} l_\varepsilon \int_{\mathbb{R}} \Phi^{p+1}(\xi)\, d\xi + \varepsilon^2 l_\varepsilon \frac{f''(0)}{4} \frac{M_p}{N_p} \int_{\mathbb{R}} \Phi^{p+1}(\xi)\, d\xi \\ &\quad + \varepsilon^2 l_\varepsilon \frac{p-1}{4(p+1)} f''(0) \int_{\mathbb{R}} \Phi^{p+1}(\xi)\xi^2\, d\xi + o(\varepsilon^2 l_\varepsilon) \\ &= \frac{p-1}{2(p+1)} l_\varepsilon \int_{\mathbb{R}} \Phi^{p+1}(\xi)\, d\xi + \varepsilon^2 l_\varepsilon \frac{f''(0)}{4} \left[M_p + \frac{p-1}{p+1} \int_{\mathbb{R}} \Phi^{p+1}(\xi)\xi^2\, d\xi\right] \\ &\quad + o(\varepsilon^2 l_\varepsilon). \end{aligned}$$

Note that

$$J_0(\Phi) = \frac{p-1}{2(p+1)} \int_{\mathbb{R}} \Phi^{p+1}(\xi)\, d\xi.$$

Therefore, $J_\varepsilon(t_\varepsilon\Phi)$ satisfies

$$J_\varepsilon(t_\varepsilon\Phi) = l_\varepsilon J_0(\Phi) + \varepsilon^2 l_\varepsilon \frac{f''(0)}{4} \left[M_p + \frac{p-1}{p+1} \int_{\mathbb{R}} \Phi^{p+1}(\xi)\xi^2\, d\xi\right] + o(\varepsilon^4 l_\varepsilon),$$

which has finished a proof of Lemma 2. □

The proof of the remaing part of Theorem 2 can be shown by two steps. First, we show the strong convergence of the least energy solution to Φ and the exponential decay in the x direction. Next, computing the energy assuming that the concentration line converges to the wrong place we have a contradiction with Lemma 2.

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Immersed, Free Boundaries in Complex Fluids

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The interaction of flexible free-to-move boundaries with non-Newtonian (complex) fluids is receiving increased attention. Accurately capturing the coupled dynamics of this intricate flow-structure interaction poses challenging mathematical and computational problems. First, the dynamics of the microstructure, responsible for the non-Newtonian (viscoelastic) behavior, couples to the macroscopic flow and consequently increases the dimensionality of the system. Second, a flexible fiber or boundary in a viscoelastic flow can experience large normal stresses (absent in Newtonian fluids) which introduce additional difficulties to the computation of the motion of non-extensible immersed boundaries.

The Immersed Boundary (IB) Method, introduced by Peskin [4], offers a flexible framework for the modeling and simulation of this type of systems. It combines a Lagrangian representation of the immersed structures with an Eulerian flow description. The immersed structures often have very stiff components and as a consequence strong forces are generated, which in turn induce severe time-step restrictions for explicit discretization [6, 5].

We will describe in this work two recent advances to IB method as well as the application of these new techniques to the investigation of some free boundary problems. First, we will focus on the 2D case and introduce two ideas which allow for a fast and robust computation free of high order stability constraints [3]. After this we will consider the 3D case, for which a straightforward generalization is not possible and new ideas had to be developed. One of these ideas is the use of treecode strategy for the fast evaluation of the flow-structure interaction [1]. Specifically, we will show that the flow-structure operator can be seen as a *multipole summation* with a suitable choice of potential. Using the Singular Value Decomposition and a new, efficient, iterative algorithm we compute L^2-optimal far field expansions of this potential to be used in an effective treecode strategy. This treecode approach allows for a very fast evaluation of the flow-structure operator. With that in hand, we solve the implicit system for the interface configuration with a Krylov subspace method, employing the treecode evaluation at every iteration.

We apply these new computational advances to the investigation of peristalsis in a viscoelastic flow [2]. Peristalsis is a mechanism for transporting fluid or immersed particles in a channel by waves of contraction. It occurs in many biological organisms as well as in several human designed systems. In this study, we investigate numerically the peristaltic pumping of an incompressible viscoelastic fluid using the simple Oldroyd-B model coupled to the Navier-Stokes equations. The pump's walls are assumed to be massless immersed fibers whose prescribed periodic motion and flow interaction is handled by our new IB Method. This allows us to explore an unprecedented range of parameter regimes, nearly all possible occlusion ratios and Weissenberg numbers in excess of 100. Our numerical investigation reveals rich, highly concentrated stress structures and new, striking dynamics. The investigation also points to the limitations of the Oldroyd B model, with a potential finite time blow-up, and to the role of numerical regularization.

Time permitting, we will also report on our progress on the modeling and simulation of free swimmers in viscoelastic flows, for both Oldroyd B and FENE models.

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Evolution of spirals by an eikonal-curvature flow equation with a single level set formulation

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In the memory of Rentaro Agemi

1. Introduction

Burton, Cabrera and Frank [BCF51] proposed a theory of crystal growth with aid of screw dislocations. They pointed out that screw dislocations supply spiral steps to a crystal surface when the screw dislocations appear on the surface. Steps evolve catching adatoms as they climb a spiral staircase, and thus the surface evolves. Burton et al. calculated the step velocity with Gibbs-Thomson effect, and derived an eikonal-curvature flow of the form

$$V = v_\infty(1 - \rho_c \kappa) \tag{1}$$

by regarding the evolution of steps as evolution of curves on the plane, where v_∞ is the velocity of straight line steps, ρ_c is the critical radius reflecting the Gibbs-Thomson effect, V is the normal velocity of the curve which denotes the location of steps, and κ is the curvature of the curve with opposite direction of V. Note that we shall use the words ‘step’, ‘curve’, and ‘spiral’ interchangeably because of the above background. One can find a complex spiral patterns on the growing crystal surface, which is caused by the evolution of spiral steps and collision with each other. Several models for this phenomena are proposed by [KP98], [Kob10] with phase field models, and by [Sme00] and [Oht03] with level set methods.

In this talk we consider the evolution of spiral curves by an eikonal-curvature flow with the level set formulation by [Oht03], and investigate behavior of spirals with mathematical results of the formulation. In particular, two characteristic problems are considered; one is behavior of a bunch of steps, which corresponds to variety of heights of the steps. In this problem one can find the crucial difference between phase field models and our formulations. The other is on the stationary solutions caused by an ‘inactive pair’, which corresponds to the stationary curve under an eikonal-curvature flow equation. Formally, the circle whose radius is ρ_c does not evolve under (1), and it is unstable. In this talk we shall find stable stationary curves like as the above.

Results on §2.2 are partly joint work with Shun’ichi Goto and Maki Nakagawa, and those on §2.3 and §3 are joint work with Yen-Hsi Richard Tsai and Yoshikazu Giga.

2. Formulation and basic properties

We here introduce a level set formulation with a single auxiliary function for evolving spirals by an eikonal-curvature flow equation. Its crucial difficulty lies in the fact that a spiral curve generally does not divide a domain into two subdomains so that the usual level set formulation

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$\{x;\ u(t,x) = 0\}$ does not work well. To overcome this difficulty, we combine a level set method and a sheet structure function due to Kobayashi [Kob10] or Karma and Plapp [KP98] in their phase field models.

2.1. Level set formulation for evolving spirals

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary. We assume that there exist $N(\geq 1)$ spiral centers denoted by $a_1, \ldots, a_N \in \Omega$, and each center may have multiple spirals. In this talk we also regard an open neighborhood U_j of a_j as a j-th center of spirals interchangeably with a_j. Set $W = \Omega \setminus (\bigcup_{j=1}^N \overline{U}_j)$, and we here consider evolving spirals Γ_t at time $t \geq 0$ on $\overline{W}$ having the direction $\mathbf{n}$ of the evolution, where $\mathbf{n}\colon \Gamma_t \to S^1$ is a continuous unit normal vector field of Γ_t. The evolution equation is the rescaled equation of (1) on time of the form

$$V = C - \kappa \quad \text{on } \Gamma_t \tag{2}$$

with a constant C, and also impose that the end points of Γ_t always stay on ∂W with the orthogonality condition

$$\Gamma_t \perp \partial W. \tag{3}$$

For multiplicity of spirals let $m_j \in \mathbb{Z} \setminus \{0\}$ be a constant denoting the number and rotational orientation of spirals associated with a_j: $|m_j|$-spirals go around a_j with anti-clockwise (resp. clockwise) rotation if $m_j > 0$ (resp. $m_j < 0$) provided that spirals have positive velocity in the direction of $\mathbf{n}$. We shall discuss in detail how to determine m_j from physical situation in §2.3.

In [Oht03] the author propose a level set formulation for spirals Γ_t for $t \geq 0$ as

$$\Gamma_t = \{x \in \overline{W};\ u(t,x) - \theta(x) \equiv 0 \mod 2\pi\mathbb{Z}\}, \quad \mathbf{n} = -\frac{\nabla(u-\theta)}{|\nabla(u-\theta)|} \tag{4}$$

with a sheet structure function

$$\theta(x) = \sum_{j=1}^N m_j \arg(x - a_j).$$

The function θ is introduced by Kobayashi [Kob10] in his phase field model. Karma and Plapp [KP98] also introduce $\theta(x) = \arg x$ for a single spiral, i.e., for the case $N = 1$, $a_1 = 0$ and $m_1 = 1$. The function θ denotes helical layer structure of atoms in a crystal with screw dislocations. From the theory of dislocation and linear elasticity the surface height $h(t,x)$ satisfies

$$h = -h_0 \mathrm{div} \delta_{\Gamma_t} \mathbf{n},$$

where h_0 is a unit height of steps (see [HL68]). One can find $h = (h_0/2\pi)\theta$ whose discontinuity is only on Γ_t satisfies the above from straightforward calculation.

Our formulation is regarded interior and exterior of the crystal as the place where $z < h(t,x)$ and $z > h(t,x)$, respectively, provided that '$z = (h_0/2\pi)\sum_{j=1}^N m_j \arg(x - a_j)$'. To describe the above exactly we now introduce the covering space $\mathfrak{X}$ as in [Oht03] of the form

$$\mathfrak{X} := \{(x, \xi) \in \overline{W} \times \mathbb{R}^N;\ \frac{x - a_j}{|x - a_j|} = (\cos \xi_j, \sin \xi_j) \quad \text{for } j = 1, \ldots, N\},$$

where ${}_i$ is such that $=({}_1,\dots,{}_N)$. Then, the interior $\widetilde{I}_t$ or the exterior $\widetilde{O}_t$ and thus the steps $\widetilde{\ }_t$ at time t 0 is described by

$$\widetilde{I}_t=\{(x,\ )\in\mathfrak{X};\ u(t,x)\quad\sum_{j=1}^{N}m_j\ {}_j>0\},\quad \widetilde{O}_t=\{(x,\ )\in\mathfrak{X};\ u(t,x)\quad\sum_{j=1}^{N}m_j\ {}_j<0\},$$

$$\widetilde{\ }_t=\{(x,\ )\in\mathfrak{X};\ u(t,x)\quad\sum_{j=1}^{N}m_j\ {}_j=0\}$$

with an auxiliary function $u\colon[0,\infty)\quad\overline{W}\to\mathbb{R}$. Then we obtain (4) from this formulation and inequalities describing the interior and exterior. The above formulae play very important role in mathematical analysis, in particular, when we investigate behavior of spirals.

Naturally the sheet structure function should be a multi-valued function in our formulation, but locally our formulation is same as the usual level set of u . Then from straightforward calculation in the usual level set method we derive

$$V=\frac{u_t}{|\nabla(u\quad)|},\quad \kappa=\quad\mathrm{div}\frac{\nabla(u\quad)}{|\nabla(u\quad)|},$$

and thus we obtain the level set equation of the form

$$u_t\quad|\nabla(u\quad)|\quad\mathrm{div}\frac{\nabla(u\quad)}{|\nabla(u\quad)|}+C\Big\}=0\quad\text{in}\quad(0,T)\quad W, \tag{5}$$

$$\langle\vec{\ },\nabla(u\quad)\rangle=0\quad\text{on}\quad(0,T)\quad\partial W \tag{6}$$

from (2)–(3), where $\vec{\ }$ is the outer unit normal vector eld of ∂W, and $\langle\ ,\ \rangle$ is the usual inner product in $\mathbb{R}^2$ (see [Gig06] for detail).

2.2. Basic properties

The equation (5)–(6) is represented by

$$u_t+F(\nabla(u\quad),\nabla^2(u\quad))=0\quad\text{in}\quad(0,T)\quad W,$$
$$B(x,\nabla(u\quad))=0\quad\text{on}\quad(0,T)\quad\partial W$$

with $F\colon(\mathbb{R}^2\setminus\{0\})\quad\mathbb{S}^2\to\mathbb{R}$, $B\colon\partial W\quad\mathbb{R}\to\mathbb{R}$ and functions $\widetilde{F}$ and $\widetilde{B}$ of the form

$$F(p,x)=\quad\mathrm{trace}\quad I\quad\frac{p\otimes p}{|p|^2}\quad X\Big\}\quad C|p|,$$
$$B(x,p)=\langle\vec{\ },p\rangle,$$

where $\mathbb{S}^2$ is the space of 2 2 real symmetric matrices.

Note that F is degenerate elliptic, and then we consider solutions of (4)–(6) in usual viscosity solution sense (see [CGG91], [CIL92] or [Gig06] for detail). The author obtained the comparison principle, and the existence and uniqueness of viscosity solutions globally in time for a continuous initial data.

Theorem 1 ([Oht03]). *Let $u, v\colon [0,T) \quad \overline{W} \to \mathbb{R}$ be a viscosity sub- and supersolution of* (5)–(6) *on $(0,T) \quad \overline{W}$. If $u^* \quad v_*$ on $\{0\} \quad \overline{W}$, then $u^* \quad v_*$ on $(0,T) \quad \overline{W}$, where u^* (resp. v_*) is an upper (resp. lower) semicontinuous envelope of u (resp. v) of the form*

$$u^*(t,x) = \lim_{r\to 0} \sup\{u(s,y);\ |(t,x) \quad (s,y)| < r\}$$
$$(\textit{resp. } v_*(t,x) = \lim_{r\to 0} \inf\{v(s,y);\ |(t,x) \quad (s,y)| < r\}).$$

Theorem 2 ([Oht03]). *For $u_0 \in C(\overline{W})$ there exists a viscosity solution $u \in C([0,\infty) \quad \overline{W})$ on $(0,\infty) \quad \overline{W}$ with $u|_{t=0} = u_0$.*

In the above analysis, in particular on the comparison, we attempt to consider $w = u \qquad$ and apply the results by [GS93] and [Sat94] directly, but it does not work well since $\quad$ is a multi-valued function. To overcome this di culty we estimate $\tilde{u}^* \quad \tilde{v}_*$ in $[0,T) \quad \mathfrak{X}$ instead of $u^* \quad v_*$, where

$$\tilde{u}(t,x,\ ) := u(t,x) \quad \sum_{j=1}^{N} m_j \ _j, \tag{7}$$

and $\tilde{v}$ is similar as the above. Note that $\tilde{u}^*(t,x,\ ) = u^*(t,x) \quad \sum_{j=1}^{N} m_j \ _j$ and $\tilde{v}_*(t,x,\ ) = v_*(t,x) \quad \sum_{j=1}^{N} m_j \ _j$. Then, we derive the above results by revising the proofs in [GS93] or [Sat94] with $\tilde{u}$ and $\tilde{v}$ instead of $u \qquad$ and $v \qquad$, respectively.

2.3. Properties on the presented level set method

To describe an evolution of spirals we execute the followings;

(i) From given $\ _0$ and $\mathbf{n}$, we construct $u_0 \in C(\overline{W})$ and $\ (x) = \sum_{j=1}^{N} m_j \arg(x \quad a_j)$ satisfying

$$\ _0 = \{x \in \overline{W};\ u_0(x) \qquad (x) \equiv 0 \mod 2\ \mathbb{Z}\}. \tag{8}$$

(ii) Solve (5)–(6) with an initial data $u|_{t=0} = u_0$.

(iii) Draw $\ _t$ de ned by (4) (and construct the height function $h(t,x)$ from u if necessary).

It remains two problems to complete the above.

(Q1) **(Construction of initial configuration)** How to construct $u_0 \in C(\overline{W})$ and determine m_j from given $\ _0$?

(Q2) **(Uniqueness of level sets)** Is $\ _t$ uniquely determined from $\ _0$?

Uniqueness of level sets is come from the fact that $u_0 \in C(\overline{W})$ satisfying (8) is not unique for given $\ _0$. However, Chen, Giga and Goto [CGG91], or Evans and Spruck [ES91] obtained the uniqueness of level sets for geometric evolution equation. Although our equation is not geometric for u, Goto, Nakagawa and the author also derived the uniqueness result with revision of the proof of [CGG91] since our equation presented is geometric for '$u \qquad$'.

Theorem 3 ([GNO08]). *Let $u, v\colon [0,T) \times \overline{W} \to \mathbb{R}$ be a viscosity sub- and supersolution of (5)–(6) in $(0,T) \times \overline{W}$. Assume that*

$$\{(x,\theta) \in \mathfrak{X};\ \tilde{u}^*(0,x,\theta) > 0\} \subset \{(x,\theta) \in \mathfrak{X};\ \tilde{v}_*(0,x,\theta) > 0\}$$
$$(\textit{resp.}\ \{(x,\theta) \in \mathfrak{X};\ \tilde{u}^*(0,x,\theta) < 0\} \supset \{(x,\theta) \in \mathfrak{X};\ \tilde{v}_*(0,x,\theta) < 0\}),$$

where $\tilde{u}^*(t,x,\theta) = u^*(t,x) - \sum_{j=1}^N m_j\theta_j$ and $\tilde{v}_*(t,x,\theta) = v_*(t,x) - \sum_{j=1}^N m_j\theta_j$. Then,

$$\{(x,\theta) \in \mathfrak{X};\ \tilde{u}^*(t,x,\theta) > 0\} \subset \{(x,\theta) \in \mathfrak{X};\ \tilde{v}_*(t,x,\theta) > 0\}$$
$$(\textit{resp.}\ \{(x,\theta) \in \mathfrak{X};\ \tilde{u}^*(t,x,\theta) < 0\} \supset \{(x,\theta) \in \mathfrak{X};\ \tilde{v}_*(t,x,\theta) < 0\}),$$

for $t \in (0,T)$.

The result in [GNO08] is obtained for continuous solutions u and v. Fortunately, their result is extended to our statement with a little revision for semicontinuous solutions.

The basic strategy of the proof of Theorem 3 is based on [CGG91], i.e., modify v to $w = G(v_* - \theta) + \theta$ with lower semicontinous and nondecreasing function G to enjoy Theorem 1 between u and w with $\{(x,\theta) \in \mathfrak{X};\ \tilde{v}_*(t,x,\theta) > 0\} \supset \{(x,\theta) \in \mathfrak{X};\ \tilde{w}(t,x,\theta) > 0\}$. The function G is defined similarly as in [CGG91] with a little revision for our problem. Although w includes the multi-valued function θ, however we also obtain $G(s+2\pi) = G(s) + 2\pi$ for sufficiently large s with the revision to our problem, and thus w is well-defined in some sense.

For the problem of initial configuration Goto, Nakagawa and the author [GNO08] proved the existence of m_j and $u_0 \in C(\overline{W})$ for suitable Γ_0, and clarify class of Γ_0.

It is convenient for the initial configuration to classify spirals as in [GNO08] into two kind of spirals depending on the feature whether or not it touches $\partial\Omega$. In the following argument let $\Gamma_0 := \{P(s);\ s \in [0,\ell]\}$ be smooth enough, and s be an arclength parameter.

Definition 4. (i) For a given $a \in \Omega$ let $U \subset\subset \Omega$ be its neighborhood, and set $W = \Omega \setminus \overline{U}$. We say Γ_0 is a simple spiral on $\overline{W}$ associated with a if $P(s)$ satisfies

- (S1) $P(s)$ is a simple arc and $|\dot{P}(s)| = |(\mathrm{d}P/\mathrm{d}s)(s)| \neq 0$ for $s \in [0,\ell]$,
- (S2) $P(0) \in \partial U$, $P(\ell) \in \partial\Omega$ and $P(s) \in W$ for $s \in (0,\ell)$.

(ii) For a given $a_1, a_2 \in \Omega$ let $U_i \subset\subset \Omega$ be a neighborhood of a_i for $i = 1,2$, and set $W = \Omega \setminus (\overline{U_1} \cup \overline{U}_2)$. Assume that $\overline{U}_1 \cap \overline{U}_2 = \emptyset$. We say Γ_0 is a connecting spiral on $\overline{W}$ between a_1 and a_2 if $P(s)$ satisfies (S1) and

- (S2)' $P(0) \in \partial U_1$, $P(\ell) \in \partial U_2$ and $P(s) \in W$ for $s \in (0,\ell)$.

In the previous section we pointed out that $m_j \in \mathbb{Z} \setminus \{0\}$ is a number of rotational orientation for spirals associated with a_j. It is defined as follows.

Definition 5. Let Γ_0 be associated with a center a at $s = 0$. We say Γ_0 is anti-clockwise (resp. clockwise) rotational orientation (with respect to a) if

$$\mathbf{n}(P(s)) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \dot{P}(s) \qquad \text{resp. } \mathbf{n}(P(s)) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \dot{P}(s) \ .$$

The signed number $m_j \in \mathbb{Z} \setminus \{0\}$ of spirals associated with a_j is defined by

$$m_j = m_j^+ - m_j^-,$$

where $m_j^+, m_j^- \in \mathbb{N}$ are numbers of anti-clockwise and clockwise rotational orientations of spirals associated with a_j, respectively.

Then, Goto, Nakagawa and the author obtained the existence of a continuous initial data for a given suitable Γ_0.

Theorem 6 ([GNO08]). *Let Γ_0 be a union of single and connecting spirals with a continuous unit normal vector field $\mathbf{n}$ on Γ_t. Then, there exists $m_j \in \mathbb{Z} \setminus \{0\}$ and $u_0 \in C(\overline{W})$ satisfying* (8).

It is obtained from the existence of a branch of θ whose discontinuity is only on Γ_0.

Lemma 7 ([GNO08]). *Under the same hypothesis in Theorem 6, there exists $\theta_{\Gamma_0} \colon \overline{W} \setminus \Gamma_0 \to \mathbb{R}$ which is a smooth branch of $\theta(x) = \sum_{j=1}^{N} m_j \arg(x - a_j)$.*

Let us consider a tubular neighborhood of Γ_0 of the form

$$\Gamma_0^\delta := \{x \in \overline{W};\ \inf_{y \in \Gamma_0} |x - y| < \delta\}.$$

Then, the signed distance function from Γ_0 is well-defined in Γ_0^δ, and thus we construct u_0 with θ_{Γ_0} on $\overline{W} \setminus \Gamma_0^\delta$, and a linear interpolation between θ_{Γ_0} and $\theta_{\Gamma_0} + 2\pi$ using the signed distance function on Γ_0^δ.

However, the above way, in particular the construction of θ_{Γ_0} and a tubular neighborhood of Γ_0 are impractical. For practicability we now introduce an additive way from initial data with less centers and multiplicity of spirals. Let $\Gamma_{0,1}$ and $\Gamma_{0,2}$ be a part of Γ_0 satisfying $\Gamma_{0,1} \cap \Gamma_{0,2} = \emptyset$ and $\Gamma_{0,1} \cup \Gamma_{0,2} = \Gamma_0$, and they are described by

$$\Gamma_{0,i} = \{x \in \overline{W};\ u_i(x) - \theta_i(x) \equiv 0 \mod 2\pi\mathbb{Z}\},$$

with auxiliary functions $u_i \in C(\overline{W})$ and $\theta_i(x) = \sum_{k=1}^{N_i} m_{i,k} \arg(x - a_{i,k})$ for $i = 1, 2$. To construct $u_0 \in C(\overline{W})$ describing Γ_0 we first modify u_i as

$$v_i(x) = \Theta_i(x) + 2\pi k_i(x) + \pi H_1(\lambda_i(u_i - (\Theta_i(x) + 2\pi k_i(x))))$$

with suitable constants $\lambda_i > 1/\pi$ determined later, where $\Theta_i(x) = \sum_{k=1}^{N_i} m_{i,k} \Theta_{i,k}(x)$, $\Theta_{i,k} \colon \overline{W} \to [0, 2\pi)$ is a principal value of $\arg(x - a_{i,k})$, $k_i \colon \overline{W} \to \mathbb{Z}$ is a function satisfying

$$u_i(x) - (\Theta_i(x) + 2\pi k_i(x)) < \pi \quad \text{for } x \in \overline{W}$$

for $i = 1, 2$, and H_1 is a function defined as

$$H_1(\sigma) = \begin{cases} -1 & \text{if } \sigma < -1, \\ \sigma & \text{if } |\sigma| \le 1, \\ 1 & \text{if } \sigma > 1. \end{cases}$$

The coe cients λ_i for $i = 1, 2$ is chosen such that

$$\bigcap_{i=1}^{2} \{x \in \overline{W};\ |v_i(x) \quad (\Theta_i(x) + 2 \quad k_i(x))| < \quad \} = \emptyset.$$

Note that v_i still describes $\ _{0,i}$ as (4) for $i = 1, 2$, and $v_i \quad \Theta_i(x) \equiv \quad \mod 2 \quad \mathbb{Z}$ on $\ _{0,j}$ if $i \neq j$. Thus we set

$$u_0(x) := v_1(x) + v_2(x) + \quad ,$$

then we have obtained a desired function describing $\ _0$ by (4). Note that simple and connecting straight lines are given by constant functions as follows;

$$\begin{aligned}
&\{a_i + r(\cos \quad, \sin \quad) \in \overline{W};\ r > 0\} = \{x \in \overline{W}; \qquad \arg(x \quad a_i) \equiv 0 \quad \mod 2 \quad \mathbb{Z}\}, \\
&\{\sigma a_i + (1 \quad \sigma)a_j \in \overline{W};\ \sigma \in (0, 1)\} \\
&\qquad = \{x \in \overline{W}; \qquad (\arg(x \quad a_i) \quad \arg(x \quad a_j)) \equiv 0 \quad \mod 2 \quad \mathbb{Z}\}.
\end{aligned}$$

Here we have assumed that each spirals are anti-clockwise rotational orientations with respect to a_i. From the above formulae and additive way we obtain $u_0 \in C(\overline{W})$ for $\ _0$ which is a union of straight lines.

3. Behavior of spirals from phenomena

Our level set formulation, in particular the results of comparison in Theorem 1 and Theorem 3 enables us to study behavior of spirals. As their applications we investigate two kinds of behavior of spirals in this talk, one is related to heights of steps, and the other is on stationary solutions.

3.1. Stability of bunched steps

There is a di erence on height of steps between the theory and physical experiments. Although we consider evolution of unit step (whose height is the diameter of an atom) in the theory, we also observe steps whose height is $O(10)$ or $O(100)$ by number of atoms in experiments. For simulations describing more exact situations the height of steps should be implied in formulations of spirals.

One of simple way to express the multiple height of steps is considering evolution of bunched steps. From this view point, it is important to investigate the stability of a bunch of steps.

For this problem we assume that there exists only one center at the origin, and $W = B_R(0) \setminus \overline{B_\rho(0)}$, where $B_\rho(a)$ is an open disc whose center is a and radius is ρ. Assume that there exist $m(\quad 1)$ evolving spirals with anti-clockwise rotational orientations. This con guration is described by

$$u_t \quad |\nabla(u \quad m \ _0)| \quad \mathrm{div}\frac{\nabla(u \quad m \ _0)}{|\nabla(u \quad m \ _0)|} + C \Big\} = 0 \quad \text{in} \quad (0, T) \quad W, \tag{9}$$

$$\langle \vec{\ }, \nabla(u \quad m \ _0)\rangle = 0 \quad \text{on} \quad (0, T) \quad \partial W, \tag{10}$$

where $\ _0(x) = \arg(x)$.

Ogiwara and Nakamura [ON03] obtained a negative result with a phase field model by Kobayashi [Kob10] and a same configuration of the domain. They proved the existence of a solution describing rotating m spirals with $1/m$-times rotational symmetric pattern. In particular, any solutions converges to the above rotating spirals with a rotation if necessary.

However, we obtain the following stability results on a bunch of steps in our formulation.

Theorem 8. *Let u be a solution of (9)–(10) in $(0,\infty) \times \overline{W}$. Assume that there exists $\zeta_0 \in C([\rho,R])$ and $\varepsilon > 0$ such that, for $j = 0,1,\ldots,m-1$, there exists $k_j \in \mathbb{Z}$ such that*

$$\{(x,\theta) \in \mathfrak{X};\ u(0,x) - m\theta = 2\pi j\} \subset \{(x,\theta) \in \mathfrak{X};\ |\theta - (\zeta_0(|x|) + 2\pi k_j)| < \varepsilon\}. \tag{11}$$

Then, there exists $\zeta \in C([0,\infty) \times [\rho,R])$ such that $w(t,x) = \zeta(t,|x|)$ is a viscosity solution of (9)–(10) with $m = 1$ satisfying $w(0,x) = \zeta_0(|x|)$, and

$$\begin{aligned}\{(x,\theta) \in \mathfrak{X};\ u(t,x) - m\theta = 2\pi j\} \subset \{(x,\theta) \in \mathfrak{X};\ |\theta - (\zeta(t,|x|) + 2\pi k_j)| < \varepsilon\}\\ \text{for } t > 0 \text{ and } j = 0,1,\ldots,m-1.\end{aligned} \tag{12}$$

Note that $\Gamma_{t,j} := \{x \in \overline{W};\ u(t,x) - m\theta = 2\pi j\}$ denotes one of continuous spiral curves in Γ_t. Thus (11) means that all curves in Γ_0 is between $\mathcal{C}_0^{\pm\varepsilon}$ of the form

$$\mathcal{C}_0^{\pm\varepsilon} := \{r(\cos(\zeta_0(r) \pm \varepsilon), \sin(\zeta_0(r) \pm \varepsilon));\ r \in [\rho,R]\},$$

which is the rotation with the angles $\pm\varepsilon$ of the curve $\mathcal{C}_0 := \{r(\cos\zeta_0(r), \sin\zeta_0(r));\ r \in [\rho,R]\}$. Consequently, Theorem 8 means that Γ_t cannot escape from the place between $\mathcal{C}_t^{\pm\varepsilon}$ of the form

$$\mathcal{C}_t^{\pm\varepsilon} := \{r(\cos(\zeta(t,r) \pm \varepsilon), \sin(\zeta(t,r) \pm \varepsilon));\ r \in [\rho,R]\}$$

for $t > 0$, and consequently we obtain the stability in the sense of Lyapunov. Moreover, the curve $\mathcal{C}_t := \{r(\cos\zeta(t,r), \sin\zeta(t,r));\ r \in [\rho,R]\}$ evolves by $V = C - \kappa$, and thus the bunch of spirals can be regarded as an evolving spiral by the same equation.

The crucial difference between our formulation and a phase field model is the type of equations; our equation is degenerate parabolic, and the phase field model is uniformly parabolic. This implies that all spiral curves evolve with the same equation since $v_j(t,x) := (u(t,x) - 2\pi j)/m$ satisfies (9)–(10) with $m = 1$, and

$$\Gamma_{t,j} = \{x \in \overline{W};\ v_j(t,x) - \theta_0(x) \equiv 0 \mod 2\pi\mathbb{Z}\}$$

for $j = 0,1,\ldots,m-1$.

The existence of ζ is derived from the rotation invariance of (9)–(10). In fact, we observe that

$$\mathcal{C}_0 = \{x \in \overline{W};\ \zeta_0(|x|) - \theta_0(x) \equiv 0 \mod 2\pi\mathbb{Z}\}.$$

Let $w(t,x)$ be a viscosity solution of (9)–(10) with $m = 1$ and $w(0,x) = \zeta_0(|x|)$. Then we obtain $w(t,x) = w(t,|x|e)(=: \zeta(t,|x|))$ for some $e \in S^1$ because of the uniqueness and $w(0,Rx) = w(0,x)$ for all rotation matrix R. This implies $\mathcal{C}_t$ and also $\mathcal{C}_t^{\pm\varepsilon}$ are solutions of (2)–(3) in the level set sense. Consequently, Theorem 8 is derived by the comparison of interior and exterior (cf. Theorem 3) between the each curves $\Gamma_{t,j}$ and $\mathcal{C}_t^{\pm\varepsilon}$.

3.2. Inactive pair

Burton, Cabrera and Frank [BCF51] pointed out that, if a pair of centers with opposite rotational orientations is closer together than the critical distance $2/C$, then this pair has no influence to the evolution of the crystal surface. They call such a pair an inactive pair. We now demonstrate the existence of an inactive pair.

For this problem we assume that $N = 2$, $a_1 = (\quad , 0)$, $a_2 = (\ , 0)$ with $\quad \in (0, 1/C)$, $U_i = B_\rho(a_1)$ with $\rho \in (0, \ )$, and $\ (x) = \arg(x \quad a_1) \quad \arg(x \quad a_2)$. Assume that Ω is large enough (its sense is clari ed later), and set $W = \Omega \setminus (\overline{U}_1 \cup \overline{U}_2)$.

Note that the circle whose radius is $1/C$ is a stationary solution of $V = C \quad \kappa$. Thus the curves satisfying the following condition should be a stationary solution of our problem.

(R1) It is a part of the circle whose radius is $1/C$.

(R2) It satis es the right angle condition between $\partial B_\rho(a_1)$ and $\partial B_\rho(a_2)$.

We now give such curves explicitly. Set

$$p_1(\sigma) = a_1 + \rho(\cos\sigma, \sin\sigma), \quad p_2(\sigma) = a_2 + \rho(\quad \cos\sigma, \sin\sigma),$$
$$q_1(\sigma) = p_1(\sigma) + \frac{1}{C}(\sin\sigma, \quad \cos\sigma), \quad q_2(\sigma) = p_2(\sigma) + \frac{1}{C}(\quad \sin\sigma, \quad \cos\sigma)$$

Then there exists $\quad > 0$ and σ_1, σ_2 such that $0 < \sigma_1 < \sigma_2 < \quad$ and

$$b_1 = (0, \quad) = q_1(\sigma_1) = q_2(\sigma_1), \quad b_2 = (0, \ ) = q_1(\sigma_2) = q_2(\sigma_2).$$

We now de ne

$$R_i = \quad r_i(\sigma) = b_i + \frac{1}{C} \quad \cos \quad \frac{}{2} + \sigma \quad , \sin \quad \frac{}{2} + \sigma \quad ; \ \sigma \in [\quad \sigma_i, \sigma_i] \Big\}$$

for $i = 1, 2$. The sense of the assumption 'Ω is large enough' means that $R_i \subset \Omega \setminus (U_1 \cup U_2)$. Then, R_i is a connecting spiral between a_1 and a_2 satisfying (R1)–(R2), and consequently R_i is our desired curve for $i = 1, 2$. Note that there are two stationary curves in our problems in general.

To demonstrate that R_i is a stationary curve for $i = 1, 2$ we have to nd a solution u describing R_i in our level set formulation. However, in usual evolution of a closed curve (i.e., $W = \Omega$ and $\quad \equiv 0$), there are no continuous solutions describing the stationary circle. Accordingly, we nd discontinuous a viscosity solution of (5)–(6) describing R_i for $i = 1, 2$.

Theorem 9. *Let R_i be given on above for $i = 1, 2$. Then, $\ _{R_i} : \overline{W} \to \mathbb{R}$ which is a branch of $\ (x) = \arg(x \quad a_1) \quad \arg(x \quad a_2)$ whose discontinuity is only on R_i is a viscosity solution of (5)–(6).*

For all $u_0 \in C(\overline{W})$ there exists $k \in \mathbb{Z}$ such that $\ _{R_i} + 2 \ k \quad u_0$ on $\overline{W}$, which implies that $\ _{R_i} + 2 \ k \quad u$ on $[0, \infty) \quad \overline{W}$ from Theorem 1, where u is a viscosity solution of (5)–(6) with $u|_{t=0} = u_0$. From the above and Theorem 3 the curves R_i plays a role of 'upper bound' for all evolution of spirals in the con guration of an inactive pair.

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Decay estimates of the Oseen flow in two-dimensional exterior domains

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1 Introduction

Let Ω be an exterior domain in $\mathbb{R}^2$ with smooth boundary $\partial\Omega$. We consider the Navier-Stokes system

$$\begin{aligned} \partial_t u + u\cdot\nabla u = \Delta u - \nabla p, \qquad \mathrm{div}\, u = 0 \quad \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \\ u \to u_\infty \quad \text{as } |x|\to\infty \end{aligned} \tag{1.1}$$

which describes a motion of a viscous incompressible fluid past an obstacle $\mathbb{R}^2\setminus\Omega$ (rigid body) that moves with translational velocity $-u_\infty$, where $u(x,t) = (u_1,u_2)$ and $p(x,t)$ respectively denote unknown velocity and pressure of the fluid, while $u_\infty \in \mathbb{R}^2\setminus\{0\}$ is a given uniform velocity. Because of the Stokes paradox, we do need to consider the problem around u_∞, so that the Oseen linearization works well as an approximation of the Navier-Stokes system. Since the Navier-Stokes system is rotationally invariant, without loss of generality, one may take

$$u_\infty = -2\alpha e_1 \qquad \text{with } \alpha\in\mathbb{R}\setminus\{0\} \text{ (Oseen parameter)} \tag{1.2}$$

where one can regard $|\alpha|$ as the Reynolds number. Then, by denoting $u - u_\infty$ by the same symbol u, (1.1) is reduced to

$$\begin{aligned} \partial_t u + u\cdot\nabla u = \Delta u + 2\alpha\partial_1 u - \nabla p, \qquad \mathrm{div}\, u = 0 \quad \text{in } \Omega, \\ u|_{\partial\Omega} = 2\alpha e_1, \\ u \to 0 \quad \text{as } |x|\to\infty. \end{aligned} \tag{1.3}$$

It is an open question to clarify the large time behavior of solutions to the initial value problem for (1.3) even when $\alpha\in\mathbb{R}\setminus\{0\}$ is small enough. Toward better understanding of this problem, it is important to study: (i) steady flows with fine decay/summability for $|x|\to\infty$; (ii) decay properties of solutions to the Oseen initial value problem, see (1.5) below, for $t\to\infty$. Concerning the first

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issue (i), it was proved by Finn and Smith [7], [8], [20] and, later on, refined by Galdi [9], [10], that if $|\alpha|$ is nonzero but sufficiently small, then (1.3) admits a steady flow (called a physically reasonable solution), $u(x) = (u_1, u_2)$ that satisfies $u(x) = O(|x|^{-1/2})$ as $|x| \to \infty$ and exhibits a parabolic wake region behind the body like the Oseen fundamenatal solution. To be precise, such an anisotropic decay structure with wake is found only for u_1, while u_2 has no wake; as a consequence, we have

$$u_1 \in L^q(\Omega) \quad \text{for } \forall q > 3; \qquad u_2 \in L^r(\Omega) \quad \text{for } \forall r > 2. \tag{1.4}$$

So far, the stability/instablity of this flow is unsolved, while we know the stability of physically reasonable solutions in 3D exterior domains, see [18] and the references therein. The diffculty in 2D is due to less summability (1.4), that is not enough to show the stablity.

This presentation is concerned with the second issue (ii) above, that is, the large time behavior of solutions to the initial value problem for the Oseen system

$$\begin{aligned}
&\partial_t u - \Delta u - 2\alpha\partial_1 u + \nabla p = 0, \qquad \text{div } u = 0 \quad \text{in } \Omega \times (0,\infty) \\
&u|_{\partial\Omega} = 0, \\
&u \to 0 \quad \text{as } |x| \to \infty, \\
&u(\cdot, 0) = f.
\end{aligned} \tag{1.5}$$

We use the standard L^q spaces, $1 < q < \infty$, of solenoidal vector fields

$$\begin{aligned}
L^q_\sigma(\Omega) &= \text{completion of } C^\infty_{0,\sigma}(\Omega) \text{ in } L^q(\Omega) \\
&= \{u \in L^q(\Omega);\ \text{div } u = 0,\ \nu \cdot u|_{\partial\Omega} = 0\}
\end{aligned}$$

where $C^\infty_{0,\sigma}(\Omega)$ consists of all smooth and divergence free vector fields with compact support, ν is the outer unit normal to the boundary $\partial\Omega$ and $\nu \cdot u|_{\partial\Omega}$ denotes the normal trace of u. It is well known that the space $L^q(\Omega)$ of vector fields admits the Helmholtz decomposition $L^q(\Omega) = L^q_\sigma(\Omega) \oplus \{\nabla p \in L^q(\Omega); p \in L^q_{loc}(\overline{\Omega})\}$ which was proved by Miyakawa [16], and Simader and Sohr [19]. By using the projection $P : L^q(\Omega) \to L^q_\sigma(\Omega)$, the Oseen operator $L = L_\alpha$ is defined by

$$\begin{cases} D(L) = \{u \in W^{2,q}(\Omega) \cap L^q_\sigma(\Omega);\ u|_{\partial\Omega} = 0\}, \\ Lu = -P[\Delta u + 2\alpha\partial_1 u]. \end{cases}$$

As in [16], by a perturbation argument from the Stokes operator $L_0 = -P\Delta$, it is easily verified that the Oseen operator generates an analytic semigroup (the Oseen semigroup) $\{e^{-tL}\}_{t\geq 0}$ in $L^q_\sigma(\Omega)$. Thus the solution of (1.5) is given by $u(\cdot, t) = e^{-tL}f$.

Our aim is to show the L^q-L^r estimates

$$\|e^{-tL}f\|_r \leq C\, t^{-(n/q-n/r)/2}\|f\|_q \qquad (1 < q \leq r \leq \infty,\ q \neq \infty) \tag{1.6}$$

$$\|\nabla e^{-tL}f\|_r \leq C\, t^{-(n/q-n/r)/2-1/2}\|f\|_q \qquad (1 < q \leq r \leq n) \tag{1.7}$$

for $t > 0$, where $n \geq 2$ is the space dimension and $\|\cdot\|_q$ stands for the L^q-norm. For the Stokes semigroup (case $\alpha = 0$), these estimates were deduced by Iwashita [13] ($n \geq 3$), Dan and Shibata [3], [4] ($n = 2$), and Maremonti and Solonnikov [15] ($n \geq 2$). We cannot avoid the restriction $r \leq n$ for (1.7), see [15] and [12] (while it is not clear whether the same restriction is essential for the case $\alpha \in \mathbb{R} \setminus \{0\}$). As for the Oseen semigroup (case $\alpha \in \mathbb{R} \setminus \{0\}$), (1.6) and (1.7) were established by Kobayashi and Shibata [14] ($n = 3$) and Enomoto and Shibata [5], [6] ($n \geq 3$), except for the case of plane exterior domains ($n = 2$), where the constant $C > 0$ above can be taken uniformly with respect to small $\alpha \in \mathbb{R} \setminus \{0\}$; that is, for each $M > 0$, we have $C = C(M; p, q)$ provided $0 < |\alpha| \leq M$. This is important in the proof of stability of 3D steady flows as an application of (1.6)–(1.7), see [18]. Unfortunately, our main result on the Oseen semigroup in 2D, see Theorem 2.1 below, does not porivide such desirable situation although I have made efforts to show the dependence of the constant $C > 0$ on the parameter α as less singlar as possible. This is because I cannot find the best way to overcome, among others, the following difficulty yet, but I believe the theorem will have to be improved in the future. The most difficulty in 2D is to control both parameters λ (spectral parameter) and α, see (1.2), in asymptotics of the Oseen resolvent. In fact, the fundamental solution in $\mathbb{R}^2$, see (5.3) below, involves the modified Bessel function of the second kind (order 0), and its leading term near the origin is given by $\log(\sqrt{\lambda + \alpha^2}|x|)$, whose convergence to $\log(|\alpha||x|)$ as $\lambda \to 0$ is not uniform with respect to small $\alpha \in \mathbb{R} \setminus \{0\}$, unlike 3D case.

2 L^q-L^r estimate

The main result on L^q-L^r estimate of the Oseen semigroup in 2D exterior domains reads as follows.

Theorem 2.1 *Let $\alpha \in \mathbb{R} \setminus \{0\}$. For $\{q, r\}$ specified below and arbitrary small $\varepsilon > 0$, there are positive constants $C_\alpha = C_\alpha(q, r, \varepsilon)$, $C'_\alpha = C'_\alpha(q, \varepsilon)$ and $C''_\alpha = C''_\alpha(q, r, \varepsilon)$ such that*

$$\|e^{-tL} f\|_r \leq C_\alpha\, t^{-1/q+1/r} \|f\|_q \qquad (1 < q \leq r < \infty) \tag{2.1}$$

$$\|e^{-tL} f\|_\infty \leq C'_\alpha\, t^{-1/q} (\log t) \|f\|_q \qquad (1 < q < r = \infty) \tag{2.2}$$

$$\|\nabla e^{-tL} f\|_r \leq C''_\alpha\, t^{-1/q+1/r-1/2} \|f\|_q \qquad (1 < q \leq r < 2 = n) \tag{2.3}$$

$$\|\nabla e^{-tL} f\|_2 \leq C''_\alpha\, t^{-1/q} (\log t) \|f\|_q \qquad (1 < q \leq r = 2 = n) \tag{2.4}$$

for $t \geq 2$ and $f \in L^q_\sigma(\Omega)$, where C_α, C'_α and C''_α behave as

$$\begin{aligned}
C_\alpha &= \begin{cases} O(|\alpha|^{-1-\varepsilon}) & 1/q - 1/r \leq 1/2 \\ O(|\alpha|^{-2-\varepsilon}) & 1/q - 1/r > 1/2 \end{cases} \\
C'_\alpha &= \begin{cases} O(|\alpha|^{-1-\varepsilon}) & q > 2 \\ O(|\alpha|^{-2-\varepsilon}) & q \leq 2 \end{cases} \\
C''_\alpha &= \begin{cases} O(|\alpha|^{-1-\varepsilon}) & q = r \\ O(|\alpha|^{-2-\varepsilon}) & q < r \end{cases}
\end{aligned} \tag{2.5}$$

when $\alpha \to 0$.

For the marginal cases, the rate of decay given in (2.2) and (2.4) is not sharp. In fact, the Stokes semigroup e^{-tL_0} satisfies

$$\|e^{-tL_0}f\|_\infty \le C\,t^{-1/q}\|f\|_q \qquad (1 < q < \infty) \tag{2.6}$$

$$\|\nabla e^{-tL_0}f\|_2 \le C\,t^{-1/q}\|f\|_q \qquad (1 < q \le 2) \tag{2.7}$$

for $t > 0$ and $f \in L^q_\sigma(\Omega)$, see [3], [4]. It is worth while noting that (2.7) with $q = 2$ can be deduced by a simple weighted energy method. One can also apply the energy method to the Oseen system (1.5) to obtain

$$\|\nabla e^{-tL}f\|_2 \le C\big(|\alpha|^{1/2}t^{-1/4} + t^{-1/2}\big)\|f\|_2 \tag{2.8}$$

with some $C > 0$ independent of α.

Set $B_R = \{x \in \mathbb{R}^2; |x| < R\}$. We fix $R_0 > 0$ such that $\mathbb{R}^2 \setminus \Omega \subset B_{R_0}$. As in [13] and [14], the essential step for the proof of Theorem 2.1 is to derive local energy decay properties in $\Omega_R = \Omega \cap B_R$ for $R \ge R_0$, see (3.2)–(3.4) below. And then, we combine (3.4) in Ω_R with decay estimates in $\Omega \setminus \Omega_R = \{|x| \ge R\}$. The latter can be deduced by means of cut-off technique with the aid of L^q-L^r estimate of the Oseen semigroup in the whole plane $\mathbb{R}^2$, that is of the explicit form

$$\big(U(t)f\big)(x) = \int_{\mathbb{R}^2} G(x + 2\alpha t e_1 - y, t) f(y)\, dy \tag{2.9}$$

where $G(x,t)$ denotes the heat kernel

$$G(x,t) = \frac{1}{4\pi t}\, e^{-|x|^2/4t}. \tag{2.10}$$

3 Local energy decay

For $1 < q < \infty$ and $d \ge R_0$ we set

$$L^q_{[d]}(\Omega) = \{f \in L^q(\Omega);\; f(x) = 0 \; a.e. \; |x| \ge d\} \tag{3.1}$$

from which the initial data are taken in the following key proposition.

Proposition 3.1 *Let $\alpha \in \mathbb{R} \setminus \{0\}$, $2 \le q < \infty$, $M > 0$, $R \ge R_0$, $d \ge R_0$ and $0 \le \theta \le 1$. Then there is a positive constant $C = C(q, M, R, d, \theta)$ such that*

$$\|e^{-tL}Pf\|_{W^{1,q}(\Omega_R)} \le \frac{C}{|\alpha|^{1+2\theta}}\; t^{-(1+\theta)}\,(\log t)^{2\theta}\|f\|_q \tag{3.2}$$

for $t \ge 2$, $f \in L^q_{[d]}(\Omega)$ and $|\alpha| \in (0, M]$.

For the Stokes semigroup (case $\alpha = 0$), the rate of local energy decay derived by Dan and Shibata [3] is $t^{-1}\,(\log t)^{-2}$. Therefore, one can expect no singularity with respect to α in (3.2) at least for the case $\theta = 0$, however, we could not remove $|\alpha|^{-1}$. When we fix $\alpha \in \mathbb{R} \setminus \{0\}$ and take $\theta = 1$ in Proposition 3.1, we find that the rate of local energy decay of the Oseen semigroup is $t^{-2}\,(\log t)^2$, which is better than that of the Stokes semigroup.

The reason why we have the restricttion $q \in [2, \infty)$ is that we are forced to employ L^2 theory in a part of the proof. But, even for the case $1 < q < 2$, it is obvious that

$$\|e^{-tL}Pf\|_{W^{1,q}(\Omega_R)} \leq \frac{C}{|\alpha|^{1+2\theta}}\, t^{-(1+\theta)}\, (\log t)^{2\theta} \|f\|_2 \tag{3.3}$$

for $f \in L^2_\sigma(\Omega)$, which is enough to proceed to the next stage on account of the smoothing effect of analytic semigroups.

In the next step, we still consider the local energy decay, however, for general data from $L^q_\sigma(\Omega)$. By using (3.2)–(3.3) with arbirary small $\theta > 0$ together with L^q-L^r estimate of (2.9) in the whole plane $\mathbb{R}^2$, a certain cut-off procedure provides the following decay property.

Proposition 3.2 *Let $\alpha \in \mathbb{R} \setminus \{0\}$, $1 < q < \infty$, $M > 0$, $R \geq R_0$, and suppose $\varepsilon > 0$ is arbitrarily small. Then there is a constant $C = C(q, M, R, \varepsilon)$ such that*

$$\|e^{-tL}f\|_{W^{1,q}(\Omega_R)} \leq \frac{C}{|\alpha|^{1+\varepsilon}}\, t^{-1/q} \|f\|_q \tag{3.4}$$

for $t \geq 2$, $f \in L^q_\sigma(\Omega)$ and $|\alpha| \in (0, M]$.

In view of (1.6) with $n = 2$ and $r = \infty$, one finds that the decay rate $t^{-1/q}$ is reasonable. We note that this rate cannot be improved even though we use (3.2)–(3.3) with $\theta = 1$. On the other hand, if we used (3.2)–(3.3) with $\theta = 0$, the decay rate in (3.4) would be $t^{-1/q} \log t$.

4 Strategy

For the proof of Proposition 3.1 we need the spectral analysis. We always consider estimates of $(\lambda + L)^{-1}Pf$ as well as $e^{-tL}Pf$ in $W^{1,q}(\Omega_R)$ under the condition $f \in L^q_{[d]}(\Omega)$. We have the Dunford integral representation formula of the semigroup in terms of the resolvent $(\lambda + L)^{-1}$. The spectrum of the Oseen operator is contained in $\{\lambda \in \mathbb{C};\ 4\alpha^2 \mathrm{Re}\, \lambda + (\mathrm{Im}\, \lambda)^2 \leq 0\}$, and thus it seems to be impossible to take the same path of integration as in [3] for the Stokes semigroup. What we need is to study the asymptotic behavior of $\partial_\lambda(\lambda+L)^{-1}Pf$ as $\lambda \to 0$ (and $\alpha \to 0$ as well), which ensures its summability near the origin. This enables us to justify the representation formula (which was used in [14] as well)

$$e^{-tL}Pf = \frac{-1}{2\pi i t} \int_{-\infty}^{\infty} e^{i\tau t}\, \partial_\tau (i\tau + L)^{-1} Pf\, d\tau \tag{4.1}$$

when we perform integration by parts and then move the path of integration to the imaginary axis. When we derive faster decay than t^{-1}, we have to study further regularity of $\partial_\lambda(\lambda + L)^{-1}Pf$ near $\lambda = 0$.

In order to carry out this strategy, we construct a parametrix of solutions to the Oseen resolvent problem

$$\left\{ \begin{array}{l} \lambda u - \Delta u - 2\alpha \partial_1 u + \nabla p = f, \qquad \mathrm{div}\ u = 0 \qquad \text{in } \Omega \\ u|_{\partial\Omega} = 0 \end{array} \right. \tag{4.2}$$

where Re $\lambda \geq 0$ ($\lambda \neq 0$). We fix a cut-off function $\psi \in C^\infty(\mathbb{R}^2; [0,1])$ such that $\psi(x) = 1$ for $|x| \leq R_0 + 1$ and $\psi(x) = 0$ for $|x| \geq R_0 + 2$, and use the Bogovskii operator B in the annulus $\{R_0 + 1 < |x| < R_0 + 2\}$, see [1], [2] and [9]. We define $\{A_\lambda^\alpha, \Pi\}$ by (4.9) the solution operator of

$$\lambda u - \Delta u - 2\alpha\partial_1 u + \nabla p = f, \qquad \text{div } u = 0 \qquad \text{in } \mathbb{R}^2 \tag{4.3}$$

and let $\{M_\lambda^\alpha, N_\lambda^\alpha\}$ be that of

$$\begin{cases} \lambda u - \Delta u - 2\alpha\partial_1 u + \nabla p = f, \qquad \text{div } u = 0 \qquad \text{in } \Omega_{R_0+3} \\ u|_{\partial\Omega_{R_0+3}} = 0 \end{cases} \tag{4.4}$$

with the side condition

$$\int_{R_0+1<|x|<R_0+2} p(x)\, dx = c_0 \tag{4.5}$$

for given $c_0 \in \mathbb{C}$. We set

$$\begin{cases} v = R_\lambda^\alpha f := (1-\psi)A_\lambda^\alpha f + \psi M_\lambda^\alpha f + B\left[(A_\lambda^\alpha f - M_\lambda^\alpha f)\cdot\nabla\psi\right] \\ \sigma = Q_\lambda^\alpha f := (1-\psi)\Pi f + \psi N_\lambda^\alpha f \end{cases} \tag{4.6}$$

where f is understood as its zero extension/restriction and the pressure $N_\lambda^\alpha f$ is chosen in such a way that $c_0 = \int_{R_0+1<|x|<R_0+2} (\Pi f)(x)\, dx$ in (4.4)–(4.5). Then a pair $\{v, \sigma\}$ should obey

$$\begin{cases} \lambda v - \Delta v - 2\alpha\partial_1 v + \nabla\sigma = f + T_\lambda^\alpha f, \qquad \text{div } v = 0 \qquad \text{in } \Omega \\ v|_{\partial\Omega} = 0 \end{cases} \tag{4.7}$$

and the operator T_λ^α is compact from $L^q_{[d]}(\Omega)$ into itself provided $d \geq R_0 + 2$, see (3.1). Since $I + T_\lambda^\alpha$ is injective by the uniqueness for (4.2), the Fredholm alternative yields the existence of $(1 + T_\lambda^\alpha)^{-1}$ as a bounded operator, so that we obtain a representation of the resolvent

$$(\lambda + L)^{-1} P f = R_\lambda^\alpha (1 + T_\lambda^\alpha)^{-1} f. \tag{4.8}$$

But the compactness argument above provides us little information about the dependence of $(\lambda + L)^{-1}Pf$ on λ and α. Thus one needs reconstruction of $(1 + T_\lambda^\alpha)^{-1}$ especially near $\lambda = 0$. To this end, we have to have a precise look at the resolvent $A_\lambda^\alpha f$ in the whole plane $\mathbb{R}^2$, which is the dominant part of the remaining term $T_\lambda^\alpha f$ when $\lambda \to 0$ and $\alpha \to 0$. Here, the solution operator of (4.3) is given by

$$u = A_\lambda^\alpha f = E_\lambda^\alpha * f, \qquad p = \Pi f = \frac{x}{2\pi|x|^2} * f \tag{4.9}$$

where $E_\lambda^\alpha(x)$ is the fundamental solution that is analyzed in the next section.

5 Fundamental solution

The fundamenatal solution of (4.3) is given by

$$\begin{aligned} E_\lambda^\alpha(x) &= \mathcal{F}^{-1}\left[\frac{|\xi|^2\mathbb{I}-\xi\otimes\xi}{(\lambda+|\xi|^2-2\alpha i\xi_1)|\xi|^2}\right](x) \\ &= \int_0^\infty e^{-\lambda t}(G\,\mathbb{I}+H)(x+2\alpha te_1,t)\,dt \end{aligned} \tag{5.1}$$

where $G(x,t)$ is the heat kernel, see (2.10), and

$$\begin{aligned} H(x,t) &= \int_t^\infty \nabla^2 G(x,s)\,ds = \int_t^\infty \frac{e^{-|x|^2/4s}}{4\pi s}\left(\frac{x\otimes x}{4s^2}-\frac{\mathbb{I}}{2s}\right)ds \\ &= \frac{-(x\otimes x)e^{-|x|^2/4t}}{4\pi t|x|^2}+\frac{1}{\pi|x|^2}\left(\frac{x\otimes x}{|x|^2}-\frac{\mathbb{I}}{2}\right)\left(1-e^{-|x|^2/4t}\right). \end{aligned} \tag{5.2}$$

Note that $G\,\mathbb{I}+H$ is the fundamental solution of unsteady Oseen system. By a lengthy calculation we obtain

$$\begin{aligned} E_\lambda^\alpha(x) = &\frac{\mathbb{I}}{2\pi}e^{-\alpha x_1}\,K_0\big(\sqrt{\lambda+\alpha^2}\,|x|\big) \\ &-\frac{\mathbb{I}}{4\pi}\int_0^1 e^{-\alpha x_1 s}\,K_0\big(\sqrt{s(\lambda+\alpha^2 s)}\,|x|\big)ds \\ &+\frac{x\otimes x}{4\pi|x|}\int_0^1 e^{-\alpha x_1 s}\sqrt{s(\lambda+\alpha^2 s)}\,K_1\big(\sqrt{s(\lambda+\alpha^2 s)}\,|x|\big)ds \\ &+\frac{\alpha(x\otimes e_1+e_1\otimes x)}{4\pi}\int_0^1 se^{-\alpha x_1 s}\,K_0\big(\sqrt{s(\lambda+\alpha^2 s)}\,|x|\big)ds \\ &+\frac{\alpha^2|x|e_1\otimes e_1}{4\pi}\int_0^1 \frac{s^2e^{-\alpha x_1 s}}{\sqrt{s(\lambda+\alpha^2 s)}}\,K_1\big(\sqrt{s(\lambda+\alpha^2 s)}\,|x|\big)ds \end{aligned} \tag{5.3}$$

for Re $\lambda \geq 0$ and $\alpha\in\mathbb{R}$, except for $(\lambda,\alpha)=(0,0)$, where

$$\begin{aligned} K_0(z) &= \frac{1}{2}\int_0^\infty \exp\left[\frac{-z}{2}\left(t+\frac{1}{t}\right)\right]\frac{dt}{t} \\ K_1(z) &= -K_0'(z) = \frac{1}{2}\int_0^\infty \exp\left[\frac{-z}{2}\left(t+\frac{1}{t}\right)\right]\frac{dt}{t^2} \end{aligned} \tag{5.4}$$

are modified Bessel functions of the second kind (order 0/order 1, respectively) for $z\in\mathbb{C}_+=\{\text{Re }\lambda>0\}$. The representation (5.3) for the case $\lambda=0$ was derived by Guenther and Thomann [11]; in this case, another representation of $E_0^\alpha(x)$ without s-integral is also available, see for instance Okamura, Shibata and Yamaguchi [17]. By making use of asymptotic expansion of the modified Bessel functions ([17])

$$\begin{aligned} K_0(z) &= -\log z+\log 2-\gamma+(\log z)O(z^2) \\ K_1(z) &= \frac{1}{z}+\frac{z}{2}\big(\log z-\log 2+\gamma-\frac{1}{2}\,\big)+(\log z)O(z^3) \end{aligned} \tag{5.5}$$

as $\mathbb{C}_+ \ni z \to 0$, where $\gamma = \lim_{m\to\infty} \big(\sum_{k=1}^{m} \frac{1}{k} - \log m\big)$ is the Euler constant, one can find the following structure of the fundamental solution:

$$E_\lambda^\alpha(x) = \underbrace{E_0^0(x) + \frac{1}{4\pi}\left[\left(\log\frac{1}{|\alpha|}\right)\ \mathbb{I} + \mathbb{J}\right] + F^\alpha(x)}_{=E_0^\alpha(x)} + S_\lambda^\alpha(x) \tag{5.6}$$

where

$$E_0^0(x) = \frac{1}{4\pi}\left[\left(\log\frac{1}{|x|}\right)\mathbb{I} + \frac{x\otimes x}{|x|^2}\right] \tag{5.7}$$

is the Stokes fundamental solution in $\mathbb{R}^2$,

$$\mathbb{I} = (\delta_{jk})_{1\le j,k\le 2}, \qquad \mathbb{J} = (\log 2 - \gamma - 1)\,\mathbb{I} + (e_1\otimes e_1)$$

are constant matrices, and

$$\begin{aligned} &\sup_{|x|\le R} |F^\alpha(x)| = O(\alpha\log|\alpha|) \qquad \text{as } \alpha\to 0, \\ &\sup_{|x|\le R} |S_\lambda^\alpha(x)| \le \rho\left(\frac{|\lambda|}{\alpha^2}\right), \qquad |\alpha|\in(0,M],\ \mathrm{Re}\,\lambda\ge 0, \end{aligned} \tag{5.8}$$

with some function $\rho = \rho_M$ satisfying $\rho(\varepsilon) = O(\varepsilon\log\varepsilon)$ as $\varepsilon\to 0$, where $R>0$ and $M>0$ are arbitralily fixed.

6 Sketch of Proof

We fix $M>0$ and let $|\alpha|\in(0,M]$. We go back to the consideration of the remaining term $T_\lambda^\alpha f$ in (4.7). By (5.6) one can write $A_\lambda^\alpha f = A_0^\alpha f + S_\lambda^\alpha * f$ with

$$A_0^\alpha f = E_0^0 * f + \frac{1}{4\pi}\left[\left(\log\frac{1}{|\alpha|}\right)\ \mathbb{I} + \mathbb{J}\right]\Gamma f + F^\alpha * f, \qquad \Gamma f = \int_{\mathbb{R}^2} f(y)\,dy \tag{6.1}$$

Since we have the logarithmic singularity only in the degenerate part of A_0^α, it is possible to show that $\|(1+T_0^\alpha)^{-1}\|_{\mathcal{L}(L^q_{[d]}(\Omega))} \le C_M$, where $C_M>0$ is independent of $|\alpha|\in(0,M]$, along the same idea as in [3]. Note that indeed $1+T_0^0$ is not injective (unlike 3D case), but $\dim\ker(1+T_0^0)\le 2$. We then regard $1+T_\lambda^\alpha$ as

$$1+T_\lambda^\alpha = \left[1 + (T_\lambda^\alpha - T_0^\alpha)(1+T_0^\alpha)^{-1}\right](1+T_0^\alpha). \tag{6.2}$$

By $(5.8)_2$ there is a constant $\delta=\delta(M)>0$ such that if $|\lambda|\le\delta\alpha^2$ (as well as $|\alpha|\in(0,M]$), then $\|(T_\lambda^\alpha - T_0^\alpha)(1+T_0^\alpha)^{-1}\|_{\mathcal{L}(L^q_{[d]}(\Omega))} \le \frac{1}{2}$ which yields a reconstruction of (4.8) via

$$\begin{aligned} (1+T_\lambda^\alpha)^{-1} &= (1+T_0^\alpha)^{-1}\left[1+(T_\lambda^\alpha - T_0^\alpha)(1+T_0^\alpha)^{-1}\right]^{-1} \\ &= (1+T_0^\alpha)^{-1}\sum_{k=0}^{\infty}\left\{-(T_\lambda^\alpha - T_0^\alpha)(1+T_0^\alpha)^{-1}\right\}^k. \end{aligned} \tag{6.3}$$

This combined with the relation

$$\begin{aligned}&\partial_\tau(i\tau+L)^{-1}Pf\\&=(\partial_\tau R^\alpha_{i\tau})(1+T^\alpha_{i\tau})^{-1}Pf-R^\alpha_{i\tau}\;(1+T^\alpha_{i\tau})^{-1}(\partial_\tau T^\alpha_{i\tau})(1+T^\alpha_{i\tau})^{-1}f\end{aligned}\tag{6.4}$$

by use of

$$\sup_{|x|\le R}|\partial_\lambda E^\alpha_\lambda(x)|\le C\left(\frac{1}{|\lambda+\alpha^2|}+\frac{1}{\alpha^2}\left|\log\left(1+\frac{\alpha^2}{\lambda}\right)\right|+\frac{1}{|\alpha|}\right)\tag{6.5}$$

for Re $\lambda\ge 0$, $0<|\lambda|\le\Lambda$ and $|\alpha|\in(0,M]$ implies

$$\int_{|\tau|\le\delta\alpha^2}\|\partial_\tau(i\tau+L)^{-1}f\|_{W^{1,q}(\Omega_R)}\,d\tau\le C\left(\big|\log|\alpha|\big|+1\right).\tag{6.6}$$

We take a cut-off function $\eta\in C^\infty(\mathbb{R};[0,1])$ such that $\eta(\tau)=1$ for $|\tau|\le 1$ and $\eta(\tau)=0$ for $|\tau|\ge 2$, and divide the integral of (4.1) into

$$\frac{-1}{2\pi it}\int_{-\infty}^{\infty}e^{i\tau t}\,\eta(\tau)\partial_\tau(i\tau+L)^{-1}Pf\,d\tau\tag{6.7}$$

and the other part which decays like t^{-2} by integration by parts once more. It remains, however, to estimate the integral for $\delta\alpha^2\le|\tau|\le 2$ in which we are forced to employ L^2-theory:

$$\begin{aligned}\|\partial_\tau(i\tau+L)^{-1}f\|_{W^{1,q}(\Omega_R)}&\le C\|\partial_\tau(i\tau+L)^{-1}f\|_{H^2(\Omega_R)}\\&\le C\left(\frac{1}{|\tau|}+\frac{\alpha^2}{|\tau|^2}\right)^{3/2}\|f\|_2\end{aligned}\tag{6.8}$$

so that the only estimate I have obtained is

$$\int_{\delta\alpha^2\le|\tau|\le 2}\|\partial_\tau(i\tau+L)^{-1}f\|_{W^{1,q}(\Omega_R)}\,d\tau\le\frac{C}{|\alpha|}\tag{6.9}$$

which is somehow worse than (6.6).

Set $g(\tau):=\eta(\tau)\partial_\tau(i\tau+L)^{-1}Pf$. For further decay of (6.7), we use

$$\left\|\int_{-\infty}^{\infty}e^{i\tau t}g(\tau)\,d\tau\right\|_{W^{1,q}(\Omega_R)}\le C\int_{-\infty}^{\infty}\left\|g\left(\tau+\frac{1}{t}\right)-g(\tau)\right\|_{W^{1,q}(\Omega_R)}\,d\tau\tag{6.10}$$

for $t\ge 2$, to which we apply the following estimate for $|h|\le 1/2$:

$$\begin{aligned}&\int_{-\infty}^{\infty}\|g(\tau+h)-g(\tau)\|_{W^{1,q}(\Omega_R)}d\tau\\\le& C\left[\frac{\big|\log|\alpha|\big|+1}{\alpha^2}\,\log\left(1+\frac{\alpha^2}{2|h|}\right)+\frac{1}{\alpha^2}\left\{\log\left(1+\frac{\alpha^2}{2|h|}\right)\right\}^2+\frac{1}{|\alpha|^3}\right]\;|h|\;\|f\|_q\end{aligned}$$

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Decay and scattering of small solutions of power NLS with a potential

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Abstract

The talk is based on the joint work [2] with Scipio Cuccagna and Nicola Visciglia.

We consider

$$(\mathrm{i}\partial_t + \triangle_V)u + \lambda|u|^{p-1}u = 0 \text{ for } t \geq 1,\ x \in \mathbb{R} \text{ and } u(1) = u_0 \tag{1}$$

with $\triangle_V := \triangle - V(x)$ and $\triangle := \partial_x^2$ and $\lambda \in \mathbb{R}\backslash\{0\}$. In this talk we focus on exponents $3 < p < 5$. V is a real valued Schwartz function and $\triangle_V$ is taken without eigenvalues.

It is well known that for $2 \leq p < 5$ the initial value problem in (1) is globally well posed in $H^1(\mathbb{R})$. Our goal is to study the asymptotic behavior of solutions with initial data $u(1) = u_0$ of size ϵ in a suitable Sobolev norm, with ϵ sufficiently small. It is natural to ask whether such solutions are asymptotically free and satisfy

$$\|u(t)\|_{L_x^\infty([1,\infty))} \leq C_0 t^{-\frac{1}{2}}\epsilon, \tag{2}$$

that is have the decay rate of the solution to the linear Schrödinger equation.

We recall some of the results for $V = 0$. For spatial dimension d, McKean and Shatah [11] answered positively to our question for $1 + \frac{2}{d} < p < 1 + \frac{4}{d}$. The case $p \geq 1 + \frac{4}{d}$ and $p < 1 + \frac{4}{d-1}$ for $d \geq 3$ was answered positively by W. Strauss [14]. W. Strauss [13] proved that the zero solution is the only asymptotically free solution when $1 < p \leq \frac{2}{d}$ for $d \geq 2$, and when $1 < p \leq 2$ for $d = 1$. This result was extended to the case $1 < p \leq 3$ and $d = 1$ by J. Barab [1], using an idea of R. Glassey [8]. The exponent $p = 1 + \frac{2}{d}$ is critical and particularly interesting. The existence and the form of the scattering operator was obtained by Ozawa [12] for $d = 1$ and by Ginibre and Ozawa [6] for $d \geq 2$. The completeness of the scattering operator and the decay estimate were obtained by Hayashi and Naumkin [9]. Completeness of the scattering operator and decay estimate, not only for small solutions, for $d = 1$ and $\lambda < 0$, where obtained by Deift and Zhou [4, 5].

Our goal in the present paper is to extend the result of McKean and Shatah [11] to the case $V \neq 0$ and $d = 1$, which to our knowledge is open. For V we assume the following hypothesis, where we refer to [3] for the definition of the transmission coefficient $T(\tau)$.

(H) The potential V is a real valued Schwartz function such that for the spectrum we have $\sigma(\triangle_V) = (-\infty, 0]$. Furthermore, V is generic, that is the transmission coefficient $T(\tau)$ satisfies $T(0) = 0$.

We denote by Σ_s the Hilbert space defined as the closure of $C_0^\infty(\mathbb{R})$ functions with respect to the norm

$$\|u\|_{\Sigma_s}^2 := \|u\|_{H^s(\mathbb{R})}^2 + \| \, |x|^s u\|_{L^2(\mathbb{R})}^2.$$

Our main result is the following

Theorem 1. *Assume that V satisfies (H), $s > 1/2$ and $p > 3$. Then there exist constants $\epsilon_0 > 0$ and $C_0 > 0$ such that for $\epsilon \in (0, \epsilon_0)$ and $\|u(1)\|_{\Sigma_s} \le \epsilon$ the solution to* (1) *satisfies the decay inequality* (2)*. Furthermore there exists $u_+ \in L^2(\mathbb{R})$ such that*

$$\lim_{t \to +\infty} \|u(t) - e^{\mathrm{i}t\triangle} u_+\|_{L^2(\mathbb{R})} = 0. \tag{3}$$

The hypothesis $\sigma(\triangle_V) = (-\infty, 0]$ is necessary since otherwise for any $s > 1/2$ there are periodic solutions $u(t,x) = e^{\mathrm{i}\lambda t}\phi_\lambda(x)$ of arbitrarily small Σ_s norm. The interesting case is for $p \in (3,5)$ since the case $p \ge 5$ follows from [7, 16]. The case $V = 0$ is due to [11].

If $\sigma(\triangle_V) = (-\infty, 0]$, the existence of wave operators intertwining $\triangle_V$ and $\triangle$ and of Strichartz and dispersive estimates for $e^{\mathrm{i}t\triangle_V}$ is well known, see [7, 16, 17]. Such estimates are not sufficient to prove Theorem 1 even in the case $V = 0$.

The argument in [11] is based on the introduction of homogeneous $\dot{\mathcal{H}}^k(t)$ norms, defined substituting the standard derivative $\frac{\partial}{\partial x_j}$ with operators $J_j(t)$. In [11] it is proved almost invariance of these norms and, by a form of the Sobolev embedding theorem, the dispersion (2). Such use of invariant norms goes back to the work on the wave equation by Klainerman, see for example [10].

The development of a theory of invariant norms in the case of non translation invariant equations such as (1) is an important technical problem. Here our main goal is to adapt the framework of [11] for $d = 1$ and to introduce appropriate surrogates $|J_V(t)|^s$ for the operators $|J(t)|^s$.

To be more precise, let's describe the definition of the perturbed generators for the perturbed Schrödinger equation. First we recall the definition for unperturbed case

$$(\mathrm{i}\partial_t + \triangle)u = 0.$$

Recall that the fundamental solution is given by $e^{\mathrm{i}t\triangle}(x,y) = \frac{e^{\mathrm{i}\frac{(x-y)^2}{4t}}}{(4\pi \mathrm{i}t)^{\frac{d}{2}}}$ for $t > 0$.

Consider the Fourier transform F and its inverse:

$$Ff(x) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{\mathrm{i}x\cdot y} f(y)dy, \tag{4}$$
$$F^{-1}f(x) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{-\mathrm{i}x\cdot y} f(y)dy.$$

We introduce also the dilation operator $D(t)\psi(x) = (2\mathrm{i}t)^{-\frac{d}{2}}\psi(\frac{x}{t})$ and the multiplier operator $M(t)\psi(x) = e^{\frac{\mathrm{i}x^2}{4t}}\psi(x)$. Then we have the following well known formula

$$e^{\mathrm{i}t\triangle} = M(t)D(t)F^{-1}M(t).$$

Let $g(x)$ be a function and denote by $g(q)$ the multiplier operator $g(q)\psi(x) := g(x)\psi(x)$. We set $p_j := \mathrm{i}\partial_{x_j}$ and $p = (p_1, ..., p_d)$. More generally, set $g(p) := F^{-1}g(q)F$. The following identity is well-known:

$$e^{\mathrm{i}t\triangle}g(q)e^{-\mathrm{i}t\triangle} = M(t)g(2tp)M(-t). \tag{5}$$

for any $g(x)$. With an abuse of notation we will denote the operator $g(q)$ by $g(x)$. Notice that we have

$$\left[\mathrm{i}\partial_t + \triangle, e^{\mathrm{i}t\triangle}g(x)e^{-\mathrm{i}t\triangle}\right] =$$
$$e^{\mathrm{i}t\triangle}\left[-\triangle, g(x)\right]e^{-\mathrm{i}t\triangle} + e^{\mathrm{i}t\triangle}\left[\triangle, g(x)\right]e^{-\mathrm{i}t\triangle} = 0,$$

so obviously the same commutation rule holds for the r.h.s. of (5). In particular for $g(x) = x_j$ we get on the r.h.s. of (5) the operators $J_j = 2t\mathrm{i}e^{\frac{\mathrm{i}x^2}{4t}}\partial_{x_j}e^{-\frac{\mathrm{i}x^2}{4t}} = 2t\mathrm{i}\partial_{x_j} + x_j$ and we have

$$[\mathrm{i}\partial_t + \triangle, J_j] = 0.$$

We introduce for any $s \geq 0$ the following two operators:

$$|J(t)|^s := M(t)(-t^2\triangle)^{\frac{s}{2}}M(-t) \tag{6}$$
$$|J_V(t)|^s := M(t)(-t^2\triangle_V)^{\frac{s}{2}}M(-t). \tag{7}$$

The operators $|J_V(t)|^s$ are used to define homogeneous spaces $\dot{\mathcal{H}}^s_V(t)$ which are then shown to be almost invariant.

The argument is more complicated than in [11] because of the presence of an additional commutator. But we can show that if $\triangle_V$ is generic, in the sense of Hypothesis (H), then the commutator can be treated by a bootstrap argument.

Another complication is that the $|J_V(t)|^s$ do not enjoy Leibnitz rule type properties like $|J(t)|^s$, which play a key role in [11]. Nonetheless, we are able to treat $|J_V(t)|^s$ by switching from $|J_V(t)|^s$ to $|J(t)|^s$, by using the Leibnitz rule for $|J(t)|^s$, and by going back to $|J_V(t)|^s$.

In the part of the argument on the Leibnitz rule, an essential role is played by the observation that $\|\cdot\|_{\dot{\mathcal{H}}^s_V(t)} \approx \|\cdot\|_{\dot{\mathcal{H}}^s(t)}$ with fixed constants independent of t when $0 \leq s < 1/2$. The proof of this equivalence is based on Paley-Littlewood decompositions associated to phase spaces both of $\triangle$ and $\triangle_V$. We are able to prove this equivalence when the transmission coefficient $T(\tau)$ is such that either $T(0) = 0$ (the generic case) or $T(0) = 1$. Notice incidentally that the inclusion of this non generic case is natural, since the fact that $T(0) = 1$ makes $\triangle_V$ more similar to $\triangle$ than the case when $T(0) = 0$ (recall that $T(0) = 1$ for $\triangle$).

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Control of blow-up set by spatial inhomogeneous coefficient for a semilinear parabolic equation

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We study the blow-up phenomena for the following initial boundary value problem:

$$\begin{cases} u_t = \Delta u + q(x)u^p & x \in \Omega, t > 0, \\ u(x,0) = u_0(x), & x \in \bar{\Omega}, \\ u(x,t) = 0, & x \in \partial\Omega,\ t > 0, \end{cases} \tag{1}$$

where Ω is bounded smooth domain, q is Hölder continuous in $\bar{\Omega}$, $q(x) \geq 0$, $q(x) \not\equiv 0$, $p > 1$ and $u_0 \geq 0$, $u_0 \not\equiv 0$ is a smooth function with $u_0|_{\partial\Omega} = 0$. We assume all zeros of $q(x)$ are included in Ω. Our aim is to consider any zero point of $q(x)$ is a blow-up point or not.

It is known that for each initial datum u_0 as above, (1) has a nonnegative classical solution u for $t \in [0, T)$ for some maximal existence time $T \in (0, \infty]$. If $T < \infty$, then we have

$$\limsup_{t \to T} \|u(\cdot, t)\|_{L^\infty(\Omega)} = \infty$$

and we say that the solution u *blows up* in finite time with the *blow-up time* T. For a given solution u that blows up at $t = T < \infty$, a point $a \in \bar{\Omega}$ is called a blow-up point if there exists a sequence $\{(x_n, t_n)\}$ in $Q_T := \Omega \times (0, T)$ such that $x_n \to a$, $t_n \uparrow T$ and $u(x_n, t_n) \to \infty$ as $n \to \infty$. The set of all blow-up points is called the *blow-up set*.

Our first theorem claims that any zero of q is not blow-up point if the solution is sufficiently large ([2]).

Theorem . *Assume that $u_t(x, 0) \geq 0$ for all $x \in \Omega$. Then $T < \infty$ and u satisfies $\|u(x,t)\|_{L^\infty(\Omega)} \leq K(T-t)^{-\frac{1}{p-1}}$ for some $K = K(p, q, \Omega, T) > 0$. Moreover, any zero point of $q(x)$ is not a blow-up point.*

For non blow-up at any zero of potential $q(x)$, we construct supersolution that does not blow-up at any zero point of $q(x)$. The method can be applied to much general $q(x)$ and the proof is much simpler than that in [1] for quenching problem. This argument was generalized to parabolic systems and a single equation with nonlinear memory ([4]) as $u_t = \Delta u + q(x) \displaystyle\int_0^t u^p(x,s)ds$.

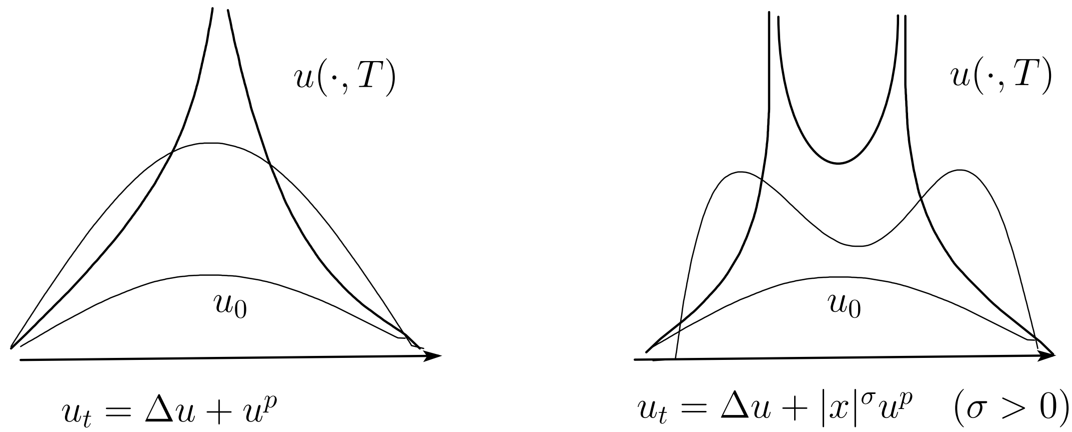
In the following, let us consider a spatially inhomogeneous equation:

$$u_t = \Delta u + |x|^\sigma u^p, \quad x \in \Omega,\ t > 0, \tag{2}$$

where $p > 1$ and $\sigma > 0$. We assume the same boundary condition as (1).

For this initial boundary value problem, we also establish several conditions that ensure the origin is not a blow-up point ([3]).

Theorem . *Let $\Omega = B_R$, $N \geq 4$ and $\sigma > (p-1)(N-1)/2$. We assume that u is a radially symmetric solution of (2) that blows up in finite time T. Then $x = 0$ is not a blow-up point of u.*



These results imply that the zero points of inhomogeneous term can control the blow-up point. Now our question is such points always can not be a blow-up point or not ? The following surprising result contradicts our intuition, since the reaction is freezed at the origin and it seems to be impossible that blow-up occurs at the origin. Our strategy is to construct the (threshold) solution that blows up at the zero point of $|x|^\sigma$. For such solution diffusion and reaction antagonize and the maximum point is attracting to the origin ([2]).

Theorem . *Let $N = 3$, $p > 5 + 2\sigma$ and $\Omega = B_R := \{x \in \mathbb{R}^N, |x| \leq R\}$. Let u_μ be the radially symmetric solution of (2) with initial value μg, where $\mu > 0$ and g is a bounded positive function. Then there exists μ^* such that the solution u^* of (2) with the initial value $u_0 = \mu^* g$ exists globally as the minimal L^1-solution but u^* blows up in finite time and the origin is a blow-up point.*

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for any $t > 0$ and $\alpha \in \mathbb{Z}_+^n$ with $|\alpha| \geq 2$. We study the following linear dissipative equation with anomalous diffusion:

$$
(1.3) \qquad \begin{cases} \partial_t u + (-\Delta)^{\theta/2} u + a(t,x)u = 0, & t > 0,\ x \in \mathbb{R}^n, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^n, \end{cases}
$$

where $n \in \mathbb{N},\ 1 < \theta \leq 2$. When $\theta = 2$, this and (1.1) are equivalent. We assume the following assumption for the coefficient:

$$
(1.4) \qquad \|x^\alpha a(t)\|_p \leq C(1+t)^{-\mu + \frac{|\alpha|}{\theta} + \frac{n}{\theta p}}
$$

for any $\alpha \in \mathbb{Z}_+^n$ with $|\alpha| < \nu,\ n/(\nu - |\alpha|) < p \leq \infty$ and $t > 0$, where C is a positive constant, and $\mu > 1 + 1/\theta$ and $\nu > 0$ are some parameters. For example, if $a(t,x)$ satisfies the inequality

$$
|a(t,x)| \leq C(1+t)^{-\mu} \left\langle (1+t)^{-1/\theta} x \right\rangle^{-\nu}
$$

for some $C > 0$ and any $(t,x) \in (0,\infty) \times \mathbb{R}^n$, then the condition (1.4) is satisfied. Here we have used the notation $\langle x \rangle := \sqrt{1+|x|^2}$. We assume that the solution to (1.3) is well-posed in $L^p(\mathbb{R}^n)$ with some $1 \leq p \leq \infty$ and satisfies

$$
(1.5) \qquad \|x^\alpha u(t)\|_p \leq C(1+t)^{-\frac{n}{\theta}(1-\frac{1}{p}) + \frac{|\alpha|}{\theta}}
$$

for any $t > 0,\ \alpha \in \mathbb{Z}_+^n$ with $|\alpha| < n + \theta$ and $n/(n+\theta-|\alpha|) < p \leq \infty$. Indeed, for some problems, we obtain this inequality. We should remark that the anomalous diffusion causes $\|x^\alpha u(t)\|_1 = +\infty$ when $|\alpha| \geq 2$. Namely a moment of the solution with high-order diverges to infinity. We consider the large-time behavior of the solution to (1.3). Especially we give the estimate on the difference between the solution and its asymptotic expansion as $t \to \infty$. For (1.1) (namely, for (1.3) with $\theta = 2$), Ishige, Ishiwata and Kawakami [6] derived the large-time behavior of the solution completely. In their asymptotic expansion, the coefficients contain the moments of the solution. Unfortunately we cannot extend this idea to our problem since the moments of the solution cannot be defined. We provide the other way to reach our goal. For some $(l,\beta) \in \mathbb{Z}_+ \times \mathbb{Z}_+^n$, we introduce the following notation:

$$
(1.6) \qquad \begin{aligned} U_{l,\beta}(t,x) := &\frac{\partial_t^l \nabla^\beta G_\theta(1+t,x)}{l!\beta!} \biggl(\int_{\mathbb{R}^n} (-1)^l (-y)^\beta u_0(y) dy \\ &- \int_0^t \int_{\mathbb{R}^n} (-1-s)^l (-y)^\beta (au)(s,y) dy ds \biggr), \end{aligned}
$$

where $G_\theta(t)$ is defined by (1.2). Then we see the following proposition.

Proposition 1.2. *Let $n \in \mathbb{N},\ 1 < \theta \leq 2,\ \mu > 1+1/\theta$ and $\nu > 0$. Let* (1.4) *and* (1.5) *be satisfied. Assume that $u_0 \in L_N^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ for $N := \max\{m \in \mathbb{Z}_+ \mid m < (\mu-1)\theta, \nu+\theta\}$ and $u(t)$ is the solution to* (1.3)*, where*

$$
(1.7) \qquad L_N^1(\mathbb{R}^n) := \left\{ \varphi \in L^1(\mathbb{R}^n) \;\middle|\; |x|^N \varphi \in L^1(\mathbb{R}^n) \right\}.
$$

Let

$$
(1.8) \qquad \bar{\Lambda}_N := \left\{ (l,\beta) \in \mathbb{Z}_+ \times \mathbb{Z}_+^n \;\middle|\; \theta l + |\beta| \leq N \right\}
$$

and the functions $\{U_{l,\beta}(t)\}_{(l,\beta)\in\bar{\Lambda}_N}$ *be given by* (1.6)*. Then the following estimate holds:*

$$\left\| u(t) - \sum_{(l,\beta)\in\bar{\Lambda}_N} U_{l,\beta}(t) \right\|_p = o\left(t^{-\frac{n}{\theta}(1-\frac{1}{p})-\frac{N}{\theta}}\right) \quad as \quad t\to\infty$$

for $1 \le p \le \infty$.

When $\theta = 2$, the asymptotic expansion of solution of this type was firstly observed by Escobedo and Zuazua. In [4], they derived the asymptotic expansion of solutions to the heat-convection equation. For the solution of the Navier-Stokes equation, the asymptotic expansion was provided by Carpio [3], and Fujigaki and Miyakawa [5]. The large-time behavior of the solution to the Keller-Segel equation in $L^p(\mathbb{R}^n)$ was considered by Nagai, Syukuinn and Umesako [9], Kato [7], and Nagai and Yamada [10].

Since the conditions (1.4), (1.5) and $N < \min\{(\mu-1)\theta, \nu+\theta\}$ are assumed, the coefficient $\int_0^\infty \int_{\mathbb{R}^n} (-1-s)^l(-y)^\beta (au)(s,y)dyds$ is uniformly integrable when $(l,\beta) \in \bar{\Lambda}_N$ holds. Hence, under this assumption, the asymptotic expansion (1.6) is well-defined. However, when $N \ge (\mu-1)\theta$ or $N \ge \nu+\theta$ is satisfied, some coefficients in the asymptotic expansion diverge to infinity. Thus, in this case, we cannot define the higher-order asymptotic expansion by the form as (1.6). Before proceeding next step, we study the decay-rates of the solution as $|x|\to\infty$. Then we obtain the following proposition.

Proposition 1.3. *Let* $n\in\mathbb{N}$, $1<\theta\le 2$, $\mu > 1+1/\theta$ *and* $\nu > 0$*. Let* (1.4) *and* (1.5) *be satisfied. Assume that* $u_0 \in L^1_N(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)$ *for* $N := \max\{m\in\mathbb{Z}_+ \mid m < (\mu-1)\theta, \nu+\theta\}$ *and* $u(t)$ *is the solution to* (1.3)*, where* $L^1_N(\mathbb{R}^n)$ *is defined by* (1.7)*. Let*

$$\Lambda_N := \left\{(l,\beta)\in\mathbb{Z}_+\times\mathbb{Z}_+^n \;\middle|\; \theta l + |\beta| < N\right\} \tag{1.9}$$

and

$$U_{\Lambda_N}(t,x) := \sum_{(l,\beta)\in\Lambda_N} U_{l,\beta}(t,x), \tag{1.10}$$

where the functions $\{U_{l,\beta}(t)\}_{(l,\beta)\in\Lambda_N}$ *be defined by* (1.6)*. Then* $u(t) - U_{\Lambda_N}(t) \in L^1_N(\mathbb{R}^n)$ *holds for any* $t>0$*. Moreover there exists a positive constant* C *such that*

$$\|x^\alpha (u(t) - U_{\Lambda_N}(t))\|_1 \le C(1+t)^{-\frac{N-|\alpha|}{\theta}}$$

for any $t>0$ *and* $\alpha\in\mathbb{Z}^n_+$ *with* $|\alpha|\le N$.

We should remark that, for any $\alpha\in\mathbb{Z}^n_+$ with $|\alpha|\ge 2$, the function $U_{\Lambda_N}(t)$ in Proposition 1.3 satisfies $\|x^\alpha U_{\Lambda_N}(t)\|_1 = \infty$. Thus this function gives the approximation of the solution as $|x|\to\infty$. Proposition 1.2 and 1.3 state that the asymptotic expansion of the solution as $t\to\infty$ and the approximation of the solution as $|x|\to\infty$ are given by the same form. We derive the higher-order asymptotic expansion of the solution by employing this proposition.
Notation. Throughout this manuscript, we use the following notation. For any $x=(x_1,\dots,x_n)$ and $y=(y_1,\dots,y_n)\in\mathbb{R}^n$, we denote $x\cdot y := \sum_{j=1}^n x_jy_j$, $|x|^2 := x\cdot x$ and $\langle x\rangle := \sqrt{1+|x|^2}$. For $1\le p\le\infty$ and $\theta>0$, $L^p(\mathbb{R}^n)$ denotes the Lebesgue spaces and $W^{\theta,p}(\mathbb{R}^n)$ denotes the Sobolev spaces. The norm of $L^p(\mathbb{R}^n)$ is represented by $\|\cdot\|_p$. For $s>0$, we define the weighted $L^1(\mathbb{R}^n)$ space by $L^1_s(\mathbb{R}^n) := \{\varphi\in L^1(\mathbb{R}^n) \mid \|\varphi\|_{L^1_s} := \int_{\mathbb{R}^n}\langle x\rangle^s|\varphi(x)|dx < \infty\}$. For $f=f(x)$ and $g=g(x)$,

we denote the convolution by $f * g(x) := \int_{\mathbb{R}^n} f(x-y)g(y)dy$. The gamma function $\Gamma = \Gamma(p)$ for $p > 0$ is provided by $\Gamma(p) := \int_0^\infty e^{-t}t^{p-1}dt$. Varias constants are simply denoted by C.

2. Preliminaries

Before stating our results, we study some basic properties of $G_\theta(t) := \mathcal{F}^{-1}[e^{-t|\xi|^\theta}]$.

Lemma 2.1. *For any $l \in \mathbb{Z}_+$ and $\alpha \in \mathbb{Z}_+^n$, there exists a positive constant $C > 0$ such that*

$$\left|\partial_t^l \nabla^\alpha G_\theta(t,x)\right| \le Ct^{-\frac{n}{\theta}-l-\frac{|\alpha|}{\theta}} \left\langle t^{-1/\theta}x\right\rangle^{-n-\theta-\theta l-|\alpha|}$$

for any $(t,x) \in (0,\infty)\times\mathbb{R}^n$.

Proof. This lemma is proved by employing [11, Theorem 3.1.]. Here we omit the proof. □

By applying Taylor's formula, Lemma 2.1 and Hausdorf-Young's inequality, we obtain the following lemma.

Lemma 2.2. *Let $N \in \mathbb{Z}_+$ and $u_0 \in L_N^1(\mathbb{R}^n)$. Then the following estimate holds for any $1 \le p \le \infty$:*

$$\left\|G_\theta(t) * u_0 - \sum_{(l,\beta)\in\bar{\Lambda}_N} \frac{\partial_t^l \nabla^\beta G_\theta(1+t)}{l!\beta!} \int_{\mathbb{R}^n} (-1)^l(-y)^\beta u_0(y)dy\right\|_p = o\left(t^{-\frac{n}{\theta}(1-\frac{1}{p})-\frac{N}{\theta}}\right)$$

as $t \to \infty$, where $\bar{\Lambda}_N$ is defined by (1.8).

When we study the decay of $G_\theta(t) * u_0(x)$ as $|x| \to \infty$, we obtain the approximation of this by the same form as in Lemma 2.2.

Lemma 2.3. *Let $N \in \mathbb{Z}_+$ and $u_0 \in L_N^1(\mathbb{R}^n)$. Then the following inequality holds for any $t > 0$ and $\alpha \in \mathbb{Z}_+^n$ with $|\alpha| \le N$:*

$$\begin{aligned}&\left\|x^\alpha\left(G_\theta(t) * u_0 - \sum_{(l,\beta)\in\Lambda_N} \frac{\partial_t^l \nabla^\beta G_\theta(1+t)}{l!\beta!} \int_{\mathbb{R}^n} (-1)^l(-y)^\beta u_0(y)dy\right)\right\|_1 \\ \le &C(1+t)^{-\frac{N-|\alpha|}{\theta}},\end{aligned}$$

where C is a positive constant and Λ_N is defined by (1.9).

3. Main Results

In order to derive the asymptotic expansion of the solution, we introduce the following corresponding integral equation:

$$u(t) = G_\theta(t) * u_0 - \int_0^t G_\theta(t-s) * (au)(s)ds. \tag{3.1}$$

The solution to (3.1) is called the mild solution of (1.3). Generally speaking a mild solution solves an original Cauchy problem if it has sufficiently high regularity. Hereafter we consider the mild solution for deriving the asymptotic expansion of the solution of (1.3). For $N = \min\{m \in \mathbb{Z}_+ \mid m < (\mu-1)\theta, \nu+\theta\}$, the nonlinear term on the right hand side of (3.1) is split as

$$\int_0^t G_\theta(t-s) * (au)(s)ds = \sum_{(l,\beta)\in\Lambda_N} J_{l,\beta}(t) + \int_0^t G_\theta(t-s) * \left(a(u-U_{\Lambda_N})\right)(s)ds,$$

where

$$J_{l,\beta}(t,x) := \int_0^t G_\theta(t-s) * (aU_{l,\beta})(s)ds \tag{3.2}$$

for $(l,\beta) \in \Lambda_N$ and $U_{l,\beta}(t)$ is defined by (1.6). We introduce the function $\tilde{U}_{l,\beta} = \tilde{U}_{l,\beta}(t,x)$ by

$$\begin{aligned} \tilde{U}_{l,\beta}(t,x) :=& \frac{\partial_t^l \nabla^\beta G_\theta(1+t,x)}{l!\beta!}\Bigg(\int_{\mathbb{R}^n} (-1)^l (-y)^\beta u_0(y)dy \\ &- \int_0^t \int_{\mathbb{R}^n} (-1-s)^l (-y)^\beta \left(a(u-U_{\Lambda_N})\right)(s,y)dyds\Bigg) \end{aligned} \tag{3.3}$$

for some $(l,\beta) \in \mathbb{Z}_+ \times \mathbb{Z}_+^n$, where $U_{\Lambda_N}(t)$ is defined by (1.10). By employing the assumption (1.4) and Proposition 1.3, we see that $\tilde{U}_{l,\beta}(t)$ is well-defined when $(l,\beta) \in \bar{\Lambda}_{2N}$ holds, where $\bar{\Lambda}_{2N}$ is defined as (1.8). Then we give our main result in the following theorems.

Theorem 3.1. *Let $n \in \mathbb{N}$, $1 < \theta \le 2$, $\mu > 1 + 1/\theta$ and $\nu > 0$. Let* (1.4) *and* (1.5) *be satisfied. Assume that $u_0 \in L^1_{2N}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ for $N := \max\{m \in \mathbb{Z}_+ \mid m < (\mu-1)\theta, \nu+\theta\}$ and $u(t)$ is the solution to* (1.3)*, where $L^1_{2N}(\mathbb{R}^n)$ is defined as* (1.7)*. Let Λ_N and $\bar{\Lambda}_{2N}$ be defined as* (1.8) *and* (1.9)*, and the functions $\{J_{k,\alpha}(t)\}_{(k,\alpha)\in\Lambda_N}$ and $\{\tilde{U}_{l,\beta}(t)\}_{(l,\beta)\in\bar{\Lambda}_{2N}}$ be given by* (3.2) *and* (3.3)*. Then the following estimate holds:*

$$\left\| u(t) - \sum_{(k,\alpha)\in\Lambda_N} J_{k,\alpha}(t) - \sum_{(l,\beta)\in\bar{\Lambda}_{2N}} \tilde{U}_{l,\beta}(t) \right\|_p = o\left(t^{-\frac{n}{\theta}(1-\frac{1}{p})-\frac{2N}{\theta}}\right) \quad as \quad t \to \infty$$

for $1 \le p \le \infty$.

Theorem 3.2. *Let $n \in \mathbb{N}$, $1 < \theta \le 2$, $\mu > 1 + 1/\theta$ and $\nu > 0$. Let* (1.4) *and* (1.5) *be satisfied. Assume that $u_0 \in L^1_{2N}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ for $N := \max\{m \in \mathbb{Z}_+ \mid m < (\mu-1)\theta, \nu+\theta\}$ and $u(t)$ is the solution to* (1.3)*, where $L^1_{2N}(\mathbb{R}^n)$ is defined as* (1.7)*. Let Λ_N and Λ_{2N} be defined as* (1.9) *and the functions $\{J_{k,\alpha}(t)\}_{(k,\alpha)\in\Lambda_N}$ and $\{\tilde{U}_{l,\beta}(t)\}_{(l,\beta)\in\Lambda_{2N}}$ be given by* (3.2) *and* (3.3)*. Then*

$$u(t) - \sum_{(k,\alpha)\in\Lambda_N} J_{k,\alpha}(t) - \sum_{(l,\beta)\in\Lambda_{2N}} \tilde{U}_{l,\beta}(t) \in L^1_{2N}(\mathbb{R}^n)$$

holds for any $t > 0$. Moreover there exists a positive constant C such that:

$$\left\| x^\alpha \left(u(t) - \sum_{(k,\alpha)\in\Lambda_N} J_{k,\alpha}(t) - \sum_{(l,\beta)\in\Lambda_{2N}} \tilde{U}_{l,\beta}(t) \right) \right\|_1 \le C(1+t)^{-\frac{2N-|\alpha|}{\theta}}$$

for any $t > 0$ and $\alpha \in \mathbb{Z}_+^n$ with $|\alpha| \le 2N$.

For $N := \max\{m \in \mathbb{Z}_+ \mid m < (\mu-1)\theta, \nu+\theta\}$, Proposition 1.2 and 1.3 give the Nth-order asymptotic expansion of the solution. Theorem 3.1 and 3.2 provide the $2N$th-order asymptotic expansion. We prove those theorems by employing Proposition 1.2 and 1.3. Similarly, by applying Theorem 3.1 and 3.2, we obtain a $3N$th-order asymptotic expansion. By repeating this procedure, we can derive an mNth-order asymptotic expansion for arbitrary large $m \in \mathbb{Z}_+$.

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