Evolution of spirals by an eikonal-curvature flow equation with a single level set formulation

Takeshi Ohtsuka[1]
Division of Mathematical Sciences, Graduate School of Engineering, Gunma University
4-2, Aramaki-machi, Maebashi-shi, Gunma 371-8310, Japan.

In the memory of Rentaro Agemi

1. Introduction

Burton, Cabrera and Frank [BCF51] proposed a theory of crystal growth with aid of screw dislocations. They pointed out that screw dislocations supply spiral steps to a crystal surface when the screw dislocations appear on the surface. Steps evolve catching adatoms as they climb a spiral staircase, and thus the surface evolves. Burton et al. calculated the step velocity with Gibbs-Thomson effect, and derived an eikonal-curvature flow of the form

$$V = v_\infty(1 - \rho_c\kappa) \tag{1}$$

by regarding the evolution of steps as evolution of curves on the plane, where v_∞ is the velocity of straight line steps, ρ_c is the critical radius reflecting the Gibbs-Thomson effect, V is the normal velocity of the curve which denotes the location of steps, and κ is the curvature of the curve with opposite direction of V. Note that we shall use the words 'step', 'curve', and 'spiral' interchangeably because of the above background. One can find a complex spiral patterns on the growing crystal surface, which is caused by the evolution of spiral steps and collision with each other. Several models for this phenomena are proposed by [KP98], [Kob10] with phase field models, and by [Sme00] and [Oht03] with level set methods.

In this talk we consider the evolution of spiral curves by an eikonal-curvature flow with the level set formulation by [Oht03], and investigate behavior of spirals with mathematical results of the formulation. In particular, two characteristic problems are considered; one is behavior of a bunch of steps, which corresponds to variety of heights of the steps. In this problem one can find the crucial difference between phase field models and our formulations. The other is on the stationary solutions caused by an 'inactive pair', which corresponds to the stationary curve under an eikonal-curvature flow equation. Formally, the circle whose radius is ρ_c does not evolve under (1), and it is unstable. In this talk we shall find stable stationary curves like as the above.

Results on §2.2 are partly joint work with Shun'ichi Goto and Maki Nakagawa, and those on §2.3 and §3 are joint work with Yen-Hsi Richard Tsai and Yoshikazu Giga.

2. Formulation and basic properties

We here introduce a level set formulation with a single auxiliary function for evolving spirals by an eikonal-curvature flow equation. Its crucial difficulty lies in the fact that a spiral curve generally does not divide a domain into two subdomains so that the usual level set formulation

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$\{x;\ u(t,x)=0\}$ does not work well. To overcome this difficulty, we combine a level set method and a sheet structure function due to Kobayashi [Kob10] or Karma and Plapp [KP98] in their phase field models.

2.1. Level set formulation for evolving spirals

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary. We assume that there exist $N(\geq 1)$ spiral centers denoted by $a_1, \ldots, a_N \in \Omega$, and each center may have multiple spirals. In this talk we also regard an open neighborhood U_j of a_j as a j-th center of spirals interchangeably with a_j. Set $W = \Omega \setminus (\bigcup_{j=1}^N \overline{U}_j)$, and we here consider evolving spirals Γ_t at time $t \geq 0$ on $\overline{W}$ having the direction $\mathbf{n}$ of the evolution, where $\mathbf{n}\colon \Gamma_t \to S^1$ is a continuous unit normal vector field of Γ_t. The evolution equation is the rescaled equation of (1) on time of the form

$$V = C - \kappa \quad \text{on } \Gamma_t \tag{2}$$

with a constant C, and also impose that the end points of Γ_t always stay on ∂W with the orthogonality condition

$$\Gamma_t \perp \partial W. \tag{3}$$

For multiplicity of spirals let $m_j \in \mathbb{Z} \setminus \{0\}$ be a constant denoting the number and rotational orientation of spirals associated with a_j: $|m_j|$-spirals go around a_j with anti-clockwise (resp. clockwise) rotation if $m_j \geq 0$ (resp. $m_j < 0$) provided that spirals have positive velocity in the direction of $\mathbf{n}$. We shall discuss in detail how to determine m_j from physical situation in §2.3.

In [Oht03] the author propose a level set formulation for spirals Γ_t for $t \geq 0$ as

$$\Gamma_t = \{x \in \overline{W};\ u(t,x) - \theta(x) \equiv 0 \mod 2\pi\mathbb{Z}\}, \quad \mathbf{n} = -\frac{\nabla(u-\theta)}{|\nabla(u-\theta)|} \tag{4}$$

with a sheet structure function

$$\theta(x) = \sum_{j=1}^N m_j \arg(x - a_j).$$

The function θ is introduced by Kobayashi [Kob10] in his phase field model. Karma and Plapp [KP98] also introduce $\theta(x) = \arg x$ for a single spiral, i.e., for the case $N=1$, $a_1 = 0$ and $m_1 = 1$. The function θ denotes helical layer structure of atoms in a crystal with screw dislocations. From the theory of dislocation and linear elasticity the surface height $h(t,x)$ satisfies

$$\Delta h = -h_0 \mathrm{div} \delta_{\Gamma_t} \mathbf{n},$$

where h_0 is a unit height of steps (see [HL68]). One can find $h = (h_0/2\pi)\theta$ whose discontinuity is only on Γ_t satisfies the above from straightforward calculation.

Our formulation is regarded interior and exterior of the crystal as the place where $z < h(t,x)$ and $z > h(t,x)$, respectively, provided that '$z = (h_0/2\pi)\sum_{j=1}^N m_j \arg(x-a_j)$'. To describe the above exactly we now introduce the covering space $\mathfrak{X}$ as in [Oht03] of the form

$$\mathfrak{X} := \{(x,\xi) \in \overline{W} \times \mathbb{R}^N;\ \frac{x-a_j}{|x-a_j|} = (\cos\xi_j, \sin\xi_j) \quad \text{for } j = 1, \ldots, N\},$$

where ξ_i is such that $\xi = (\xi_1, \dots, \xi_N)$. Then, the interior $\widetilde{I}_t$ or the exterior $\widetilde{O}_t$ and thus the steps $\widetilde{\Gamma}_t$ at time $t \geq 0$ is described by

$$\widetilde{I}_t = \{(x,\xi) \in \mathfrak{X};\ u(t,x) - \sum_{j=1}^{N} m_j \xi_j > 0\}, \quad \widetilde{O}_t = \{(x,\xi) \in \mathfrak{X};\ u(t,x) - \sum_{j=1}^{N} m_j \xi_j < 0\},$$
$$\widetilde{\Gamma}_t = \{(x,\xi) \in \mathfrak{X};\ u(t,x) - \sum_{j=1}^{N} m_j \xi_j = 0\}$$

with an auxiliary function $u\colon [0,\infty) \times \overline{W} \to \mathbb{R}$. Then we obtain (4) from this formulation and inequalities describing the interior and exterior. The above formulae play very important role in mathematical analysis, in particular, when we investigate behavior of spirals.

Naturally the sheet structure function should be a multi-valued function in our formulation, but locally our formulation is same as the usual level set of $u - \theta$. Then from straightforward calculation in the usual level set method we derive

$$V = \frac{u_t}{|\nabla(u-\theta)|}, \quad \kappa = -\mathrm{div}\frac{\nabla(u-\theta)}{|\nabla(u-\theta)|},$$

and thus we obtain the level set equation of the form

$$u_t - |\nabla(u-\theta)| \left\{ \mathrm{div}\frac{\nabla(u-\theta)}{|\nabla(u-\theta)|} + C \right\} = 0 \quad \text{in} \quad (0,T) \times W, \tag{5}$$
$$\langle \vec{\nu}, \nabla(u-\theta) \rangle = 0 \quad \text{on} \quad (0,T) \times \partial W \tag{6}$$

from (2)–(3), where $\vec{\nu}$ is the outer unit normal vector field of ∂W, and $\langle \cdot, \cdot \rangle$ is the usual inner product in $\mathbb{R}^2$ (see [Gig06] for detail).

2.2. Basic properties

The equation (5)–(6) is represented by

$$u_t + F(\nabla(u-\theta), \nabla^2(u-\theta)) = 0 \quad \text{in} \quad (0,T) \times W,$$
$$B(x, \nabla(u-\theta)) = 0 \quad \text{on} \quad (0,T) \times \partial W$$

with $F\colon (\mathbb{R}^2 \setminus \{0\}) \times \mathbb{S}^2 \to \mathbb{R}$, $B\colon \partial W \times \mathbb{R} \to \mathbb{R}$ and functions $\widetilde{F}$ and $\widetilde{B}$ of the form

$$F(p,x) = -\mathrm{trace}\left\{ \left(I - \frac{p \otimes p}{|p|^2} \right) X \right\} - C|p|,$$
$$B(x,p) = \langle \vec{\nu}, p \rangle,$$

where $\mathbb{S}^2$ is the space of 2×2 real symmetric matrices.

Note that F is degenerate elliptic, and then we consider solutions of (4)–(6) in usual viscosity solution sense (see [CGG91], [CIL92] or [Gig06] for detail). The author obtained the comparison principle, and the existence and uniqueness of viscosity solutions globally in time for a continuous initial data.

Theorem 1 ([Oht03])**.** *Let $u, v\colon [0,T)\times\overline{W}\to\mathbb{R}$ be a viscosity sub- and supersolution of* (5)–(6) *on $(0,T)\times\overline{W}$. If $u^*\le v_*$ on $\{0\}\times\overline{W}$, then $u^*\le v_*$ on $(0,T)\times\overline{W}$, where u^* (resp. v_*) is an upper (resp. lower) semicontinuous envelope of u (resp. v) of the form*

$$u^*(t,x)=\lim_{r\to 0}\sup\{u(s,y);\ |(t,x)-(s,y)|<r\}$$
$$(\textit{resp. } v_*(t,x)=\lim_{r\to 0}\inf\{v(s,y);\ |(t,x)-(s,y)|<r\}).$$

Theorem 2 ([Oht03])**.** *For $u_0\in C(\overline{W})$ there exists a viscosity solution $u\in C([0,\infty)\times\overline{W})$ on $(0,\infty)\times\overline{W}$ with $u|_{t=0}=u_0$.*

In the above analysis, in particular on the comparison, we attempt to consider $w=u-\theta$ and apply the results by [GS93] and [Sat94] directly, but it does not work well since θ is a multi-valued function. To overcome this difficulty we estimate $\tilde{u}^*-\tilde{v}_*$ in $[0,T)\times\mathfrak{X}$ instead of u^*-v_*, where

$$\tilde{u}(t,x,\xi):=u(t,x)-\sum_{j=1}^{N}m_j\xi_j, \tag{7}$$

and $\tilde{v}$ is similar as the above. Note that $\tilde{u}^*(t,x,\xi)=u^*(t,x)-\sum_{j=1}^{N}m_j\xi_j$ and $\tilde{v}_*(t,x,\xi)=v_*(t,x)-\sum_{j=1}^{N}m_j\xi_j$. Then, we derive the above results by revising the proofs in [GS93] or [Sat94] with $\tilde{u}$ and $\tilde{v}$ instead of $u-\theta$ and $v-\theta$, respectively.

2.3. Properties on the presented level set method

To describe an evolution of spirals we execute the followings;

(i) From given Γ_0 and $\mathbf{n}$, we construct $u_0\in C(\overline{W})$ and $\theta(x)=\sum_{j=1}^{N}m_j\arg(x-a_j)$ satisfying

$$\Gamma_0=\{x\in\overline{W};\ u_0(x)-\theta(x)\equiv 0 \mod 2\pi\mathbb{Z}\}. \tag{8}$$

(ii) Solve (5)–(6) with an initial data $u|_{t=0}=u_0$.

(iii) Draw Γ_t defined by (4) (and construct the height function $h(t,x)$ from u if necessary).

It remains two problems to complete the above.

(Q1) **(Construction of initial configuration)** How to construct $u_0\in C(\overline{W})$ and determine m_j from given Γ_0?

(Q2) **(Uniqueness of level sets)** Is Γ_t uniquely determined from Γ_0?

Uniqueness of level sets is come from the fact that $u_0\in C(\overline{W})$ satisfying (8) is not unique for given Γ_0. However, Chen, Giga and Goto [CGG91], or Evans and Spruck [ES91] obtained the uniqueness of level sets for geometric evolution equation. Although our equation is not geometric for u, Goto, Nakagawa and the author also derived the uniqueness result with revision of the proof of [CGG91] since our equation presented is geometric for '$u-\theta$'.

Theorem 3 ([GNO08]). *Let $u, v\colon [0,T) \times \overline{W} \to \mathbb{R}$ be a viscosity sub- and supersolution of (5)–(6) in $(0,T) \times \overline{W}$. Assume that*

$$\{(x,\xi) \in \mathfrak{X};\ \tilde{u}^*(0,x,\xi) > 0\} \subset \{(x,\xi) \in \mathfrak{X};\ \tilde{v}_*(0,x,\xi) > 0\}$$
$$(\textit{resp. } \{(x,\xi) \in \mathfrak{X};\ \tilde{u}^*(0,x,\xi) < 0\} \supset \{(x,\xi) \in \mathfrak{X};\ \tilde{v}_*(0,x,\xi) < 0\}),$$

where $\tilde{u}^*(t,x,\xi) = u^*(t,x) - \sum_{j=1}^N m_j\xi_j$ and $\tilde{v}_*(t,x,\xi) = v_*(t,x) - \sum_{j=1}^N m_j\xi_j$. Then,

$$\{(x,\xi) \in \mathfrak{X};\ \tilde{u}^*(t,x,\xi) > 0\} \subset \{(x,\xi) \in \mathfrak{X};\ \tilde{v}_*(t,x,\xi) > 0\}$$
$$(\textit{resp. } \{(x,\xi) \in \mathfrak{X};\ \tilde{u}^*(t,x,\xi) < 0\} \supset \{(x,\xi) \in \mathfrak{X};\ \tilde{v}_*(t,x,\xi) < 0\}),$$

for $t \in (0,T)$.

The result in [GNO08] is obtained for continuous solutions u and v. Fortunately, their result is extended to our statement with a little revision for semicontinuous solutions.

The basic strategy of the proof of Theorem 3 is based on [CGG91], i.e., modify v to $w = G(v_*-\theta)+\theta$ with lower semicontinous and nondecreasing function G to enjoy Theorem 1 between u and w with $\{(x,\xi) \in \mathfrak{X};\ \tilde{v}_*(t,x,\xi) > 0\} \supset \{(x,\xi) \in \mathfrak{X};\ \tilde{w}(t,x,\xi) > 0\}$. The function G is defined similarly as in [CGG91] with a little revision for our problem. Although w includes the multi-valued function θ, however we also obtain $G(s+2\pi) = G(s)+2\pi$ for sufficiently large s with the revision to our problem, and thus w is well-defined in some sense.

For the problem of initial configuration Goto, Nakagawa and the author [GNO08] proved the existence of m_j and $u_0 \in C(\overline{W})$ for suitable Γ_0, and clarify class of Γ_0.

It is convenient for the initial configuration to classify spirals as in [GNO08] into two kind of spirals depending on the feature whether or not it touches $\partial\Omega$. In the following argument let $\Gamma_0 := \{P(s);\ s \in [0,\ell]\}$ be smooth enough, and s be an arclength parameter.

Definition 4. (i) For a given $a \in \Omega$ let $U \subset\subset \Omega$ be its neighborhood, and set $W = \Omega \setminus \overline{U}$. We say Γ_0 is a simple spiral on $\overline{W}$ associated with a if $P(s)$ satisfies

(S1) $P(s)$ is a simple arc and $|\dot{P}(s)| = |(\mathrm{d}P/\mathrm{d}s)(s)| \neq 0$ for $s \in [0,\ell]$,

(S2) $P(0) \in \partial U$, $P(\ell) \in \partial\Omega$ and $P(s) \in W$ for $s \in (0,\ell)$.

(ii) For a given $a_1, a_2 \in \Omega$ let $U_i \subset\subset \Omega$ be a neighborhood of a_i for $i = 1, 2$, and set $W = \Omega \setminus (\overline{U_1} \cup \overline{U}_2)$. Assume that $\overline{U}_1 \cap \overline{U}_2 = \emptyset$. We say Γ_0 is a connecting spiral on $\overline{W}$ between a_1 and a_2 if $P(s)$ satisfies (S1) and

(S2)' $P(0) \in \partial U_1$, $P(\ell) \in \partial U_2$ and $P(s) \in W$ for $s \in (0,\ell)$.

In the previous section we pointed out that $m_j \in \mathbb{Z} \setminus \{0\}$ is a number of rotational orientation for spirals associated with a_j. It is defined as follows.

Definition 5. Let Γ_0 be associated with a center a at $s = 0$. We say Γ_0 is anti-clockwise (resp. clockwise) rotational orientation (with respect to a) if

$$\mathbf{n}(P(s)) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \dot{P}(s) \quad \left(\text{resp. } \mathbf{n}(P(s)) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \dot{P}(s)\right).$$

The signed number $m_j \in \mathbb{Z} \setminus \{0\}$ of spirals associated with a_j is defined by

$$m_j = m_j^+ - m_j^-,$$

where $m_j^+, m_j^- \in \mathbb{N}$ are numbers of anti-clockwise and clockwise rotational orientations of spirals associated with a_j, respectively.

Then, Goto, Nakagawa and the author obtained the existence of a continuous initial data for a given suitable Γ_0.

Theorem 6 ([GNO08]). *Let Γ_0 be a union of single and connecting spirals with a continuous unit normal vector field $\mathbf{n}$ on Γ_t. Then, there exists $m_j \in \mathbb{Z} \setminus \{0\}$ and $u_0 \in C(\overline{W})$ satisfying* (8).

It is obtained from the existence of a branch of θ whose discontinuity is only on Γ_0.

Lemma 7 ([GNO08]). *Under the same hypothesis in Theorem 6, there exists $\theta_{\Gamma_0} \colon \overline{W} \setminus \Gamma_0 \to \mathbb{R}$ which is a smooth branch of $\theta(x) = \sum_{j=1}^N m_j \arg(x - a_j)$.*

Let us consider a tubular neighborhood of Γ_0 of the form

$$\Gamma_0^\delta := \{x \in \overline{W};\ \inf_{y \in \Gamma_0} |x - y| < \delta\}.$$

Then, the signed distance function from Γ_0 is well-defined in Γ_0^δ, and thus we construct u_0 with θ_{Γ_0} on $\overline{W} \setminus \Gamma_0^\delta$, and a linear interpolation between θ_{Γ_0} and $\theta_{\Gamma_0} + 2\pi$ using the signed distance function on Γ_0^δ.

However, the above way, in particular the construction of θ_{Γ_0} and a tubular neighborhood of Γ_0 are impractical. For practicability we now introduce an additive way from initial data with less centers and multiplicity of spirals. Let $\Gamma_{0,1}$ and $\Gamma_{0,2}$ be a part of Γ_0 satisfying $\Gamma_{0,1} \cap \Gamma_{0,2} = \emptyset$ and $\Gamma_{0,1} \cup \Gamma_{0,2} = \Gamma_0$, and they are described by

$$\Gamma_{0,i} = \{x \in \overline{W};\ u_i(x) - \theta_i(x) \equiv 0 \mod 2\pi\mathbb{Z}\},$$

with auxiliary functions $u_i \in C(\overline{W})$ and $\theta_i(x) = \sum_{k=1}^{N_i} m_{i,k} \arg(x - a_{i,k})$ for $i = 1, 2$. To construct $u_0 \in C(\overline{W})$ describing Γ_0 we first modify u_i as

$$v_i(x) = \Theta_i(x) + 2\pi k_i(x) + \pi H_1(\lambda_i(u_i - (\Theta_i(x) + 2\pi k_i(x))))$$

with suitable constants $\lambda_i > 1/\pi$ determined later, where $\Theta_i(x) = \sum_{k=1}^{N_i} m_{i,k}\Theta_{i,k}(x)$, $\Theta_{i,k} \colon \overline{W} \to [0, 2\pi)$ is a principal value of $\arg(x - a_{i,k})$, $k_i \colon \overline{W} \to \mathbb{Z}$ is a function satisfying

$$-\pi \le u_i(x) - (\Theta_i(x) + 2\pi k_i(x)) < \pi \quad \text{for } x \in \overline{W}$$

for $i = 1, 2$, and H_1 is a function defined as

$$H_1(\sigma) = \begin{cases} -1 & \text{if } \sigma < -1, \\ \sigma & \text{if } |\sigma| \le 1, \\ 1 & \text{if } \sigma > 1. \end{cases}$$

The coefficients λ_i for $i = 1, 2$ is chosen such that

$$\bigcap_{i=1}^{2} \{x \in \overline{W};\ |v_i(x) - (\Theta_i(x) + 2\pi k_i(x))| < \pi\} = \emptyset.$$

Note that v_i still describes $\Gamma_{0,i}$ as (4) for $i = 1, 2$, and $v_i - \Theta_i(x) \equiv \pi \mod 2\pi\mathbb{Z}$ on $\Gamma_{0,j}$ if $i \neq j$. Thus we set

$$u_0(x) := v_1(x) + v_2(x) + \pi,$$

then we have obtained a desired function describing Γ_0 by (4). Note that simple and connecting straight lines are given by constant functions as follows;

$$\begin{aligned}
&\{a_i + r(\cos\alpha, \sin\alpha) \in \overline{W};\ r > 0\} = \{x \in \overline{W};\ \alpha - \arg(x - a_i) \equiv 0 \mod 2\pi\mathbb{Z}\},\\
&\{\sigma a_i + (1-\sigma)a_j \in \overline{W};\ \sigma \in (0,1)\}\\
&\qquad = \{x \in \overline{W};\ \pi - (\arg(x - a_i) - \arg(x - a_j)) \equiv 0 \mod 2\pi\mathbb{Z}\}.
\end{aligned}$$

Here we have assumed that each spirals are anti-clockwise rotational orientations with respect to a_i. From the above formulae and additive way we obtain $u_0 \in C(\overline{W})$ for Γ_0 which is a union of straight lines.

3. Behavior of spirals from phenomena

Our level set formulation, in particular the results of comparison in Theorem 1 and Theorem 3 enables us to study behavior of spirals. As their applications we investigate two kinds of behavior of spirals in this talk, one is related to heights of steps, and the other is on stationary solutions.

3.1. Stability of bunched steps

There is a difference on height of steps between the theory and physical experiments. Although we consider evolution of unit step (whose height is the diameter of an atom) in the theory, we also observe steps whose height is $O(10)$ or $O(100)$ by number of atoms in experiments. For simulations describing more exact situations the height of steps should be implied in formulations of spirals.

One of simple way to express the multiple height of steps is considering evolution of bunched steps. From this view point, it is important to investigate the stability of a bunch of steps.

For this problem we assume that there exists only one center at the origin, and $W = B_R(0) \setminus \overline{B_\rho(0)}$, where $B_\rho(a)$ is an open disc whose center is a and radius is ρ. Assume that there exist $m (\geq 1)$ evolving spirals with anti-clockwise rotational orientations. This configuration is described by

$$u_t - |\nabla(u - m\theta_0)| \left\{ \operatorname{div} \frac{\nabla(u - m\theta_0)}{|\nabla(u - m\theta_0)|} + C \right\} = 0 \quad \text{in} \quad (0,T) \times W, \tag{9}$$

$$\langle \vec{\nu}, \nabla(u - m\theta_0) \rangle = 0 \quad \text{on} \quad (0,T) \times \partial W, \tag{10}$$

where $\theta_0(x) = \arg(x)$.

Ogiwara and Nakamura [ON03] obtained a negative result with a phase field model by Kobayashi [Kob10] and a same configuration of the domain. They proved the existence of a solution describing rotating m spirals with $1/m$-times rotational symmetric pattern. In particular, any solutions converges to the above rotating spirals with a rotation if necessary.

However, we obtain the following stability results on a bunch of steps in our formulation.

Theorem 8. *Let u be a solution of (9)–(10) in $(0,\infty)\times\overline{W}$. Assume that there exists $\zeta_0 \in C([\rho,R])$ and $\alpha>0$ such that, for $j=0,1,\ldots,m-1$, there exists $k_j\in\mathbb{Z}$ such that*

$$\{(x,\xi)\in\mathfrak{X};\ u(0,x)-m\xi=2\pi j\}\subset\{(x,\xi)\in\mathfrak{X};\ |\xi-(\zeta_0(|x|)+2\pi k_j)|<\alpha\}. \tag{11}$$

Then, there exists $\zeta\in C([0,\infty)\times[\rho,R])$ such that $w(t,x)=\zeta(t,|x|)$ is a viscosity solution of (9)–(10) with $m=1$ satisfying $w(0,x)=\zeta_0(|x|)$, and

$$\begin{aligned}\{(x,\xi)\in\mathfrak{X};\ u(t,x)-m\xi=2\pi j\}\subset\{(x,\xi)\in\mathfrak{X};\ |\xi-(\zeta(t,|x|)+2\pi k_j)|<\alpha\}\\ \text{for } t>0 \text{ and } j=0,1,\ldots,m-1.\end{aligned} \tag{12}$$

Note that $\Gamma_{t,j}:=\{x\in\overline{W};\ u(t,x)-m\xi=2\pi j\}$ denotes one of continuous spiral curves in Γ_t. Thus (11) means that all curves in Γ_0 is between $\mathcal{C}_0^{\pm\alpha}$ of the form

$$\mathcal{C}_0^{\pm\alpha}:=\{r(\cos(\zeta_0(r)\pm\alpha),\sin(\zeta_0(r)\pm\alpha));\ r\in[\rho,R]\},$$

which is the rotation with the angles $\pm\alpha$ of the curve $\mathcal{C}_0:=\{r(\cos\zeta_0(r),\sin\zeta_0(r));\ r\in[\rho,R]\}$. Consequently, Theorem 8 means that Γ_t cannot escape from the place between $\mathcal{C}_t^{\pm\alpha}$ of the form

$$\mathcal{C}_t^{\pm\alpha}:=\{r(\cos(\zeta(t,r)\pm\alpha),\sin(\zeta(t,r)\pm\alpha));\ r\in[\rho,R]\}$$

for $t>0$, and consequently we obtain the stability in the sense of Lyapunov. Moreover, the curve $\mathcal{C}_t:=\{r(\cos\zeta(t,r),\sin\zeta(t,r));\ r\in[\rho,R]\}$ evolves by $V=C-\kappa$, and thus the bunch of spirals can be regarded as an evolving spiral by the same equation.

The crucial difference between our formulation and a phase field model is the type of equations; our equation is degenerate parabolic, and the phase field model is uniformly parabolic. This implies that all spiral curves evolve with the same equation since $v_j(t,x):=(u(t,x)-2\pi j)/m$ satisfies (9)–(10) with $m=1$, and

$$\Gamma_{t,j}=\{x\in\overline{W};\ v_j(t,x)-\theta_0(x)\equiv 0 \mod 2\pi\mathbb{Z}\}$$

for $j=0,1,\ldots,m-1$.

The existence of ζ is derived from the rotation invariance of (9)–(10). In fact, we observe that

$$\mathcal{C}_0=\{x\in\overline{W};\ \zeta_0(|x|)-\theta_0(x)\equiv 0 \mod 2\pi\mathbb{Z}\}.$$

Let $w(t,x)$ be a viscosity solution of (9)–(10) with $m=1$ and $w(0,x)=\zeta_0(|x|)$. Then we obtain $w(t,x)=w(t,|x|e)(=:\zeta(t,|x|))$ for some $e\in S^1$ because of the uniqueness and $w(0,Rx)=w(0,x)$ for all rotation matrix R. This implies $\mathcal{C}_t$ and also $\mathcal{C}_t^{\pm\alpha}$ are solutions of (2)–(3) in the level set sense. Consequently, Theorem 8 is derived by the comparison of interior and exterior (cf. Theorem 3) between the each curves $\Gamma_{t,j}$ and $\mathcal{C}_t^{\pm\alpha}$.

3.2. Inactive pair

Burton, Cabrera and Frank [BCF51] pointed out that, if a pair of centers with opposite rotational orientations is closer together than the critical distance $2/C$, then this pair has no influence to the evolution of the crystal surface. They call such a pair an inactive pair. We now demonstrate the existence of an inactive pair.

For this problem we assume that $N = 2$, $a_1 = (-\alpha, 0)$, $a_2 = (\alpha, 0)$ with $\alpha \in (0, 1/C)$, $U_i = B_\rho(a_1)$ with $\rho \in (0, \alpha)$, and $\theta(x) = \arg(x - a_1) - \arg(x - a_2)$. Assume that Ω is large enough (its sense is clarified later), and set $W = \Omega \setminus (\overline{U}_1 \cup \overline{U}_2)$.

Note that the circle whose radius is $1/C$ is a stationary solution of $V = C - \kappa$. Thus the curves satisfying the following condition should be a stationary solution of our problem.

(R1) It is a part of the circle whose radius is $1/C$.

(R2) It satisfies the right angle condition between $\partial B_\rho(a_1)$ and $\partial B_\rho(a_2)$.

We now give such curves explicitly. Set

$$p_1(\sigma) = a_1 + \rho(\cos\sigma, \sin\sigma), \quad p_2(\sigma) = a_2 + \rho(-\cos\sigma, \sin\sigma),$$
$$q_1(\sigma) = p_1(\sigma) + \frac{1}{C}(\sin\sigma, -\cos\sigma), \quad q_2(\sigma) = p_2(\sigma) + \frac{1}{C}(-\sin\sigma, -\cos\sigma)$$

Then there exists $\beta > 0$ and σ_1, σ_2 such that $0 < \sigma_1 < \sigma_2 < \pi$ and

$$b_1 = (0, -\beta) = q_1(\sigma_1) = q_2(\sigma_1), \quad b_2 = (0, \beta) = q_1(\sigma_2) = q_2(\sigma_2).$$

We now define

$$R_i = \left\{ r_i(\sigma) = b_i + \frac{1}{C}\left(\cos\left(\frac{\pi}{2} + \sigma\right), \sin\left(\frac{\pi}{2} + \sigma\right)\right) \; ; \; \sigma \in [-\sigma_i, \sigma_i] \right\}$$

for $i = 1, 2$. The sense of the assumption 'Ω is large enough' means that $R_i \subset \Omega \setminus (U_1 \cup U_2)$. Then, R_i is a connecting spiral between a_1 and a_2 satisfying (R1)–(R2), and consequently R_i is our desired curve for $i = 1, 2$. Note that there are two stationary curves in our problems in general.

To demonstrate that R_i is a stationary curve for $i = 1, 2$ we have to find a solution u describing R_i in our level set formulation. However, in usual evolution of a closed curve (i.e., $W = \Omega$ and $\theta \equiv 0$), there are no continuous solutions describing the stationary circle. Accordingly, we find discontinuous a viscosity solution of (5)–(6) describing R_i for $i = 1, 2$.

Theorem 9. *Let R_i be given on above for $i = 1, 2$. Then, $\theta_{R_i} : \overline{W} \to \mathbb{R}$ which is a branch of $\theta(x) = \arg(x - a_1) - \arg(x - a_2)$ whose discontinuity is only on R_i is a viscosity solution of (5)–(6).*

For all $u_0 \in C(\overline{W})$ there exists $k \in \mathbb{Z}$ such that $\theta_{R_i} + 2\pi k \geq u_0$ on $\overline{W}$, which implies that $\theta_{R_i} + 2\pi k \geq u$ on $[0, \infty) \times \overline{W}$ from Theorem 1, where u is a viscosity solution of (5)–(6) with $u|_{t=0} = u_0$. From the above and Theorem 3 the curves R_i plays a role of 'upper bound' for all evolution of spirals in the configuration of an inactive pair.

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