ON WELL-POSEDNESS OF INCOMPRESSIBLE TWO-PHASE FLOWS WITH PHASE TRANSITIONS

SENJO SHIMIZU
SHIZUOKA UNIVERSITY

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1. The model

In this talk, we consider a free boundary problem of incompressible two-phase flows with phase transitions in the framework of L_p-theory with nearly flat interface represented as a graph over $\mathbb{R}^{n-1}$, namely in the regions

$$\Omega_\pm(t) = \{(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : \ x_n \gtrless h(t, x), \ t \geq 0\}$$

with interface

$$\Gamma(t) = \{(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : \ x_n = h(t, x), \ t \geq 0\}.$$

We set $\Omega_0 = \Omega_+(0) \cup \Omega_-(0)$ and ν_0 be the outer normal of $\Omega_-(0)$.

Let u denote the velocity field, π the pressure field, $T(u, \pi, \theta)$ the stress tensor, $D(u) = (\nabla u + [\nabla u]^{\mathsf{T}})/2$ the rate of deformation tensor, θ the (absolute) temperature field, ν_Γ the outer normal of $\Omega_-(t)$, u_Γ the interface velocity, $V_\Gamma = u_\Gamma \cdot \nu_\Gamma$ the normal velocity of $\Gamma(t)$, $H_\Gamma = H(\Gamma(t)) = -\mathrm{div}_\Gamma \nu_\Gamma$ the curvature of $\Gamma(t)$, j the phase flux, and

$$[\![v]\!] = \big(v|_{\Omega_+(t)} - v|_{\Omega_-(t)}\big)\big|_{\Gamma(t)}$$

the jump of a quantity v across $\Gamma(t)$.

Let $\rho_\pm > 0$ denote the densities of $\Omega_\pm(t)$. In order to economize our notation, we set

$$\rho = \rho_+ \chi_{\Omega_+(t)} + \rho_- \chi_{\Omega_-(t)},$$

where χ_D denotes the indicator function of a set D, and this notation is employed for μ, κ, d, etc. as well. We just keep in mind that the coefficients depend on the phases.

By an *Incompressible Two-Phase Flow with Phase Transition* we mean the following problem: Find a family of closed hypersurfaces $\{\Gamma(t)\}_{t \geq 0}$ and appropriately

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smooth functions $u : \mathbb{R}_+ \times \bar{\Omega} \to \mathbb{R}^n$, and $\pi, \theta : \mathbb{R}_+ \times \bar{\Omega} \to \mathbb{R}$ satisfy (1.1) - (1.3):

$$\left.\begin{aligned} \rho(\partial_t u + u\cdot\nabla u) - \operatorname{div} T(u,\pi,\theta) &= 0 && \text{in } \Omega(t),\ t>0,\\ T(u,\pi,\theta) = 2\mu(\theta)D(u) - \pi I, \quad \operatorname{div} u &= 0 && \text{in } \Omega(t),\ t>0,\\ [\![\frac{1}{\rho}]\!] j^2 \nu_\Gamma - [\![T(u,\pi,\theta)\nu_\Gamma]\!] - \sigma H_\Gamma \nu_\Gamma &= 0 && \text{on } \Gamma(t),\ t>0,\\ [\![u]\!] - [\![\frac{1}{\rho}]\!] j\nu_\Gamma &= 0 && \text{on } \Gamma(t),\ t>0,\\ u(0) &= u_0 && \text{in } \Omega_0, \end{aligned}\right\} \tag{1.1}$$

$$\left.\begin{aligned} \rho\kappa(\theta)(\partial_t\theta + u\cdot\nabla\theta) - \operatorname{div}(d(\theta)\nabla\theta) - 2\mu(\theta)|D(u)|_2^2 &= 0 && \text{in } \Omega(t),\ t>0,\\ l(\theta)j + [\![d(\theta)\partial_{\nu_\Gamma}\theta]\!] &= 0 && \text{on } \Gamma(t),\ t>0,\\ [\![\theta]\!] &= 0 && \text{on } \Gamma(t),\ t>0,\\ \theta(0) &= \theta_0 && \text{in } \Omega_0, \end{aligned}\right\} \tag{1.2}$$

$$\left.\begin{aligned} [\![\psi(\theta)]\!] + [\![\frac{1}{2\rho^2}]\!] j^2 - [\![\frac{T(u,\pi,\theta)\nu_\Gamma\cdot\nu_\Gamma}{\rho}]\!] &= 0 && \text{on } \Gamma(t),\ t>0,\\ V_\Gamma - u\cdot\nu_\Gamma + \frac{1}{\rho} j &= 0 && \text{on } \Gamma(t),\ t>0,\\ \Gamma(0) &= \{x\in\mathbb{R}^n \mid x_n = h_0(x)\}. \end{aligned}\right\} \tag{1.3}$$

Several quantities are derived from the specific *free energy* $\psi_\pm(\theta)$ in $\Omega_\pm(t)$ as follows.

- $\epsilon_\pm(\theta) := \psi_\pm(\theta) + \theta\eta_\pm(\theta)$ the internal energy,
- $\eta_\pm(\theta) := -\psi'_\pm(\theta)$ the entropy,
- $\kappa_\pm(\theta) := \epsilon'_\pm(\theta) = -\theta\psi''_\pm(\theta) > 0$ the heat capacity,
- $l(\theta) := \theta[\![\psi'(\theta)]\!] = -\theta[\![\eta(\theta)]\!]$ the latent heat.

Further $d_\pm(\theta) > 0$ denotes the coefficient of heat conduction in Fourier's law, $\mu_\pm(\theta) > 0$ the viscosity in Newton's law, and $\sigma > 0$ the constant coefficient of surface tension.

Concerning the second equation of (1.3), we remind that balance of mass across $\Gamma(t)$ requires $[\![\rho(u - u_\Gamma)]\!]\cdot\nu_\Gamma = 0$, which implies

$$j = \rho_+(u_+ - u_\Gamma)\cdot\nu_\Gamma = \rho_-(u_- - u_\Gamma)\cdot\nu_\Gamma,$$

and so

$$u_+\cdot\nu_\Gamma - \frac{1}{\rho_+}j = u_-\cdot\nu_\Gamma - \frac{1}{\rho_-}j.$$

Therefore this equation is well-defined on $\Gamma(t)$.

This model is derived from balance of mass, balance of momentum, balance of energy under the assumption of no entropy production on the interface and of constitutive laws, which is explained in more detail in [12]. It has been recently proposed by Anderson et al. [1], see also the monographs by Ishii [7] and Ishii and Takashi [8], and it is thermodynamically consistent in the sense that in absence of exterior forces and heat sources, the total energy is preserved and the total entropy is nondecreasing, see [12]. It is in some sense the simplest sharp interface model for incompressible Newtonian two-phase flows taking into account phase transitions driven by temperature.

Note that in the case of equal densities, since $[\![\rho]\!] = [\![1/\rho]\!] = 0$, the phase flux j does not enter (1.1). So in this case we obtain essentially a Stefan problem with surface tension, which is only weakly coupled to the standard two-phase Navier-Stokes problem via temperature dependent viscosities. We call this case *temperature dominated*. But in the case of different densities, the phase flux j causes a jump in the velocity field on the interface, which leads to so called Stefan currents which are convections driven by phase transitions. In this situation it turns out that the heat problem (1.2) is only weakly coupled to (1.1) and (1.3), we call this case *velocity dominated*. The resulting two-phase Navier-Stokes problem is non-standard, therefore it requires a new analysis.

The analytical properties of the problem appear to be different in these two cases. The spaces for well-posedness are not the same. In the temperature dominated case the phase flux j can be eliminated by solving the second equation in (1.2) for j. This yields

$$j = -[\![d(\theta)\partial_\nu\theta]\!]/l(\theta),$$

as long as $l(\theta) \neq 0$; this is the essential well-posedness condition in this case. Then the equation describing the evolution of the interface becomes

$$V_\Gamma = u_\Gamma \cdot \nu_\Gamma + [\![d(\theta)\partial_\nu\theta]\!]/\rho l(\theta).$$

On the other hand, in the velocity determined case $[\![\rho]\!] \neq 0$ we can eliminate j by taking the inner product of the fourth equation in (1.1) with ν_Γ to the result

$$j = [\![u \cdot \nu_\Gamma]\!]/[\![1/\rho]\!].$$

In this case the equation for V_Γ becomes

$$V_\Gamma = [\![\rho u \cdot \nu_\Gamma]\!]/[\![\rho]\!],$$

which does not contain temperature, in contrast to the first case. Therefore the analysis for these two cases necessarily is different, too.

There is a large literature on isothermal incompressible Newtonian two-phase flows without phase transitions, and also on the two-phase Stefan problem with surface tension modeling temperature driven phase transitions. On the other hand, mathematical work on two-phase flow problems including phase transitions are rare. In this direction, we only know the papers by Hoffmann and Starovoitov [5, 6] dealing with a simplified two-phase flow model, and Kusaka and Tani [10, 11] which is two-phase for temperature but only one phase is moving. The papers of Di Benedetto and Friedman [2] and Di Benedetto and O'Leary[3] deal with weak solutions of conduction-convection problems with phase change. However, none of these papers considers models which are consistent with thermodynamics.

2. Results

2.1. **The case of equal densities.** In this case, the equilibrium state is

$$u_\infty = 0,\ \theta_\infty = \text{const.},\ \pi_\infty = \text{const.},\ [\![\pi_\infty]\!] = 0,\ j = 0,$$
$$[\![\psi(\theta_\infty)]\!] = 0,\ \Gamma_\infty = \{x \in \mathbb{R}^n \mid x_n = 0\}.$$

The main result in the case of equal densities is the local well-posedness of (1.1)-(1.3) close to the equilibrium state.

Theorem 2.1. *Let $p > n+2$, $\sigma > 0$, $\rho_+ = \rho_- > 0$ be constants, and suppose $\psi_\pm \in C^3(0,\infty)$, $\mu_\pm$, $d_\pm \in C^2(0,\infty)$ such that*

$$\kappa_\pm(s) = -s\psi_\pm''(s) > 0, \quad \mu_\pm(s) > 0, \quad d_\pm(s) > 0, \quad s \in (0,\infty).$$

Let the initial interface Γ_0 be given by a graph $x \mapsto (x, h_0(x))$ and let θ_∞ be the constant temperature at infinity.

Then given any finite interval $J = [0,T]$, there exists $\eta > 0$ such that (1.1)-(1.3) *admits a unique L_p-solution on J provided the smallness conditions:*

$$\|u_0\|_{W_p^{2-2/p}(\Omega_0)} + \|\theta_0 - \theta_\infty\|_{W_p^{2-2/p}(\Omega_0)} + \|h_0\|_{W_p^{4-3/p}(\mathbb{R}^{n-1})} \le \eta,$$

the compatibility conditions[1]*:*

$$\begin{aligned} \operatorname{div} u_0 = 0 \quad &\text{in} \quad \Omega_0,\\ [\![u_0]\!] = [\![\theta_0]\!] = 0, \quad P_{\Gamma_0}[\![\mu(\theta_0)D(u_0)\nu_0]\!] = 0 \quad &\text{on} \quad \Gamma_0,\\ [\![\psi(\theta_0)]\!] + \sigma H_{\Gamma_0} = 0 \quad &\text{on} \quad \Gamma_0,\\ [\![d(\theta_0)\partial_{\nu_0}\theta_0]\!] \in W_p^{2-6/p}(\Gamma_0),& \end{aligned}$$

and the well-posedness conditions:

$$l(\theta_\infty) \ne 0 \ \text{ on } \ \Gamma_0 \quad \text{and} \quad \theta_\infty > 0 \ \text{ on } \ \mathbb{R}^n$$

are satisfied.

For a proof of this result we show maximal regularity for the linear part of the problem and finally employ the contraction mapping principle to solve the nonlinear problem.

We set $\dot{\mathbb{R}}^n = \mathbb{R}^n_+ \cup \mathbb{R}^n_-$ and $\nu = e_n = (0,\dots,0,1)^\mathsf{T}$. The principal part of the linearization of (1.1)-(1.3) reads as follows.

$$\left.\begin{aligned} \rho\partial_t u - \mu_\infty \Delta u + \nabla\pi = f_u \quad &\text{in} \quad \dot{\mathbb{R}}^n, \quad t > 0,\\ \operatorname{div} u = f_d \quad &\text{in} \quad \dot{\mathbb{R}}^n, \quad t > 0,\\ -2[\![\mu_\infty D(u)\nu]\!] + [\![\pi]\!]\nu - \sigma(\Delta_{x'}h)\nu = g_u \quad &\text{on} \quad \mathbb{R}^{n-1},\ t > 0,\\ [\![u]\!] = 0 \quad &\text{on} \quad \mathbb{R}^{n-1},\ t > 0\\ u(0) = u_0 \quad &\text{in} \quad \dot{\mathbb{R}}^n, \end{aligned}\right\} \tag{2.1}$$

[1] $P_{\Gamma_0} = I - \nu_0 \otimes \nu_0$ denotes the projection onto the tangent bundle of Γ_0.

$$\left.\begin{aligned} \rho\kappa_\infty\partial_t\theta - d_\infty\Delta\theta &= f_\theta && \text{in} && \dot{\mathbb{R}}^n, && t>0,\\ [\![\theta]\!] &= 0 && \text{on} && \mathbb{R}^{n-1}, && t>0,\\ \theta(0) &= \theta_0 && \text{in} && \dot{\mathbb{R}}^n, && \end{aligned}\right\} \tag{2.2}$$

$$\left.\begin{aligned} (l_\infty/\theta_\infty)\theta + (\sigma/\rho)\Delta_{x'}h &= g_\theta && \text{on} && \mathbb{R}^{n-1}, && t>0\\ \partial_t h - [\![d_\infty\partial_n\theta]\!]/\rho l_\infty &= g_h && \text{on} && \mathbb{R}^{n-1}, && t>0,\\ h(0) &= h_0 && \text{on} && \mathbb{R}^{n-1}. && \end{aligned}\right\} \tag{2.3}$$

Here $\mu_{\infty\pm} = \mu_\pm(\theta_\infty)$, $d_{\infty\pm} = d_\pm(\theta_\infty)$, $\kappa_{\infty\pm} = \kappa_\pm(\theta_\infty)$, $l_\infty = l(\theta_\infty)$ are constants. Observe that the term $u\cdot\nu_\Gamma$ in the equation for h is of lower order as it enjoys more regularity than the trace of θ on $\mathbb{R}^{n-1}$. Since u does neither appear in (2.2) nor in (2.3), (2.1) decouples from the remaining problem. (2.1) is the two-phase Stokes problem with surface tension which was studied by Prüss and Simonett [14, 15] and Shibata and Shimizu [17]. The latter system comprises the linearized Stefan problem with surface tension which has been studied by Prüss, Simonett and Zacher [16]. Therefore the linearized problem (2.1)-(2.3) has the property of maximal L_p-regularity.

Before stating maximal regularity results of linear problems, let us introduce the relevant function spaces. Let $\Omega\subset\mathbb{R}^m$ be open and X be an arbitrary Banach space.By $L_p(\Omega;X)$ and $H^s_p(\Omega;X)$, for $1\le p\le\infty$, $s\in\mathbb{R}$, we denote the X-valued Lebesgue and the X-valued Bessel potential spaces of order s, respectively. We will also make use of the fractional Sobolev-Slobodeckij spaces $W^s_p(\Omega;X)$, $1\le p<\infty$, $s>0, s\notin\mathbb{N}$ with norm

$$\|g\|_{W^s_p(\Omega;X)} = \|g\|_{W^{[s]}_p(\Omega;X)} + \sum_{|\alpha|=[s]}\left(\int_\Omega\int_\Omega \frac{\|\partial^\alpha g(x)-\partial^\alpha g(y)\|^p_X}{|x-y|^{m+(s-[s])p}}\,dxdy\right)^{1/p},$$

where $[s]$ denotes the largest integer smaller than s. We remind that $H^k_p = W^k_p$ for $k\in\mathbb{N}$ and $1<p<\infty$, and that $W^s_p = B^s_{pp}$ for $s>0, s\notin\mathbb{N}$.

For $s\in\mathbb{R}$ and $1<p<\infty$, $\dot{H}^s_p(\mathbb{R}^n)$ denotes the homogeneous Bessel-potentianl spaces. For $s\in\mathbb{R}\setminus\mathbb{Z}$, the homogeneous Sobolev-Slobodeckij spaces $\dot{W}^s_p(\mathbb{R}^n)$ of fractional order can be obtained by real interpolation as

$$\dot{W}^s_p(\mathbb{R}^n) := (\dot{H}^k_p(\mathbb{R}^n), \dot{H}^{k+1}_p(\mathbb{R}^n))_{s-k,p}, \quad k<s<k+1,$$

where $(\cdot,\cdot)_{\theta,p}$ is the real interpolation functor.

To state the result we introduce appropriate function spaces. We set

$$\begin{aligned} \mathbb{E}_u(J) &= H^1_p(J;L_p(\mathbb{R}^n))^n \cap L_p(J;H^2_p(\dot{\mathbb{R}}^n))^n\\ &\quad\cap\{[\![u_n]\!]\in H^1_p(J;\dot{W}^{-1/p}_p(\mathbb{R}^{n-1}))\}\cap\{[\![u]\!]=0\},\\ \mathbb{E}_\pi(J) &= L_p(J;\dot{H}^1_p(\dot{\mathbb{R}}^n)),\\ \mathbb{E}_{\gamma\pi}(J) &= W^{1/2-1/2p}_p(J;L_p(\mathbb{R}^{n-1}))^2\cap L_p(J;W^{1-1/p}_p(\mathbb{R}^{n-1}))^2,\\ \mathbb{E}_\theta(J) &= H^1_p(J;L_p(\mathbb{R}^n))\cap L_p(J;H^2_p(\dot{\mathbb{R}}^n))\cap\{[\![\theta]\!]=0\}, \end{aligned}$$

$$\mathbb{E}_h(J) = W_p^{3/2-1/2p}(J; L_p(\mathbb{R}^{n-1})) \cap W_p^{1-1/2p}(J; H_p^2(\mathbb{R}^{n-1})) \\ \cap L_p(J; W_p^{4-1/p}(\mathbb{R}^{n-1})),$$

and define the solution space for (2.1)-(2.3) as

$$\mathbb{E}(J) = \mathbb{E}_u(J) \times \mathbb{E}_\pi(J) \times \mathbb{E}_{\gamma\pi}(J) \times \mathbb{E}_\theta(J) \times \mathbb{E}_h(J).$$

We denote by $\gamma\pi$ the two one-sided traces of π on $\mathbb{R}^{n-1}$. $\mathbb{E}(J)$ is a Banach space with its natural norm, and the generic elements of $\mathbb{E}(J)$ are functions $(u, \pi, \gamma\pi, \theta, h)$. Moreover we set

$$\begin{aligned}
&\mathbb{F}_u(J) := L_p(J; L_p(\mathbb{R}^n))^n,\\
&\mathbb{F}_d(J) := H_p^1(J; \dot{H}_p^{-1}(\mathbb{R}^n)) \cap L_p(J; H_p^1(\dot{\mathbb{R}}^n)),\\
&\mathbb{G}_u(J) := W_p^{1/2-1/2p}(J; L_p(\mathbb{R}^{n-1})^n \cap L_p(J; W_p^{1-1/p}(\mathbb{R}^{n-1}))^n,\\
&\mathbb{F}_\theta(J) := L_p(J; L_p(\mathbb{R}^n)),\\
&\mathbb{G}_\theta(J) := W_p^{1-1/2p}(J; L_p(\mathbb{R}^{n-1})) \cap L_p(J; W_p^{2-1/p}(\mathbb{R}^{n-1})),\\
&\mathbb{G}_h(J) := W_p^{1/2-1/2p}(J; L_p(\mathbb{R}^{n-1})) \cap L_p(J; W_p^{1-1/p}(\mathbb{R}^{n-1})),
\end{aligned}$$

and define the regularity of the data space for (2.4)-(2.6) as

$$\mathbb{F}(J) := \mathbb{F}_u(J) \times \mathbb{F}_d(J) \times \mathbb{G}_u(J) \times \mathbb{F}_\theta(J) \times \mathbb{G}_\theta(J) \times \mathbb{G}_h(J).$$

$\mathbb{F}(J)$ is a Banach space with its natural norm, and the generic elements of $\mathbb{F}(J)$ are functions $(f_u, f_d, g_u, f_\theta, g_\theta, g_h)$. Finally, we define the time trace space X_γ of $\mathbb{E}(J)$ as

$$X_\gamma := W_p^{2-2/p}(\dot{\mathbb{R}}^n)^n \times W_p^{2-2/p}(\dot{\mathbb{R}}^n) \times W_p^{4-3/p}(\mathbb{R}^{n-1}).$$

Then the main result on the linearized problem (2.1)-(2.3) is stated as

Theorem 2.2. *Let $1 < p < \infty$, $p \neq 3/2, 3$ and assume that $\sigma > 0$, $\rho_+ = \rho_- > 0$, $\mu_{\infty\pm}$, $d_{\infty\pm}$ $\kappa_{\infty\pm} > 0$. Then the linear problem (2.1)-(2.3) admits a unique solution $(u, \pi, \gamma\pi, \theta, h) \in \mathbb{E}(J)$ if and only if the data (u_0, θ_0, h_0) and $(f_u, f_d, g_u, f_\theta, g_\theta, g_h)$ satisfy the regularity conditions:*

$$(u_0, \theta_0, h_0) \in X_\gamma, \quad (f_u, f_d, g_u, f_\theta, g_\theta, g_h) \in \mathbb{F}(J),$$

and the compatibility conditions:

$$\begin{aligned}
\operatorname{div} u_0 &= f_d(0) && \text{in} \quad \dot{\mathbb{R}}^n,\\
[\![u_0]\!] = 0, \quad -P_{\mathbb{R}^{n-1}}[\![\mu_\infty(\nabla u_0 + [\nabla u_0]^{\mathsf{T}})\nu]\!] &= P_{\mathbb{R}^{n-1}} g_u(0), && \text{on} \quad \mathbb{R}^{n-1},\\
[\![\theta_0]\!] = 0, \quad (l_\infty/\theta_\infty)\theta_0 + \sigma\Delta_{x'} h_0 &= g_\theta(0) && \text{on} \quad \mathbb{R}^{n-1}\\
g_h(0) + [\![d_\infty \partial_n \theta_0]\!]/\rho l_\infty &\in W_p^{2-6/p}(\mathbb{R}^{n-1}),
\end{aligned}$$

and the well-posedness conditions:

$$l_\infty \neq 0 \quad \text{on } \mathbb{R}^{n-1} \quad \text{and} \quad \theta_\infty > 0 \quad \text{on } \mathbb{R}^n.$$

The solution map $[(u_0, \theta_0, h_0, f_u, f_d, g_u, f_\theta, g_\theta, g_h) \mapsto (u, \pi, \gamma\pi, \theta, h)]$ is continuous between the corresponding spaces.

2.2. **The case of non-equal densities.** The equilibrium state is the same as the equal density case except

$$[\![\psi(\theta_\infty)]\!] + [\![\pi_\infty/\rho]\!] = 0$$

replaces $[\![\psi(\theta_\infty)]\!] = 0$.

The main result in the case of non-equal densities is the local well-posedness of (1.1)-(1.3) close to the equilibrium state.

Theorem 2.3. *Let $p > n+2$, $\rho_+, \rho_-, \sigma > 0$ be constant, $\rho_+ \neq \rho_-$, and suppose $\psi_\pm \in C^3(0,\infty)$, $\mu_\pm$, $d_\pm \in C^2(0,\infty)$ are such that*

$$\kappa_\pm(s) = -s\psi_\pm''(s) > 0, \quad \mu_\pm(s) > 0, \quad d_\pm(s) > 0 \quad s \in (0,\infty).$$

Let the initial interface Γ_0 be given by a graph $x \mapsto (x', h_0(x'))$, and let $\theta_\infty > 0$ be the constant temperature at infinity.

Then given any finite interval $J = [0,T]$, there exists $\eta > 0$ such that (1.1)-(1.3) *admits a unique L_p-solution on J provided the smallness conditions:*

$$\|u_0\|_{W_p^{2-2/p}(\Omega_0)} + \|\theta_0 - \theta_\infty\|_{W_p^{2-2/p}(\Omega_0)} + \|h_0\|_{W_p^{3-2/p}(\mathbb{R}^{n-1})} \le \eta,$$

and the compatibility conditions:

$$\begin{aligned} \operatorname{div} u_0 = 0 \quad &\text{in } \Omega_0,\\ P_{\Gamma_0}[\![\mu(\theta_0)D(u_0)\nu_0]\!] = 0, \quad P_{\Gamma_0}[\![u_0]\!] = 0 \quad &\text{on } \Gamma_0,\\ [\![\theta_0]\!] = 0, \quad (l(\theta_0)/[\![1/\rho]\!])[\![u_0\cdot\nu_0]\!] + [\![d(\theta_0)\partial_{\nu_0}\theta_0]\!] = 0 \quad &\text{on } \Gamma_0, \end{aligned}$$

are satisfied.

For a proof of this result we show maximal regularity for the linear part of the problem and finally employ the contraction mapping principle to solve the nonlinear problem.

The principal part of the linearized problem in the case of a nearly flat initial interface reads as follows

$$\left.\begin{aligned} \rho\partial_t u - \mu_\infty\Delta u + \nabla\pi = f_u \quad &\text{in } \dot{\mathbb{R}}^n, \quad t>0,\\ \operatorname{div} u = f_d \quad &\text{in } \dot{\mathbb{R}}^n, \quad t>0,\\ -2[\![\mu_\infty D(u)\nu]\!] + [\![\pi]\!]\nu - \sigma(\Delta_x h)\nu = g_u \quad &\text{on } \mathbb{R}^{n-1}, \ t>0,\\ [\![u']\!] = g_j \quad &\text{on } \mathbb{R}^{n-1}, \ t>0,\\ u(0) = u_0 \quad &\text{in } \dot{\mathbb{R}}^n, \end{aligned}\right\} \tag{2.4}$$

$$\left.\begin{aligned} \rho\kappa_\infty\partial_t\theta - d_\infty\Delta\theta = f_\theta \quad &\text{in } \dot{\mathbb{R}}^n, \quad t>0,\\ -[\![d_\infty\partial_n\theta]\!] = g_\theta \quad &\text{on } \mathbb{R}^{n-1}, \ t>0,\\ [\![\theta]\!] = 0 \quad &\text{on } \mathbb{R}^{n-1}, \ t>0,\\ \theta(0) = \theta_0 \quad &\text{in } \dot{\mathbb{R}}^n, \end{aligned}\right\} \tag{2.5}$$

$$\left.\begin{aligned} -2[\![\frac{\mu_\infty D(u)\nu\cdot\nu}{\rho}]\!] + [\![\frac{\pi}{\rho}]\!] &= g_\pi && \text{on } \mathbb{R}^{n-1}, \quad t>0,\\ \partial_t h - [\![\rho u_n]\!]/[\![\rho]\!] &= g_h && \text{on } \mathbb{R}^{n-1}, \quad t>0,\\ h(0) &= h_0 && \text{on } \mathbb{R}^{n-1}, \end{aligned}\right\} \tag{2.6}$$

where μ_∞, d_∞, κ_∞, are positive constants and $\nu = e_n$. We assume in this subsection $[\![\rho]\!] = \rho_+ - \rho_- \neq 0$. Apparently, (2.5) decouples from the remaining problem and it is well-known that this problem has maximal L_p-regularity (cf. Escher, Prüss and Simonett [4]), we concentrate on the remaining one. The resulting two-phase Navier-Stokes problem is non-standard, it requires a new analysis. Therefore the analysis of the coupled system (2.4) and (2.6) is the most important part through this work.

To state the result we introduce appropriate function spaces. We set

$$\begin{aligned}
\mathbb{E}_u(J) &= H^1_p(J; L_p(\mathbb{R}^n))^n \cap L_p(J; H^2_p(\dot{\mathbb{R}}^n))^n \cap \{[\![u_n]\!] \in H^1_p(J; \dot{W}^{-1/p}_p(\mathbb{R}^{n-1}))\},\\
\mathbb{E}_\pi(J) &= L_p(J; \dot{H}^1_p(\dot{\mathbb{R}}^n)),\\
\mathbb{E}_{\gamma\pi}(J) &= W^{1/2-1/2p}_p(J; L_p(\mathbb{R}^{n-1}))^2 \cap L_p(J; W^{1-1/p}_p(\mathbb{R}^{n-1}))^2,\\
\mathbb{E}_\theta(J) &= H_p(J; L_p(\mathbb{R}^n) \cap L_p(J; H^2_p(\dot{\mathbb{R}}^n)) \cap \{[\![\theta]\!] = 0\},\\
\mathbb{E}_h(J) &= W^{2-1/2p}_p(J; L_p(\mathbb{R}^{n-1})) \cap H^1_p(J; W^{2-1/p}_p(\mathbb{R}^{n-1}))\\
&\qquad \cap L_p(J; W^{3-1/p}_p(\mathbb{R}^{n-1})),
\end{aligned}$$

and define the solution space for (2.4)-(2.6) as

$$\mathbb{E}(J) := \mathbb{E}_u(J) \times \mathbb{E}_\pi(J) \times \mathbb{E}_{\gamma\pi}(J) \times \mathbb{E}_\theta(J) \times \mathbb{E}_h(J).$$

We denote by $\gamma\pi$ the two one-sided traces of π on $\mathbb{R}^{n-1}$. $\mathbb{E}(J)$ is a Banach space with its natural norm, and the generic elements of $\mathbb{E}(J)$ are functions $(u, \pi, \gamma\pi, \theta, h)$. Moreover we set

$$\begin{aligned}
\mathbb{F}_u(J) &:= L_p(J; L_p(\mathbb{R}^n))^n,\\
\mathbb{F}_d(J) &:= H^1_p(J; \dot{H}^{-1}_p(\mathbb{R}^n)) \cap L_p(J; H^1_p(\dot{\mathbb{R}}^n)),\\
\mathbb{G}_u(J) &:= W^{1/2-1/2p}_p(J; L_p(\mathbb{R}^{n-1})^n \cap L_p(J; W^{1-1/p}_p(\mathbb{R}^{n-1}))^n,\\
\mathbb{G}_j(J) &:= W^{1-1/2p}_p(J; L_p(\mathbb{R}^{n-1}))^{n-1} \cap L_p(J; W^{2-1/p}_p(\mathbb{R}^{n-1}))^{n-1},\\
\mathbb{F}_\theta(J) &:= L_p(J; L_p(\mathbb{R}^n)),\\
\mathbb{G}_\theta(J) &:= W^{1/2-1/2p}_p(J; L_p(\mathbb{R}^{n-1})) \cap L_p(J; W^{1-1/p}_p(\mathbb{R}^{n-1})),\\
\mathbb{G}_\pi(J) &:= \mathbb{G}_\theta(J),\\
\mathbb{G}_h(J) &:= W^{1-1/2p}_p(J; L_p(\mathbb{R}^{n-1})) \cap L_p(J; W^{2-1/p}_p(\mathbb{R}^{n-1})),
\end{aligned}$$

and define the regularity of the data space for (2.4)-(2.6) as

$$\mathbb{F}(J) := \mathbb{F}_u(J) \times \mathbb{F}_d(J) \times \mathbb{G}_u(J) \times \mathbb{G}_j(J) \times \mathbb{F}_\theta(J) \times \mathbb{G}_\theta(J) \times \mathbb{G}_\pi(J) \times \mathbb{G}_h(J).$$

$\mathbb{F}(J)$ is a Banach space with its natural norm, and the generic elements of $\mathbb{F}(J)$ are functions $(f_u, f_d, g_u, g_j, f_\theta, g_\theta, g_\pi, g_h)$. Finally, we define the time trace space

X_γ of $\mathbb{E}(J)$ as

$$X_\gamma := W_p^{2-2/p}(\dot{\mathbb{R}}^n)^n \times W_p^{2-2/p}(\dot{\mathbb{R}}^n) \times W_p^{3-2/p}(\mathbb{R}^{n-1}).$$

We obtain the maximal L_p regularity results of (2.4) and (2.6) as the following way. Let λ and ξ' be dual parameters of the Laplace transform w.r.t. t and of the Fourier transform w.r.t. x', respectively. Solving the coupled system (2.4) and (2.6), we express the second equation of (2.6) by

$$s(\lambda, |\xi'|)\hat{h} = \hat{g}_h.$$

We set $\tau = |\xi'|$. The boundary symbol $s(\lambda, \tau)$ is written by

$$s(\lambda, \tau) = \lambda + \frac{\sigma\tau}{[\![\rho]\!]^2} m(z)$$

with $z = \lambda/\tau^2$, where the holomorphic function $m(z)$ satisfies

$$|m(z)| \le \frac{M}{1+|z|}, \quad z \in \Sigma_\phi \cup B_r(0)$$

for each $\phi \le \pi/2 + \eta$ and some $r > 0$. If λ_0 is chosen large enough, the boundary symbol is estimated as

$$c_\eta(|\lambda| + |\tau|) \le |s(\lambda, \tau)| \le C_\eta(|\lambda| + |\tau|), \quad \lambda \in \Sigma_{\pi/2+\eta},\ \tau \in \Sigma_\eta,\ |\lambda| \ge \lambda_0.$$

By this estimte, the operator-valued $\mathcal{H}^\infty$-calculus allows for an application of the Kalton and Weis theorem [9, Theorem 4.4], which shows $Sh = g_h$ has a unique solution in the right regularity class.

The main result on the linearized problem (2.4)-(2.6) now can be stated as

Theorem 2.4. *Let $1 < p < \infty$ be fixed, $p \ne 3/2, 3$, and assume that $\rho_+ \ne \rho_-$ and $\mu_{\infty\pm}$, $\kappa_{\infty\pm}$, $d_{\infty\pm} > 0$. Then the linear problem* (2.4)-(2.6) *admits a unique solution $(u, \pi, \gamma\pi, \theta, h) \in \mathbb{E}(J)$ if and only if the data (u_0, θ_0, h_0) and $(f_u, f_d, g_u, g_j, f_\theta, g_\theta, g_\pi, g_h)$ satisfy the regularity conditions:*

$$(u_0, \theta_0, h_0) \in X_\gamma, \quad (f_u, f_d, g_u, g_j, f_\theta, g_\theta, g_\pi, g_h) \in \mathbb{F}(J),$$

and the compatibility conditions:

$$\begin{aligned} \operatorname{div} u_0 = f_d(0) \quad &\text{in } \dot{\mathbb{R}}^n, \\ -P_{\mathbb{R}^{n-1}}[\![\mu_\infty(\nabla u_0 + [\nabla u_0]^\mathsf{T})\nu]\!] = P_{\mathbb{R}^{n-1}} g_u(0), \quad [\![u_0']\!] = g_j(0) \quad &\text{on } \mathbb{R}^{n-1}, \\ [\![\theta_0]\!] = 0, \quad -[\![d_\infty \partial_n \theta_0]\!] = g_\theta(0) \quad &\text{on } \mathbb{R}^{n-1}. \end{aligned}$$

The solution map $[(f_u, f_d, g_u, g_j, f_\theta, g_\theta, g_\pi, g_h, u_0, \theta_0, h_0) \mapsto (u, \pi, \gamma\pi, \theta, h)]$ is continuous between the corresponding spaces.

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Department of Mathematics, Shizuoka University, 422-8529 Shizuoka, Japan.
E-mail address: ssshimi@ipc.shizuoka.ac.jp